Arithmetic compactifications of PEL-type Shimura varieties

A dissertation presented
by

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to

The Department of Mathematics

in partial fulfilment of the requirements
for the degree of
Doctor of Philosophy
in the subject of
Mathematics

Harvard University
Cambridge, Massachusetts

May 2008
(Revised March 14, 2021)

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Abstract

In this thesis, we constructed minimal (Satake-Baily-Borel) compactifications and smooth toroidal compactifications of integral models of *general* PEL-type Shimura varieties (defined as in Kottwitz [76]), with descriptions of stratifications and local structures on them extending the well-known ones in the complex analytic theory. This carries out a program initiated by Chai, Faltings, and some other people more than twenty years ago. The approach we have taken is to redo the Faltings-Chai theory [42] in full generality, with as many details as possible, but without any substantial case-by-case study. The essential new ingredient in our approach is the emphasis on *level structures*, leading to a crucial Weil pairing calculation that enables us to avoid unwanted boundary components in naive constructions.

Contents

Acknowledgments

First of all, I would like to thank my advisor Richard Taylor for all his guidance and encouragement during the work on this project. He is very capable in identifying the essential difficulties in an efficient way, and in providing practical and thought-provoking viewpoints leading to solutions of problems. Having the opportunity to see his ways of understanding is really a great experience. Without his influence, it would be extremely difficult for me to approach the elusive points of this theory without getting lost or making fundamental mistakes. I certainly cannot feel more grateful to him.

Many mathematicians have helped and encouraged me in various ways, including (in alphabetical order) Valery Alexeev, Ching-Li Chai, Wen-Chen Chi, Brian Conrad, Ellen Eischen, Benedict Gross, Michael Harris, Ming-Lun Hsieh, Tetsushi Ito, Kazuya Kato, Robert Kottwitz, King Fai Lai, Michael Larsen, Dong Uk Lee, Elena Mantovan, Barry Mazur, Yoichi Mieda, Andrea Miller, James Milne, Sophie Morel, Iku Nakamura, Marc-Hubert Nicole, Martin Olsson, George Pappas, Gopal Prasad, Dinakar Ramakrishnan, Nicholas Shepherd-Barron, Sug Woo Shin, Christopher Skinner, Ki-Seng Tan, Jacques Tilouine, Eric Urban, Adrian Vasiu, Andrew Wiles, Teruyoshi Yoshida, and Chia-Fu Yu. Some of them kindly answered my questions regardless of my ill-formulations; some of them patiently allowed me to explain my work in lengthy details and provided me with useful feedbacks; some of them bravely read parts of my ridiculous drafts and pointed out typos and mistakes; some of them have generously shared their viewpoints on the subject and related topics; some of them asked me ingenious questions that greatly enhanced my understanding; and most of them have given me stimulating comments and invaluable encouragements. I would like to thank all of them heartily.

Several former and present students including me at Harvard have been promoting seminars in an intensive but informal style (the "alcove meetings" or "alcove seminars"). I would like to thank all the participants over the years, including (in alphabetical order) Thomas Barnet-Lamb, Florian Herzig, Jesse Kass, Abhinav Kumar, Chung Pang Mok, Jay Pottharst, Michael Schein, Sug Woo Shin, Alexander Stasinski, Teruyoshi Yoshida, and Jeng-Daw Yu, for their devotedness, their unreservedness, and (practically speaking) their patience. In particular, I would like to thank Sug Woo Shin and Teruyoshi Yoshida, for sharing their working knowledge on Shimura varieties and related topics with passion and precision, and for being willing to receive my passion in return, often without comparable quality. The importance of these inspiring experiences could hardly be overestimated.

During the different stages in the preparation of this work, I have been giving lectures in various formats at Harvard University, National Taiwan University, California Institute of Technology, National Center for Theoretical Sciences (Taipei Office), Kyoto University, Hokkaido University, Academia Sinica, Korea Institute of Advanced Study, Indiana University, Princeton University, University of Michigan, and Michigan State University. I would like to thank all the organizers and the participants, whose names I could regretfully not list exhaustively.

My thesis writing has benefited enormously from the various kinds of supports provided by the Harvard Mathematics Department. Apart from the friendly and helpful people there, the timely arrival of the new printer `lw1` (in the winter of the year 2006), just a few days before I finished the main calculation and was about to start the typesetting, can hardly be more appreciated. I would like to thank everybody in the department for maintaining such a pleasant environment.

Finally, I would like to thank all my friends, teachers, and family members, who have encouraged, tolerated, and supported me in various stages of my life.

謹以此文，
獻給我的父母。

Introduction

Here we give a soft introduction to the background and the status of this work. This is not a summary of the results. To avoid introducing a heavy load of notations and concepts, we shall not attempt to give any precise mathematical statement. Please refer to the main body of the work for more precise information.

Complex Theory

It is classically known, especially since the work of Shimura, that complex abelian varieties with so-called PEL structures (polarizations, endomorphisms, and level structures) can be parameterized by unions of arithmetic quotients of (connected) Hermitian symmetric spaces. Simple examples include modular curves as quotients of the Poincaré upper-half plane, Hilbert modular spaces as quotients of products of the Poincaré upper-half plane, and the Siegel moduli spaces as quotients of Siegel upper-half spaces. (In this introduction, we shall not try to include the historical details of the modular or Hilbert modular cases related to only GL_2.)

Thanks to Baily and Borel [17], each such arithmetic quotient can be given an algebro-geometric structure because it can be embedded as a Zariski open subvariety of a canonically associated complex normal projective variety called the *Satake–Baily–Borel* or *minimal compactification*. Thus the above-mentioned parameter spaces can be viewed as unions of complex quasi-projective varieties. These parameter spaces are called *PEL-type Shimura varieties*. They admit canonical models over number fields, as investigated by Shimura and many others (see, in particular, [33] and [34]). Since abelian varieties (with additional structures) make sense over rings of algebraic integers localized at some precise sets of *good primes*, we obtain *integral models* of these PEL-type Shimura varieties by defining suitable moduli problems of abelian varieties. Moreover, the precise sets of good primes can be chosen so that the moduli problems are representable by smooth schemes with nonempty fibers. (See, for example, [80], [117], and [76].)

Although the minimal compactifications mentioned above are normal and canonical, Igusa [64] and others have discovered that minimal compactifications are in general highly singular. In [16], Mumford and his coworkers constructed a large class of (noncanonical) compactifications in the category of complex algebraic spaces, called *toroidal compactifications*. Within this class, there are plenty of nonsingular compactifications, many among them are projective, hence providing a theory of smooth compactifications for the PEL-type Shimura varieties over the complex numbers. Based on the work of many people since Shimura, it is known that both minimal compactifications and toroidal compactifications admit canonical models over the same number fields over which the Shimura varieties are defined (see Pink's thesis [102], and also [62]).

Integral Theory

In [42], Faltings and Chai studied the theory of degeneration for polarized abelian varieties over complete adic rings satisfying certain reasonable normality conditions, and constructed smooth toroidal compactifications of the integral models of Siegel modular varieties (parameterizing principally polarized abelian schemes over base schemes over which the primes dividing the level are invertible). The key point in their construction is the *gluing process* in the étale topology. Such a process is feasible because there exist local charts over which the sheaves of differentials can be explicitly calculated and compared. The above-mentioned theory of degeneration and the theory of toroidal embeddings over arbitrary bases play a major role in the construction of these local charts.

As a by-product, they obtained the minimal compactifications of the integral models of Siegel modular varieties using the graded algebra generated by automorphic forms of various (parallel) weights, extending the ones over the complex numbers. We would like to remark that, although the local charts for the minimal compactifications can also be written down explicitly, the fact that we do not have a good way to compare the local structures between different local charts makes the gluing process very difficult in practice. This explains the main difference between the complex analytic and the arithmetic geometric stories, and explains why the toroidal compactifications were constructed before the minimal compactifications in the latter case.

These toroidal and minimal compactifications of the integral models of the Siegel modular varieties are the prototypes of our arithmetic compactifications of PEL-type Shimura varieties. It is not surprising that the existence of such integral models are important for arithmetic applications of Shimura varieties.

In Larsen's thesis [81] (see also [82]), he applied the techniques of Faltings and Chai and constructed arithmetic compactifications of integral models of Picard modular varieties, namely, Shimura varieties associated to unitary groups defined by Hermitian pairings of real signature $(2, 1)$ over imaginary quadratic fields. This is the so-called $\mathrm{GU}(2, 1)$-case. In this case, there is a unique toroidal compactification for each Shimura variety one considers. (The same phenomenon occurs whenever each of the $\mathbb{Q}$-simple factors of the adjoint quotient of the corresponding algebraic group has $\mathbb{R}$-rank no greater than one.) His compactification theory has been used in the main results of the Montréal volume [79].

Before moving on, let us mention that there is also the unpublished revision of Fujiwara's master's thesis [46] on the arithmetic compactifications of PEL-type Shimura varieties involving simple components of only types A and C. The main difference between his work and Faltings–Chai's is his ingenious use of rigid-analytic methods in the gluing process. The point is that his gluing method has the potential to be generalized for nonsmooth moduli problems. However, as far as we can understand, there are important steps in his boundary construction (*before* the gluing step) that are not fully justified.

What Is New?

In this work, our goal is to carry out the theory of arithmetic compactifications for smooth integral models of PEL-type Shimura varieties, as defined in Kottwitz's paper [76], with no other special restriction on the Hermitian pairings or the groups involved. Our construction is based on a generalization of Faltings and Chai's in [42]. It is a very close imitation from the perspective of algebraic geometry. Thus our work can be viewed as a long student exercise justifying the claims in [42, pp. 95–96 and 137] that their method works for general PEL-type Shimura varieties.

However, there do exist some differences coming from linear algebra and related issues, as long as the readers believe that the theory of modules and pairings over $\mathbb{Z}$ is simpler than the analogous theory over orders in some (not necessarily commutative) semisimple algebras, and that solving equations like $0 = 0$ is easier than solving any other linear equations (which might not have solutions) in a (not necessarily torsion-free) noncommutative algebra. Let us make this more precise.

The main issue is about the level structures. In the work of Faltings and Chai, the moduli problem for Siegel modular varieties is defined for abelian schemes with *principal* polarizations. In particular, they only have to work with self-dual lattices. Moreover, the additional fact that $\mathbb{Q}$ does not have a nontrivial involution makes the isotropic submodules of the lattice modulo n always *liftable* to characteristic zero. With some care, the first assumption alone can be harmless, as in the earlier work of Rapoport [103], and the second assumption can continue to be assumed in the Hilbert modular cases. But these convenient assumptions simply do not hold in general. A symplectic isomorphism between modules modulo n may not lift to a symplectic isomorphism between the original modules. We need the notion of *symplectic-liftability* to translate the adelic definition of level structures correctly into the language of finite étale group schemes. Accordingly, we need to find the right way to assign degeneration data to level structures, namely symplectic-liftable isomorphisms between finite étale group schemes.

Thus, the main objective of our approach is to formulate certain liftability and pairing conditions on the degeneration data, so that the combination of these two conditions can predict the existence of level structures of a prescribed type on the generic fiber of the corresponding degenerating families. This involves a Weil-pairing calculation that we believe has never been mentioned in the past literature. After this important step, we have to construct boundary charts parameterizing the degeneration data we need. We have to incorporate the additional liftability and pairing conditions that are absent in the work of Faltings and Chai. We would like to point out that naive generalizations of their construction, essentially the only one available in the past literature, lead to unwanted additional components along the boundary. We can think of these additional components as belonging to some different Shimura varieties. The question of avoiding these unwanted components is certainly another complication. Fortunately, our calculation mentioned above suggests that there is a rather elementary and algebro-geometric way to identify and to give meanings to the correct components. Finally, with the correct components, the approximating and gluing steps are exactly the same as in the original work of Faltings and Chai. We shall not pretend that we have any invention in this respect.

We would like to mention that our own approach (with emphasis on conditions on the level structures) emerged from our initial attempts on the case of unitary groups of ranks higher than $\mathrm{GU}(2,1)$. At that time our more naive and rather ad hoc generalization of the method of Faltings and Chai could only handle the cases of unitary groups defined over an imaginary quadratic field *of odd discriminant*. The cases of even discriminants remained problematic for a long while. After some trials in vain, we realized that the essential difficulty is not special for these particular unitary cases. As soon as we have obtained the right approach for the even discriminant cases, it seemed clear to us that it could also work for all other PEL-type cases, without avoiding any particular one. After all, it is important that the strategy we have thus obtained does not require any previous studies on special cases.

Note that there are inevitably some inaccuracies in the main results of [42, Ch. II and III]. As far as we can understand, most of the existing theories of arithmetic compactifications over an integral base scheme depend logically on the theory of degeneration for abelian varieties and on Mumford's construction, both of which have only been sufficiently explained in their full generality in [42, Ch. II and III]. Hence it seems desirable for us to rework through these most fundamental machineries, even if such an effort does not involve any novelty in mathematical ideas. We do not believe that our work can replace or even become partially comparable with the monumental contributions of Mumford, Faltings, and Chai. We do not think there should be any reason to cast any doubt on the importance of their works. But at least we would like to try to make their brilliant ideas more consolidated after they have appeared for so many years. Alongside with other small corrections, modifications, or justifications that we have attempted to offer, we hope that our unoriginal, nonconstructive, and uninspiring effort is not totally redundant even for cases which might be considered *well known*. (We believe it is sensible to justify the existing works before proceeding to more general cases, anyway.)

At this moment, there are people who are working on also cases of nonsmooth integral models of Shimura varieties. Let us explain why we do not consider this further generality in our work. The main reason is not about the theory of degeneration data or the techniques of constructing local boundary charts. It is rather about the definition of the Shimura varieties themselves, and the expectation of the results to have. In some cases, there seem to be more than one reasonable way of stating them. This certainly does not mean that it is impossible to compactify a particular nonsmooth integral model. However, it would probably be more sensible if we know why we compactify it, and if we know why it should be compactified in a particular way. It is not easy to provide systematic answers to these questions when the objects we consider are not smooth. (The best we can hope is probably to answer these questions for integral models of Shimura varieties that are flat and regular, as aimed in [104], [100], [101], and sequels to them.) Since we have a much better understanding of the general smooth cases, we shall be content with treating only them in this work.

Finally, we would like to mention that we are not working along the lines of the *canonical compactifications* constructed by Alexeev and Nakamura [1], or by Olsson [98], because it is not plausible that one could define general Hecke actions on their canonical compactifications. Let us explain the reason as follows. The main component in their compactifications can be related to the toroidal compactifications constructed using some particular choices of cone decompositions. However, the collection of such cone decompositions is not invariant under conjugation by rational elements in the group naturally arising from the Hecke actions. It is possible that their definition could be useful for the construction of minimal compactifications (with Hecke actions), but we believe that the argument will be forced to be indirect. Nevertheless, we would like to emphasize that their compactifications are described by moduli problems allowing deformation-theoretic considerations along the boundary. Hence their compactifications might be more useful for applications to algebraic geometry.

Structure of The Exposition

In Chapter 1, we lay down the foundations and give the definition of the moduli problems we consider. The moduli problems we define parameterize *isomorphism classes* of abelian schemes over integral bases with additional structures of the above-mentioned types, which are equivalent to the moduli problems defining integral models of PEL-type Shimura varieties using *isogeny classes* as in [76]. Therefore, as already explained in [76], the complex fibers of these moduli problems contain the (complex) Shimura varieties associated to the reductive groups mentioned above.

In Chapter 2, we elaborate on the representability of the moduli problems defined in Chapter 1. Our treatment is biased towards the prorepresentability of local moduli and Artin's criterion of algebraic stacks. We do not need geometric invariant theory or the theory of Barsotti–Tate groups. The argument is very elementary and might be considered outdated by the experts in this area. (Indeed, it may not be enough for the study of bad reductions.) Although readers might want to skip this chapter as they might be willing to believe the representability of moduli problems, there are still some reasons to include this chapter. For example, the Kodaira–Spencer morphisms of abelian schemes with PEL structures are of fundamental importance in our argument for the gluing of boundary charts (in Chapter 6), and they are best understood via the study of deformation theory. Furthermore, the proof of the formal smoothness of local moduli functors illustrates how the linear algebraic assumptions are used. Some of the linear algebraic facts are used again in the construction of boundary components, and it is an interesting question whether one can propose a satisfactory intuitive explanation of this coincidence.

In Chapter 3, we explain well-known notions important for the study of semi-abelian schemes, such as groups of multiplicative type and torsors under them, biextensions, cubical structures, semi-abelian schemes, Raynaud extensions, and certain *dual objects* for the last two notions extending the notion of dual abelian varieties. Our main references for these are [40], [57], and in particular [93].

In Chapter 4, we reproduce the theory of degeneration data for polarized abelian varieties, as elaborated in the first three chapters of [42]. In the main theorems (of Faltings and Chai) that we present, we have made some modifications to the statements according to our own understanding of the proofs. Notably, we have provided weakened statements in the main definitions and theorems, because we do not need their original stronger versions in [42] for our main result. Examples of this sort include, in particular, Definitions 4.2.1.1 and 4.5.1.2, Theorems 4.2.1.14 and 4.4.16, and Remarks 4.2.1.2, 4.2.1.16, and 4.5.1.4.

In Chapter 5, we supply a theory of degeneration data for endomorphism structures, Lie algebra conditions, and level structures, based on the theory of degeneration in Chapter 4. People often claim that the degeneration theory for general PEL-type structures is just a straightforward consequence of the functoriality of the merely polarized case. However, the Weil-pairing calculation carried out in this chapter may suggest that it is not necessarily the case. As far as we can see, functoriality does not seem to imply properties about pairings in an explicit way. There are conceptual details to be understood beyond simple implications of functoriality. However, we are able to present in this chapter a theory of degeneration data for abelian varieties with PEL structures, together with the notion of cusp labels.

In Chapter 6, we explain the algebraic construction of toroidal compactifications. For this purpose we need one more basic tool, namely, the theory of toroidal embeddings for torsors under groups of multiplicative type. Based on this theory, we begin the general construction of local charts on which degeneration data for PEL structures are tautologically associated. The key ingredient in these constructions is the construction of the tautological PEL structures, including particularly the level structures. The construction depends heavily on the way we classify the degeneration data and cusp labels developed in Chapter 5. As explained above, there are complications that are not seen in special cases such as Faltings and Chai's work. The next important step is the description of good formal models, and good algebraic models approximating them. The correct formulation of necessary properties and the actual construction of these good algebraic models are the key to the gluing process in the étale topology. In particular, this includes the comparison of local structures using the Kodaira–Spencer morphisms mentioned above. As a result of gluing, we obtain the arithmetic toroidal compactifications in the category of algebraic stacks. The chapter is concluded by a study of Hecke actions on towers of arithmetic toroidal compactifications.

In Chapter 7, we first study the automorphic forms that are defined as global sections of certain invertible sheaves on the toroidal compactifications. The local structures of toroidal compactifications lead naturally to the theory of Fourier–Jacobi expansions and the Fourier–Jacobi expansion principle. As in the case of Siegel modular schemes, we obtain also the algebraic construction of arithmetic minimal compactifications, which are normal schemes defined over the same integral bases as the moduli problems are. As a by-product of codimension counting, we obtain Koecher's principle for arithmetic automorphic forms (of naive parallel weights). Furthermore, following the generalization in [25, Ch. IV] and [42, Ch. V, §5] of Tai's result in [16, Ch. IV, §2] to Siegel moduli schemes in mixed characteristics, we can show the projectivity of a large class of arithmetic toroidal compactifications by realizing them as normalizations of blowups of the corresponding minimal compactifications. The results in this chapter parallel part of those in [42, Ch. V].

For the convenience of the readers, we have included two appendices containing basic information about algebraic stacks and Artin's criterion for them. There is also an index of notations and terminologies at the end of the document. We hope they will be useful for the readers.

Our overall treatment might seem unreasonably lengthy, and some of the details might have made the arguments more clumsy than they should be. Even so, we have tried to provide sufficient information, so that readers should have no trouble correcting any of the foolish mistakes, or improving any of the unnecessarily inefficient arguments. It is our belief that it is the right of the reader, but not the author, to skip details. At least, we hope that readers will not have to repeat some of the elementary but tedious tasks we have gone through.

Apology

Due to limitation of time and energy, the proofreading process might not have achieved a satisfactory status at the time that this work is sent to print, and there might be non-mathematical or mathematical typos that are very difficult to correct from the readers' side. We apologize for this inconvenience due to our incompetence. Please contact the author whenever there are unclear or incorrect statements that require justifications or modifications.

Comparison With Submitted Version

The content in this volume is very close to the version submitted to Harvard University. However, some changes or corrections made after the submission have been incorporated, and it is obvious that the formatting of this document is drastically different from any other existing versions. It is important to keep in mind that numbering of results in this document may differ from the official submitted version.

Notation and Conventions

All rings, commutative or not, will have an identity element. All left or right module structures, or algebra structures, will preserve the identity elements. Unless the violation is clear from the context, or unless otherwise specified, every ring homomorphism will send the identity to the identity element. Unless otherwise specified, all modules will be assumed to be left modules by default. An exception is ideals in noncommutative rings, in which case we shall always describe precisely whether it is a left ideal, a right ideal, or a two-sided ideal. All involutions in this work are anti-automorphisms of order two. The dual of a left module is naturally equipped with a left module structure over the opposite ring, and hence over the same ring if the ring admits an involution (which is an anti-isomorphism from the ring to its opposite ring).

We shall use the notation $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{A}$, $\mathbb{A}^\infty$, and $\hat{\mathbb{Z}}$ to denote respectively the ring of rational integers, rational numbers, real numbers, complex numbers, adeles, finite adeles, and integral adeles, without any further explanation.

More generally, for each set $\Box$ of rational primes, which can be either finite or infinite in cardinality, or even empty, we denote by $\mathbb{Z}_{(\Box)}$ the unique localization of $\mathbb{Z}$ (at the multiplicative subset of $\mathbb{Z}$ generated by nonzero integers prime-to-$\Box$) having $\Box$ as its set of nonzero height-one primes, and denote by $\hat{\mathbb{Z}}^\Box$ (resp. $\mathbb{A}^{\infty,\Box}$, resp. $\mathbb{A}^\Box$) the integral adeles (resp. finite adeles, resp. adeles) away from $\Box$. (When $\Box$ is empty, we have $\mathbb{Z}_{(\Box)} = \mathbb{Q}$, $\mathbb{A}^{\infty,\Box} = \mathbb{A}^\infty$, and $\mathbb{A}^\Box = \mathbb{A}$.)

We say that an integer m is prime-to-$\Box$ if m is not divisible by any prime number in $\Box$. In this case, we write $\Box \nmid m$.

These conventions and notation are designed so that results would be compatible if $\Box$ were literally just a prime number.

The notation $[A : B]$ will mean either the index of B in A as a subgroup when we work in the category of lattices, or the degree of A over B when we work in the category of finite-dimensional algebras over a field. We allow this ambiguity because there is no interesting overlap of these two usages.

The notation δ_{ij}, when i and j are indices, means the Kronecker delta, which is 1 when $i = j$ and 0 when $i \neq j$, as usual.

In our exposition, *schemes* will almost always mean *quasi-separated preschemes*, unless otherwise specified (see Remark A.2.5 and Lemma A.2.6). All *algebraic stacks* that we will encounter are *Deligne–Mumford stacks* (cf. [36], [42, Ch. I, §4], [83], and Appendix A).

The notion of *relative schemes* over a ringed topos can be found in [61], which, in particular, is necessary when we talk about relative schemes over formal schemes. We shall generalize this notion tacitly to relative schemes over formal algebraic stacks.

By a *normal* scheme we mean a scheme whose local rings are all integral and integrally closed in its fraction field. A ring R is normal if $\mathrm{Spec}(R)$ is normal. We do not need R to be integral and/or noetherian in such a statement.

We shall almost always interpret *points* as *functorial points*, and hence fibers as fibers over functorial points. By a *geometric point* of a point we mean a morphism from an algebraically closed field to the scheme we consider. We will often use the relative notion of various scheme-theoretic concepts without explicitly stating the convention.

We will use the notation $\mathbf{G}_{\mathrm{m}}$, $\mathbf{G}_{\mathrm{a}}$, and $\boldsymbol{\mu}_n$ to denote, respectively, the *multiplicative group*, the *additive group*, and the group scheme kernel of $[n] : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ over $\mathrm{Spec}(\mathbb{Z})$. Their base change to other base schemes S will often be denoted by $\mathbf{G}_{\mathrm{m},S}$, $\mathbf{G}_{\mathrm{a},S}$, and $\boldsymbol{\mu}_{n,S}$, respectively.

For each scheme S and each set X, we denote by X_S the sheaf of locally constant functions over S valued in X. When X carries additional structures such as being an algebra or a module over some ring, then X_S is a sheaf also carrying such additional structures. (We can also interpret X_S as a scheme over S defined by the disjoint union of copies of S indexed by elements in X.)

For each scheme S and each $\mathscr{O}_S$-algebra (resp. graded $\mathscr{O}_S$-algebra) $\mathcal{A}$, we denote by $\underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathcal{A})$ (resp. $\underline{\mathrm{Proj}}_{\mathscr{O}_S}(\mathcal{A})$) the spectrum (resp. homogeneous spectrum) of $\mathcal{A}$ over S. This is often denoted by $\mathbf{Spec}(\mathcal{A})$ (resp. $\mathbf{Proj}(\mathcal{A})$), or by $\mathrm{Spec}(\mathcal{A})$ (resp. $\mathrm{Proj}(\mathcal{A})$) as in [59, II, 1.3.1, resp. 3.1.3]. Our underlined notation is compatible with our other notation $\underline{\mathrm{Hom}}$, $\underline{\mathrm{Isom}}$, $\underline{\mathrm{Pic}}$, etc. for sheafified objects.

Throughout the text, there will be relative objects such as sheaves, group schemes, torsors, extensions, biextensions, cubical structures, etc. These objects are equipped with their base schemes or algebraic stacks (or their formal analogues) by definition. Unless otherwise specified, morphisms between objects with these structures will be given by morphisms respecting the bases, unless otherwise specified. (We will make this clear when there is room for ambiguity.) We will be more explicit when the structures are defined or studied, but will tacitly maintain this convention afterwards.

The typesetting of this work will be sensitive to small differences in notation. Although no difficult simultaneous comparison between similar symbols will be required, the differences should not be overlooked when looking for references. More concretely, we have used all the following fonts: A (normal), A (Roman), $\mathbf{A}$ (boldface), $\mathbb{A}$ (blackboard boldface), A (sans serif), $\mathtt{A}$ (typewriter), $\mathcal{A}$ (calligraphic), $\mathfrak{A}$ (Fraktur), and $\mathscr{A}$ (Ralph Smith's formal script). The tiny difference between A (normal) and *A* (italic) in width, which does exist, seems to be extremely difficult to see. So we shall never use both of them. We distinguish between A and $\underline{A}$, where the latter almost always means the relative version of A (as a sheaf or functor, etc.). We distinguish between Greek letters in each of the pairs ϵ and ε, ρ and ϱ, σ and ς, ϕ and φ, and π and ϖ. The musical symbols $^\flat$ (flat), $^\natural$ (natural), and $^\sharp$ (sharp) will be used following Grothendieck (cf., for example, [57, IX]) and some other authors. The difference in each of the pairs $^\flat$ and ^{b}, and $^\sharp$ and $^\#$, should not lead to any confusion. The notation $^\heartsuit$ and $^\diamondsuit$ are used, respectively, for Mumford families and good formal models, where the convention for the former follows from [42]. We distinguish between the two star signs * and *. The two dagger forms † and ‡ are used as superscripts. The differences between v, ν, υ, and the dual sign $^\vee$ should not be confusing because they are never used for similar purposes. The same is true for i, ι, $\imath$, and $\jmath$. Since we will never need calculus in this work, the symbols ∂, $\int$, and $\oint$ are used as variants of d or S.

Finally, unless it comes with "resp.", the content of each set of parentheses in text descriptions is *not an option*, but rather a reminder, a remark, or a supplement of information.

Chapter 1

Definition of Moduli Problems

In this chapter, we give the definition of the moduli problems providing integral models of PEL-type Shimura varieties that we will compactify.

Just to make sure that potential logical problems do not arise in our use of categories, we assume that a pertinent choice of a *universe* has been made (see Section A.1.1 for more details). This is harmless for our study, and we shall not mention it again in our work.

The main objective in this chapter is to state Definition 1.4.1.4 with justifications. In order to explain the relation between our definition and those in the literature, we include also Definition 1.4.2.1 (which, in particular, agrees with the definition in [76, §5] when specialized to the same bases), and compare our two definitions. All sections preceding them are preparatory in nature. Technical results worth noting are Propositions 1.1.2.20, 1.1.5.17, 1.2.2.3, 1.2.3.7, 1.2.3.11, 1.2.5.15, 1.2.5.16, and 1.4.3.4. Theorem 1.4.1.11 (on the representability of our moduli problems in the category of algebraic stacks) is stated in Section 1.4, but its proof will be carried out in Chapter 2. The representability of our moduli problem as schemes (when the level is neat) will be deferred until Corollary 7.2.3.10, after we have accomplished the construction of the minimal compactifications.

1.1 Preliminaries in Algebra

1.1.1 Lattices and Orders

For the convenience of readers, we shall summarize certain basic definitions and important properties of lattices over an order in a (possibly noncommutative) finite-dimensional algebra over a Dedekind domain. Our main reference for this purpose will be [107].

Let us begin with the most general setting. Let R be a (commutative) noetherian integral domain with fractional field $\mathrm{Frac}(R)$.

Definition 1.1.1.1. *An R-**lattice** M is a finitely generated R-module M with no nonzero R-torsion. Namely, for every nonzero $m \in M$, there is no nonzero element $r \in R$ such that $rm = 0$.*

Note that in this case we have an embedding from M to $M \underset{R}{\otimes} \mathrm{Frac}(R)$.

Definition 1.1.1.2. *Let V be any finite-dimensional $\mathrm{Frac}(R)$-vector space. A **full R-lattice** M in V is a finitely generated submodule M of V such that $\mathrm{Frac}(R) \cdot M = V$. In other words, M contains a $\mathrm{Frac}(R)$-basis of V.*

Let A be a (possibly noncommutative) finite-dimensional algebra over $\mathrm{Frac}(R)$.

Definition 1.1.1.3. *An R-**order** $\mathcal{O}$ in the $\mathrm{Frac}(R)$-algebra A is a subring of A having the same identity element as A, such that $\mathcal{O}$ is also a full R-lattice in A.*

Here are two familiar examples of orders:

1. If R is a Dedekind domain, and if $A = L$ is a finite separable field extension of $\mathrm{Frac}(R)$, then the integral closure $\mathcal{O}$ of R in L is an R-order in A. In particular, if $R = \mathbb{Z}$, then the rings of algebraic integers $\mathcal{O} = \mathcal{O}_L$ in L is a $\mathbb{Z}$-order in L.

2. If $A = \mathrm{M}_n(\mathrm{Frac}(R))$, then $\mathcal{O} = \mathrm{M}_n(R)$ is an R-order in A.

Definition 1.1.1.4. *A **maximal** R-**order** in A is an R-order not properly contained in another R-order in A.*

Proposition 1.1.1.5 ([107, Thm. 8.7])**.**

1. *If the integral closure of R in A is an R-order, then it is automatically maximal.*

2. *If $\mathcal{O}$ is a maximal R-order in A, then $\mathrm{M}_n(\mathcal{O})$ is a maximal R-order in $\mathrm{M}_n(A)$ for each integer $n \geq 1$. In particular, if R is normal (namely, integrally closed in $\mathrm{Frac}(R)$), then $\mathrm{M}_n(R)$ is a maximal R-order in $\mathrm{M}_n(\mathrm{Frac}(R))$.*

Suppose moreover that A is a *separable* $\mathrm{Frac}(R)$-algebra. By definition, A is Artinian and semisimple. For simplicity, we shall suppress the modifier *reduced* from traces and norms when talking about such algebras. By [107, Thm. 9.26], the assumption that A is a (finite-dimensional) separable $\mathrm{Frac}(R)$-algebra implies that the (reduced) trace pairing $\mathrm{Tr}_{A/\mathrm{Frac}(R)} : A \times A \to \mathrm{Frac}(R)$ is *nondegenerate* (as pairings on $\mathrm{Frac}(R)$-vector spaces).

An important invariant of an order defined by the trace pairing is the discriminant.

Definition 1.1.1.6. *Let $t = [A : \mathrm{Frac}(R)]$. The **discriminant***
$$\mathrm{Disc} = \mathrm{Disc}_{\mathcal{O}/R}$$
is the ideal of R generated by the set of elements
$$\{\mathrm{Det}_{\mathrm{Frac}(R)}(\mathrm{Tr}_{A/\mathrm{Frac}(R)}(x_i x_j))_{1 \leq i \leq t, 1 \leq j \leq t} : x_1, \ldots, x_t \in \mathcal{O}\}.$$

Remark 1.1.1.7. If $\mathcal{O}$ has a free R-basis $e_1, \ldots, e_t$, then each of the t elements $x_1, \ldots, x_t$ can be expressed as an R-linear combination of $e_1, \ldots, e_t$. Hence in this case Disc is generated by a single element
$$\mathrm{Det}_{\mathrm{Frac}(R)}(\mathrm{Tr}_{A/\mathrm{Frac}(R)}(e_i e_j))_{1 \leq i \leq t, 1 \leq j \leq t}.$$

Another important invariant is the following definition:

Definition 1.1.1.8. *The **inverse different***
$$\mathrm{Diff}^{-1} = \mathrm{Diff}^{-1}_{\mathcal{O}/R}$$
of $\mathcal{O}$ over R is defined by
$$\mathrm{Diff}^{-1}_{\mathcal{O}/R} := \{x \in A : \mathrm{Tr}_{A/\operatorname{Frac}(R)}(xy) \subset R \ \forall y \in \mathcal{O}\}.$$

It is clear from the definition that $\mathrm{Diff}^{-1}_{\mathcal{O}/R}$ is a two-sided ideal in A, and that the formation of inverse differents is compatible with localizations.

Lemma 1.1.1.9. *Suppose $\mathcal{O}$ is locally free as an R-module. Then Diff^{-1} is locally free as an R-module, and $\mathrm{Tr}_{A/\operatorname{Frac}(R)}$ induces a perfect pairing*
$$\mathrm{Tr}_{A/\operatorname{Frac}(R)} : \mathcal{O} \times \mathrm{Diff}^{-1} \to R.$$
Moreover, if $\{e_i\}_{1\le i\le t}$ is any R_1-basis of $\mathcal{O} \underset{R}{\otimes} R_1$ for some localization R_1 of R, then there exists a unique dual R_1-basis $\{f_i\}_{1\le i\le t}$ of $\mathrm{Diff}^{-1} \underset{R}{\otimes} R_1$ such that $\mathrm{Tr}_{A/\operatorname{Frac}(R)}(e_i f_j) = \delta_{ij}$ for all $1 \le i \le t$ and $1 \le j \le t$.

Proof. We may localize R and assume that $\mathcal{O}$ is free over R. Let $\{e_i\}_{1\le i\le t}$ be any basis of $\mathcal{O}$ over R. Then $\{e_i\}_{1\le i\le t}$ is also a basis of A over $\operatorname{Frac}(R)$. By nondegeneracy of the trace pairing $\mathrm{Tr}_{A/\operatorname{Frac}(R)} : A \times A \to \operatorname{Frac}(R)$, there exists a unique basis $\{f_i\}_{1\le i\le t}$ of A over $\operatorname{Frac}(R)$, which is dual to $\{e_i\}_{1\le i\le t}$ in the sense that $\mathrm{Tr}_{A/\operatorname{Frac} R}(e_i f_j) = \delta_{ij}$ for all $1 \le i \le t$ and $1 \le j \le t$. If $y = \sum_{1\le j\le t} c_j f_j \in A$ satisfies $\mathrm{Tr}_{A/\operatorname{Frac}(R)}(xy) \in R$ for all $x \in \mathcal{O}$, then in particular, $c_j = \mathrm{Tr}_{A/\operatorname{Frac}(R)}(e_j y) \in R$ for all $1 \le j \le t$. Thus $\{f_j\}_{1\le j\le t}$ is also a basis of Diff^{-1}. This shows the perfectness of the pairing $\mathrm{Tr}_{A/\operatorname{Frac}(R)} : \mathcal{O} \times \mathrm{Diff}^{-1} \to R$ and the existence of the dual bases, as desired. □

Suppose R is a noetherian *normal* domain, and A is a finite-dimensional *separable* $\operatorname{Frac}(R)$-algebra.

Proposition 1.1.1.10 ([107, Cor. 10.4])**.** *Let R be a noetherian normal domain, and let A be a finite-dimensional separable $\operatorname{Frac}(R)$-algebra. Then every R-order in A is contained in a maximal R-order in A. There exists at least one maximal R-order in A.*

For each ideal $\mathfrak{p}$ of R, we denote by $R_\mathfrak{p}$ the localization of R at $\mathfrak{p}$, and by $\hat{R}_\mathfrak{p}$ the completion of $R_\mathfrak{p}$ with respect to its maximal ideal $\mathfrak{p}R_\mathfrak{p}$. (The slight deviation of this convention is that we shall denote by $\mathbb{Z}_{(p)}$ the localization of $\mathbb{Z}$ at (p), and by $\mathbb{Z}_p$ the completion of $\mathbb{Z}_{(p)}$.) If R is local, then we denote simply by $\hat{R}$ its completion at its maximal ideal.

Definition 1.1.1.11. *Let R be a noetherian normal domain, and let $\mathfrak{p}$ be a prime ideal of R. We say that an R-order $\mathcal{O}$ in A is **maximal at** $\mathfrak{p}$ if $\mathcal{O} \underset{R}{\otimes} R_\mathfrak{p}$ is maximal in A.*

Proposition 1.1.1.12 ([107, Thm. 11.1, Cor. 11.2])**.** *Let R be a noetherian normal domain. An R-order $\mathcal{O}$ in A is maximal if and only if $\mathcal{O}$ is maximal at every prime ideal of R, or equivalently at every maximal ideal of R.*

Proposition 1.1.1.13 ([107, Thm. 11.5])**.** *Let R be a **local** noetherian normal domain. Suppose $\hat{R}$ is a noetherian domain. Then an R-order $\mathcal{O}$ in A is maximal if and only if $\mathcal{O} \underset{R}{\otimes} \hat{R}$ is an $\hat{R}$-maximal order in $A \underset{\operatorname{Frac}(R)}{\otimes} \operatorname{Frac}(\hat{R})$.*

Remark 1.1.1.14. The statement and proof of [107, Thm. 11.5] make sense only when $\hat{R}$ is a noetherian integral domain.

Corollary 1.1.1.15. *Let R be a noetherian normal domain, and let $\mathfrak{p}$ be a prime ideal of R. Suppose $\hat{R}_\mathfrak{p}$ is a noetherian domain. Then an R-order $\mathcal{O}$ is maximal at $\mathfrak{p}$ if and only if $\mathcal{O} \underset{R}{\otimes} \hat{R}_\mathfrak{p}$ is maximal in $A \underset{\operatorname{Frac}(R)}{\otimes} \operatorname{Frac}(\hat{R}_\mathfrak{p})$.*

Now suppose R is a *Dedekind domain*. In particular, R is noetherian and normal, and the completions of R at localizations of its prime ideals are noetherian domains.

Proposition 1.1.1.16. *Let*
$$\mathrm{Diff} = \mathrm{Diff}_{\mathcal{O}/R} := \{z \in A : z\,\mathrm{Diff}^{-1}_{\mathcal{O}/R} \subset \mathcal{O}\}$$
be the inverse ideal of $\mathrm{Diff}^{-1}_{\mathcal{O}/R}$. Then this is a two-sided ideal of $\mathcal{O}$, and the discriminant $\mathrm{Disc}_{\mathcal{O}/R}$ is related to $\mathrm{Diff}^{-1}_{\mathcal{O}/R}$ by
$$\mathrm{Disc}_{\mathcal{O}/R} = \mathrm{Norm}_{A/\operatorname{Frac}(R)}(\mathrm{Diff}_{\mathcal{O}/R}) = [\mathrm{Diff}^{-1}_{\mathcal{O}/R} : \mathcal{O}]_R. \tag{1.1.1.17}$$

If $\mathcal{O}$ is a maximal order, then this is just [107, Thm. 25.2]. The same proof via localizations works in the case where $\mathcal{O}$ is not maximal as well:

Proof of Proposition 1.1.1.16. By replacing R with its localizations, we may assume that every R-lattice is free over R. Let $t = [A : \operatorname{Frac}(R)]$. Let $\{e_i\}_{1\le i\le t}$ be any R-basis of $\mathcal{O}$, and let $\{f_i\}_{1\le i\le t}$ be the dual R-basis of Diff^{-1} given by Lemma 1.1.1.9. Since $\mathcal{O} \subset \mathrm{Diff}^{-1}$, we may express each e_i as $e_i = \sum_{1\le j\le t} a_{ij} f_j$ for some $a_{ij} \in R$. By definition,
$$\mathrm{Norm}_{A/\operatorname{Frac}(R)}(\mathrm{Diff}) = [\mathrm{Diff}^{-1} : \mathcal{O}]_R = (\mathrm{Det}_{\operatorname{Frac}(R)}(a_{ij})).$$
On the other hand,
$$\mathrm{Tr}_{A/\operatorname{Frac}(R)}(e_i e_j) = \sum_{1\le k\le t} a_{ik}\,\mathrm{Tr}_{A/\operatorname{Frac}(R)}(f_k e_j) = a_{ij}.$$
Hence $\mathrm{Disc} = (\mathrm{Det}_{\operatorname{Frac}(R)}(\mathrm{Tr}_{A/\operatorname{Frac}(R)}(e_i x_j))) = (\mathrm{Det}_{\operatorname{Frac}(R)}(a_{ij}))$, verifying equation (1.1.1.17) as desired. □

Definition 1.1.1.18. *We say that the prime ideal $\mathfrak{p}$ of R is **unramified** in $\mathcal{O}$ if $\mathfrak{p} \nmid \mathrm{Disc}_{\mathcal{O}/R}$.*

Proposition 1.1.1.19 ([107, Thm. 25.3])**.** *Every two maximal R-orders $\mathcal{O}_1$ and $\mathcal{O}_2$ in A have the same discriminant over R.*

Therefore it makes sense to say,

Definition 1.1.1.20. *An ideal $\mathfrak{p}$ of R is unramified in A if it is **unramified** in one (and hence every) maximal R-order of A.*

For each global field K, we shall denote the rings of integers in K by $\mathcal{O}_K$. This is in conflict with the notation $\mathcal{O}$ with no subscripts, but the correct interpretation should be clear from the context.

Proposition 1.1.1.21. *Let R be a Dedekind domain such that $\operatorname{Frac}(R)$ is a **global field**, let A be a finite-dimensional $\operatorname{Frac}(R)$-algebra with center E, and let $\mathcal{O}$ be an R-order in A. Suppose $\mathfrak{p}$ is a **nonzero** prime ideal of R such that $\mathfrak{p} \nmid \mathrm{Disc}_{\mathcal{O}/R}$. Then,*

1. *$\mathcal{O}$ is maximal at $\mathfrak{p}$;*

2. *$\mathcal{O} \underset{R}{\otimes} \hat{R}_{\mathfrak{p}}$ is isomorphic to a product of matrix algebras containing $\mathcal{O}_E \underset{R}{\otimes} \hat{R}_{\mathfrak{p}}$ as its center;*
3. *$\mathfrak{p}$ is unramified in A and in E.*

Proof. If $\mathcal{O} \subset \mathcal{O}'$ are two orders, then necessarily

$$\mathcal{O} \subset \mathcal{O}' \subset \mathrm{Diff}^{-1}_{\mathcal{O}'/R} \subset \mathrm{Diff}^{-1}_{\mathcal{O}/R}.$$

In particular, if $\mathfrak{p}$ is a prime ideal of R such that $\mathfrak{p} \nmid \mathrm{Disc}_{\mathcal{O}/R}$, then the relation $\mathcal{O} \underset{R}{\otimes} R_{\mathfrak{p}} = \mathrm{Diff}^{-1}_{\mathcal{O}/R} \underset{R}{\otimes} R_{\mathfrak{p}}$ forces $\mathcal{O} \underset{R}{\otimes} R_{\mathfrak{p}}$ to be maximal. This proves the first statement.

According to [107, Thm. 10.5], $\mathcal{O} \underset{R}{\otimes} R_{\mathfrak{p}}$ is a maximal $R_{\mathfrak{p}}$-order if and only if it is a maximal $\mathcal{O}_E \underset{R}{\otimes} R_{\mathfrak{p}}$-order. Then [107, Thm. 25.7] implies that $\mathcal{O} \underset{R}{\otimes} \hat{R}_{\mathfrak{p}}$ is a product of matrix algebras containing $\mathcal{O}_E \underset{R}{\otimes} \hat{R}_{\mathfrak{p}}$ as its center. This is the second statement.

Finally, by taking the matrix with only one element on the diagonal, we see that $\mathcal{O} \underset{R}{\otimes} R_{\mathfrak{p}} = \mathrm{Diff}^{-1}_{\mathcal{O}/R} \underset{R}{\otimes} R_{\mathfrak{p}}$ forces $\mathcal{O}_E \underset{R}{\otimes} R_{\mathfrak{p}} = \mathrm{Diff}^{-1}_{\mathcal{O}_E/R} \underset{R}{\otimes} R_{\mathfrak{p}}$. That is, $\mathfrak{p} \nmid \mathrm{Disc}_{\mathcal{O}/R}$ forces $\mathfrak{p} \nmid \mathrm{Disc}_{\mathcal{O}_E/R}$. Then the third statement follows from Proposition 1.1.1.19 and Definition 1.1.1.20. □

Definition 1.1.1.22. *A (left) $\mathcal{O}$-module M is called an **$\mathcal{O}$-lattice** if it is an R-lattice. Namely, it is finitely generated and torsion-free as an R-module.*

Proposition 1.1.1.23 (see [107, Thm. 21.4 and Cor. 21.5])**.** *Every maximal order $\mathcal{O}$ over a Dedekind domain is **hereditary** in the sense that all $\mathcal{O}$-lattices are projective $\mathcal{O}$-modules.*

Proposition 1.1.1.24 (see [107, Cor. 21.5 and Thm. 2.44])**.** *Every projective module over a maximal order is a direct sum of left ideals.*

Remark 1.1.1.25. Propositions 1.1.1.23 and 1.1.1.24 imply that, although $\mathcal{O}$-lattices might not be projective, their localizations or completions become projective as soon as $\mathcal{O}$ itself becomes maximal after localization or completion.

1.1.2 Determinantal Conditions

Let C be a finite-dimensional *separable* algebra over a field k. By definition (such as [107, p. 99]), the center E of C is a commutative finite-dimensional separable algebra over k. Let K be a (possibly infinite) field extension of k. Unless otherwise specified, all the homomorphisms below will be k-linear.

Fix a separable closure K^{sep} of K, and consider the possible k-algebra homomorphisms τ from E to K^{sep}. Note that $\mathrm{Hom}_k(E, K^{\mathrm{sep}})$ has cardinality $[E : k]$, because E is separable over k. The $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-orbits $[\tau]$ of such homomorphisms $\tau : E \to K^{\mathrm{sep}}$ can be classified in the following way: Consider the equivalence classes of pairs of the form (K_τ, τ), where K_τ is isomorphic over K to the composite of K and the image of τ in K^{sep}, and where τ is the induced homomorphism from E to K_τ. Here K_τ is necessarily separable over K with degree at most $[E : k]$. Two such pairs (K_{τ_1}, τ_1) and (K_{τ_2}, τ_2) are considered equivalent if there is an isomorphism $\sigma : K_{\tau_1} \xrightarrow{\sim} K_{\tau_2}$ over K such that $\tau_2 = \sigma \circ \tau_1$. We shall denote such an equivalence class by $[\tau] : E \to K_{[\tau]}$. By abuse of notation, this will also mean an actual representative $\tau : E \to K_\tau$, which can be considered as a homomorphism $\tau : E \to K^{\mathrm{sep}}$ as well. Note that

$$K_{[\tau]} \underset{K}{\otimes} K^{\mathrm{sep}} \cong \prod_{\tau' \in [\tau]} K^{\mathrm{sep}}_{\tau'},$$

where each $K^{\mathrm{sep}}_{\tau'}$ means a copy of K^{sep} with $\tau' : E \to K^{\mathrm{sep}}$ in the equivalence class $[\tau]$.

Lemma 1.1.2.1. *We have a decomposition*

$$E \underset{k}{\otimes} K \cong \prod_{[\tau]: E \to K_{[\tau]}} E \underset{E,[\tau]}{\otimes} K_{[\tau]} \cong \prod_{[\tau]: E \to K_{[\tau]}} K_{[\tau]}$$

into a product of separable extensions $E_{[\tau]}$ of K.

Corollary 1.1.2.2. *We have a decomposition*

$$C \underset{k}{\otimes} K \cong C \underset{E}{\otimes} (E \underset{k}{\otimes} K) \cong \prod_{[\tau]: E \to K} C \underset{E,[\tau]}{\otimes} K_{[\tau]} \tag{1.1.2.3}$$

into simple K-algebras.

Let us quote the following weaker form of the Noether–Skolem theorem:

Lemma 1.1.2.4 (see, for example, [63, Lem. 4.3.2])**.** *Let C' be a simple Artinian algebra. Then all simple C'-modules are isomorphic to each other.*

A useful reformulation of Lemma 1.1.2.4 is as follows:

Corollary 1.1.2.5. *Let C' be a simple Artinian algebra with center E'. Then an irreducible representation of C' with coefficients in some field K' is determined up to isomorphism by its restriction to E'.*

Applying Corollary 1.1.2.5 to the simple factors of $C \underset{k}{\otimes} K$ as in Corollary 1.1.2.2, we obtain:

Corollary 1.1.2.6.

1. *There is a unique simple $C \underset{k}{\otimes} K$-module $W_{[\tau]}$ on which E acts via the homomorphism $[\tau] : E \to K_{[\tau]}$. The semisimple algebra $C \underset{k}{\otimes} K$ acts K-linearly on $W_{[\tau]}$ via its projection to the simple subalgebra $C \underset{E,[\tau]}{\otimes} K_{[\tau]}$ given by Corollary 1.1.2.2.*
2. *Each finitely generated $C \underset{k}{\otimes} K$-module is of the form*
$$M \cong \bigoplus_{[\tau]: E \to K_{[\tau]}} W_{[\tau]}^{\oplus m_{[\tau]}}$$
for some integers $m_{[\tau]}$. When applied to the case $M = C \underset{k}{\otimes} K$, this decomposition agrees with the one given in Corollary 1.1.2.2.
3. *The isomorphism class of the $C \underset{k}{\otimes} K$-module M is determined by the set of integers $\{m_{[\tau]}\}_{[\tau]: E \to K_{[\tau]}}$.*

Definition 1.1.2.7. *Let M_0 be any finitely generated $C \underset{k}{\otimes} K$-module. The **field of definition** K_0 of M_0 is the subfield of K^{sep} consisting of elements fixed by every σ in $\mathrm{Aut}(K^{\mathrm{sep}}/k)$ such that $M_0 \underset{K}{\otimes} K^{\mathrm{sep}} \cong (M_0 \underset{K}{\otimes} K^{\mathrm{sep}}) \underset{K^{\mathrm{sep}},\sigma}{\otimes} K^{\mathrm{sep}}$ as $C \underset{k}{\otimes} K^{\mathrm{sep}}$-modules.*

Remark 1.1.2.8. Since M_0 is an object defined over K, this K_0 must be contained in K and independent of the choice of K^{sep}.

Remark 1.1.2.9. Even if we say that the field of definition of a finitely generated $C \underset{k}{\otimes} K$-module M_0 is K_0, it is not necessarily true that there exists a $C \underset{k}{\otimes} K_0$-module M_{00} such that $M_{00} \underset{K_0}{\otimes} K^{\mathrm{sep}} \cong M_0 \underset{K}{\otimes} K^{\mathrm{sep}}$ as $C \underset{k}{\otimes} K^{\mathrm{sep}}$-modules. This is incompatible with some conventions in representation theory, and we must point this out for the sake of clarity.

By replacing K with K^{sep} in Corollary 1.1.2.6, we see that there is a unique simple $C \underset{k}{\otimes} K^{\mathrm{sep}}$-module W_τ for each $\tau : E \to K^{\mathrm{sep}}$. Each $\sigma \in \mathrm{Aut}(K^{\mathrm{sep}}/k)$ modifies the E-action on W_τ by composition with σ, and hence we have a canonical isomorphism

$$W_\tau \underset{K^{\mathrm{sep}},\sigma}{\otimes} K^{\mathrm{sep}} \cong W_{\sigma\circ\tau}$$

for each $\tau : E \to K^{\mathrm{sep}}$. By checking the E-action, we see that there is a decomposition

$$W_{[\tau]} \underset{K}{\otimes} K^{\mathrm{sep}} \cong \underset{\tau' \in [\tau]}{\oplus} W_{\tau'}^{\oplus s_{[\tau]}} \tag{1.1.2.10}$$

for some integer $s_{[\tau]} \geq 1$. We have $s_{[\tau]} = 1$ when $C \underset{k}{\otimes} K$ (or rather the factor $C \underset{E,[\tau]}{\otimes} K_{[\tau]}$ in (1.1.2.3)) is a product of matrix algebras. Moreover, $W_{[\tau]} \underset{K}{\otimes} K^{\mathrm{sep}} \cong W_{[\sigma\circ\tau]} \underset{K}{\otimes} K^{\mathrm{sep}}$ as a $C \underset{k}{\otimes} K^{\mathrm{sep}}$-module only when $[\tau] = [\sigma \circ \tau]$. As a result, if M is any finitely generated $C \underset{k}{\otimes} K$-module with decomposition $M \cong \underset{[\tau]:E\to K_{[\tau]}}{\oplus} W_{[\tau]}^{\oplus m_{[\tau]}}$ as in Corollary 1.1.2.6, then

$$(M \underset{K}{\otimes} K^{\mathrm{sep}}) \underset{K^{\mathrm{sep}},\sigma}{\otimes} K^{\mathrm{sep}} \cong M \underset{K}{\otimes} K^{\mathrm{sep}}$$

if and only if

$$s_{[\tau]} m_{[\tau]} = s_{[\sigma\circ\tau]} m_{[\sigma\circ\tau]}$$

for all τ. The following corollary is a useful observation:

Corollary 1.1.2.11. *The field of definition for W_τ (as a $C \underset{k}{\otimes} K^{\mathrm{sep}}$-module) is contained in $\tau(E) \subset K^{\mathrm{sep}}$.*

Therefore,

Corollary 1.1.2.12. *Let E^{Gal} denote the **Galois closure** E^{Gal} of E in K^{sep}, namely, the composite field of $\tau(E)$ for all possible $\tau : E \to K^{\mathrm{sep}}$. Then the field of definition K_0 for each finitely generated $C \underset{k}{\otimes} K$-module M_0 is contained in the intersection $E^{\mathrm{Gal}} \cap K$.*

Proof. Homomorphisms $E \to K^{\mathrm{sep}}$ are unchanged under the action of $\mathrm{Aut}(K^{\mathrm{sep}}/E^{\mathrm{Gal}})$. □

It is desirable to have a way to detect whether two $C \underset{k}{\otimes} K$-modules are isomorphic, without having to go through the comparison of the $m_{[\tau]}$'s. In characteristic zero, it is classical to use the *trace* to classify representations:

Lemma 1.1.2.13. *Suppose* $\mathrm{char}(k) = 0$. *Then the maps $C \to K : x \mapsto \mathrm{Tr}_K(x|W_{[\tau]})$, for all $[\tau]$ as above, are linearly independent over K.*

Proof. It suffices to show this by restricting the maps to E. Then, for all $e \in E$, we have

$$\begin{aligned}\mathrm{Tr}_K(e|W_{[\tau]}) &= \mathrm{Tr}_{K^{\mathrm{sep}}}(e|W_{[\tau]} \underset{K}{\otimes} K^{\mathrm{sep}}) \\ &= s_{[\tau]} \sum_{\tau' \in [\tau]} \mathrm{Tr}_{K^{\mathrm{sep}}}(e|W_{\tau'}) = s_{[\tau]} d_{[\tau]} \sum_{\tau' \in [\tau]} \tau'(e)\end{aligned}$$

by (1.1.2.10), where $d_{[\tau]}^2$ is the degree of $C \underset{E,[\tau]}{\otimes} K_{[\tau]}$ over its center $K_{[\tau]}$. This is $s_{[\tau]} d_{[\tau]}$ times a sum of homomorphisms from E to K^{sep}, which all factor through a single subfield of E as they are all in the same $\mathrm{Gal}(K^{\mathrm{sep}}/K)$-orbit of a single homomorphism from E to K^{sep}. Now, it is a classical lemma of Dedekind's that distinct homomorphisms from a first field to a second field are linearly independent over the second field. □

Corollary 1.1.2.14. *Suppose* $\mathrm{char}(k) = 0$. *Then two finitely generated $C \underset{k}{\otimes} K$-modules M_1 and M_2 are isomorphic if and only if $\mathrm{Tr}_K(x|M_1) = \mathrm{Tr}_K(x|M_2)$ for all $x \in C$.*

Proof. If $M_1 \cong M_2$ as $C \underset{k}{\otimes} K$-modules, then clearly the traces are the same. Conversely, suppose that $\mathrm{Tr}_K(x|M_1) = \mathrm{Tr}_K(x|M_2)$ for all $x \in C$. In particular, $\mathrm{Tr}_K(e|M_1) = \mathrm{Tr}_K(e|M_2)$ for all $e \in E$. For $i = 1, 2$, let us decompose $M_i \cong \underset{[\tau]:E\to K_{[\tau]}}{\oplus} W_{[\tau]}^{\oplus m_{[\tau],i}}$ as in Corollary 1.1.2.6. By Lemma 1.1.2.13,

$$\begin{aligned}\mathrm{Tr}_K(e|M_1) &= \sum_{[\tau]:E\to K_{[\tau]}} m_{[\tau],1} \mathrm{Tr}_K(e|W_{[\tau]}) \\ &= \sum_{[\tau]:E\to K_{[\tau]}} m_{[\tau],2} \mathrm{Tr}_K(e|W_{[\tau]}) = \mathrm{Tr}_K(e|M_2)\end{aligned}$$

for all $e \in E$ if and only if $m_{[\tau],1} = m_{[\tau],2}$ for all $[\tau]$, or equivalently $M_1 \cong M_2$ as $C \underset{k}{\otimes} K$-modules. □

Remark 1.1.2.15. Note that $\mathrm{char}(k) = 0$ is used in an essential way. In positive characteristics, we cannot expect the trace comparison to work in general.

Corollary 1.1.2.16. *Suppose* $\mathrm{char}(k) = 0$. *Then the field of definition K_0 of a finitely generated $C \underset{k}{\otimes} K$-module M_0 is*

$$K_0 = k(\mathrm{Tr}_K(x|M_0) : x \in C) = k(\mathrm{Tr}_K(x|M_0) : x \in C).$$

Proof. Let σ be any element in $\mathrm{Gal}(K^{\mathrm{sep}}/k)$. We would like to show that $(M_0 \underset{K}{\otimes} K^{\mathrm{sep}}) \underset{K^{\mathrm{sep}},\sigma}{\otimes} K^{\mathrm{sep}} \underset{E,[\tau]}{\otimes} K_{[\tau]} \cong M_0 \underset{K}{\otimes} K^{\mathrm{sep}}$ as $C \underset{k}{\otimes} K^{\mathrm{sep}}$-modules if and only if σ leaves $\mathrm{Tr}(x|M_0)$ invariant for all $x \in C$. But this follows from Corollary 1.1.2.14 as soon as we observe that

$$\begin{aligned}\sigma \,\mathrm{Tr}_K(x|M_0) &= \sigma \,\mathrm{Tr}_{K^{\mathrm{sep}}}(x|M_0 \underset{K}{\otimes} K^{\mathrm{sep}}) \\ &= \mathrm{Tr}_{K^{\mathrm{sep}}}(x|(M_0 \underset{K}{\otimes} K^{\mathrm{sep}}) \underset{K^{\mathrm{sep}},\sigma}{\otimes} K^{\mathrm{sep}})\end{aligned}$$

for all $x \in C$. □

To classify finitely generated $C \underset{k}{\otimes} K$-modules without the assumption that $\mathrm{char}(k) = 0$, let us introduce the *determinantal conditions* used by Kottwitz in the fundamental paper [76]. To avoid dependence on the choice of basis elements, we shall give a definition similar to the one in [104, 3.23(a)].

Definition 1.1.2.17. *Let L_0 be any finitely generated locally free module over a commutative ring R_0, and let $L_0^\vee := \mathrm{Hom}_{R_0}(L_0, R_0)$ be the dual module of L_0 over R_0. Define*

$$R_0[L_0^\vee] := \underset{k\geq 0}{\oplus} \mathrm{Sym}_{R_0}^k(L_0^\vee),$$

and consider the associated vector bundle
$$\mathbb{V}_{L_0} := \mathrm{Spec}(R_0[L_0^\vee])$$
over $\mathrm{Spec}(R)$. *Then, for every* R_0*-algebra* R*, we have canonical isomorphisms*
$$\begin{aligned}\mathbb{V}_{L_0}(R) &\cong \mathrm{Hom}_{\mathrm{Spec}(R_0)}(\mathrm{Spec}(R), \mathbb{V}_{L_0}) \cong \mathrm{Hom}_{R_0\text{-alg.}}(R_0[L_0^\vee], R)\\ &= \mathrm{Hom}_{R_0\text{-alg.}}(\bigoplus_{k\geq 0} \mathrm{Sym}^k_{R_0}(L_0^\vee), R) \cong \mathrm{Hom}_{R_0\text{-mod.}}(L_0^\vee, R) \cong L_0 \underset{R_0}{\otimes} R.\end{aligned}$$
In other words, $\mathbb{V}_{L_0}$ *represents the functor that assigns to each* R_0*-algebra* R *the locally free* R*-module* $L_0 \underset{R_0}{\otimes} R$.

This construction sheafifies and associates with each coherent locally free sheaf $\mathscr{L}$ *over a scheme* S *the dual sheaf* $\mathscr{L}^\vee := \underline{\mathrm{Hom}}_{\mathscr{O}_S}(\mathscr{L}, \mathscr{O}_S)$*, the graded* $\mathscr{O}_S$*-algebra*
$$\mathscr{O}_S[\mathscr{L}^\vee] := \bigoplus_{k\geq 0} \mathrm{Sym}^k_{\mathscr{O}_S}(\mathscr{L}^\vee),$$
and the vector bundle
$$\mathbb{V}_{\mathscr{L}} := \underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathscr{O}_S[\mathscr{L}^\vee])$$
over S.

Definition 1.1.2.18. *Let* $\{\alpha_1, \ldots, \alpha_t\}$ *be any* k*-basis of* C*. Let* $\{\alpha_1^\vee, \ldots, \alpha_t^\vee\}$ *be the* k*-basis of* $C^\vee := \mathrm{Hom}_k(C, k)$ *dual to* $\{\alpha_1, \ldots, \alpha_t\}$ *in the sense that* $\alpha_i^\vee(\alpha_j) = \delta_{ij}$ *for all* $1 \leq i \leq t$ *and* $1 \leq j \leq t$*. Let* $\{X_1, \ldots, X_t\}$ *be a set of free variables over* k*. Then we have a canonical* k*-isomorphism* $k[C^\vee] \cong k[X_1, \ldots, X_t]$ *defined by sending* $\alpha_i^\vee$ *to* X_i *for each* $1 \leq i \leq t$*. For each finitely generated* $C \underset{k}{\otimes} K$*-module* M*, we define a homogeneous polynomial*
$$\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|M} \in K[X_1, \ldots, X_t]$$
by
$$\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|M}(X_1, \ldots, X_t) := \mathrm{Det}_K(X_1\alpha_1 + \cdots + X_t\alpha_t|M),$$
which corresponds to an element
$$\mathrm{Det}_{C|M} \in K[C^\vee] := k[C^\vee] \underset{k}{\otimes} K$$
under the canonical isomorphism $K[C^\vee] \cong K[X_1, \ldots, X_t]$*. This element* $\mathrm{Det}_{C|M}$ *is independent of the choice of the* k*-basis* $\{\alpha_1, \ldots, \alpha_t\}$.

Lemma 1.1.2.19. *If* $[\tau_1] \neq [\tau_2]$*, then* $\mathrm{Det}_{C|W_{[\tau_1]}} \neq \mathrm{Det}_{C|W_{[\tau_2]}}$ *as elements in* $K[C^\vee]$*. Furthermore, they have no common irreducible factors in the unique factorization domain* $K^{\mathrm{sep}}[C^\vee]$*. (Here* $K^{\mathrm{sep}}[C^\vee]$ *is a unique factorization domain because it is isomorphic to a polynomial algebra over* K^{sep}*.)*

Proof. Let $t := [C : k]$, let $t_0 := [E : k]$, and let us take any k-basis $\{\alpha_1, \ldots, \alpha_t\}$ of C such that $\{\alpha_1, \ldots, \alpha_{t_0}\}$ is a k-basis of E. (This is always possible up to a k-linear change of coordinates, which does not affect the statement.) Under the canonical isomorphisms $K[C^\vee] \cong K[X_1, \ldots, X_t]$ and $K[E^\vee] \cong K[X_1, \ldots, X_{t_0}]$, the canonical surjection of K-algebras $K[C^\vee] \twoheadrightarrow K[E^\vee]$ (defined by the canonical embedding $E \hookrightarrow C$ of the center E in C) is identified with the canonical surjection of polynomial algebras $K[X_1, \ldots, X_t] \twoheadrightarrow K[X_1, \ldots, X_{t_0}]$ defined by setting $X_s = 0$ for $s > t_0$. By definition, for $i = 1, 2$, this surjection sends the homogeneous polynomial $\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{[\tau_i]}}$ to the homogeneous polynomial $\mathrm{Det}^{\alpha_1,\ldots,\alpha_{t_0}}_{E|W_{[\tau_i]}}$.

If a nonunit (i.e., nonconstant) element in the polynomial algebra $K^{\mathrm{sep}}[X_1, \ldots, X_t]$ divides both $\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{[\tau_1]}}$ and $\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{[\tau_2]}}$, then this element is homogeneous, and its image under the canonical surjection $K[X_1, \ldots, X_t] \twoheadrightarrow K[X_1, \ldots, X_{t_0}]$ is again nonunit and divides both $\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{[\tau_1]}}(X_1, \ldots, X_{t_0})$ and $\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{[\tau_2]}}(X_1, \ldots, X_{t_0})$. Therefore, to verify the lemma, it suffices to show that the polynomials $\mathrm{Det}^{\alpha_1,\ldots,\alpha_{t_0}}_{E|W_{[\tau_1]}}(X_1, \ldots, X_{t_0})$ and $\mathrm{Det}^{\alpha_1,\ldots,\alpha_{t_0}}_{E|W_{[\tau_2]}}(X_1, \ldots, X_{t_0})$ have no common irreducible factors in the unique factorization domain $K^{\mathrm{sep}}[X_1, \ldots, X_{t_0}]$.

Let $d_{[\tau_i]}$ be the degree of $C \underset{E,[\tau_i]}{\otimes} K_{[\tau_i]}$ over its center $K_{[\tau_i]}$. By (1.1.2.10), we have
$$\begin{aligned}\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{[\tau_i]}}(X_1, \ldots, X_{t_0}) &= \mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{[\tau_i]} \underset{K}{\otimes} K^{\mathrm{sep}}}(X_1, \ldots, X_{t_0})\\ &= \prod_{\tau_i' \in [\tau_i]} \mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{\tau_i'}}(X_1, \ldots, X_{t_0})^{s_{[\tau_i]}}\\ &= \prod_{\tau_i' \in [\tau_i]} (X_1\tau_i'(\alpha_1) + \cdots + X_{t_0}\tau_i'(\alpha_{t_0}))^{s_{[\tau_i]}d_{[\tau_i]}}\end{aligned}$$
in $K^{\mathrm{sep}}[X_1, \ldots, X_{t_0}]$, for $i = 1, 2$. By reason of degree, each homogeneous linear factor
$$X_1\tau_i'(\alpha_1) + \cdots + X_{t_0}\tau_i'(\alpha_{t_0})$$
is irreducible. Suppose we have $\tau_1' \in [\tau_1]$ and $\tau_2' \in [\tau_2]$ such that
$$X_1\tau_1'(\alpha_1) + \cdots + X_{t_0}\tau_1'(\alpha_{t_0}) = X_1\tau_2'(\alpha_1) + \cdots + X_{t_0}\tau_2'(\alpha_{t_0}).$$
Then in particular, $\tau_1' = \tau_2'$ because
$$c_1\tau_1'(\alpha_1) + \cdots + c_{t_0}\tau_1'(\alpha_{t_0}) = c_1\tau_2'(\alpha_1) + \cdots + c_{t_0}\tau_2'(\alpha_{t_0})$$
for all $c_1, \ldots, c_{t_0} \in k$. Since $[\tau_1] \neq [\tau_2]$ are disjoint orbits, this is a contradiction. As a result, $\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{\tau_1}}$ and $\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{C|W_{\tau_2}}$ have no common irreducible factors in the unique factorization domain $K^{\mathrm{sep}}[X_1, \ldots, X_{t_0}]$, as desired. □

Proposition 1.1.2.20. *Two finitely generated* $C \underset{k}{\otimes} K$*-modules* M_1 *and* M_2 *are isomorphic if and only if* $\mathrm{Det}_{C|M_1} = \mathrm{Det}_{C|M_2}$.

Proof. Let us decompose $M_i = \bigoplus_{[\tau]:E\to K_{[\tau]}} W_{[\tau]}^{\oplus m_{[\tau],i}}$ for $i = 1, 2$, as in Corollary 1.1.2.6. Then we have
$$\mathrm{Det}_{C|M_i} = \prod_{[\tau]:E\to K_{[\tau]}} \mathrm{Det}^{m_{[\tau],i}}_{C|W_{[\tau]}}$$
for $i = 1, 2$. By Lemma 1.1.2.19, different factors $\mathrm{Det}_{C|W_{[\tau_1]}}$ and $\mathrm{Det}_{C|W_{[\tau_2]}}$ have no common irreducible factors in the unique factorization domain $K^{\mathrm{sep}}[C^\vee]$. Therefore, $\mathrm{Det}_{C|M_1} = \mathrm{Det}_{C|M_2}$ if and only if $m_{[\tau],1} = m_{[\tau],2}$ for all $[\tau]$, or equivalently if $M_1 \cong M_2$ as $C \underset{k}{\otimes} K$-modules. □

Now suppose R_0 is a commutative noetherian integral domain with fraction field $\mathrm{Frac}(R_0)$. Let $\mathcal{O}$ be an R_0-order in some finite-dimensional $\mathrm{Frac}(R_0)$-algebra A with center F. Suppose the underlying R_0-module $\mathcal{O}$ is free.

Definition 1.1.2.21. *Let* S *be a scheme over* $\mathrm{Spec}(R_0)$*, and let* $\mathscr{M}$ *be any locally free* $\mathscr{O}_S$*-module of finite rank on which* $\mathcal{O}$ *acts by morphisms of* $\mathscr{O}_S$*-modules. Let* $\{\alpha_1, \ldots, \alpha_t\}$ *be any free* R_0*-basis of* $\mathcal{O}$*. Let* $\{\alpha_1^\vee, \ldots, \alpha_t^\vee\}$ *be the free* R_0*-basis of* $\mathcal{O}^\vee = \mathrm{Hom}_{R_0}(\mathcal{O}, R_0)$ *dual to* $\{\alpha_1, \ldots, \alpha_t\}$ *in the sense that* $\alpha_i^\vee(\alpha_j) = \delta_{ij}$ *for all* $1 \leq i \leq t$ *and* $1 \leq j \leq t$*. Let* $\{X_1, \ldots, X_t\}$ *be a set of free variables over* k*. Then we have a canonical* R_0*-isomorphism* $R_0[\mathcal{O}^\vee] \cong R_0[X_1, \ldots, X_t]$ *defined by sending* $\alpha_i^\vee$ *to* X_i *for* $1 \leq i \leq t$*. Define a homogeneous polynomial function*
$$\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{\mathcal{O}|\mathscr{M}} \in \mathscr{O}_S[X_1, \ldots, X_t]$$
by
$$\mathrm{Det}^{\alpha_1,\ldots,\alpha_t}_{\mathcal{O}|\mathscr{M}}(X_1, \ldots, X_t) := \mathrm{Det}_{\mathscr{O}_S}(X_1\alpha_1 + \cdots + X_t\alpha_t|\mathscr{M}),$$

which corresponds to an element

$$\mathrm{Det}_{\mathcal{O}|\mathscr{M}} \in \mathscr{O}_S[\mathcal{O}^\vee] := R_0[\mathcal{O}^\vee] \underset{R_0}{\otimes} \mathscr{O}_S$$

under the canonical isomorphism $\mathscr{O}_S[\mathcal{O}^\vee] \cong \mathscr{O}_S[X_1, \ldots, X_t]$. *This element* $\mathrm{Det}_{\mathcal{O}|\mathscr{M}}$ *is independent of the choice of the free* R_0*-basis* $\{\alpha_1, \ldots, \alpha_t\}$.

Remark 1.1.2.22. Definition 1.1.2.21 works in particular when $S = \mathrm{Spec}(R)$ and R is a commutative algebra over R_0. In this case we may consider the same definition for a locally free module M over R, and write $\mathrm{Det}_{\mathcal{O}|M} \in R[\mathcal{O}^\vee] := R_0[\mathcal{O}^\vee] \underset{R_0}{\otimes} R$ instead of $\mathrm{Det}_{\mathcal{O}|\mathscr{M}} \in \mathscr{O}_S[\mathcal{O}^\vee]$. If $S = \mathrm{Spec}(k)$, where k is a field, where $C := \mathcal{O} \underset{R_0}{\otimes} k$ is semisimple over k, and where $E := F \underset{R_0}{\otimes} k$ is a separable k-algebra, then Definition 1.1.2.21 agrees with Definition 1.1.2.18 if we consider C, E, and k as before with $K = k$.

1.1.3 Projective Modules

Lemma 1.1.3.1. *Let* R_0 *be a commutative noetherian integral domain with fraction field* $\mathrm{Frac}(R_0)$*, and let* $\mathcal{O}$ *be an* R_0*-order in some finite-dimensional* $\mathrm{Frac}(R_0)$*-algebra* A *with center* F*, such that the underlying* R_0*-module* $\mathcal{O}$ *is projective. Let* R *be a noetherian local* R_0*-algebra with residue field* k*. Let* M_1 *and* M_2 *be two finitely generated* $\mathcal{O} \underset{R_0}{\otimes} R$*-modules such that* M_1 *is projective as an* $\mathcal{O} \underset{R_0}{\otimes} R$*-module and* M_2 *is projective as an* R*-module. Then* $M_1 \cong M_2$ *as* $\mathcal{O} \underset{R_0}{\otimes} R$*-modules if and only if* $M_1 \underset{R}{\otimes} k \cong M_2 \underset{R}{\otimes} k$ *as* $\mathcal{O} \underset{R_0}{\otimes} k$*-modules.*

Proof. The direction from R to k is obvious. Conversely, suppose there is an isomorphism $\bar{f} : M_1 \underset{R}{\otimes} k \cong M_2 \underset{R}{\otimes} k$ of $\mathcal{O} \underset{R_0}{\otimes} k$-modules. Since M_1 is projective as an $\mathcal{O} \underset{R_0}{\otimes} R$-module, we have a morphism $f : M_1 \to M_2$ of $\mathcal{O} \underset{R_0}{\otimes} R$-modules such that $f \underset{R}{\otimes} k = \bar{f}$. Note that this is, in particular, a morphism of R-modules. Since the underlying R_0-module $\mathcal{O}$ is projective over R_0, it is a direct summand of a free R_0-module. Therefore, being projective as an $\mathcal{O} \underset{R_0}{\otimes} R$-module, or equivalently, being a direct summand of a free $\mathcal{O} \underset{R_0}{\otimes} R$-module, implies being a direct summand of a free R-module, or equivalently, being projective as a R-module. Now, the projectivity of the two R-modules implies that f is an isomorphism by the usual Nakayama's lemma for R-modules. (There is, nevertheless, a noncommutative version of Nakayama's lemma. See [107, Thm. 6.11]. The proof is the same well-known one.) ☐

Remark 1.1.3.2. For $\mathcal{O} \underset{R_0}{\otimes} R$-modules, being projective, namely, being a direct summand of a free module, is not equivalent to being locally free.

Suppose R_0 is the ring of integers in a number field. In particular, R_0 is an excellent Dedekind domain, and the underlying R_0-module $\mathcal{O}$ is projective over R_0. Let $\mathrm{Disc} = \mathrm{Disc}_{\mathcal{O}/R_0}$ be the discriminant of $\mathcal{O}$ over R_0 (see Definition 1.1.1.6). Let k be either a field of characteristic $p = 0$ or a finite field of characteristic $p > 0$, together with a fixed nonzero ring homomorphism $R_0 \to k$ of kernel a prime ideal $\mathfrak{p}$ of R_0. Let Λ be the noetherian complete local R_0-algebra with residue field k, such that a noetherian complete local algebra with residue field k is a local R_0-algebra if and only if it is a local Λ-algebra (with compatible structural morphisms to k) (see, for example, Lemma B.1.1.11). Concretely, $\Lambda = k$ if $p = 0$, and Λ is the unique unramified extension with residue field k of the completion of the localization of R_0 at $\mathfrak{p}$, if $p > 0$. (In particular, $\Lambda = W(k)$ if $R_0 = \mathbb{Z}$ and $p > 0$.)

Suppose $\mathfrak{p}$ is unramified in $\mathcal{O}$. By Proposition 1.1.1.21, we know that $\mathcal{O} \underset{R_0}{\otimes} k$ is a separable algebra over k, and $\mathcal{O}_\Lambda$ is a maximal order over Λ. By [107, Thm. 10.5] (see also Section 1.1.2), we have decompositions

$$\mathcal{O}_F \underset{R_0}{\otimes} \Lambda \cong \prod_\tau \mathcal{O}_{F_\tau}$$

and

$$\mathcal{O} \underset{R_0}{\otimes} \Lambda \cong \prod_\tau \mathcal{O}_\tau$$

into simple factors, where τ is parameterized by orbits of embeddings of F into a separable closure of $\mathrm{Frac}(\Lambda)$. In the first decomposition, each simple factor $\mathcal{O}_{F_\tau}$ is the maximal Λ-order in some separable field extension F_τ of $\mathrm{Frac}(\Lambda)$.

If $p > 0$, then by Proposition 1.1.1.21 again, we know that each $\mathcal{O}_\tau$ is isomorphic to $\mathrm{M}_{d_\tau}(\mathcal{O}_{F_\tau})$ for some integer $d_\tau \geq 1$. We may identify $\mathcal{O}_\tau$ with $\mathrm{End}_{\mathcal{O}_{F,\tau}}(M_\tau)$, where $M_\tau := \mathcal{O}_{F_\tau}^{\oplus d_\tau}$ can be considered as an $\mathcal{O}_\Lambda$-module via the projection from $\mathcal{O}_\Lambda$ to $\mathcal{O}_\tau$. If $p = 0$, we take M_τ to be the unique simple module of $\mathcal{O}_\tau$ given by Corollary 1.1.2.6, although we can no longer assume that $\mathcal{O}_\tau$ is a matrix algebra.

Let R be any noetherian local Λ-algebra with residue field k. We shall allow two different interpretations of subscripts R:

Convention 1.1.3.3.
1. *For objects such as* $\mathcal{O}$, $\mathcal{O}_F$, Diff^{-1}, *etc. that are defined over* R_0*, subscripts such as* $\mathcal{O}_R$, $\mathcal{O}_{F,R}$, $(\mathrm{Diff}^{-1})_R$*, etc. will stand for base changes from* R_0 *to* R*.*
2. *For objects such as* $\mathcal{O}_\tau$, $\mathcal{O}_{F_\tau}$, M_τ*, etc. that are defined over* Λ*, subscripts such as* $\mathcal{O}_{\tau,R}$, $\mathcal{O}_{F_\tau,R}$, $M_{\tau,R}$*, etc. will stand for base changes from* Λ *to* R*.*

Lemma 1.1.3.4. *With assumptions as above, every finitely generated projective* $\mathcal{O}_R$*-module* M *is isomorphic to* $\underset{\tau}{\oplus} M_{\tau,R}^{\oplus m_\tau}$ *for some uniquely determined integers* $m_\tau \geq 0$*.*

Proof. By Lemma 1.1.3.1, we may replace M with $M \underset{R}{\otimes} k$ and reduce the problem to the classification of finite-dimensional modules over a finite-dimensional semisimple algebra with separable center over a field. This is already addressed in Corollary 1.1.2.6, with the W_τ there replaced with $M_{\tau,R} \underset{R}{\otimes} k$. ☐

Motivated by Lemma 1.1.3.4,

Definition 1.1.3.5. *With assumptions as above, the* $\mathcal{O}_R$***-multirank*** *of a finitely generated projective* $\mathcal{O}_R$*-module* M *is defined to be the tuple* $(m_\tau)_\tau$ *of integers appearing in the decomposition* $M \cong \underset{\tau}{\oplus} M_{\tau,R}^{\oplus m_\tau}$ *in Lemma 1.1.3.4.*

It is useful to have the following generalized form of the Noether–Skolem theorem in our context:

Lemma 1.1.3.6. *With the setting as above, suppose* M *is any finitely generated projective* $\mathcal{O}_R$*-module. Let* C *be any* $\mathcal{O}_{F,R}$*-subalgebra of* $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ *containing the image of* $\mathcal{O}_{F,R}$*. Then each* C*-automorphism of* $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ *(namely, an automorphism inducing the identity on* C*) is an inner automorphism* $\mathrm{Int}(a)$ *for some invertible element* a *in* $\mathrm{End}_C(M)$*.*

The proof we give here is an imitation of the proof in [107, Thm. 7.21].

Proof of Lemma 1.1.3.6. For simplicity, let us denote $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ by E, and denote the image of $\mathcal{O}_{F,R}$ in E by Z. Let $\varphi : E \xrightarrow{\sim} E$ be any C-automorphism of E.

By definition, M is an E-module. Let M' be the E-module with the same elements as M, but with the E-action twisted by φ. Namely, for $b \in E$ and $m \in M$, we replace the action $m \mapsto bm$ with $m \mapsto \varphi(b)m$. Since φ is C-linear, M and M' are isomorphic as C-modules. By assumption, and by Lemma 1.1.3.4, E is the base change $\tilde{\mathcal{O}}_R$ of a product $\tilde{\mathcal{O}}$ of matrix algebras over $\mathcal{O}_{F,\Lambda}$. By Proposition 1.1.1.5, $\tilde{\mathcal{O}}$ is a maximal order over Λ. By Lemma 1.1.3.1, M and M' are isomorphic as E-modules if $M \underset{R}{\otimes} k$ and $M' \underset{R}{\otimes} k$ are isomorphic as $E \underset{R}{\otimes} k$-modules. Since the prime ideal $\mathfrak{p} = \ker(R_0 \to k)$ of R_0 is unramified in $\mathcal{O}$, which in particular implies that $\mathfrak{p}$ is unramified in $\mathcal{O}_F$, we know that the center $Z \underset{R}{\otimes} k$ of the matrix algebra $E \underset{R}{\otimes} k$ is separable over k. In particular, the classification in Section 1.1.2 (based on Lemma 1.1.2.4) shows that $M \underset{R}{\otimes} k$ and $M' \underset{R}{\otimes} k$ are isomorphic as $E \underset{R}{\otimes} k$-modules if they are isomorphic as $Z \underset{R}{\otimes} k$-modules. This is true simply because M and M' are isomorphic as C-modules, and because C contains Z. As a result, we see that there is an isomorphism $\theta : M \xrightarrow{\sim} M'$ of E-modules, which by definition satisfies $\theta(bm) = \varphi(b)\theta(m)$ for all $b \in E$ and $m \in M$. Since M and M' are identical as $\mathcal{O}_{F,R}$-modules, we may interpret θ as an element a in $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$. Since $\varphi(b) = b$ for all $b \in C$, we see that a lies in $\mathrm{End}_C(M)$. Since θ is an isomorphism, we see that a is invertible. Finally, $\theta(bm) = \varphi(b)\theta(m)$ for all $m \in M$ means $ab = \varphi(b)a$, or $\varphi(b) = aba^{-1} = \mathrm{Int}(a)(b)$, for all $b \in E$. This shows that $\varphi = \mathrm{Int}(a)$, as desired. □

1.1.4 Generalities on Pairings

In this section, let R_0 be a commutative noetherian integral domain with fraction field $\mathrm{Frac}(R_0)$, and let $\mathcal{O}$ be an R_0-order in some finite-dimensional $\mathrm{Frac}(R_0)$-algebra A with center F. Suppose moreover that A is equipped with an *involution* $\star$ sending $\mathcal{O}$ to itself. Then it is automatic that $\star$ sends the center F of A to itself. The elements in F fixed by $\star$ form a subalgebra, which we shall denote by F^+.

Let R be any commutative R_0-algebra, and let M be any R-module. We shall adopt Convention 1.1.3.3 in this section, so that for example, $\mathcal{O}_R$ stands for $\mathcal{O} \underset{R_0}{\otimes} R$.

Let $\mathrm{Diff}^{-1} = \mathrm{Diff}^{-1}_{\mathcal{O}/R_0}$ (see Definition 1.1.1.8). By definition, the restriction of $\mathrm{Tr}_{A/\mathrm{Frac}(R_0)} : A \to \mathrm{Frac}(R_0)$ to Diff^{-1} defines a morphism

$$\mathrm{Tr}_{\mathcal{O}/R_0} : \mathrm{Diff}^{-1} \to R_0$$

of R_0-modules. We shall denote by the same notation,

$$\mathrm{Tr}_{\mathcal{O}/R_0} : M_0 \underset{R_0}{\otimes} \mathrm{Diff}^{-1} \to M_0,$$

its natural base change to each R_0-module M_0.

Lemma 1.1.4.1. *Suppose $\mathcal{O}$ is locally free as an R_0-module. Let M_0 be any R_0-module, and let z be any element of $M_0 \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$. If $\mathrm{Tr}_{\mathcal{O}/R_0}(bz) = 0$ for all $b \in \mathcal{O}$, then $z = 0$.*

Proof. By Lemma 1.1.1.9, we may localize and assume that both $\mathcal{O}$ and Diff^{-1} are free as R_0-modules. Let $\{e_i\}_{1\leq i\leq t}$ be any R_0-basis of $\mathcal{O}$, and let $\{f_i\}_{1\leq i\leq t}$ be the dual R_0-basis of Diff^{-1}. The element z in $M_0 \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$ can be written uniquely in the form $z = \sum_{1\leq j\leq t} z_j \otimes f_j$, where $x_j \in M_0$. By assumption, $z_i = \mathrm{Tr}_{\mathcal{O}/R_0}(e_i z) = 0$ for all $1 \leq i \leq t$. This shows that z is zero, as desired. □

Definition 1.1.4.2. *Let R be a commutative R_0-algebra, and let M and N be two R-modules.*

1. *An R-bilinear pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to N$ is called **symmetric** if $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in M$.*
2. *An R-bilinear pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to N$ is called **skew-symmetric** if $\langle x, y\rangle = -\langle y, x\rangle$ for all $x, y \in M$.*
3. *An R-bilinear pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to N$ is called **alternating** if $\langle x, x\rangle = 0$ for all $x \in M$.*
4. *An R-bilinear pairing $(\!|\,\cdot\,,\,\cdot\,|\!) : M \times M \to N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$ is called **Hermitian** if $(\!|x, y|\!) = (\!|y, x|\!)^\star$ and $(\!|x, by|\!) = b(\!|x, y|\!)$ for all $x, y \in M$ and $b \in \mathcal{O}$. Here $\star$ and $\mathcal{O}$ act only on the second factor Diff^{-1} of $N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$.*
5. *An R-bilinear pairing $(\!|\,\cdot\,,\,\cdot\,|\!) : M \times M \to N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$ is called **skew-Hermitian** if $(\!|x, y|\!) = -(\!|y, x|\!)^\star$ and $(\!|x, by|\!) = b(\!|x, y|\!)$ for all $x, y \in M$ and $b \in \mathcal{O}$. Here $\star$ and $\mathcal{O}$ act only on the second factor Diff^{-1} of $N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$.*

Remark 1.1.4.3. An alternating form is always skew-symmetric, but the converse might not be true when 2 is a zero-divisor.

Definition 1.1.4.4. *Let ϵ be either $+1$ or -1. An R-bilinear pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to N$ is called **ϵ-symmetric** if $\langle x, y\rangle = \varepsilon\langle y, x\rangle$ for all $x, y \in M$. An R-bilinear pairing $(\!|\,\cdot\,,\,\cdot\,|\!) : M \times M \to N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$ is called **ϵ-Hermitian** if $(\!|x, y|\!) = \epsilon(\!|y, x|\!)^\star$ and $(\!|x, by|\!) = b(\!|x, y|\!)$ for all $x, y \in M$ and $b \in \mathcal{O}$.*

Lemma 1.1.4.5. *Let ϵ be either $+1$ or -1, let M be a finitely generated $\mathcal{O}_R$-module, and let N be a finitely generated R-module. Suppose $\mathcal{O}$ is locally free over R_0. Then there is a one–one correspondence between the set of ϵ-Hermitian pairings*

$$(\!|\,\cdot\,,\,\cdot\,|\!) : M \times M \to N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$$

(which is $\mathcal{O}_R$-linear in the second variable according to our definition) and the set of ϵ-symmetric pairings

$$\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to N,$$

such that $\langle bx, y\rangle = \langle x, b^\star y\rangle$ for all $x, y \in M$ and $b \in \mathcal{O}_R$. The assignment in one direction can be given explicitly as follows: We associate with each ϵ-Hermitian pairing $(\!|\,\cdot\,,\,\cdot\,|\!) : M \times M \to N \underset{R_0}{\otimes} \mathrm{Diff}^{-1}$ the ϵ-symmetric pairings $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to N$ defined by

$$\langle\,\cdot\,,\,\cdot\,\rangle := \mathrm{Tr}_{\mathcal{O}/R_0}(\!|\,\cdot\,,\,\cdot\,|\!),$$

where $\mathrm{Tr}_{\mathcal{O}/R_0}$ here means $\mathrm{Tr}_{\mathcal{O}/R_0} : N \underset{R_0}{\otimes} \mathrm{Diff}^{-1} \to N$ (by abuse of notation), the natural base change of $\mathrm{Tr}_{\mathcal{O}/R_0} : \mathrm{Diff}^{-1} \to R_0$.

Proof. It is obvious that if $(\!|\cdot,\cdot|\!)$ is ϵ-Hermitian, then the associated $\langle\cdot,\cdot\rangle := \mathrm{Tr}_{\mathcal{O}/R_0}(\!|\cdot,\cdot|\!)$ is ϵ-symmetric and satisfies $\langle bx,y\rangle = \langle x,b^\star y\rangle = \epsilon\langle b^\star y,x\rangle$ for all $x,y\in L$ and $b\in\mathcal{O}$.

If $(\!|\cdot,\cdot|\!)' : M\times M \to N\underset{R_0}{\otimes}\mathrm{Diff}^{-1}$ is another ϵ-Hermitian pairing such that $\mathrm{Tr}_{\mathcal{O}/R_0}(\!|x,y|\!) = \mathrm{Tr}_{\mathcal{O}/R_0}(\!|x,y|\!)'$ for all $x,y\in M$, then we have $\mathrm{Tr}_{\mathcal{O}/R_0}[b(\!|x,y|\!)] = \mathrm{Tr}_{\mathcal{O}/R_0}(\!|x,by|\!) = \mathrm{Tr}_{\mathcal{O}/R_0}(\!|x,by|\!)' = \mathrm{Tr}_{\mathcal{O}/R_0}[b(\!|x,y|\!)']$ for all $x,y\in M$ and $b\in\mathcal{O}$. By Lemma 1.1.4.1, this implies that $(\!|x,y|\!) = (\!|x,y|\!)'$ for all $x,y\in L$. This shows the injectivity of the association using only $\mathcal{O}_R$-linearity in the second variable.

In the remaining proof, let us assume that $\mathcal{O}_R$ is free over R by localization. If the result is true after all localizations, then it is also true before localization, because the modules M and N we consider are finitely generated over R.

Let $\{e_i\}_{1\le i\le t}$ be any R_0-basis of $\mathcal{O}$, and let $\{f_i\}_{1\le i\le t}$ be the dual R_0-basis of Diff^{-1} given by Lemma 1.1.1.9. Then we can write each $(\!|\cdot,\cdot|\!) : M\times M\to N\underset{R_0}{\otimes}\mathrm{Diff}^{-1}$ as a sum $(\!|\cdot,\cdot|\!) = \sum\limits_{1\le i\le t}[\langle\cdot,\cdot\rangle_i\ f_i]$, where $\langle\cdot,\cdot\rangle_i : M\times M\to N$ is determined by taking $\langle x,y\rangle_i = \mathrm{Tr}_{\mathcal{O}/R_0}[e_i(\!|x,y|\!)]$ for all $x,y\in M$. By $\mathcal{O}_R$-linearity of $(\!|\cdot,\cdot|\!)$ in the second variable, this means $\langle x,y\rangle_i = \mathrm{Tr}_{\mathcal{O}/R_0}(\!|x,e_iy|\!) = \langle x,e_iy\rangle$, and so $(\!|x,y|\!) = \sum\limits_{1\le i\le t}[\langle x,e_iy\rangle\ f_i]$ for all $x,y\in M$.

Now we are ready to show the surjectivity: If $\langle\cdot,\cdot\rangle := \mathrm{Tr}_{\mathcal{O}/R_0}(\!|\cdot,\cdot|\!)$ is any ϵ-symmetric pairing such that $\langle bx,y\rangle = \langle x,b^\star y\rangle$ for all $x,y\in M$ and $b\in\mathcal{O}_R$. Consider $(\!|\cdot,\cdot|\!) : M\times M\to N\underset{R}{\otimes}(\mathrm{Diff}^{-1})_R$ defined by $(\!|x,y|\!) = \sum\limits_{1\le i\le t}[\langle x,e_iy\rangle\ f_i]$ for all $x,y\in M$. Suppose $1 = \sum\limits_{1\le k\le t}u_ke_k$, where $u_k\in R$. Then $\mathrm{Tr}_{\mathcal{O}/R_0} f_i = \mathrm{Tr}_{\mathcal{O}/R_0}(\sum\limits_{1\le k\le t}u_ke_kf_i) = u_i$, and we have $\mathrm{Tr}_{\mathcal{O}/R_0}(\!|x,y|\!) = \sum\limits_{1\le i\le t}[\langle x,e_iy\rangle\,\mathrm{Tr}_{\mathcal{O}/R_0} f_i] = \langle x,[\sum\limits_{1\le i\le t}u_ie_i]y\rangle = \langle x,y\rangle$.

Moreover, $(\!|\cdot,\cdot|\!)$ is $\mathcal{O}_R$-linear in the second variable: Given any $b\in\mathcal{O}_R$, set $a_{ij} := \mathrm{Tr}_{\mathcal{O}/R_0}(e_ibf_j)$ for each $1\le i\le t$ and $1\le j\le t$. Then $e_ib = \sum\limits_{1\le j\le t}a_{ij}e_j$ for all $1\le i\le t$, and $bf_j = \sum\limits_{1\le i\le t}a_{ij}f_i$ for all $1\le j\le t$. Consequently, $(\!|x,by|\!) = \sum\limits_{1\le i\le t}\langle x,e_iby\rangle f_i = \sum\limits_{1\le i\le t,1\le j\le t}\langle x,e_jy\rangle a_{ij}f_i = \sum\limits_{1\le j\le t}\langle x,ye_j\rangle bf_j = b(\!|x,y|\!)$.

Finally, since $\{e_i^\star\}_{1\le i\le t}$ is also a basis for $\mathcal{O}_R$ over R with $\{f_i^\star\}_{1\le i\le t}$ its dual basis for $(\mathrm{Diff}^{-1})_R$ over R respect to $\mathrm{Tr}_{\mathcal{O}/R_0}$, we can also consider $(\!|\cdot,\cdot|\!)' : M\times M\to N\underset{R}{\otimes}(\mathrm{Diff}^{-1})_R$ defined by $(\!|x,y|\!)' = \sum\limits_{1\le i\le t}[\langle x,e_i^\star y\rangle\ f_i^\star]$ for all $x,y\in M$, which then also satisfies $\mathrm{Tr}_{\mathcal{O}/R_0}(\!|\cdot,\cdot|\!)' = \langle\cdot,\cdot\rangle$ and $\mathcal{O}_R$-linearity in the second variable. By the injectivity above (using only $\mathcal{O}_R$-linearity in the second variable), we have $(\!|\cdot,\cdot|\!) = (\!|\cdot,\cdot|\!)'$. As a result, we have $(\!|y,x|\!) = (\!|y,x|\!)' = \sum\limits_{1\le i\le t}[\langle y,e_i^\star x\rangle f_i^\star] = \sum\limits_{1\le i\le t}[\langle e_iy,x\rangle f_i^\star] = \epsilon\sum\limits_{1\le i\le t}\langle x,e_iy\rangle f_i^\star = \epsilon(\!|x,y|\!)^\star$. □

Definition 1.1.4.6. *Suppose $\mathcal{O}$ is locally free over R_0. Let R be a commutative R_0-algebra, let M be a finitely generated $\mathcal{O}_R$-module, and let N be a finitely generated R-module. An R-bilinear pairing $\langle\cdot,\cdot\rangle : M\times M\to N$ is called an **$(\mathcal{O}_R,\star)$-pairing**, or simply an $\mathcal{O}_R$-pairing, if it satisfies $\langle bx,y\rangle = \langle x,b^\star y\rangle$ for all $x,y\in M$ and $b\in\mathcal{O}_R$.*

Definition 1.1.4.7. *Suppose $\mathcal{O}$ is locally free over R_0. Let R be a commutative R_0-algebra. A **symplectic $\mathcal{O}_R$-module** $(M,\langle\cdot,\cdot\rangle,N)$ is a $\mathcal{O}_R$-module M together with an **alternating** $\mathcal{O}_R$-pairing*

$$\langle\cdot,\cdot\rangle : M\times M\to N$$

(see Definitions 1.1.4.2 and 1.1.4.6), where M and N are both finitely generated R-modules.

*Suppose N is locally free of rank one over R. We say $(M,\langle\cdot,\cdot\rangle,N)$ is **nondegenerate** (resp. **self-dual**) if the pairing $\langle\cdot,\cdot\rangle$ is **nondegenerate** (resp. **perfect**), in the sense that the R-module morphism*

$$M\to\mathrm{Hom}_R(M,N) : x\mapsto(y\mapsto\langle x,y\rangle)$$

induced by the pairing is an injection (resp. an isomorphism).

If $N=R$, or if it is clear from the context, then we often omit N from the notation, and simply write $(M,\langle\cdot,\cdot\rangle)$.

Definition 1.1.4.8. *A **symplectic morphism***

$$f : (M_1,\langle\cdot,\cdot\rangle_1,N_1)\to(M_2,\langle\cdot,\cdot\rangle_2,N_2)$$

is a pair of morphisms

$$(f : M_1\to M_2, \nu(f) : N_1\to N_2)$$

*such that $\langle f(x),f(y)\rangle_2 = \nu(f)\langle x,y\rangle_1$ for all $x,y\in M_1$. A symplectic morphism $(f,\nu(f))$ is a **symplectic isomorphism** if both f and $\nu(f)$ are isomorphisms.*

Remark 1.1.4.9. The datum of a symplectic isomorphism consists of not only the morphism f between the underlying modules, but also the morphism $\nu(f)$ between the values (which is not always determined by f). We are enforcing an abuse of notation here.

Definition 1.1.4.10. *Conventions as above, assume moreover that R is a noetherian integral domain. A **symplectic $\mathcal{O}_R$-lattice** $(M,\langle\cdot,\cdot\rangle,N)$ is a symplectic $\mathcal{O}_R$-module whose underlying $\mathcal{O}_R$-module M is an R-lattice.*

Definition 1.1.4.11. *Let $(M,\langle\cdot,\cdot\rangle,N)$ be a nondegenerate symplectic $\mathcal{O}_R$-lattice with values in a locally free sheaf N of rank one over a noetherian integral domain R. The **dual lattice** $M^\#$ (with respect to $\langle\cdot,\cdot\rangle$ and N) is defined by*

$$M^\# := \{x\in M\underset{R}{\otimes}\mathrm{Frac}(R) : \langle x,y\rangle\in N\ \forall y\in M\}.$$

By definition, the dual lattice contains M as a sublattice.

Definition 1.1.4.12. *Let M_1 and M_2 be two finitely generated $\mathcal{O}_R$-modules with two respective $\mathcal{O}_R$-pairings $\langle\cdot,\cdot\rangle_i : M_i\times M_i\to N$, where $i=1,2$, with images in the same finitely generated R-module N. For simplicity, let us use the same notation $(M_i,\langle\cdot,\cdot\rangle_i,N)$ as in the case of symplectic $\mathcal{O}_R$-modules. The **orthogonal direct sum** of $(M_i,\langle\cdot,\cdot\rangle_i,N)$, for $i=1,2$, denoted by*

$$(M_1,\langle\cdot,\cdot\rangle_1,N)\overset{\perp}{\oplus}(M_2,\langle\cdot,\cdot\rangle_2,N),$$

is a triple $(M,\langle\cdot,\cdot\rangle,N)$ whose underlying $\mathcal{O}_R$-module M is $M_1\oplus M_2$ and whose pairing

$$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_1\overset{\perp}{\oplus}\langle\cdot,\cdot\rangle_2$$

is defined such that

$$\langle(x_1,x_2),(y_1,y_2)\rangle = \langle x_1,y_1\rangle_1 + \langle x_2,y_2\rangle_2$$

for all $(x_1,x_2),(y_1,y_2)\in M_1\overset{\perp}{\oplus}M_2$.

If $N=R$, or if it is clear from the context, then we often omit N from the notation.

Lemma 1.1.4.13. *Let M be any $\mathcal{O}_R$-module. Let $M^\vee := \operatorname{Hom}_R(M, R)$ denote the dual module of M with an $\mathcal{O}_R$-action given by $(bf)(y) = f(b^\star y)$ for all $b \in \mathcal{O}_R$ and $f \in \operatorname{Hom}_R(M, R)$. Then the canonical pairing*

$$M \times M^\vee \to R : (x, f) \mapsto f(x)$$

defines a canonical pairing

$$\langle \cdot , \cdot \rangle_{\text{can.}} : (M \oplus M^\vee) \times (M \oplus M^\vee) \to R : ((x_1, f_1), (x_2, f_2)) \mapsto (f_2(x_1) - f_1(x_2))$$

which gives $M \oplus M^\vee$ the canonical structure of a symplectic $\mathcal{O}_R$-module. If the canonical morphism $M \to (M^\vee)^\vee$ is an isomorphism, then $(M \oplus M^\vee, \langle \cdot , \cdot \rangle_{\text{can.}})$ is self-dual.

It is useful to have an interpretation of the pairings we shall consider in terms of anti-automorphisms of the endomorphism algebra of $\mathcal{O}_R$-modules. Let M be a finitely generated $\mathcal{O}_R$-module, and let N be locally free of rank one over R. Having an R-bilinear pairing

$$\langle \cdot , \cdot \rangle : M \times M \to N$$

(which for the moment we allow to be symmetric, skew-symmetric, or neither) is equivalent to having an R-linear morphism

$$\langle \cdot , \cdot \rangle^* : M \to \operatorname{Hom}_R(M, N) : x \mapsto (y \mapsto \langle x, y \rangle).$$

To say that we have a *perfect* pairing $\langle \cdot , \cdot \rangle$ is equivalent to requiring that $\langle \cdot , \cdot \rangle^*$ is an *isomorphism*. Once we know that $\langle \cdot , \cdot \rangle^*$ is an isomorphism, we can define an anti-automorphism $\maltese$ of $\operatorname{End}_R(M)$ by sending an endomorphism $b : M \to M$ to $b^\maltese$ defined by the composition

$$b^\maltese := (\langle \cdot , \cdot \rangle^*)^{-1} \circ b^\vee \circ \langle \cdot , \cdot \rangle^*.$$

In other words, we have $\langle x, by \rangle = \langle b^\maltese x, y \rangle$ for all $x, y \in M$ and $b \in \operatorname{End}_R(M)$.

If we equip $\operatorname{Hom}_R(M, N)$ with an action of $\mathcal{O}_R$ given by $(bf)(y) = f(b^\star y)$ for all $b \in \mathcal{O}_R$ and $f \in \operatorname{Hom}_R(M, N)$, then the condition that $\langle bx, y \rangle = \langle x, b^\star y \rangle$ means exactly that $\langle \cdot , \cdot \rangle^*$ is $\mathcal{O}_R$-linear. In this case, we have $b \in \mathcal{O}_R$ and $b^\maltese = b^\star \in \mathcal{O}_R$. Hence $\maltese$ maps the image of $\mathcal{O}_R$ in $\operatorname{End}_R(M)$ to itself and induces the same involution as that induced by $\star$. Since $\mathcal{O}_{F,R}$ is the center of $\mathcal{O}_R$, and since $\operatorname{End}_{\mathcal{O}_R}(M)$ and $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$ are the respective centralizers of the images of $\mathcal{O}_R$ and $\mathcal{O}_{F,R}$, we see that each of them is mapped to itself under $\maltese$. For simplicity, we shall denote the restrictions of the anti-automorphism $\maltese$ to $\operatorname{End}_{\mathcal{O}_R}(M)$ and $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$ by the same notation.

The general structure of $\operatorname{End}_{\mathcal{O}_R}(M)$ and $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$ for arbitrary $\mathcal{O}_R$-modules can be rather complicated. However, when M is finitely generated and projective, and when $\mathcal{O}_R$ satisfies certain reasonably strong conditions, there is a nice classification of pairings (with values in R) in terms of involutions on $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$. We shall explore this classification in the next section.

1.1.5 Classification of Pairings by Involutions

With the setting as in Section 1.1.4, suppose moreover that R_0 is the ring of integers in a number field. Let k be either a field of characteristic $p = 0$ or a finite field of characteristic $p > 0$, together with a fixed nonzero ring homomorphism $R_0 \to k$ with kernel a prime ideal $\mathfrak{p}$ of R_0. Let Λ be the noetherian complete local R_0-algebra with residue field k, such that a noetherian complete local algebra with residue field k is a local R_0-algebra if and only if it is a local Λ-algebra, as in Section 1.1.3. Suppose $\mathfrak{p}$ is unramified in $\mathcal{O}$.

Let R be any noetherian local Λ-algebra with residue field k, with Convention 1.1.3.3 as in Sections 1.1.3 and 1.1.4. Let M be a finitely generated *projective* $\mathcal{O}_R$-module. For convenience, we shall denote by $\overline{\mathcal{O}}_R$ (resp. $\overline{\mathcal{O}}_{F,R}$) the image of $\mathcal{O}_R$ (resp. $\mathcal{O}_{F,R}$) in $\operatorname{End}_R(M)$. Then $\overline{\mathcal{O}}_R$ (resp. $\overline{\mathcal{O}}_{F,R}$) is the product of those $\mathcal{O}_{\tau,R}$ (resp. $\mathcal{O}_{F_\tau,R}$) acting faithfully on M.

Since R is local, every locally free module N of rank one over R is isomorphic to R. Therefore, for the purpose of classifying perfect pairings $\langle \cdot , \cdot \rangle : M \times M \to N$ up to isomorphism, it suffices to assume that $N = R$.

Definition 1.1.5.1. *With assumptions as above, we say that two perfect $\mathcal{O}_R$-pairings $\langle \cdot , \cdot \rangle_i : M \times M \to R$, where $i = 1, 2$, are **weakly isomorphic** if $\langle \cdot , \cdot \rangle_1^*$ and $\langle \cdot , \cdot \rangle_2^*$ differ only up to multiplication by an element in $\overline{\mathcal{O}}_{F,R}^\times$. We say they are **weakly symplectic isomorphic** if the pairings are alternating pairings.*

Lemma 1.1.5.2. *With assumptions as above, two perfect $\mathcal{O}_R$-pairings $\langle \cdot , \cdot \rangle_i : M \times M \to R$, where $i = 1, 2$, are weakly isomorphic if and only if the anti-automorphisms $\maltese_i$ of $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$ determined by the pairings are identical: $\maltese_1 = \maltese_2$.*

Proof. Consider $a := (\langle \cdot , \cdot \rangle_1^*)^{-1} \circ \langle \cdot , \cdot \rangle_2^*$ as an element in $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$. The inner automorphism $\operatorname{Int}(a)$ of $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$ defined by $b \mapsto a \circ b \circ a^{-1}$ satisfies $\operatorname{Int}(a) = \maltese_1 \circ (\maltese_2)^{-1}$. Thus, $\maltese_1 = \maltese_2$ if and only if $\operatorname{Int}(a) = \operatorname{Id}_{\operatorname{End}_{\mathcal{O}_{F,R}}(M)}$, which is the case exactly when a is (an invertible element) in the center $\overline{\mathcal{O}}_{F,R}$ of $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$. $\square$

Note that we have made a choice of the two variables when defining $\langle \cdot , \cdot \rangle^*$: Alternatively, if we take $x \mapsto (y \mapsto \langle y, x \rangle)$, then we obtain an anti-automorphism $\maltese'$ such that $\langle b(y), x \rangle = \langle y, b^{\maltese'}(x) \rangle$ for all $x, y \in M$ and $b \in \operatorname{End}_R(M)$. Then we have $\langle x, b(y) \rangle = \langle b^\maltese(x), y \rangle = \langle x, (b^\maltese)^{\maltese'}(y) \rangle$ for all $x, y \in M$ and $b \in \operatorname{End}_R(M)$. In other words, we have $\maltese' \circ \maltese = \operatorname{Id}_{\operatorname{End}_R(M)}$.

If $(\maltese)^2 = \operatorname{Id}_{\operatorname{End}_R(M)}$, namely, if $\maltese$ is an involution of $\operatorname{End}_R(M)$, then $\maltese' = \maltese$, and hence, if we repeat the proof of Lemma 1.1.5.2 with $\mathcal{O}_R$ replaced with R, there is some $\gamma \in R^\times$ such that $\langle x, y \rangle = \langle y, \gamma x \rangle$ for all $x, y \in M$. Then $\langle x, y \rangle = \langle y, \gamma x \rangle = \langle \gamma x, \gamma y \rangle = \langle x, \gamma^2 y \rangle$ for all $x, y \in M$ implies $\gamma^2 = 1$. As a result, we see that the anti-automorphism $\maltese$ it induces on $\operatorname{End}_R(M)$ is an involution if and only if there is an element $\gamma \in R^\times$ such that $\gamma^2 = 1$ and $\langle x, y \rangle = \langle y, \gamma x \rangle$ for all $x, y \in M$. (The converse is clear.)

If we only consider the restriction of the anti-automorphism $\maltese$ to $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$, then we may only conclude in the argument above that $\maltese$ is an involution if and only if there is some $\gamma \in \mathcal{O}_{F,R}^\times$ such that $\gamma^\star \gamma = 1$ and $\langle x, y \rangle = \langle y, \gamma x \rangle$ for all $x, y \in M$.

If $F = F^+$, in which case $\star$ acts trivially on F, this implies as above that $\langle x, y \rangle = \langle y, \gamma x \rangle$ for some $\gamma \in \overline{\mathcal{O}}_{F,R}^\times$ with $\gamma^2 = 1$. If we write $\mathcal{O}_{F,R} = \prod\limits_\tau \mathcal{O}_{F_\tau}$ and write γ accordingly as $\gamma = (\gamma_\tau)$, then we see that $\gamma_\tau^2 = 1$ for all τ. The case where $\gamma_\tau = 1$ (resp. $\gamma_\tau = -1$) for all τ implies, in particular, that $\langle \cdot , \cdot \rangle$ is symmetric (resp. alternating). More generally,

Definition 1.1.5.3. *Let $\epsilon \in \overline{\mathcal{O}}_{F,R}^\times$ be an element such that $\epsilon^2 = 1$. Then we say that an $\mathcal{O}_R$-pairing $\langle \cdot , \cdot \rangle : M \times M \to R$ is ϵ-symmetric if it satisfies $\langle x, y \rangle = \langle y, \epsilon x \rangle$ for all $x, y \in M$.*

Then we see that a perfect $\mathcal{O}_R$-pairing $\langle \cdot , \cdot \rangle$ induces an involution of $\operatorname{End}_{\mathcal{O}_{F,R}}(M)$ if and only if $\langle \cdot , \cdot \rangle$ is ϵ-symmetric for some $\epsilon \in \overline{\mathcal{O}}_{F,R}^\times$ such that $\epsilon^2 = 1$.

Moreover, if $\langle\cdot,\cdot\rangle$ is ϵ-symmetric, then $\langle x, ry\rangle = \langle ry, \epsilon x\rangle = \langle y, r\epsilon x\rangle = \langle y, \epsilon r x\rangle$ shows that every perfect pairing that is weakly isomorphic to $\langle\cdot,\cdot\rangle$ is also ϵ-symmetric. Therefore it makes sense to consider the following:

Definition 1.1.5.4 (cf. the classification in [72] in the case of algebras)**.** *With assumptions as above, suppose moreover that $F = F^+$. Then we say that an involution $\maltese$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ is **of ϵ-symmetric type** (resp. **of symplectic type**, resp. **of orthogonal type**) if there exists a perfect ϵ-symmetric (resp. alternating, resp. symmetric) $\mathcal{O}_R$-pairing $\langle\cdot,\cdot\rangle : M \times M \to R$ inducing $\maltese$.*

Lemma 1.1.5.5. *Suppose $F = F^+$. Then an alternating $\mathcal{O}_R$-pairing $\langle\cdot,\cdot\rangle : M \times M \to R$ satisfies $\langle x, rx\rangle = 0$ for all $x \in M$ and $r \in \mathcal{O}_{F,R}$.*

Proof. By replacing k with a sufficiently large finite separable field extension, and by replacing Λ accordingly, we may assume that in the product $\mathcal{O}_{F,\Lambda} \cong \prod_\tau \mathcal{O}_{F_\tau}$ we have $\mathcal{O}_{F_\tau} = \Lambda$ for all τ. Then $\mathcal{O}_{F,R} \cong \prod_\tau \mathcal{O}_{F_\tau,R}$ is a product of copies of R, and the lemma is obvious. □

Remark 1.1.5.6. By Lemmas 1.1.5.2 and 1.1.5.5, the definitions of being of ϵ-symmetric, symplectic, and orthogonal types do not depend on the particular perfect pairing we choose that induces the involution.

If * is nontrivial on $\mathcal{O}_F$, we need a different approach. For simplicity, let us assume that F^+ is simple in this case, so that $[F : F^+] = 2$.

Lemma 1.1.5.7. *With assumptions as above, there is an element $e \in \mathcal{O}_{F,R}$ such that $e + e^\star = 1$.*

Proof. By Proposition 1.1.1.21, the assumption that $\mathfrak{p} = \ker(R_0 \to k)$ is unramified in $\mathcal{O}$ implies that $\mathrm{Diff}^{-1}_{\mathcal{O}_{F,\Lambda}/\mathcal{O}_{F^+,\Lambda}} = \mathcal{O}_{F,\Lambda}$. In particular, there exists some $e \in \mathcal{O}_{F,\Lambda}$ such that $\mathrm{Tr}_{\mathcal{O}_{F,\Lambda}/\mathcal{O}_{F^+,\Lambda}}(e) = e + e^\star = 1$. □

Corollary 1.1.5.8. *With assumptions as in Lemma 1.1.5.7, if $\gamma \in \mathcal{O}_{F,R}$ satisfies $\gamma^\star + \gamma = 0$, then there is an element $\delta \in \mathcal{O}_{F,R}$ such that $\gamma = \delta - \delta^\star$.*

Proof. If we take e as in Lemma 1.1.5.7 and take $\delta = e\gamma$, then $\delta - \delta^\star = e\gamma - e^\star\gamma^\star = (e + e^\star)\gamma = \gamma$, as desired. □

Lemma 1.1.5.9. *With assumptions as in Lemma 1.1.5.7, suppose moreover that R is a **complete** noetherian local R_0-algebra, and that the extension F/F^+ is split over k when $\mathrm{char}(k) = p = 0$. Let $\epsilon = \pm 1$, so that $x^\epsilon = x$ or x^{-1} depending on whether $\epsilon = 1$ or -1. If $\gamma \in \mathcal{O}^\times_{F,R}$ is an element such that $\gamma^\star = \gamma^\epsilon$, then $\gamma = \delta(\delta^\star)^\epsilon$ for some $\delta \in \mathcal{O}^\times_{F,R}$.*

The proof we give here is essentially the same as the one for $R = \Lambda$.

Proof. Let us investigate this situation for each τ in the decomposition $\mathcal{O}_{F,R} \cong \prod_\tau \mathcal{O}_{F_\tau}$.

If the involution interchanges $\mathcal{O}_{F_\tau}$ and $\mathcal{O}_{F_{\tau'}}$, and the two factors of $\gamma \in \mathcal{O}^\times_{F,R}$ are of the form $(\gamma_\tau, \gamma_{\tau'}) \in \mathcal{O}_{F_\tau,R} \times \mathcal{O}_{F_{\tau'},R}$, then the condition that $\gamma^\star = \gamma^\epsilon$ shows that $\gamma_{\tau'} = \gamma^\epsilon_\tau$, and hence we may take $\delta \in \mathcal{O}^\times_{F,R}$ with the two factors $(\gamma_\tau, 1)$.

If the involution is nontrivial on F_τ, then it is a nontrivial degree-two unramified extension of some local field F^+_τ. By our assumption that the extension F/F^+ is split over $\Lambda = k$ when $p = 0$, this can happen only when $p > 0$, in which case k is a finite field by assumption. Let $N := \mathrm{Norm}_{\mathcal{O}_{F_\tau,R}/\mathcal{O}_{F^+_\tau,R}} : \mathcal{O}^\times_{F_\tau,R} \to \mathcal{O}^\times_{F^+_\tau,R} : x \mapsto xx^\star$ be the *norm map*, and let $D : \mathcal{O}^\times_{F_\tau,R} \to \mathcal{O}^\times_{F_\tau,R} : x \mapsto x(x^\star)^{-1}$. Hence our goal is to show that $\mathrm{image}(N) = \ker(D)$ and $\mathrm{image}(D) = \ker(N)$.

Let $\mathfrak{m}$ be the maximal ideal of R. Since F_τ is unramified over Λ, we see that $\mathfrak{m}$ generates the maximal ideals in $\mathcal{O}_{F_\tau,R}$ and $\mathcal{O}_{F^+_\tau,R}$. Let k_τ and k^+_τ be the respective residue fields of $\mathcal{O}_{F_\tau,R}$ and $\mathcal{O}_{F^+_\tau,R}$. Let $U^i := \mathcal{O}^\times_{F_\tau,R} \cap [1 + (\mathfrak{m}\cdot\mathcal{O}_{F_\tau,R})^i]$, $U^i_D := \ker(D) \cap U^i = \mathcal{O}^\times_{F^+_\tau,R} \cap [1 + (\mathfrak{m}\cdot\mathcal{O}_{F^+_\tau,R})^i]$, and $U^i_N := \ker(N) \cap U^i$. Let $\mathrm{Gr}^i_U := U_i/U_{i+1}$, $\mathrm{Gr}^i_{U_D} := U^i_D/U^{i+1}_D$, and $\mathrm{Gr}^i_{U_N} := U^i_N/U^{i+1}_N$. Then $\mathrm{Gr}^0_U \cong k^\times_\tau$ and $\mathrm{Gr}^0_{U_D} \cong (k^+_\tau)^\times$. If we identify the multiplication $(1 + x)(1 + y) \equiv 1 + x + y \pmod{\mathfrak{m}^{i+1}}$ with the addition $x + y$, then Gr^i_U, $\mathrm{Gr}^i_{U_D}$, and $\mathrm{Gr}^i_{U_N}$ are all vector spaces over k. The map N (resp. D) sends U^i to U^i_D (resp. to U^i_N) and induces a map $N_i : \mathrm{Gr}^i_U \to \mathrm{Gr}^i_{U_D}$ (resp. $D_i : \mathrm{Gr}^i_U \to \mathrm{Gr}^i_{U_N}$) for all $i \geq 0$.

Let us claim that $N_i : \mathrm{Gr}^i_U \to \mathrm{Gr}^i_{U_D}$ and $D_i : \mathrm{Gr}^i_U \to \mathrm{Gr}^i_{U_N}$ are surjective for all $i \geq 0$. Suppose q is the cardinality of k^+_τ. Then $\mathrm{Gr}^0_U \cong k^\times_\tau$ can be identified with the *cyclic group* of solutions to $x^{q^2-1} = 1$, and the involution $x \mapsto x^\star$ can be identified with $x \mapsto x^q$. From these we see that $N_0(x) = x^{1+q}$ and $D_0(x) = x^{1-q}$, and the assertion follows simply by counting: $(1 + q)(1 - q) = 1 - q^2$. Now suppose $i \geq 1$. Let us first treat the case of N_i. By flatness of $\mathcal{O}_{F_\tau}$ and $\mathcal{O}_{F^+_\tau}$ over Λ, we have $\mathrm{Gr}^i_U \cong (\mathcal{O}_{F_\tau} \underset{\Lambda}{\otimes} \mathfrak{m}^i)/(\mathcal{O}_{F_\tau} \underset{\Lambda}{\otimes} \mathfrak{m}^{i+1}) \cong \mathcal{O}_{F_\tau} \underset{\Lambda}{\otimes} (\mathfrak{m}^i/\mathfrak{m}^{i+1})$, and similarly $\mathrm{Gr}^i_{U_D} \cong \mathcal{O}_{F^+_\tau} \underset{\Lambda}{\otimes} (\mathfrak{m}^i/\mathfrak{m}^{i+1})$. Hence we may reinterpret $N_i = \mathrm{Tr}_{\mathcal{O}_{F_\tau}/\mathcal{O}_{F^+_\tau}} \underset{\Lambda}{\otimes} (\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \mathrm{Tr}_{k_\tau/k^+_\tau} \underset{k}{\otimes} (\mathfrak{m}^i/\mathfrak{m}^{i+1})$ as the base change of $\mathrm{Tr}_{\mathcal{O}_{F_\tau}/\mathcal{O}_{F^+_\tau}}$ from Λ to k to $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. Then the surjectivity of N_i onto $\mathrm{Gr}^i_{U_D}$ follows from the surjectivity of $\mathrm{Tr}_{\mathcal{O}_{F_\tau}/\mathcal{O}_{F^+_\tau}}$ (by assumption that $\mathfrak{p} = \ker(R_0 \to k)$ is unramified in $\mathcal{O}$). On the other hand, consider any element $e \in \mathcal{O}_{F,R}$ given by Lemma 1.1.5.7 such that $e + e^\star = 1$. Let x be any element in $\mathrm{Gr}^i_{U_N}$, which by definition is an element in Gr^i_U such that $N_i(x) = x + x^\star = 0$. Then $D_i(ex) = ex - (ex)^\star = (e + e^\star)x = x$. This shows D_i is surjective onto $\mathrm{Gr}^i_{U_N}$. Hence the claim follows.

Since R is complete, $U^0 = \mathcal{O}_{F_\tau,R}$, $U^0_D = \mathcal{O}_{F^+_\tau,R}$, and U^0_N are all complete with respect to their topologies defined by $\mathfrak{m}$. By successive approximation (as in, for example, [110, Ch. V, Lem. 2]), the surjectivity of N (resp. D) follows from the surjectivity of N_i (resp. D_i) for all $i \geq 0$, as desired. □

Corollary 1.1.5.10. *With assumptions as in Lemma 1.1.5.9, the anti-automorphism $\maltese$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ induced by $\langle\cdot,\cdot\rangle$ is an involution if and only if there is an element $\delta \in \mathcal{O}^\times_{F,R}$ (resp. $\delta' \in \mathcal{O}^\times_{F,R}$) such that the pairing $\langle\cdot,\cdot\rangle'$ defined by $\langle x, y\rangle' := \langle x, \delta y\rangle$ (resp. by $\langle x, y\rangle' := \langle x, \delta' y\rangle$) is symmetric (resp. skew-symmetric).*

Proof. If we take δ as in Lemma 1.1.5.9 (with $\epsilon = -1$) such that $\delta(\delta^\star)^{-1} = \gamma$, then $\langle x, \delta y\rangle = \langle \delta y, \gamma x\rangle = \langle y, \delta^\star\gamma x\rangle = \langle y, \delta x\rangle$ for all $x, y \in M$. If we take δ' such that $\delta'((\delta')^\star)^{-1} = -\gamma$, then $\langle x, \delta' y\rangle = \langle \delta' y, \gamma x\rangle = \langle y, (\delta')^\star\gamma x\rangle = -\langle y, \delta' x\rangle$ for all

$x, y \in M$. □

Definition 1.1.5.11. *With assumptions as in Lemma 1.1.5.9, we say that an involution $\maltese$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ is* ***of unitary type*** *if there exists a perfect symmetric $\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to R$ inducing $\maltese$.*

Remark 1.1.5.12. By Corollary 1.1.5.10, we may replace symmetric pairings with skew-symmetric ones in Definition 1.1.5.11 without changing the class of involutions we consider.

To proceed further, let us record a consequence of Lemma 1.1.3.6:

Corollary 1.1.5.13. *Suppose $\mathcal{O}$ is maximal at $\mathfrak{p} = \ker(R_0 \to k)$, and suppose M is any finitely generated projective $\mathcal{O}_R$-module. Then each two involutions of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ inducing the same involution * on $\mathcal{O}_R$ are conjugate to each other by an element $a \in \mathrm{End}_{\mathcal{O}_R}(M)$.*

Proof. Let $\maltese_1$ and $\maltese_2$ be two such involutions. Then $\maltese_2 \circ (\maltese_1)^{-1}$ is an $\mathcal{O}_R$-automorphism of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$. By Lemma 1.1.3.6, with $C = \mathcal{O}_R$, there is an invertible element $a \in \mathrm{End}_{\mathcal{O}_R}(M)$ such that $\mathrm{Int}(a) = \maltese_2 \circ (\maltese_1)^{-1}$, which means $\mathrm{Int}(a) \circ (\maltese_1) = \maltese_2$, as desired. □

Corollary 1.1.5.14. *With assumptions on R as at the beginning of Section 1.1.5, suppose M is any finitely generated projective $\mathcal{O}_R$-module. Let $\maltese_i$, where $i = 1, 2$, be two involutions of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ inducing the same involution * on $\overline{\mathcal{O}}_R$. Suppose $\maltese_1$ is induced by some perfect $\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle_1 : M \times M \to R$. Then there is an invertible element $a \in \mathrm{End}_{\mathcal{O}_R}(M)$ such that $\maltese_2$ is induced by the perfect $\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle_2 := \langle\,\cdot\,,\,\cdot\,\rangle_1 \circ (a \times \mathrm{Id})$.*

Proof. By Corollary 1.1.5.13, there is an invertible $a \in \mathrm{End}_{\mathcal{O}_R}(M)$ such that $\maltese_1 = \mathrm{Int}(a) \circ (\maltese_2)$. In this case, the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_3$ defined by $\langle x, y\rangle_3 := \langle a(x), y\rangle_1$ satisfies $\langle\,\cdot\,,\,\cdot\,\rangle_3^* = \langle\,\cdot\,,\,\cdot\,\rangle_1^* \circ a$, and hence the involution $\maltese_3$ it induces on $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ satisfies $\maltese_1 = \mathrm{Int}(a) \circ (\maltese_3)$. That is, $\maltese_2 = \maltese_3$. Then $\maltese_2$ is induced by $\langle\,\cdot\,,\,\cdot\,\rangle_3$, as desired. □

Remark 1.1.5.15. With assumptions on R as at the beginning of Section 1.1.5, let $M_{\tau,R}$ be as in Lemma 1.1.3.4. If we denote the restriction of * to $\mathcal{O}_F$ by c, then $\mathrm{Hom}_R(M_{\tau,R}, R) \cong M_{\tau\circ c,R}$, because its $\mathcal{O}_{F,R}$-action is twisted by *. This shows that for our purpose of studying pairings we need to consider τ and τ' at the same time only when $\tau' = \tau \circ c$.

Lemma 1.1.5.16. *With assumptions on R as at the beginning of Section 1.1.5, suppose M is any finitely generated projective $\mathcal{O}_R$-module that decomposes as $M \cong \underset{\tau}{\oplus} M_{\tau,R}^{\oplus m_\tau}$ as in Lemma 1.1.3.4. Suppose $m_\tau = m_{\tau\circ c}$ for all τ. Then there exists a perfect $\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to R$ that induces an involution on $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$. (Conversely, the condition $m_\tau = m_{\tau\circ c}$ is automatic if there exists any perfect bilinear $\mathcal{O}_R$-pairing on M.)*

Proof. By forming orthogonal direct sums as in Definition 1.1.4.12, it suffices to construct an $\mathcal{O}_R$-module isomorphism $M_{\tau,R} \to M^\vee_{\tau\circ c,R}$ for each τ, which is possible by Remark 1.1.5.15. □

Combining Lemma 1.1.5.2, Corollary 1.1.5.14, and Lemma 1.1.5.16 (with justifications above for Definitions 1.1.5.4 and 1.1.5.11), we obtain the following proposition:

Proposition 1.1.5.17. *With assumptions on R as at the beginning of Section 1.1.5, suppose moreover that R is a noetherian complete R_0-algebra. Suppose M is any finitely generated projective $\mathcal{O}_R$-module. Consider the association of anti-automorphisms $\maltese$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ with weak isomorphism classes of perfect $\mathcal{O}_R$-pairings $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to R$ on M.*

1. *Suppose that $F = F^+$. Then, for each $\epsilon \in \mathcal{O}_{F,R}^\times$ such that $\epsilon^2 = 1$, there is a well-defined bijection from weak equivalence classes containing at least one ϵ-symmetric (resp. alternating, resp. symmetric) $\mathcal{O}_R$-pairing to involutions of ϵ-symmetric type (resp. of symplectic type, resp. of orthogonal type).*

2. *Suppose that F^+ is simple, that $[F : F^+] = 2$, and that the extension F/F^+ is split over k when $\mathrm{char}(k) = p = 0$. Then there is a well-defined bijection from weak equivalence classes containing at least one symmetric $\mathcal{O}_R$-pairing to involutions of unitary type. The same statement is true if we consider instead classes containing at least one skew-symmetric pairing, or classes containing at least one alternating pairing.*

In both cases, the images of the bijections exhaust all possible involutions of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ that induce * on $\overline{\mathcal{O}}_R$. By decomposing F^+ as a product of simple factors, the general cases also decompose as products of corresponding factors, each of which belongs to one of these two cases (cf. Remark 1.1.5.15).

1.2 Linear Algebraic Data

1.2.1 PEL-Type $\mathcal{O}$-Lattices

Let B be a finite-dimensional semisimple algebra over $\mathbb{Q}$ with positive involution * and center F. Here *positivity* of * means $\mathrm{Tr}_{B/\mathbb{Q}}(xx^\star) > 0$ for all $x \neq 0$ in B.

Let $\mathcal{O}$ be an order in B mapped to itself under *. Then $\mathcal{O}$ has an involution given by the restriction of *. Let $\mathrm{Disc} = \mathrm{Disc}_{\mathcal{O}/\mathbb{Z}}$ be the discriminant of $\mathcal{O}$ over $\mathbb{Z}$ (see Definition 1.1.1.6).

Let

$$\mathbb{Z}(1) := \ker(\exp : \mathbb{C} \to \mathbb{C}^\times),$$

which is a free $\mathbb{Z}$-module of rank one. Each choice $\sqrt{-1}$ of a square root of -1 in $\mathbb{C}$ determines an isomorphism

$$\frac{1}{2\pi\sqrt{-1}} : \mathbb{Z}(1) \xrightarrow{\sim} \mathbb{Z}, \tag{1.2.1.1}$$

but there is no canonical isomorphism between $\mathbb{Z}(1)$ and $\mathbb{Z}$. For each commutative $\mathbb{Z}$-algebra R, we denote by $R(1)$ the module $R \underset{\mathbb{Z}}{\otimes} \mathbb{Z}(1)$.

For reasons that will become clear in Section 1.3.4, let us introduce the following structure on symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-lattices:

Definition 1.2.1.2. *Let R be a noetherian subring of $\mathbb{R}$ and let $(M, \langle\,\cdot\,,\,\cdot\,\rangle_M, R(1))$ be a symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-lattice (see Definition 1.1.4.7). A* ***polarization*** *of $(M, \langle\,\cdot\,,\,\cdot\,\rangle_M, R(1))$ is an $\mathbb{R}$-algebra homomorphism*

$$h : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(M \underset{R}{\otimes} \mathbb{R})$$

such that the following two conditions are satisfied:

1. *For all $z \in \mathbb{C}$ and $x, y \in M \underset{R}{\otimes} \mathbb{R}$, we have*

$$\langle h(z)x, y\rangle_M = \langle x, h(z^c)y\rangle_M,$$

where $\mathbb{C} \to \mathbb{C} : z \mapsto z^c$ is the complex conjugation.

2. *For each choice of $\sqrt{-1}$ in $\mathbb{C}$ defining an isomorphism $\mathbb{Z}(1) \xrightarrow{\sim} \mathbb{Z}$ as in (1.2.1.1), the $\mathbb{R}$-bilinear pairing*
$$\frac{1}{2\pi\sqrt{-1}} \circ \langle \cdot , h(\sqrt{-1}) \cdot \rangle_M : (M \underset{R}{\otimes} \mathbb{R}) \times (M \underset{R}{\otimes} \mathbb{R}) \to \mathbb{R}$$
is symmetric and positive definite. (This last condition forces $\langle \cdot , \cdot \rangle_M$ to be nondegenerate.)

*We say in this case that $(M, \langle \cdot , \cdot \rangle_M, h)$ is a **polarized symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-lattice**, suppressing $R(1)$ from the notation.*

*Two polarized symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-lattices $(M_1, \langle \cdot , \cdot \rangle_{M_1}, h_1)$ and $(M_2, \langle \cdot , \cdot \rangle_{M_2}, h_2)$ are **isomorphic** if the underlying symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-lattices $(M_1, \langle \cdot , \cdot \rangle_{M_1})$ and $(M_2, \langle \cdot , \cdot \rangle_{M_2})$ are isomorphic, and if there exists a symplectic isomorphism $(M_1 \underset{R}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle_{M_1}) \xrightarrow{\sim} (M_2 \underset{R}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle_{M_2})$ (over $\mathbb{R}$) matching h_1 with h_2.*

Definition 1.2.1.3. *A **PEL-type $\mathcal{O}$-lattice** $(L, \langle \cdot , \cdot \rangle, h)$ is a polarized symplectic $\mathcal{O}$-lattice $(L, \langle \cdot , \cdot \rangle, h)$ in Definition 1.2.1.2 (with $R = \mathbb{Z}$).*

Remark 1.2.1.4. The datum $(\mathcal{O}, \star, L, \langle \cdot , \cdot \rangle, h)$ is an integral version of the datum $(B, \star, V, \langle \cdot , \cdot \rangle, h)$ in [76] and related works.

Remark 1.2.1.5. We shall suppress h from the notation when the polarization is not needed. In this case, we shall denote the underlying symplectic $\mathcal{O}$-lattice by $(L, \langle \cdot , \cdot \rangle)$, suppressing the target $\mathbb{Z}(1)$ of the pairing from the notation. Similarly, for each (commutative) $\mathbb{Z}$-algebra R, we shall denote the base change of $(L, \langle \cdot , \cdot \rangle)$ to R as $(L \underset{\mathbb{Z}}{\otimes} R, \langle \cdot , \cdot \rangle)$, suppressing $R(1)$ from the notation.

Definition 1.2.1.6. *Let $(L, \langle \cdot , \cdot \rangle, h)$ be a PEL-type $\mathcal{O}$-lattice as in Definition 1.2.1.3. For each $\mathbb{Z}$-algebra R, set*
$$\mathrm{G}(R) := \left\{ \begin{array}{l} (g, r) \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(L \underset{\mathbb{Z}}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R) : \\ \langle gx, gy \rangle = r \langle x, y \rangle \ \forall x, y \in L \underset{\mathbb{Z}}{\otimes} R \end{array} \right\}.$$
In other words, $\mathrm{G}(R)$ is the group of symplectic automorphisms of $L \underset{\mathbb{Z}}{\otimes} R$ over R (see Definition 1.1.4.8). For each $\mathbb{Z}$-algebra homomorphism $R \to R'$, we have by definition a natural homomorphism $\mathrm{G}(R) \to \mathrm{G}(R')$, making G a group functor (or, in fact, an affine group scheme) over $\mathbb{Z}$.

*The projection to the second factor $(g, r) \mapsto r$ defines a homomorphism $\nu : \mathrm{G} \to \mathbf{G}_{\mathrm{m}}$, which we call the **similitude character**. For simplicity, we shall often denote elements (g, r) in G simply by g, and denote by $\nu(g)$ the value of r when we need it. (If $L \neq \{0\}$ and R is flat over $\mathbb{Z}$, then the value of r is uniquely determined by g. Hence there is little that we lose when suppressing r from the notation. However, this suppression is indeed an abuse of notation in general. For example, when $L = \{0\}$, we have $\mathrm{G} = \mathbf{G}_{\mathrm{m}}$.)*

Remark 1.2.1.7. The polarization h is not needed in Definition 1.2.1.6.

Remark 1.2.1.8. For a general nonflat $\mathbb{Z}$-algebra R, the pairing induced by $\langle \cdot , \cdot \rangle$ on $L \underset{\mathbb{Z}}{\otimes} R$ is not necessarily nondegenerate (see Definition 1.1.4.7). This suggests that G is not necessarily a smooth functor over the whole base $\mathbb{Z}$.

Remark 1.2.1.9. This gives the definitions for $\mathrm{G}(\mathbb{Q})$, $\mathrm{G}(\mathbb{A}^{\infty,\Box})$, $\mathrm{G}(\mathbb{A}^\infty)$, $\mathrm{G}(\mathbb{R})$, $\mathrm{G}(\mathbb{A}^\Box)$, $\mathrm{G}(\mathbb{A})$, $\mathrm{G}(\mathbb{Z})$, $\mathrm{G}(\mathbb{Z}/n\mathbb{Z})$, $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$, $\mathrm{G}(\hat{\mathbb{Z}})$,
$$\Gamma(n) := \ker(\mathrm{G}(\mathbb{Z}) \to \mathrm{G}(\mathbb{Z}/n\mathbb{Z})),$$
$$\mathcal{U}^\Box(n) := \ker(\mathrm{G}(\hat{\mathbb{Z}}^\Box) \to \mathrm{G}(\hat{\mathbb{Z}}^\Box/n\hat{\mathbb{Z}}^\Box) = \mathrm{G}(\mathbb{Z}/n\mathbb{Z}))$$
for each integer $n \geq 1$ prime-to-$\Box$, and
$$\mathcal{U}(n) := \ker(\mathrm{G}(\hat{\mathbb{Z}}) \to \mathrm{G}(\hat{\mathbb{Z}}/n\hat{\mathbb{Z}}) = \mathrm{G}(\mathbb{Z}/n\mathbb{Z})).$$

Now let us take a closer look at the pairs $(B, \star)$ that we are considering. As in Section 1.1.2, the $\mathbb{Q}$-algebra F decomposes into a product
$$F \cong \prod_{[\tau]: F \to \mathbb{Q}_{[\tau]}} F_{[\tau]}$$
of fields, finite-dimensional over $\mathbb{Q}$, giving the Galois orbits of homomorphisms $F \to \mathbb{Q}^{\mathrm{sep}}$, and we obtain accordingly a decomposition
$$B \cong \prod_{[\tau]: F \to \mathbb{Q}_{[\tau]}} B_{[\tau]}, \tag{1.2.1.10}$$
where B_τ is the simple factor of B containing F_τ as its center.

Lemma 1.2.1.11. *Every simple factor of B is mapped by $\star$ to itself.*

Proof. If any of the simple factors of B is mapped to a different simple factor, then every nonzero element x in the former simple factor satisfies $xx^\star = 0$, which contradicts the positivity condition that $\mathrm{Tr}_{B/\mathbb{Q}}(xx^\star) > 0$. □

Remark 1.2.1.12. It is clear that modules over semisimple algebras can be decomposed as a direct sum of modules over its simple factors. By Lemma 1.2.1.11, we see that we can decompose positive involutions, and hence symplectic modules (in a way compatible with the involutions), into products over simple factors. Note however that the group of similitudes defined by a general semisimple datum (as in Definition 1.2.1.6) is only a subgroup of a product of groups of similitudes defined by simple data, because the similitude factors have to be identical in all factors.

By Wedderburn's structure theorem (see, for example, [107, Thm. 7.4]), each finite-dimensional simple algebra B over $\mathbb{Q}$ is of the form $\mathrm{M}_k(D)$ for some integer k and some division algebra D over $\mathbb{Q}$. Let us record the fundamental classification of division algebras with positive involutions exhibited in [94, §21], originally due to Albert:

Proposition 1.2.1.13 (Albert)**.** *Suppose D is a finite-dimensional **division algebra** over $\mathbb{Q}$ with a positive involution $^\diamond$. Then the elements in the center F invariant under $^\diamond$ form a totally real extension F^+ of $\mathbb{Q}$, and there are only four possibilities:*

1. *$D = F = F^+$ is totally real.*

2. *$F = F^+$ is totally real, and $D \underset{F,\tau}{\otimes} \mathbb{R}$ is isomorphic to $\mathrm{M}_2(\mathbb{R})$ for every embedding $\tau : F \hookrightarrow \mathbb{R}$, with the involution $^\diamond$ given by conjugating the natural involution $x \mapsto x' := \mathrm{Tr}_{D/F}(x) - x$ by some element $a \in D$ such that $a^\diamond = -a$. In this case, $a^2 = -aa^\diamond$ is totally negative in F.*

3. *$F = F^+$ is totally real, and $D \underset{F,\tau}{\otimes} \mathbb{R}$ is isomorphic to the real Hamilton quaternion algebra $\mathbb{H}$ for every embedding $\tau : F \hookrightarrow \mathbb{R}$, with the natural involution $^\diamond$ given by $x \mapsto x^\diamond := \mathrm{Tr}_{D/F}(x) - x$.*

4. *F is totally imaginary over the totally real F^+, with complex conjugation c, and D satisfies the condition that if $v = v \circ c$ then $\mathrm{inv}_v(D) = 0$, and if $v \neq v \circ c$ then $\mathrm{inv}_v(D) + \mathrm{inv}_{v \circ c}(D) = 0$.*

A rough analogue of Proposition 1.2.1.13 for simple algebras (which nevertheless suffices for our purpose) can be given as follows:

Proposition 1.2.1.14. *Suppose B is a finite-dimensional **simple algebra** over $\mathbb{Q}$ with a positive involution $\star$. Then the elements in F invariant under $\star$ form a totally real extension F^+ of $\mathbb{Q}$, and there are only four possibilities:*

1. *$F = F^+$ is totally real, and $B \cong \mathrm{M}_k(F)$ for some integer k, with the involution $\star$ given by conjugating the natural involution $x \mapsto {}^t x$ by some element $a \in B$ such that ${}^t a = a$ and such that a is totally positive in the sense that $a = {}^t bb$ for some element $b \in B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$.*

2. *$F = F^+$ is totally real, and $B \cong \mathrm{M}_k(D)$ for some integer k and some quaternion division algebra D over $\mathbb{Q}$, with $D \underset{F,\tau}{\otimes} \mathbb{R}$ isomorphic to $\mathrm{M}_2(\mathbb{R})$ for every embedding $\tau : F \hookrightarrow \mathbb{R}$. In this case, $B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ is a product of copies of $\mathrm{M}_{2k}(\mathbb{R})$ indexed by embeddings $\tau : F \hookrightarrow \mathbb{R}$, and the involution $\star$ is given by conjugating $x \mapsto {}^t x$ by some element $a \in B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ that is totally positive in the sense that $a = {}^t bb$ for some element $b \in B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$.*

3. *$F = F^+$ is totally real, and $B \cong \mathrm{M}_k(D)$ for some integer k and some quaternion division algebra D over $\mathbb{Q}$, with $D \underset{F,\tau}{\otimes} \mathbb{R}$ isomorphic to the real Hamilton quaternion algebra $\mathbb{H}$ for every embedding $\tau : F \hookrightarrow \mathbb{R}$. Let us denote by $\diamond$ the standard involution $x \mapsto \mathrm{Tr}_{D/F}\, x - x$ on D. Then $B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$ is a product of copies of $\mathrm{M}_k(\mathbb{H})$ indexed by embeddings $\tau : F \hookrightarrow \mathbb{R}$, and the involution $\star$ is given by conjugating $x \mapsto {}^t x^\diamond$ by some element $a \in B$ such that ${}^t a^\diamond = a$ and such that a is totally positive in the sense that $a = {}^t b^\diamond b$ for some element $b \in B \underset{\mathbb{Q}}{\otimes} \mathbb{R} \cong \mathrm{M}_k(\mathbb{H})$.*

4. *F is totally imaginary over the totally real F^+.*

Note that Proposition 1.2.1.14 is not as comprehensive as Proposition 1.2.1.13 when we specialize to the case that B is a division algebra.

Proof of Proposition 1.2.1.14. Following the classification for division algebras with positive involutions in [94, §21], the case $F = F^+$ implies (for a general simple algebra) that B is isomorphic to its opposite algebra in the Brauer group over F. This shows that $B = \mathrm{M}_k(D)$ for some division algebra over F that is either F or quaternion over F. For the statements about involutions over $\mathbb{R}$, combine Lemma 1.1.3.6 with the classification of real positive involutions in [76, §2, especially Lem. 2.11]. □

Combining Proposition 1.2.1.14 with Lemma 1.2.1.11, we obtain all finite-dimensional semisimple algebras over $\mathbb{Q}$ with positive involutions.

Definition 1.2.1.15. *Let B be a finite-dimensional semisimple over $\mathbb{Q}$ with a positive involution $\star$. Let $B \cong \prod_{[\tau]: F \to \mathbb{Q}_{[\tau]}} B_{[\tau]}$ be the decomposition into simple factors as in (1.2.1.10). We say that B involves a simple factor of **type** C (resp. **type** D, resp. **type** A) if, for some homomorphism $\tau : F \to \mathbb{R}$ (resp. $\tau : F \to \mathbb{R}$, resp. $\tau : F \to \mathbb{C}$ such that $\tau(F) \not\subset \mathbb{R}$), we have an isomorphism $B \underset{F,\tau}{\otimes} \mathbb{R} \cong \mathrm{M}_k(\mathbb{R})$ (resp. $B \underset{F,\tau}{\otimes} \mathbb{R} \cong \mathrm{M}_k(\mathbb{H})$, resp. $B \underset{F,\tau}{\otimes} \mathbb{C} \cong \mathrm{M}_k(\mathbb{C})$), for some integer $k \geq 1$, respecting the positive involutions. In this case, we say that the factor $B_{[\tau]}$ with $[\tau] : F \to \mathbb{Q}_{[\tau]}$ determined by $\tau : F \to \mathbb{R}$ (resp. $\tau : F \to \mathbb{R}$, resp. $\tau : F \to \mathbb{C}$) is of type C (resp. type D, resp. type A).*

Remark 1.2.1.16. Definition 1.2.1.15 will be justified in Proposition 1.2.3.11 below, which implies that, for G defined by $(L, \langle \cdot, \cdot \rangle, h)$ as in Definition 1.2.1.6, $\mathrm{G}^{\mathrm{ad}}(\mathbb{C})$ has a simple factor of type C (resp. type D, resp. type A) if B involves a simple factor of type C (resp. type D, resp. type A) acting nontrivially on L. (An explanation using only algebras over $\mathbb{R}$ can be found in [76, §5].)

Remark 1.2.1.17. Though providing convenient terminologies for the classification of $(B, \star)$, knowledge of smooth geometric fibers of G (such as classification of $\mathrm{G}^{\mathrm{ad}}(\mathbb{C})$) will never be needed (and in fact has good reason not to be helpful) in any of our main theorems.

Definition 1.2.1.18. *If B involves any simple factor of type D (see Definition 1.2.1.15), then we set $\mathrm{I}_{\mathrm{bad}} := 2$. Otherwise we set $\mathrm{I}_{\mathrm{bad}} := 1$.*

Remark 1.2.1.19. The invariant $\mathrm{I}_{\mathrm{bad}}$ will be used (in Definition 1.4.1.1) to describe the set of *bad primes* for our moduli problems. Its (ad hoc) definition will be justified by the calculations in Sections 1.2.2, 1.2.3, and 1.2.5, which are basic to the proofs of Theorem 1.4.1.11 and Theorem 6.4.1.1.

By Lemma 1.1.2.4, each simple factor $B_{[\tau]}$ of B in (1.2.1.10) has only one unique simple module $W_{[\tau]}$. As a result, it makes sense to classify finite-dimensional B-modules W over $\mathbb{Q}$ by its *B-multirank*, namely, the integers $(m_{[\tau]})_{[\tau]}$ such that

$$W \cong \bigoplus_{[\tau]: F \to \mathbb{Q}_{[\tau]}} W_{[\tau]}^{\oplus m_{[\tau]}}. \tag{1.2.1.20}$$

(This is consistent with Definition 1.1.3.5, with $R = k = \mathbb{Q}$ there.) We would like to introduce the notion of *$\mathcal{O}$-multirank* for $\mathcal{O}$-lattices, even if $\mathcal{O}$-lattices are not necessarily projective.

Definition 1.2.1.21. *The **$\mathcal{O}$-multirank** $(m_{[\tau]})_{[\tau]}$ of an $\mathcal{O}$-lattice M is the B-multirank of its induced B-module $W := M \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (as explained above). If B is simple, then the multirank of an $\mathcal{O}$-lattice M is given by a single integer. We call this single integer the **$\mathcal{O}$-rank** of M.*

Remark 1.2.1.22. Even if B is simple, $\mathcal{O}$ is not necessarily of $\mathcal{O}$-rank one.

Definition 1.2.1.23. *Let R be a commutative ring. An $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-module is called **integrable** if it is isomorphic to $M \underset{\mathbb{Z}}{\otimes} R$ for some **$\mathcal{O}$-lattice** M.*

Note that an integrable $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-module is finitely presented over $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$.

Lemma 1.2.1.24. *If R is a noetherian (commutative) ring **flat** over $\mathbb{Z}$, and if M is an integrable $\mathcal{O}\underset{\mathbb{Z}}{\otimes}R$-module such that $M \cong M_{[\mathbb{Z}]}\underset{\mathbb{Z}}{\otimes}R$, then $M_{[\mathbb{Q}]} := M_{[\mathbb{Z}]}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ is uniquely determined by M and independent of the choice of $M_{\mathbb{Z}}$.*

Proof. Since $M\underset{\mathbb{Z}}{\otimes}\mathbb{Q} \cong M_{[\mathbb{Q}]}\underset{\mathbb{Z}}{\otimes}R$, it suffices to show that if $M_{[\mathbb{Q}]}$ and $M'_{[\mathbb{Q}]}$ are two finite-dimensional B-modules of different B-multirank, then $M_{[\mathbb{Q}]}\underset{\mathbb{Z}}{\otimes}R \not\cong M'_{[\mathbb{Q}]}\underset{\mathbb{Z}}{\otimes}R$ as $B\underset{\mathbb{Z}}{\otimes}R$-modules. By decomposing $M_{[\mathbb{Q}]}$ and $M_{[\mathbb{Q}]}$ as in (1.2.1.20), we may assume that B is simple. Then it suffices to treat the case that $M'_{[\mathbb{Q}]} \subsetneq M_{[\mathbb{Q}]}$, which follows from the assumption that R is flat over $\mathbb{Z}$. □

Definition 1.2.1.25. *If R is a noetherian (commutative) ring **flat over** $\mathbb{Z}$, then the $\mathcal{O}$-multirank of an integrable $\mathcal{O}\underset{\mathbb{Z}}{\otimes}R$-module M, which is by definition isomorphic to $M_{[\mathbb{Z}]}\underset{\mathbb{Z}}{\otimes}R$ for some $\mathcal{O}$-lattice $M_{[\mathbb{Z}]}$, is defined to be the $\mathcal{O}$-multirank of $M_{[\mathbb{Z}]}$. (This is justified by Lemma 1.2.1.24.)*

Suppose now that R is a noetherian complete local ring with residue field k, and let $p := \mathrm{char}(k)$. Let us adopt Convention 1.1.3.3 with $R_0 = \mathbb{Z}$.

Suppose $p \nmid \mathrm{Disc}$. We set $\Lambda := k$ if $p = 0$, and set $\Lambda := W(k)$ if $p > 0$. This is consistent with the setting in Section 1.1.3, where we have introduced the notion of $\mathcal{O}_R$-multiranks for finitely generated projective $\mathcal{O}_R$-modules. As a special case of Lemma 1.1.3.4, we have the following lemma:

Lemma 1.2.1.26. *With assumptions as above, given any finitely generated projective $\mathcal{O}_R$-module M, if M_τ is defined as in Lemma 1.1.3.4, and if M has $\mathcal{O}_R$-multirank $(m_\tau)_\tau$ as a projective $\mathcal{O}_R$-module (see Definition 1.1.3.5), then we have a decomposition*

$$M \cong \underset{\tau}{\oplus} M_{\tau,R}^{\oplus m_\tau}. \tag{1.2.1.27}$$

In particular, there exists a unique $\mathcal{O}_\Lambda$-lattice $M_{[\Lambda]}$ (of $\mathcal{O}_\Lambda$-rank $(m_\tau)_\tau$) such that $M \cong M_{[\Lambda]} \underset{\Lambda}{\otimes} R$.

The decomposition (1.2.1.20) (resp.(1.2.1.27)) is indexed by homomorphisms $[\tau] : F \to \mathbb{Q}_{[\tau]}$ (resp. $\tau : F \to \mathrm{Frac}(\Lambda)_\tau$), which should be interpreted as a Galois orbit of homomorphisms $F \to \mathbb{Q}^{\mathrm{sep}}$ (resp. $F \to \mathrm{Frac}(\Lambda)^{\mathrm{sep}}$). Each orbit $\tau : F \to \mathrm{Frac}(\Lambda)_\tau$ determines a unique orbit $[\tau] : F \to \mathbb{Q}_{[\tau]}$. Let us write this symbolically as $\tau \in [\tau]$.

Definition 1.2.1.28. *For each $[\tau]$, let $s_{[\tau]} \geq 1$ be the integer such that $W_{[\tau]} \underset{\mathbb{Q}}{\otimes} k \cong \underset{\tau\in[\tau]}{\oplus} M_{\tau,k}^{\oplus s_{[\tau]}}$ when $p = \mathrm{char}(k) = 0$, and let $s_{[\tau]} = 1$ when $p > 0$.*

Our definition for the case $p > 0$ makes sense because $\mathcal{O}_\Lambda$ is a product of matrix algebras, so that no higher multiplicities as in (1.1.2.10) appear after a base change.

Lemma 1.2.1.29. *With assumptions as above, if an $\mathcal{O}_R$-module M is integrable, namely, if $M \cong M_{[\mathbb{Z}]}\underset{\mathbb{Z}}{\otimes}R$ for some $\mathcal{O}$-lattice $M_{[\mathbb{Z}]}$, then it is finitely generated and projective as an $\mathcal{O}_R$-module, and the $\mathcal{O}$-multirank of $M_{[\mathbb{Z}]}$ depends only on M (but not on the choice of $M_{[\mathbb{Z}]}$). If M has $\mathcal{O}_R$-multirank $(m_\tau)_\tau$, then $M_{[\mathbb{Z}]}$ has $\mathcal{O}$-multirank $(m_{[\tau]})_{[\tau]}$ with $m_\tau = s_{[\tau]}m_{[\tau]}$ for $\tau \in [\tau]$.*

Proof. If $M \cong M_{[\mathbb{Z}]}\underset{\mathbb{Z}}{\otimes}R$, then $M \cong (M_{[\mathbb{Z}]} \underset{\mathbb{Z}}{\otimes} \Lambda) \underset{\Lambda}{\otimes} R$. Since $M_{[\mathbb{Z}]} \underset{\mathbb{Z}}{\otimes} \Lambda$ is an $\mathcal{O}_\Lambda$-lattice, it is finitely generated and projective as an $\mathcal{O}_\Lambda$-module (by Propositions 1.1.1.21 and 1.1.1.23). Hence M is finitely generated and projective as an $\mathcal{O}_R$-module. By Lemma 1.2.1.24 and by the flatness of Λ over $\mathbb{Z}$, the $\mathcal{O}$-multirank of $M_{[\mathbb{Z}]}$ depends only on $M_{[\mathbb{Z}]} \underset{\mathbb{Z}}{\otimes} \Lambda$. Now we can conclude the proof by applying Lemma 1.2.1.26. □

Definition 1.2.1.30. *With assumptions as above, the $\mathcal{O}$-multirank of an integral $\mathcal{O}_R$-module M is defined to be the $\mathcal{O}$-multirank of any $\mathcal{O}$-lattice $M_{[\mathbb{Z}]}$ such that $M \cong M_{[\mathbb{Z}]}\underset{\mathbb{Z}}{\otimes}R$. (This is compatible with Definition 1.2.1.25 when R is flat over $\mathbb{Z}$.)*

Lemma 1.2.1.31. *With assumptions as above, let M_τ be defined as in Lemma 1.1.3.4, let*

$$M_{[\tau],R} := \underset{\tau\in[\tau]}{\oplus} M_{\tau,R},$$

and let $\mathcal{O}_{[\tau],R}$ be the image of $\mathcal{O}_R$ in $\mathrm{End}_{\mathcal{O}_{F,R}}(M_{[\tau],R})$. Then

$$\mathcal{O}_R \cong \prod_{[\tau]} \mathcal{O}_{[\tau],R}, \tag{1.2.1.32}$$

and there exists an element $x_{[\tau]}$ in $M_{[\tau],R}$ such that $M_{[\tau],R} = (\mathcal{O}_{[\tau],R})x_{[\tau]}$.

Proof. It suffices to treat the universal case $R = \Lambda$. Then the lemma is clear from the decomposition $\mathcal{O}_{F,\Lambda} \cong \prod\limits_{[\tau]} \mathcal{O}_{F_\tau}$ inducing all other decompositions accordingly. For each fixed $[\tau]$, we may take an explicit choice of $x_{[\tau]} = (x_\tau)_{\tau\in[\tau]}$ with $x_\tau \in M_\tau$ satisfying $M_\tau = \mathcal{O}_\tau x_\tau$, as follows: If $p > 0$, then $\mathcal{O}_\tau$ is a matrix algebra, and we can take x_τ to the vector $(1, 0, 0, \ldots, 0)$. If $p = 0$, then any nonzero element x_τ in M_τ would suffice. □

Lemma 1.2.1.33. *With assumptions as above, let M be any finitely generated $\mathcal{O}_R$-module. Then the following statements are equivalent:*

1. *The $\mathcal{O}_R$-module M is integrable of $\mathcal{O}$-multirank $(m_{[\tau]})_{[\tau]}$.*
2. *The $\mathcal{O}_R$-module M is finitely generated and projective of $\mathcal{O}_R$-multirank $(m_\tau)_\tau$, where $m_\tau/s_{[\tau]} = m_{\tau'}/s_{[\tau']}$ is the same integer for every τ and τ' that determine the same orbit $[\tau] = [\tau']$.*
3. *The $\mathcal{O}_R$-module M is the direct sum of copies of modules of the form of $M_{[\tau],R}^{\oplus s_{[\tau]}}$.*

Moreover, the implications can be checked modulo the maximal ideal of R.

Proof. The equivalences among the statements follow from Lemma 1.2.1.29. Since R is noetherian local, and since M is finitely generated, the statement that the equivalences can be checked modulo the maximal ideal of R follows from Lemma 1.1.3.1. □

1.2.2 Torsion of Universal Domains

Let us continue with the setting in Section 1.2.1.

Proposition 1.2.2.1. *Let k be either a field of characteristic $p = 0$ or a finite field of characteristic $p > 0$, such that $p \nmid \mathrm{GCD}(2, \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc})$. Let $\Lambda = k$ when $p = 0$, and let $\Lambda = W(k)$ when $p > 0$. Let R be a noetherian local Λ-algebra with residue field k. Let $x \in \mathcal{O}_R := \mathcal{O}\underset{\mathbb{Z}}{\otimes}R$ be any element such that $x = -x^\star$. When B involves any* *simple factor of type* C *(see Definition 1.2.1.15), we assume moreover that* 2 *is not a zero-divisor in* R*. Then* x *is equal to* $z - z^\star$ *for some* $z \in \mathcal{O}_R$*.*

Proof. Throughout the proof, the subscript Λ will mean tensor product with Λ, and the subscript R will mean tensor product with R.

If $p \neq 2$, then there is an element e in Λ such that $2e = 1$. Then by taking $z = ex$, we have $x = 2z$ and $z = -z^\star$, and hence $x = 2z = z - z^\star$, as desired.

If $p = 2$, then the assumption is that $p \nmid \mathrm{I}_{\mathrm{bad}} \,\mathrm{Disc}$. By Lemmas 1.2.1.11 and 1.2.1.31, we have a decomposition

$$\mathcal{O}_\Lambda \cong \prod_{[\tau]} \mathcal{O}_{[\tau],\Lambda}, \tag{1.2.2.2}$$

and the involution * maps each factor $\mathcal{O}_{[\tau],\Lambda}$ in (1.2.2.2) into itself. The corresponding facts over R (as in (1.2.1.32)) are induced by base change. Therefore it suffices to prove the proposition for each factor $\mathcal{O}_{[\tau],\Lambda}$. For simplicity of notation, let us assume that B is simple. According to Proposition 1.2.1.14, we have four cases of simple B with its positive involution.

Suppose $F = F^+$ is a totally real field over $\mathbb{Q}$. Then B is either of type C or of type D (see Definition 1.2.1.15). Since $p = 2 \nmid \mathrm{I}_{\mathrm{bad}}$, we see that B is of type C, which implies that 2 is not a zero divisor in R by assumption.

Since $p \nmid \mathrm{Disc}$, we may assume that $\mathcal{O}_\Lambda \cong \mathrm{M}_k(\mathcal{O}_{F,\Lambda})$ for some integer k. There is another involution of $\mathcal{O}_\Lambda$ given by $x \mapsto {}^t x$. By Lemma 1.1.3.6, there exists an invertible element $c \in \mathcal{O}_\Lambda$ such that $x^\star = c\,{}^t x c^{-1}$ for all $x \in \mathcal{O}_\Lambda$. Then, as in [94, §21, p. 195], since $x = (x^\star)^\star = c\,{}^t(c\,{}^t x c^{-1})c^{-1} = c\,{}^t c^{-1} x\,{}^t c c^{-1}$ for all $x \in \mathcal{O}_\Lambda$, we must have $c\,{}^t c^{-1} = e \in \mathcal{O}_{F,\Lambda}$ for some e. Then $c = e\,{}^t c = e^2 c$ implies $e = \pm 1$. Note that the value of e is independent of the choice of c and the isomorphisms involved. More importantly, it is unchanged if we replace Λ with a larger ring.

If we are in the case that $B \cong \mathrm{M}_k(F)$ for some integer k, with involution given by $x^\star = a\,{}^t x a^{-1}$ for some $a \in B$ such that ${}^t a = a$. Comparing with $x^\star = c\,{}^t x c^{-1}$ in $B \underset{\mathbb{Q}}{\otimes} \Lambda = \mathcal{O}_\Lambda \underset{\Lambda}{\otimes} \mathrm{Frac}(\Lambda)$, we obtain $ac^{-1} \in F \underset{\mathbb{Q}}{\otimes} \Lambda = \mathcal{O}_{F,\Lambda} \underset{\Lambda}{\otimes} \mathrm{Frac}(\Lambda)$, and hence ${}^t a = a$ implies ${}^t c = c$.

If we are in the case that $B \cong \mathrm{M}_k(D)$ for some integer k and some quaternion division algebra D over F, then $D \underset{F,\tau}{\otimes} \mathbb{R} \cong \mathrm{M}_2(\mathbb{R})$ for every embedding $\tau : F \hookrightarrow \mathbb{R}$, because B is of type C. Over $B \underset{F,\tau}{\otimes} \mathbb{R} \cong \mathrm{M}_{2k}(\mathbb{R})$, the involution * is given by conjugating $x \mapsto {}^t x$ by some element $b \in B \underset{F,\tau}{\otimes} \mathbb{R}$ such that ${}^t b = b$. Note that this b is defined in the same way as c with $\mathbb{R}$ in place of Λ. Using the fact that the e defined above is *unchanged* if we replace Λ with a larger ring, and that it is independent of the choices involved, if we work in a field containing both $\mathbb{R}$ and Λ, then the comparison between the relations $x^\star = b\,{}^t x b^{-1}$ and $x^\star = c\,{}^t x c^{-1}$ shows that ${}^t c = c$.

Now that we know $x^\star = c\,{}^t x c^{-1}$ for some $c = {}^t c$, the relation $x = -x^\star = -c\,{}^t x c^{-1}$ implies that $xc = -c\,{}^t x = -{}^t c\,{}^t x = -{}^t(xc)$. In this case, all diagonal entries of xc are zero, because we assume that 2 is not a zero divisor in R. Set y to be the element in $\mathcal{O}_R$ with only the upper-triangle entries of xc, so that $xc = y - {}^t y$. Then $x = (y - {}^t y)c^{-1} = yc^{-1} - c(c^{-1}\,{}^t y)c^{-1} = yc^{-1} - c\,{}^t(yc^{-1})c^{-1} = yc^{-1} - (yc^{-1})^\star = z - z^\star$ for $z = yc^{-1}$, as desired.

Finally, suppose $[F : F^+] = 2$. By Lemma 1.1.5.7, there exists some element $e \in \mathcal{O}_{F,\Lambda}$ such that $\mathrm{Tr}_{F_\Lambda/F^+_\Lambda}(e) = e + e^\star = 1$. Since $p \nmid \mathrm{Disc}$, $\mathcal{O}_\Lambda$ contains $\mathcal{O}_{F,\Lambda}$ in its center. Therefore $\mathcal{O}_\Lambda$ contains e. Then $x = -x^\star$ implies that $x = (e + e^\star)x = ex - x^\star e^\star = ex - (ex)^\star = z - z^\star$ for $z = ex$. □

Now suppose that Λ is either $\mathbb{Z}$, or a field of characteristic zero, or $W(k)$ for some finite field k of characteristic $p > 0$. Suppose we have two $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \Lambda$-lattices L_1 and L_2 and an embedding $\varrho : L_1 \hookrightarrow L_2$ with a cokernel of finite cardinality. Let us denote by $[L_2 : \varrho(L_1)]$ the cardinality of this cokernel. Let $\epsilon = 1$ or 0. Let us define a finitely generated Λ-module $\mathbf{Sym}^\epsilon_\varrho(L_1, L_2)$ by

$$\mathbf{Sym}^\epsilon_\varrho(L_1, L_2) := (L_1 \underset{\Lambda}{\otimes} L_2) / \begin{pmatrix} x \otimes \varrho(y) - y \otimes \varrho(x) \\ (bx) \otimes z - x \otimes (b^\star z) \end{pmatrix}_{\substack{x,y \in L_1,\\ z \in L_2, b \in \mathcal{O}}}$$

when $\epsilon = 1$ and by

$$\mathbf{Sym}^\epsilon_\varrho(L_1, L_2) := (L_1 \underset{\Lambda}{\otimes} L_2) / \begin{pmatrix} x \otimes \varrho(x) \\ (bx) \otimes z - x \otimes (b^\star z) \end{pmatrix}_{\substack{x,y \in L_1,\\ z \in L_2, b \in \mathcal{O}}}$$

when $\epsilon = 0$.

Proposition 1.2.2.3. *Let the assumptions on* Λ *be as above. Let* L_3 *be any* Λ*-module. When* $\epsilon = 1$ *(resp.* $\epsilon = 0$*), the* Λ*-module* $\mathrm{Hom}_\Lambda(\mathbf{Sym}^\epsilon_\varrho(L_1, L_2), L_3)$ *is isomorphic to the* Λ*-module of symmetric (resp. alternating)* Λ*-bilinear pairings*

$$\langle \cdot, \cdot \rangle : L_1 \times L_2 \to L_3$$

such that

$$\langle bx, y \rangle = \langle x, b^\star y \rangle$$

for all $x \in L_1$*,* $y \in L_2$*, and* $b \in \mathcal{O}$*. This* $\mathbf{Sym}^\epsilon_\varrho(L_1, L_2)$ *is the so-called* ***universal domain*** *of such pairings. Then the* Λ*-module* $\mathbf{Sym}^\epsilon_\varrho(L_1, L_2)$ *can have nonzero* p*-torsion only when* Λ *is not a field, and when* $p | \mathrm{I}^\epsilon_{\mathrm{bad}} \,\mathrm{Disc}[L_2 : \varrho(L_1)]$*.*

Here $\mathrm{I}^\epsilon_{\mathrm{bad}}$ is given its literal meaning: $\mathrm{I}^\epsilon_{\mathrm{bad}} = \mathrm{I}_{\mathrm{bad}}$ when $\epsilon = 1$ and $\mathrm{I}^\epsilon_{\mathrm{bad}} = 1$ when $\epsilon = 0$.

Proof. The universality of $\mathbf{Sym}^\epsilon_\varrho(L_1, L_2)$ is clear from its definition. The statement is also clear if Λ is a field. If $\Lambda = \mathbb{Z}$, then it suffices to check that if $p \nmid \mathrm{I}^\epsilon_{\mathrm{bad}} \,\mathrm{Disc}[L_2 : \varrho(L_1)]$, then $\mathbf{Sym}^\epsilon_\varrho(L_1, L_2) \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is p-torsion-free. Thus it suffices to treat the remaining case that $\Lambda = W(k)$ for some finite field k of characteristic $p > 0$ such that $p \nmid \mathrm{I}^\epsilon_{\mathrm{bad}} \,\mathrm{Disc}[L_2 : \varrho(L_1)]$. We shall use a subscript Λ to denote whenever we form a tensor product with Λ.

Since $p \nmid [L_2 : \varrho(L_1)]$, we have an isomorphism $\varrho_\Lambda : L_{1,\Lambda} \overset{\sim}{\to} L_{2,\Lambda}$. Therefore we may set $L_\Lambda := L_{1,\Lambda} \overset{\sim}{\underset{\varrho_\Lambda}{\to}} L_{2,\Lambda}$ and consider the Λ-module $\mathbf{Sym}^\epsilon(L_\Lambda)$ defined by

$$\mathbf{Sym}^\epsilon(L_\Lambda) := (L_\Lambda \underset{\Lambda}{\otimes} L_\Lambda) / \begin{pmatrix} x \otimes y - y \otimes x \\ (bx) \otimes z - x \otimes (b^\star z) \end{pmatrix}_{x,y,z \in L_\Lambda, b \in \mathcal{O}_\Lambda}$$

when $\epsilon = 1$, and

$$\mathbf{Sym}^\epsilon(L_\Lambda) := (L_\Lambda \underset{\Lambda}{\otimes} L_\Lambda) / \begin{pmatrix} x \otimes x \\ (bx) \otimes z - x \otimes (b^\star z) \end{pmatrix}_{x,y,z \in L_\Lambda, b \in \mathcal{O}_\Lambda}$$

when $\epsilon = 0$.

The $\mathcal{O}_\Lambda$-lattice L_Λ, being projective by Proposition 1.1.1.23, factors by Lemma 1.1.3.4 as a finite sum $L_\Lambda = \underset{1 \leq i \leq t}{\oplus} M_i$, where for each $1 \leq i \leq t$, M_i is isomorphic to some M_τ. Then we see that the Λ-span of the images of $(b_1 x_i) \otimes (b_2 x_j)$, for all $1 \leq i \leq t$, $1 \leq j \leq t$, $x_i \in M_i$, $x_j \in M_j$, and $b_1, b_2 \in \mathcal{O}_\Lambda$, is the whole module $\mathbf{Sym}^\epsilon(L_\Lambda)$. In the definition of $\mathbf{Sym}^\epsilon(L_\Lambda)$, the first relation shows that we only need those $1 \leq i \leq j \leq t$, and the second relation shows that we can replace each $(b_1 x_i) \otimes (b_2 x_j)$ with $(b_2^\star b_1 x_i) \otimes x_j$. Hence we only need the Λ-span of elements of the form $(bx_i) \otimes x_j$, for $1 \leq i \leq j \leq t$ and $b \in \mathcal{O}_p$. Let us denote by c the restriction of * to $\mathcal{O}_{F,\Lambda}$. If $M_i \cong M_\tau$ and $M_j \cong M_{\tau'}$ but $\tau' \neq \tau \circ c$, then $(b_1 x_i) \otimes (b_2 x_j) = (b_2^\star b_1 x_i) \otimes x_j = 0$ for $b_1 \in \mathcal{O}_\tau$ and $b_2 \in \mathcal{O}_{\tau'}$ shows that

$M_i \otimes M_j = 0$. On the other hand, if $\tau' = \tau \circ c$, then $M_i \otimes M_j$ can be identified with the Λ-span of $(b_1 x_i) \otimes (b_2 x_j)$ for $b_1, b_2 \in \mathcal{O}_\tau$. Writing respectively $\mathcal{O}_{ij} = 0$ or $\mathcal{O}_{ij} := \mathcal{O}_\tau$ in these two cases, we have arrived at a finite sum of Λ-modules

$$\mathbf{Sym}^\epsilon(L_\Lambda) \cong \Big[\bigoplus_{1 \leq i < j \leq t} \mathcal{O}_{ij}\Big] \oplus \Big[\bigoplus_{1 \leq i \leq t} \mathbf{Sym}^\epsilon(\mathcal{O}_{ii})\Big],$$

in which $\mathbf{Sym}^\epsilon(\mathcal{O}_{ii}) := \mathcal{O}_{ii}/(b - b^\star)_{b \in \mathcal{O}_{ii}}$ when $\epsilon = 1$, and $\mathbf{Sym}^\epsilon(\mathcal{O}_{ii}) := \mathcal{O}_{ii}/(b)_{b \in \mathcal{O}_{ii}} = 0$ when $\epsilon = 0$. These $\mathbf{Sym}^\epsilon(\mathcal{O}_{ii})$, for $1 \leq i \leq t$, are the only possible sources of nonzero torsion of $\mathbf{Sym}^\epsilon(L_\Lambda)$. This completes the proof when $\epsilon = 0$.

It remains to consider the case that $\epsilon = 1$. Since each $\mathbf{Sym}^\epsilon(\mathcal{O}_{ij})$ is either 0 or a direct factor of $\mathbf{Sym}^\epsilon(\mathcal{O}_\Lambda) := \mathcal{O}_\Lambda/(b - b^\star)_{b \in \mathcal{O}_\Lambda}$, it suffices to show that $\mathbf{Sym}^\epsilon(\mathcal{O}_\Lambda)$ is torsion-free. If $x \in \mathcal{O}_\Lambda$ is mapped to any nonzero torsion element in $\mathbf{Sym}^\epsilon(\mathcal{O}_\Lambda)$, then $rx = y - y^\star$ for some $y \in \mathcal{O}_\Lambda$ and some nonzero r in Λ. This implies $rx = -rx^\star$, and hence $x = -x^\star$ in $\mathcal{O}_\Lambda$. By Proposition 1.2.2.1, there is some element z in $\mathcal{O}_\Lambda$ such that $x = z - z^\star$. This means x is also mapped to 0 in $\mathbf{Sym}^\epsilon(\mathcal{O}_\Lambda)$. This shows that $\mathbf{Sym}^\epsilon(\mathcal{O}_\Lambda)$ is torsion-free and completes the proof. □

1.2.3 Self-Dual Symplectic Modules

Let us continue with the setting in Section 1.2.2. Let k and Λ be either of the following two types:

1. k is a field of characteristic $p = 0$, and $\Lambda = k$. In this case we assume moreover that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$ is a product of matrix algebras.

2. k is a finite field of characteristic $p > 0$, and $\Lambda = W(k)$.

Suppose $p \nmid \mathrm{Disc}$. Let R be a noetherian complete local Λ-algebra. Throughout this section, the subscript Λ will mean tensor product with Λ, and the subscripts of R will have the two possible meanings as in Convention 1.1.3.3.

Let M be a projective $\mathcal{O}_R$-module of $\mathcal{O}_R$-multirank $(m_\tau)_\tau$ (see Definition 1.1.3.5), and let M_0 be the projective $\mathcal{O}_R$-module with $\mathcal{O}_R$-multirank $(1)_\tau$. Namely, $M \cong \underset{\tau}{\oplus} M_{\tau,R}^{\oplus m_\tau}$ and $M_0 \cong \underset{\tau}{\oplus} M_{\tau,R}$ as in Lemma 1.1.3.4. If we replace $\mathcal{O}_R$ with $\mathcal{O}_{F,R}$ in Lemma 1.1.3.4, then projective $\mathcal{O}_{F,R}$-modules N also admit decompositions as $N \cong \underset{\tau}{\oplus} \mathcal{O}_{F_\tau,R}^{\oplus n_\tau}$, and it is straightforward that $M_0 \underset{\mathcal{O}_{F,R}}{\otimes} N \cong \underset{\tau}{\oplus} M_{\tau,R}^{\oplus n_\tau}$ in this case. Conversely, it is straightforward to have the following:

Lemma 1.2.3.1. *With assumptions on R, M, M_0 as above, there is a unique projective $\mathcal{O}_{F,R}$-module N such that $M \cong M_0 \underset{\mathcal{O}_{F,R}}{\otimes} N$, with its $\mathcal{O}_{F,R}$-action given by the first tensor factor alone. Explicitly, $N \cong \underset{\tau}{\oplus} \mathcal{O}_{F_\tau,R}^{\oplus m_\tau}$ if $(m_\tau)_\tau$ is the $\mathcal{O}_R$-multirank of M. Using the definition of M_0 and the explicit description of N, we have the following canonical isomorphisms:*

$$\mathrm{End}_{\mathcal{O}_{F,R}}(M) \cong \mathrm{End}_{\mathcal{O}_{F,R}}(M_0) \underset{\mathcal{O}_{F,R}}{\otimes} \mathrm{End}_{\mathcal{O}_{F,R}}(N) \cong \mathcal{O}_R \underset{\mathcal{O}_{F,R}}{\otimes} \mathrm{End}_{\mathcal{O}_{F,R}}(N).$$

(Here, for the isomorphism $\mathrm{End}_{\mathcal{O}_{F,R}}(M_0) \cong \mathcal{O}_R$, we used the fact that $\mathcal{O}_R$ is a product of matrix algebras under our assumption.)

Suppose $\langle \cdot, \cdot \rangle : M \times M \to R$ is any perfect $\mathcal{O}_R$-pairing that induces an involution $\maltese$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$, such that the involution $\maltese$ sends $\mathcal{O}_R$ to itself and induces $\star$ on $\mathcal{O}_R$. (For our purpose it suffices to consider perfect pairings with values in R because locally free rank-one modules over R are automatically free.) Then the composition

$$(\maltese) \circ [(\star) \otimes (\mathrm{Id}_{\mathrm{End}_{\mathcal{O}_{F,R}}(N)})]$$

is an involution of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ that restricts to the identity on $\overline{\mathcal{O}}_R$. Hence, by Lemma 1.2.3.1, it defines an involution $\maltese_N$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(N)$ from which we obtain the decomposition

$$\maltese = (\star) \otimes (\maltese_N).$$

As a result, by Lemma 1.2.1.11, Remark 1.2.1.12, and Proposition 1.1.5.17, the classification of those perfect $\mathcal{O}_R$-pairings $\langle \cdot, \cdot \rangle$ on M that do induce involutions on $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ can be reduced to the analogous problem of $\mathcal{O}_{F,R}$-pairings $\langle \cdot, \cdot \rangle_N$ on N.

Lemma 1.2.3.2. *With the setting of M and N as above, assume that $p \nmid \mathrm{I}_{\mathrm{bad}} \mathrm{Disc}$. Suppose that $\maltese$ is induced by some alternating $\mathcal{O}_R$-pairing $\langle \cdot, \cdot \rangle : M \times M \to R$ (cf. Proposition 1.1.5.17), which decomposes as $\maltese = (\star) \otimes (\maltese_N)$ as above. Suppose that B is **simple**. Then we have the following cases corresponding to the classification in Proposition 1.2.1.14 and Definition 1.2.1.15:*

1. *Suppose B is of type* C. *Then the classification of involutions $\maltese$ of symplectic type is the same as the classification of involutions $\maltese_N$ of symplectic type.*

2. *Suppose B is of type* D. *Then the classification of involutions $\maltese$ of symplectic type is the same as the classification of involutions $\maltese_N$ of orthogonal type.*

3. *Suppose B is of type* A, *in which case F and F^+ are simple and $[F : F^+] = 2$. Then the classification of involutions $\maltese$ of unitary type is the same as the classification of involutions $\maltese_N$ of unitary type.*

Proof. Suppose B is of type C. Then we saw in the proof of Proposition 1.2.2.1 that we may assume that $\mathcal{O}_R = \mathrm{M}_k(\mathcal{O}_{F,R})$ for some integer k, and that the involution $\star$ is given by $x \mapsto x^\star = c\,{}^t x c^{-1}$ for some $c \in \mathrm{M}_k(\mathcal{O}_{F,R})$ such that ${}^t c = c$. Since ${}^t(bx)c^{-1}y = {}^t x\,{}^t b c^{-1} y = {}^t x c^{-1}(c\,{}^t b c^{-1}) y = {}^t x c^{-1}(b^\star y)$ for all $x, y, b \in \mathcal{O}_R$, this involution $\star$ is induced by the perfect symmetric bilinear pairing on the column vectors $\mathcal{O}_{F,R}^{\oplus k}$ given by $(x, y) \mapsto \mathrm{Tr}_{\mathcal{O}_{F,R}/R}({}^t x c^{-1} y)$ for $x, y \in \mathcal{O}_{F,R}^{\oplus k}$. Hence we see that $\maltese_N$ has to be induced by an alternating pairing in this case, which by definition is of symplectic type.

Suppose B is of type D. Since we assume that $p \nmid \mathrm{I}_{\mathrm{bad}} \mathrm{Disc}$, and $\mathrm{I}_{\mathrm{bad}} = 2$ exactly in this case, we see that $p \neq 2$ and hence 2 is not a zero divisor. In this case, a skew-symmetric pairing is always alternating and never symmetric. Since $p \nmid \mathrm{Disc}$, we may assume that $\mathcal{O}_R = \mathrm{M}_{2k}(\mathcal{O}_{F,R}) \cong \mathrm{M}_k(\mathrm{M}_2(\mathcal{O}_{F,R}))$ for some integer k, that the involution $\diamond$ of $\mathrm{M}_2(\mathcal{O}_{F,R})$ can be described explicitly as $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} \delta & -\beta \\ \gamma & \alpha \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) {}^t\!\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and that the involution $\star$ is given explicitly by conjugating $x \mapsto {}^t x^\diamond$ by an element $c \in \mathrm{M}_{2k}(\mathcal{O}_{F,R})$ such that ${}^t c^\diamond = c$. By conjugating $\mathrm{M}_2(\mathcal{O}_{F,R})$-entries by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in c, we may assume that the involution $\star$ is the conjugate of $x \mapsto {}^t x$ by an element $d \in \mathrm{M}_{2k}(\mathcal{O}_{F,R})$ such that ${}^t d = -d$. Therefore, the involution $\star$ can be induced by the perfect alternating pairing on $\mathcal{O}_{F,R}^{\oplus 2k}$ given by $(x, y) \mapsto \mathrm{Tr}_{\mathcal{O}_{F,R}/R}({}^t x d^{-1} y)$ for $x, y \in \mathcal{O}_{F,R}^{\oplus 2k}$, and we see that $\maltese_N$ has to be induced by a symmetric pairing in this case, which by definition is of orthogonal type.

There is nothing to prove in the case that B is of type A. □

Let us introduce some special forms of self-dual projective $\mathcal{O}_R$-modules. By Lemma 1.2.1.31, there exists an element $x_0 = (x_{[\tau]})_{[\tau]} \in \mathcal{O}_R$ such that $\mathcal{O}_R x_0$ is isomorphic to an integrable $\mathcal{O}_R$-module of $\mathcal{O}$-multirank $(1)_{[\tau]}$. Let us fix a choice of such a generator x_0. (The actual choice is immaterial.)

Definition 1.2.3.3. *Let α be any element in $\mathcal{O}_R$ such that $\alpha = \pm\alpha^\star$. The pair $(A_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha)$ is defined by the integrable $\mathcal{O}_R$-module*

$$A_\alpha := (\mathcal{O}_R x_0)$$

of $\mathcal{O}$-multirank $(1)_{[\tau]}$ spanned by some element x_0, together with the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\alpha$ associated (by Lemma 1.1.4.5) with the skew-Hermitian or Hermitian pairing $(\!|\,\cdot\,,\,\cdot\,|\!)_\alpha$ given by the relation $(\!|x_0, x_0|\!) = \alpha$. If $p \nmid \mathrm{Disc}$, in which case $(\mathrm{Diff}^{-1})_R = \mathcal{O}_R$, this is a perfect pairing if and only if α is a unit in $\mathcal{O}_R$.

Definition 1.2.3.4. *The integrable symplectic $\mathcal{O}_R$-module $(H, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std}})$ is defined by the $\mathcal{O}_R$-module*

$$H := (\mathcal{O}_R x_0) \oplus (\mathcal{O}_R x_0)$$

of $\mathcal{O}$-multirank $(2)_{[\tau]}$ spanned by $x_1 := (x_0, 0)$ and $x_2 := (0, x_0)$, together with the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std}}$ associated (by Lemma 1.1.4.5) with the skew-Hermitian pairing $(\!|\,\cdot\,,\,\cdot\,|\!)_{\mathrm{std}}$ given by the relations $(\!|x_1, x_1|\!) = (\!|x_2, x_2|\!) = 0$ and $(\!|x_1, x_2|\!) = 1$. This is always a perfect pairing when $p \nmid \mathrm{Disc}$.

For each integer $n \geq 0$, we define the integrable symplectic $\mathcal{O}_R$-module $(H_n, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},n})$ to be the orthogonal direct sum (see Definition 1.1.4.12) of n copies of $(H, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std}})$.

Suppose that we are given a finitely generated self-dual projective $\mathcal{O}_R$-module $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$. Let us decompose M as $M \cong \underset{\tau}{\oplus} M_{\tau,R}^{\oplus m_\tau}$ as in (1.2.1.27). The $\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to R$ is uniquely determined by an isomorphism $\langle\,\cdot\,,\,\cdot\,\rangle^* : M \to M^\vee$ of $\mathcal{O}_R$-modules. By Remark 1.1.5.15 and Lemma 1.1.5.16, there exists an isomorphism $M_{\tau,R}^\vee \cong M_{\tau\circ c,R}$ of $\mathcal{O}_R$-modules, and the existence of an isomorphism $\langle\,\cdot\,,\,\cdot\,\rangle^* : M \xrightarrow{\sim} M$ forces $m_\tau = m_{\tau\circ c}$ in the decomposition $M \cong \underset{\tau}{\oplus} M_{\tau,R}^{\oplus m_\tau}$. Let us define $[\tau]_c$ to be the equivalence class of τ under the action of c, and set $M_{[\tau]_c,R} = \underset{\tau\in[\tau]_c}{\oplus} M_{\tau,R}$, which is $M_{\tau,R}$ when $\tau = \tau\circ c$ and $M_{\tau,R} \oplus M_{\tau\circ c,R}$ when $\tau \neq \tau \circ c$. Accordingly, the action of $\mathcal{O}$ (resp. $\mathcal{O}_F$) on $M_{[\tau]_c,R}$ factors through $\mathcal{O}_{[\tau]_c,R}$ (resp. $\mathcal{O}_{F_{[\tau]_c},R}$), which is $\mathcal{O}_{\tau,R}$ (resp. $\mathcal{O}_{F_\tau,R}$) when $\tau = \tau\circ c$ and $\mathcal{O}_{\tau,R} \times \mathcal{O}_{\tau\circ c,R}$ (resp. $\mathcal{O}_{F_\tau,R} \times \mathcal{O}_{F_{\tau\circ c},R}$) when $\tau \neq \tau\circ c$. Let $m_{[\tau]_c} := m_\tau$ for each $\tau \in [\tau]_c$. As pointed out in Remark 1.1.5.15, for the purpose of studying pairings we may decompose $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ as an orthogonal direct sum

$$(M, \langle\,\cdot\,,\,\cdot\,\rangle) \cong \overset{\perp}{\underset{[\tau]_c}{\oplus}} (M_{[\tau]_c,R}^{\oplus m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c}). \tag{1.2.3.5}$$

Each $[\tau]_c$ determines a unique $[\tau] : F \to \mathbb{Q}_{[\tau]}$, which corresponds to a unique simple factor $B_{[\tau]}$ of B as in (1.2.1.10).

Now let us focus on *alternating* pairings:

Definition 1.2.3.6. *A symplectic $\mathcal{O}_R$-module $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ is said to be **of standard type** if every component $(M_{[\tau]_c,R}^{\oplus m_\tau}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$ in the decomposition (1.2.3.5) can be described as follows (cf. Definition 1.2.1.15):*

1. *If $B_{[\tau]}$ is of type C, then $(M_{[\tau]_c,R}^{\oplus m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$ is isomorphic to $(H_n, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},n}) \underset{\mathcal{O}_R}{\otimes} \mathcal{O}_{[\tau]_c,R}$ for some integer $n \geq 0$.*
2. *If $B_{[\tau]}$ is of type D, then $(M_{[\tau]_c,R}^{\oplus m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$ is isomorphic to the orthogonal direct sum of modules of the form $(A_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha)$, where α is a unit in $\mathcal{O}_{F_{[\tau]_c},R}$ satisfying $\alpha = -\alpha^\star$.*
3. *If $B_{[\tau]}$ is of type A, then $(M_{[\tau]_c,R}^{\oplus m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle)$ is isomorphic to the orthogonal direct sum of modules of the form $(A_\alpha, \langle\,\cdot\,,\,\cdot\,\rangle_\alpha)$, where α is a unit in $\mathcal{O}_{F_{[\tau]_c},R}$ satisfying $\alpha = -\alpha^\star$.*

Proposition 1.2.3.7. *Suppose $p \nmid \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}$ (and suppose $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$ is a product of matrix algebras when $p = 0$). Then every finitely generated self-dual projective symplectic $\mathcal{O}_R$-module $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ is symplectic isomorphic to some symplectic $\mathcal{O}_R$-module of standard type.*

This is a generalization of [118, Lem. 3.4]; [76, Lem. 7.2] is a similar result. The proof we give here follows more closely the one in [118, Lem. 3.4]. We shall proceed by induction on the $\mathcal{O}_R$-multiranks, based on the following basic lemmas:

Lemma 1.2.3.8. *Suppose $p \nmid \mathrm{Disc}$. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be a symmetric or skew-symmetric $\mathcal{O}_R$-pairing on M, and let $(\!|\,\cdot\,,\,\cdot\,|\!)$ be the associated Hermitian or skew-Hermitian pairing (with values in $(\mathrm{Diff}^{-1})_R = \mathcal{O}_R$) as in Lemma 1.1.4.5. Suppose x is an element in M such that the projection $(\!|x, x|\!)_\tau$ of $(\!|x, x|\!) \in \mathcal{O}_R$ to $\mathcal{O}_{\tau,R}$ is a unit in $\mathcal{O}_{\tau,R}$ for some τ. Let M_1 be the $\mathcal{O}_{\tau,R}$-span of x in M. Then the $\mathcal{O}_R$-module morphism*

$$\phi : M \to M_1 : z \mapsto (\!|x, z|\!)(\!|x, x|\!)_\tau^{-1} x$$

is surjective with kernel $M_1^\perp$. Hence, the $\mathcal{O}_R$-module isomorphism

$$M \xrightarrow{\sim} M_1 \oplus M_1^\perp : z \mapsto (\phi(z), z - \phi(z))$$

identifies M as the orthogonal direct sum of M_1 and $M_1^\perp$ (with the pairings given by the restrictions of $\langle\,\cdot\,,\,\cdot\,\rangle$). In particular, M_1 and $M_1^\perp \cong M/M_1$ are projective $\mathcal{O}_R$-submodules of M.

Proof. The morphism ϕ is surjective because $\phi(rx) = rx$ for all $r \in \mathcal{O}_{\tau,R}$. If $rx = 0$ for some $r \in \mathcal{O}_{\tau,R}$, then $r(\!|x, x|\!) = (\!|x, rx|\!) = 0$, which shows that $r = 0$. Therefore, an element $z \in M$ satisfies $\phi(z) = 0$ if and only if $(\!|x, z|\!) = 0$, which shows that $\ker(\phi) = M_1^\perp$. ☐

Lemma 1.2.3.9. *Suppose $p \nmid \mathrm{Disc}$. (Here we do not need the assumption that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$ is a product of matrix algebras when $p = 0$.) Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be a skew-symmetric $\mathcal{O}_R$-pairing on M, and let $(\!|\,\cdot\,,\,\cdot\,|\!)$ be the associated skew-Hermitian pairing (with values in $(\mathrm{Diff}^{-1})_R = \mathcal{O}_R$) as in Lemma 1.1.4.5. Suppose x and y are elements in M such that $(\!|x, x|\!) = (\!|y, y|\!) = 0$, and such that the projection $(\!|x, y|\!)_\tau$ of $(\!|x, y|\!) \in \mathcal{O}_R$ to $\mathcal{O}_{\tau,R}$ is a unit in $\mathcal{O}_{\tau,R}$ for some τ. Let M_1 be the $\mathcal{O}_{\tau\circ c,R}$-span of x in M, let M_2 be the $\mathcal{O}_{\tau,R}$-span of y in M. Then $M_1 \cap M_2 = 0$ (and so the sum $M_1 + M_2$ is direct), and the $\mathcal{O}_R$-module morphism*

$$\phi : M \to M_1 \oplus M_2 : z \mapsto -(\!|y, z|\!)(\!|x, y|\!)_\tau^{-1} x + (\!|x, z|\!)(\!|x, y|\!)_\tau^{-1} y$$

is surjective with kernel $(M_1 \oplus M_2)^\perp$. Hence, the $\mathcal{O}_R$-module isomorphism

$$M \xrightarrow{\sim} (M_1 \oplus M_2) \oplus (M_1 \oplus M_2)^\perp : z \mapsto (\phi(z), z - \phi(z))$$

identifies M as the orthogonal direct sum of $(M_1 \oplus M_2)$ and $(M_1 \oplus M_2)^\perp$ (with the pairings given by the restrictions of $\langle\,\cdot\,,\,\cdot\,\rangle$). In particular, M_1, M_2, and $(M_1 \oplus M_2)^\perp \cong M/(M_1 \oplus M_2)$ are projective $\mathcal{O}_R$-submodules of M.

Proof. If there exist $a \in \mathcal{O}_{\tau\circ c,R}$ and $b \in \mathcal{O}_{\tau,R}$ such that $ax = by \in M_1 \cap M_2$, then $a^\star \llparenthesis x, y \rrparenthesis = \llparenthesis ax, y \rrparenthesis = \llparenthesis by, y \rrparenthesis = 0$ and $b \llparenthesis x, y \rrparenthesis = \llparenthesis x, by \rrparenthesis = \llparenthesis x, ax \rrparenthesis = 0$ force both $a = 0$ and $b = 0$. Hence $M_1 \cap M_2 = 0$. Moreover, the argument shows that $ax = by = 0$ is possible only when $a = 0$ and $b = 0$. Therefore, an element $z \in M$ satisfies $\phi(z) = 0$ if and only if $\llparenthesis x, z \rrparenthesis = 0$ and $\llparenthesis y, z \rrparenthesis = 0$, which shows that $\ker(\phi) = (M_1 \oplus M_2)^\perp$. Finally, the morphism ϕ is surjective because $\phi(ax) = ax$ and $\phi(by) = by$ for all $a \in \mathcal{O}_{\tau\circ c,R}$ and $b \in \mathcal{O}_{\tau,R}$. □

Proof of Proposition 1.2.3.7. By Lemmas 1.2.1.11 and 1.2.1.31, and the same argument as in the proof of Proposition 1.2.2.1 based on the decomposition (1.2.2.2), we may assume that B is simple.

Let us first classify the pairings up to weak symplectic isomorphism (see Definition 1.1.5.1).

By Lemma 1.2.3.2, we may replace $\mathcal{O}_R$ with $\mathcal{O}_{F,R}$, and replace $(M, \langle \cdot , \cdot \rangle)$ with $(N, \langle \cdot , \cdot \rangle_N)$, where $\langle \cdot , \cdot \rangle_N$ is an alternating $\mathcal{O}_{F,R}$-pairing on N except when B is of type D, in which case we consider *symmetric* $\mathcal{O}_{F,R}$-pairings instead. For simplicity, let us retain the notation $\mathcal{O}_R$, M, and $\langle \cdot , \cdot \rangle$ for the various objects. Moreover, we shall use $\llparenthesis \cdot , \cdot \rrparenthesis$ to denote the Hermitian or skew-Hermitian pairing associated with $\langle \cdot , \cdot \rangle$ by Lemma 1.1.4.5. Since $p \nmid \mathrm{Disc}$, we have $(\mathrm{Diff}^{-1})_R = \mathcal{O}_R$ by definition. Therefore the target of the pairing $\llparenthesis \cdot , \cdot \rrparenthesis$ is given by $\mathcal{O}_R = \mathcal{O}_{F,R}$. Although not logically necessary, it is often more convenient to do calculations with $\llparenthesis \cdot , \cdot \rrparenthesis$.

By replacing $(M, \langle \cdot , \cdot \rangle)$ with a factor $(M_{[\tau],R}^{\oplus m_\tau}, \langle \cdot , \cdot \rangle_{[\tau]})$ in (1.2.3.5) (which works for both alternating and symmetric pairings), we may assume from now on that the $\mathcal{O}_R$-multirank $(m_\tau)_\tau$ of M has either only one nonzero entry m_τ with $\tau = \tau \circ c$, or only two nonzero entries m_τ and $m_{\tau\circ c}$ with $\tau \neq \tau \circ c$.

In the case $\tau \neq \tau \circ c$, there exist invertible elements $f_1 \in \mathrm{End}_{\mathcal{O}_R}(M_{\tau,R}^{\oplus m_\tau})$ and $f_2 \in \mathrm{End}_{\mathcal{O}_R}(M_{\tau\circ c,R}^{\oplus m_\tau})$ such that $\langle (x_1, x_2), (y_1, y_2) \rangle = f_1(x_1)(y_2) + f_2(x_2)(y_1)$ for all $x_1, y_1 \in M_{\tau,R}^{\oplus m_\tau}$ and $x_2, y_2 \in M_{\tau\circ c,R}^{\oplus m_\tau}$. The condition that $\langle \cdot , \cdot \rangle$ is alternating shows that $f_1(x_1)(x_2) + f_2(x_2)(x_1) = 0$ for all $(x_1, x_2) \in M$. In other words, $f_2 = -f_1^\vee$ is uniquely determined by f_1. If we conjugate the pairing $\langle \cdot , \cdot \rangle$ by the automorphism $f_1 \times \mathrm{Id}$ of M, then

$$\begin{aligned} \langle (f_1^{-1}(x_1), x_2), (f_1^{-1}(y_1), y_2) \rangle &= x_1(y_2) - f_1^\vee(x_2)(f_1^{-1}(y_1)) \\ &= x_1(y_2) - x_2((f_1 \circ f_1^{-1})(y_1)) = x_1(y_2) - x_2(y_1). \end{aligned}$$

This argument shows that every two self-dual alternating $\mathcal{O}_R$-pairings on M are isomorphic to each other. Explicitly, the pairing is isomorphic to $(A_\alpha, \langle \cdot , \cdot \rangle_\alpha)^{\overset{\perp}{\oplus} m_\tau}$ with $\alpha = (1, -1) \in \mathcal{O}_{F_\tau,R}^\times \times \mathcal{O}_{F_{\tau\circ c},R}^\times$.

In the remainder of the proof, let us assume that $\tau = \tau \circ c$. Then the action of $\mathcal{O}_R = \mathcal{O}_{F,R}$ on M factors through $\overline{\mathcal{O}}_R = \mathcal{O}_{F_\tau,R}$. In this case, it makes sense to speak of $\mathcal{O}_R$*-ranks* (rather than $\mathcal{O}_R$*-multiranks*) of M and its nontrivial submodules, because there is a unique nonzero number in each $\mathcal{O}_R$-multirank. For simplicity, let us replace $(H, \langle \cdot , \cdot \rangle_{\mathrm{std}})$ and $(A_\alpha, \langle \cdot , \cdot \rangle_\alpha)$ (see Definitions 1.2.3.4 and 1.2.3.3) with $(H, \langle \cdot , \cdot \rangle_{\mathrm{std}}) \underset{\mathcal{O}_R}{\otimes} \overline{\mathcal{O}}_R$ and $(A_\alpha, \langle \cdot , \cdot \rangle_\alpha) \underset{\mathcal{O}_R}{\otimes} \overline{\mathcal{O}}_R$, respectively, and replace $\mathcal{O}_R$ with $\overline{\mathcal{O}}_R$.

We claim that $(M, \langle \cdot , \cdot \rangle)$ is an orthogonal direct sum of submodules of the form $(H, \langle \cdot , \cdot \rangle_{\mathrm{std}})$ or $(A_\alpha, \langle \cdot , \cdot \rangle_\alpha)$, depending on the type of B. We shall proceed by induction on the $\mathcal{O}_R$-rank of M. If the $\mathcal{O}_R$-rank of M is zero, then the claim is automatic.

Suppose there exists $x \in M$ such that $\alpha := \llparenthesis x, x \rrparenthesis$ is a unit. (This does not happen when B is of type C.) Let M_1 be the $\mathcal{O}_R$-span of x in M. By Lemma 1.2.3.8, M is the orthogonal direct sum of M_1 and $M_1^\perp$, with the pairings being the restrictions of $\langle \cdot , \cdot \rangle$. Hence the claim follows by induction because $(M_1, \langle \cdot , \cdot \rangle|_{M_1}) \cong (A_\alpha, \langle \cdot , \cdot \rangle_\alpha)$.

Otherwise, assume that there does not exist $z \in M$ such that $\llparenthesis z, z \rrparenthesis$ is a unit. By perfectness of the pairing, there exist $x, y \in M$ such that $\llparenthesis x, y \rrparenthesis = 1$, and such that neither $\llparenthesis x, x \rrparenthesis$ nor $\llparenthesis y, y \rrparenthesis$ is a unit in $\mathcal{O}_R = \mathcal{O}_{F_\tau,R}$.

Suppose B is of type D, in which case $\langle \cdot , \cdot \rangle$ is symmetric. Since $p \nmid I_{\mathrm{bad}} \mathrm{Disc}$, there exists $\beta \in \Lambda$ such that $2\beta = 1$. Set $z = x + \beta y$. Then $\llparenthesis z, z \rrparenthesis = 1 + (\text{nonunit})$ is a unit. This is a contradiction.

Suppose B is of type A, in which case $[F : F^+] = 2$. Let β be any element in $\mathcal{O}_F$ such that $\mathcal{O}_{F,\Lambda} = \mathcal{O}_{F^+,\Lambda} + \mathcal{O}_{F^+,\Lambda}\beta$. Since $\mathrm{Tr}_{F/F^+}((\beta - \beta^\star)^{-1}) = 0$ and $\mathrm{Tr}_{F/F^+}((\beta - \beta^\star)^{-1}\beta) = 1$, we see that $(\beta - \beta^\star)^{-1} \in (\mathrm{Diff}_{\mathcal{O}_F/\mathcal{O}_{F^+}}^{-1})_\Lambda = \mathcal{O}_{F,\Lambda}$ (because $p \nmid \mathrm{Disc}$). Hence $\alpha := \beta - \beta^\star$ is a unit in $\mathcal{O}_{F,\Lambda}$. Set $z = -\beta x + y$. Then $\llparenthesis z, z \rrparenthesis = \beta^\star\beta \llparenthesis x, x \rrparenthesis - \beta^\star \llparenthesis x, y \rrparenthesis - \beta \llparenthesis y, x \rrparenthesis + \llparenthesis y, y \rrparenthesis = \alpha + (\text{nonunit})$, which is a unit. This is a contradiction too.

Hence we may assume that B is of type C in the remainder of the proof of the claim, in which case $\langle \cdot , \cdot \rangle$ is alternating. By Lemmas 1.1.4.5 and 1.1.5.5, this implies $\llparenthesis x, x \rrparenthesis = \llparenthesis y, y \rrparenthesis = 0$ (under the simplified assumption that $\mathcal{O}_R = \mathcal{O}_{F,R}$). Let M_1 (resp. M_2) be the $\mathcal{O}_R$-spans of x (resp. y). By Lemma 1.2.3.9, the sum $M_1 + M_2$ is direct, and M is the orthogonal direct sum of $(M_1 \oplus M_2)$ and $(M_1 \oplus M_2)^\perp$, with the pairings being the restrictions of $\langle \cdot , \cdot \rangle$. Hence the claim follows by induction because $(M_1 \oplus M_2, \langle \cdot , \cdot \rangle|_{M_1 \oplus M_2}) \cong (H, \langle \cdot , \cdot \rangle_{\mathrm{std}})$.

Summarizing what we have obtained (under the simplified assumption),

1. If B is of type C, then $(M, \langle \cdot , \cdot \rangle)$ is isomorphic to $(H_{\frac{m_\tau}{2}}, \langle \cdot , \cdot \rangle_{\mathrm{std}, \frac{m_\tau}{2}})$.
2. If B is of type D, then $(M, \langle \cdot , \cdot \rangle)$ (as a symmetric pairing) is isomorphic to the orthogonal direct sum of modules of the form $(A_\alpha, \langle \cdot , \cdot \rangle_\alpha)$, each α being a unit in $\mathcal{O}_{F,R}$.
3. If B is of type A, then $(M, \langle \cdot , \cdot \rangle)$ (as an alternating pairing) is isomorphic to the orthogonal direct sum of modules of the form $(A_\alpha, \langle \cdot , \cdot \rangle_\alpha)$, each α being a unit in $\mathcal{O}_{F,R}$ satisfying $\alpha = -\alpha^\star$.

Then we can conclude the proof by tensoring the pairing $(M, \langle \cdot , \cdot \rangle)$ over $\mathcal{O}_{F,R}$ with any self-dual pairing on $\mathcal{O}$ inducing the involution * as in the proof of Lemma 1.2.3.2. The difference between weak symplectic isomorphisms and symplectic isomorphisms is immaterial, because multiplying by an element in $\mathcal{O}_{F,R}^\times$ in the second factor of the pairing does not affect the classification. □

Corollary 1.2.3.10. *Suppose $p \nmid I_{\mathrm{bad}} \mathrm{Disc}$. Then every two finitely generated self-dual projective symplectic $\mathcal{O}_R$-modules $(M_1, \langle \cdot , \cdot \rangle_1)$ and $(M_2, \langle \cdot , \cdot \rangle_2)$ such that M_1 and M_2 have the same $\mathcal{O}_R$-multirank are isomorphic after some finite étale extension $R \to R'$. If $p > 0$ and B does not involve any simple factor of type* D*, then we may take $R' = R$.*

Proof. It suffices to show that, in the latter two cases in the summary at the end of the proof of Proposition 1.2.3.7, the submodules $(A_\alpha, \langle \cdot , \cdot \rangle_\alpha)$ appearing in the orthogonal direct sum are all isomorphic to each other over some $R \to R'$.

Suppose B is of type D, in which case $F = F^+$. It suffices to show that, for each τ, each unit α in the complete local ring $\mathcal{O}_{F_\tau,R}$ (with residue field k) is the square of some unit α' in $\mathcal{O}_{F_\tau,R'}$ over some finite étale extension $R \to R'$. Consider

$f(X) := X^2 - \alpha$. Since $p \neq 2$, the reduction of $f'(X) = 2X$ in k has nonzero values for nonzero inputs of X. Let k' be the finite separable extension of k over which $f(X)$ has a (necessarily nonzero) solution, and let R' be the unique finite étale extension of R such that $R' \underset{R}{\otimes} k \cong k'$. Then it follows from Hensel's lemma (see, for example, [41, Thm. 7.3]) that $f(X)$ has a solution α' in $\mathcal{O}_{F,R'}$, as desired.

Suppose B is of type A, in which case $[F : F^+] = 2$. When $\mathrm{char}(k) = 0$, we replace R with a finite étale extension over which F/F^+ is split. Then the claim is true because, by Lemma 1.1.5.9 (with $\epsilon = 1$), if $\gamma := \alpha(\alpha')^{-1} \in \mathcal{O}_{F,R}^\times$ satisfies $\gamma = \gamma^\star$, then $\gamma = \delta\delta^\star$ for some unit $\delta \in \mathcal{O}_{F,R}$. □

Proposition 1.2.3.11. *Suppose $p \nmid \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}$. Let $B \cong \prod_{[\tau]} B_{[\tau]}$ be the decomposition of B into its simple factors as in (1.2.1.10). Let $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ be any self-dual integrable symplectic $\mathcal{O}_R$-module. Then $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ decomposes accordingly as*

$$(M, \langle\,\cdot\,,\,\cdot\,\rangle) \cong \underset{[\tau]}{\oplus} (M_{[\tau],R}^{\oplus m_{[\tau]}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]}),$$

where $M_{[\tau],R}$ is defined as in Lemma 1.2.1.31, and where $(m_{[\tau]})_{[\tau]}$ is the $\mathcal{O}$-multirank of M as in Definition 1.2.1.25. Define a group functor H over R by setting

$$\mathrm{H}(R') := \left\{ \begin{array}{c} (g, r) \in \mathrm{GL}_{\mathcal{O}_{R'}}(M_{R'}) \times \mathbf{G}_{\mathrm{m}}(R') : \\ \langle gx, gy\rangle = r\langle x, y\rangle \;\forall x, y \in M_{R'} \end{array} \right\},$$

or, when $M \neq 0$, by setting equivalently

$$\mathrm{H}(R') := \{g \in \mathrm{End}_{\mathcal{O}_{R'}}(M_{R'}) : \nu(g) := g^{\maltese} g \in (R')^\times\}.$$

Let k^{sep} be a separable closure of k, and let $\tilde{R}$ be the strict Henselization of $R \twoheadrightarrow k \to k^{\mathrm{sep}}$. Then the group $\mathrm{H} \underset{R}{\otimes} \tilde{R}$ depends (up to isomorphism) only on the $\mathcal{O}$-multirank $(m_{[\tau]})_{[\tau]}$ of the underlying integrable $\mathcal{O}_R$-module M, and is independent of the perfect alternating $\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ on M that we use. Moreover, based on the classification of the simple factors $B_{[\tau]}$ as in Proposition 1.2.1.14 and Definition 1.2.1.15, we have the following descriptions of simple factors of $\mathrm{H}^{\mathrm{ad}}(k^{\mathrm{sep}})$:

1. *If $B_{[\tau]}$ is of type C, then the existence of $\langle\,\cdot\,,\,\cdot\,\rangle$ forces $m_{[\tau]}$ to be even, and $(M_{[\tau],R}^{\oplus m_{[\tau]}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]})$ defines simple factors of $\mathrm{H}^{\mathrm{ad}}(k^{\mathrm{sep}})$ that are of type C and rank $\frac{m_{[\tau]}}{2}$.*

2. *If $B_{[\tau]}$ is of type D, then $(M_{[\tau],R}^{\oplus m_{[\tau]}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]})$ defines simple factors of $\mathrm{H}^{\mathrm{ad}}(k^{\mathrm{sep}})$ that are either of type B and rank $\frac{m_{[\tau]}-1}{2}$ when $m_{[\tau]}$ is odd, or of type D and rank $\frac{m_{[\tau]}}{2}$ when $m_{[\tau]}$ is even.*

3. *If $B_{[\tau]}$ is of type A, then $(M_{[\tau],R}^{\oplus m_{[\tau]}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]})$ defines simple factors of $\mathrm{H}^{\mathrm{ad}}(k^{\mathrm{sep}})$ that are type A of rank $m_{[\tau]} - 1$.*

Every simple factor of $\mathrm{H}^{\mathrm{ad}}(k^{\mathrm{sep}})$ is contributed by some simple factor $B_{[\tau]}$ as described above.

In particular, this applies to the group functor $\mathrm{G} \underset{\mathbb{Z}}{\otimes} R$ (cf. Definition 1.2.1.6).

Proof of Proposition 1.2.3.11. We may assume that $M \neq 0$. Since this is a question about simple factors of $\mathrm{H}^{\mathrm{ad}}(k^{\mathrm{sep}})$, we can ignore the similitude factors (cf. Remark 1.2.1.12) and assume as in the proof of Proposition 1.2.3.7 that B is simple. (When $p = 0$, the assumption that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$ is a product of matrix algebras is automatic if we replace k with k^{sep}.)

By Lemma 1.2.3.1, there is an $\mathcal{O}_F$-rank-m integrable $\mathcal{O}_{F,R}$-lattice N such that $M \cong M_0 \underset{\mathcal{O}_{F,R}}{\otimes} N$, where M_0 is the unique $\mathcal{O}$-rank-one integrable $\mathcal{O}_R$-lattice (see Definition 1.2.1.30 and Lemma 1.2.1.31). As we saw in Lemma 1.2.3.2, $^{\maltese}$ decomposes as $^{\maltese} = (^\star) \otimes (^{\maltese_N})$ for some involution $^{\maltese_N}$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(N)$ induced by some perfect pairing $\langle\,\cdot\,,\,\cdot\,\rangle_N$ on N. Since the isomorphism $\mathrm{End}_{\mathcal{O}_R}(M) \cong \mathrm{End}_{\mathcal{O}_{F,R}}(N)$ carries the restriction of the involution $^{\maltese}$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(M)$ to the involution $^{\maltese_N}$ of $\mathrm{End}_{\mathcal{O}_{F,R}}(N)$, we obtain

$$\mathrm{H}(R') \cong \left\{ g \in \mathrm{End}_{\mathcal{O}_{F,R'}}(N_{R'}) : \nu(g) := g^{\maltese_N} g \in (R')^\times \right\}$$

for every R-algebra R'. For simplicity, let us assume that $\mathcal{O}_R = \mathcal{O}_{F,R}$ and $(M, \langle\,\cdot\,,\,\cdot\,\rangle) = (N, \langle\,\cdot\,,\,\cdot\,\rangle_N)$ in the remainder of the proof.

For our purpose, we may replace k with any sufficiently large separable extension such that $\mathrm{Frac}(\Lambda)$ contains all the images of $\tau : F \hookrightarrow \mathrm{Frac}(\Lambda)^{\mathrm{sep}}$ (for any choice of separable closure $\mathrm{Frac}(\Lambda)^{\mathrm{sep}}$ of $\mathrm{Frac}(\Lambda)$). Then we may assume that the structural homomorphism $\Lambda \to \mathcal{O}_{F_\tau,\Lambda}$ is an isomorphism for all τ. As in (1.2.3.5) and in the proof of Proposition 1.2.3.7, we have an orthogonal direct sum

$$(M, \langle\,\cdot\,,\,\cdot\,\rangle) \cong \underset{[\tau]_c}{\overset{\perp}{\oplus}} (M_{[\tau]_c,R}^{\oplus m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$$

of finitely generated projective modules, with all the $m_{[\tau]_c}$ given by the same integer m, the $\mathcal{O}$-rank of the integrable $\mathcal{O}_R$-module M. Let $\mathrm{H}_{[\tau]_c}$ be the algebraic group defined by $(M_{[\tau]_c,R}^{\oplus m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$ as above. Then H is isomorphic to the subgroup of $\prod_{[\tau]_c} \mathrm{H}_{[\tau]_c}$ consisting of elements having the same similitude factors in all $\mathrm{H}_{[\tau]_c}$. Hence it suffices to classify each of the symplectic $\mathcal{O}_R$-modules $(M_{[\tau]_c,R}^{\oplus m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$.

Now let us replace everything with its base change from R to $\tilde{R}$, the strict Henselization of $R \to k \to k^{\mathrm{sep}}$. (Certainly, it is enough to work over some finite étale extension R' of R that splits everything.) By Proposition 1.2.3.7 and Corollary 1.2.3.10, the classification of $(M_{[\tau]_c,\tilde{R}}^{\oplus m_{[\tau]_c}}, \langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c})$ is completely known:

- If B is of type C, then we may identify $M_{[\tau]_c,\tilde{R}}$ with $\mathcal{O}_{F_\tau,\tilde{R}} \cong \tilde{R}$, and identify the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c}$ explicitly with
$$\tilde{R}^{\oplus m} \oplus \tilde{R}^{\oplus m} \to \tilde{R} : (x, y) \mapsto {}^t x \begin{pmatrix} & 1_{\frac{m}{2}} \\ -1_{\frac{m}{2}} & \end{pmatrix} y.$$
This alternating pairing defines GSp_m over $\tilde{R}$.

- If B is of type D, then we may identify $M_{[\tau]_c,\tilde{R}}$ with $\mathcal{O}_{F_\tau,\tilde{R}} \cong \tilde{R}$, and identify the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c}$ explicitly with
$$\tilde{R}^{\oplus m} \oplus \tilde{R}^{\oplus m} \to \tilde{R} : (x, y) \mapsto {}^t x y.$$
This symmetric pairing defines the orthogonal group GO_m over $\tilde{R}$.

- If B is of type A, then we may identify $M_{[\tau]_c,\tilde{R}} = M_{\tau,\tilde{R}} \times M_{\tau\circ c,\tilde{R}}$ with $\mathcal{O}_{F_\tau,\tilde{R}} \times \mathcal{O}_{F_{\tau\circ c},\tilde{R}} \cong \tilde{R} \times \tilde{R}$, and identify the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{[\tau]_c}$ explicitly with
$$(\tilde{R}^{\oplus m} \times \tilde{R}^{\oplus m}) \oplus (\tilde{R}^{\oplus m} \times \tilde{R}^{\oplus m}) \to \tilde{R} :$$
$$((x_1, x_2), (y_1, y_2)) \mapsto {}^t x_1 y_2 - {}^t x_2 y_1.$$

Let $g = (g_1, g_2) \in \mathrm{End}_{\mathcal{O}_{\tilde{R}}}(M_{[\tau]_c, \tilde{R}}) \cong \mathrm{M}_m(\tilde{R}) \times \mathrm{M}_m(\tilde{R})$. Then

$$\begin{aligned}&\langle (x_1, x_2), g(y_1, y_2)\rangle \\ &= \langle (x_1, x_2), (g_1(y_1), g_2(y_2))\rangle = {}^t x_1 g_2(y_2) - {}^t x_2 g_1(y_1) \\ &= {}^t({}^t g_2 x_1) y_2 - {}^t({}^t g_1 x_2) y_1 = \langle ({}^t g_2 x_1, {}^t g_1 x_2), (y_1, y_2)\rangle\end{aligned}$$

for all $(x_1, x_2), (y_1, y_2) \in \tilde{R}^{\oplus m} \times \tilde{R}^{\oplus m}$, and hence $g^{\maltese} = ({}^t g_2, {}^t g_1)$. Therefore the condition that $\nu(g) := g^{\maltese} g \in \tilde{R}^\times$ implies that $g_1 \in \mathrm{GL}_m(\tilde{R})$ and $g_2 = \nu(g)^{-1}\, {}^t g_1^{-1}$, and we have an isomorphism $\mathrm{H}_{[\tau]}(\tilde{R}) \xrightarrow{\sim} \mathrm{GL}_m(\tilde{R}) \times \mathbf{G}_{\mathrm{m}}(\tilde{R}) : g = (g_1, g_2) \mapsto (g_1, \nu(g))$.

All the above identifications remain valid if we replace $\tilde{R}$ with an $\tilde{R}$-algebra, and they are compatible in a functorial way. In each of the three cases, if we form the quotient of H by its center, then we obtain the product of the quotients of $\mathrm{H}_{[\tau]_c}$ by their centers, as desired. □

As a by-product of these explicit identifications,

Corollary 1.2.3.12. *With assumptions on k, Λ, and R as above, the group* H *defined in Proposition 1.2.3.11 is smooth over R.*

In particular, the group functor $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \Lambda$ (see Definition 1.2.1.6) is smooth over Λ.

1.2.4 Gram–Schmidt Process

Let us retain the assumptions and notation of Section 1.2.3 in this section, except that we no longer need the assumption that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$ is a product of matrix algebras when $p = 0$.

Definition 1.2.4.1. *An alternating $\mathcal{O}_R$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to R$ is called **sufficiently alternating** if it satisfies $\langle x, rx\rangle = 0$ for all $x \in M$ and all $r \in \mathcal{O}_\Lambda$ such that $r = r^\star$. Accordingly, a symplectic $\mathcal{O}_R$-module $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ is called **sufficiently symplectic** if the alternating pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ is sufficiently alternating.*

Lemma 1.2.4.2. *If there exists a noetherian complete local Λ-algebra R' in which* 2 *is not a zero divisor, and a symplectic $\mathcal{O}_{R'}$ module $(M', \langle\,\cdot\,,\,\cdot\,\rangle')$ such that $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ is the pullback of $(M', \langle\,\cdot\,,\,\cdot\,\rangle')$ under some homomorphism $R' \to R$, then $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ is automatically sufficiently symplectic.*

Proof. If suffices to show that $(M', \langle\,\cdot\,,\,\cdot\,\rangle')$ is sufficiently symplectic. Over R', we have $\langle x, rx\rangle' = \langle r^\star x, x\rangle' = \langle rx, x\rangle' = -\langle x, rx\rangle'$ for all $x \in M'$ and all $r \in \mathcal{O}_\Lambda$ such that $r = r^\star$, which forces $\langle x, rx\rangle' = 0$, as desired. □

Lemma 1.2.4.3. *Suppose $p \nmid \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}$. Let $\langle\,\cdot\,,\,\cdot\,\rangle : M \times M \to R$ be an alternating $\mathcal{O}_R$-pairing, and let $(\!|\,\cdot\,,\,\cdot\,|\!) : M \times M \to \mathcal{O}_R$ be the associated skew-Hermitian pairing as in Lemma 1.1.4.5. In the case that $p = 2$ and B involves any simple factor of type* C, *we assume moreover that the alternating pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ is sufficiently alternating. Then, for every $x \in M$, there exists some element $b \in \mathcal{O}_R$ such that $(\!|x, x|\!) = b - b^\star$.*

Proof. Since $(\!|\,\cdot\,,\,\cdot\,|\!)$ is skew-Hermitian, we know that $a := (\!|x, x|\!)$ satisfies $a = -a^\star$. Then Proposition 1.2.2.1 implies that there exists an element β such that $a = b - b^\star$, unless we are in the case that B involves some simple factor of type C, that $p = 2$, and that 2 is a zero divisor in R.

As in the proof of Proposition 1.2.2.1, we may assume that B is simple. (The assumption now is that B is simple of type C, that $p = 2$, and that 2 is a zero divisor in R.) Moreover, we may assume that $\mathcal{O}_\Lambda \cong \mathrm{M}_k(\mathcal{O}_{F,\Lambda})$ for some integer k, and that the involution $\star$ is given by $x \mapsto c\,{}^t x c^{-1}$ for some $c \in \mathrm{M}_k(\mathcal{O}_{F,\Lambda})$ such that ${}^t c = c$. Then $a = -a^\star = c\,{}^t a c^{-1}$ implies $ac = -\,{}^t(ac)$, and we may represent ac as an element (a_{ij}) in $\mathrm{M}_k(\mathcal{O}_{F,R})$ such that $a_{ij} = -a_{ji}$ for all $1 \leq i \leq k$ and $1 \leq j \leq k$. For every $d \in \mathrm{M}_k(\mathcal{O}_{F,\Lambda})$ such that ${}^t d = d$, we have $(cd)^\star = d^\star c^\star = c\,{}^t d c^{-1} c\,{}^t c c^{-1} = cd$. Let us consider the special case that $d = \mathrm{diag}(d_i)$ is a diagonal matrix. Since $\langle\,\cdot\,,\,\cdot\,\rangle$ is sufficiently alternating, we have $\langle x, cdx\rangle = \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}((cd)a) = \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(d(ac)) = \mathrm{Tr}_{\mathcal{O}_F/\mathbb{Z}}(\sum\limits_{1 \leq i \leq k} d_i a_{ii}) = 0$ for all choices of $d_i \in \mathcal{O}_{F,\Lambda}$. This forces $a_{ii} = 0$ for all i, and hence $ac = e - {}^t e$ with $e = (e_{ij})$ given by $e_{ij} = a_{ij}$ if $i < j$ and $e_{ij} = 0$ if $i \geq j$. Set $b := ec^{-1}$. Then $a = (ac)c^{-1} = (e - {}^t e)c^{-1} = ec^{-1} - c\,{}^t(ec^{-1})c^{-1} = b - b^\star$, as desired. □

Lemma 1.2.4.4. *Suppose $p \nmid \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}$. Let $\langle\,\cdot\,,\,\cdot\,\rangle$ be a sufficiently alternating $\mathcal{O}_R$-pairing on M (see Definition 1.2.4.1). Let M_1 be a totally isotropic projective submodule of M such that M/M_1 is projective, and such that the restriction of $\langle\,\cdot\,,\,\cdot\,\rangle^* : M \to M^\vee$ to M_1 is an injection. Then $M_1^\vee$ is embedded as a totally isotropic projective submodule of M, and there is a symplectic isomorphism*

$$(M, \langle\,\cdot\,,\,\cdot\,\rangle) \to (M_1 \oplus M_1^\vee, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}) \oplus ((M_1 \oplus M_1^\vee)^\perp, \langle\,\cdot\,,\,\cdot\,\rangle|_{(M_1 \oplus M_1^\vee)^\perp}).$$

Proof. Since M_1 is finitely generated and projective, it decomposes as $M \cong M_{\tau,R}^{\oplus m_\tau}$ for some $\mathcal{O}_R$-multirank $m = (m_\tau)_\tau$. Let us proceed by induction on $|m| := \sum\limits_\tau m_\tau$. There is nothing to prove when $|m| = 0$.

When $|m| \geq 1$, we may decompose $M_1 \cong M_{1,0} \oplus M_1'$, where $M_{1,0}$ and M_1' are finitely generated projective $\mathcal{O}_R$-modules such that $M_{1,0} \cong M_{\tau,R}$ for some τ with $m_\tau \geq 1$. By assumption, M_1 is totally isotropic, and hence $M_{1,0}$ and M_1' are totally isotropic as well. By assumption that restriction of $\langle\,\cdot\,,\,\cdot\,\rangle^*$ to M_1 is an injection, the morphism $M/M_{1,0}^\perp \to M_{1,0}^\vee$ induced by $\langle\,\cdot\,,\,\cdot\,\rangle^*$ is a *surjection*. Since $M_{1,0} \cong M_{\tau,R}$, its dual $M_{1,0}^\vee \cong M_{\tau\circ c,R}$ is projective as well. Hence the surjection $M/M_{1,0}^\perp \twoheadrightarrow M_{1,0}^\vee$ splits.

Let x be any element spanning $M_{1,0}$ in the sense that $M_{1,0} = (\mathcal{O}_{\tau,R})x$. The statement above that the surjection $M/M_{1,0}^\perp \twoheadrightarrow M_{1,0}^\vee$ splits implies that there exists an element y in M such that $\langle x, y\rangle = 1_{\tau\circ c}$, the identity element in $\mathcal{O}_{\tau\circ c,R}$. By Lemma 1.2.4.3 and the assumption that $\langle\,\cdot\,,\,\cdot\,\rangle$ is sufficiently alternating, there is an element $b \in \mathcal{O}_R$ such that $b - b^\star = (\!|y, y|\!)$. Then $(\!|bx + y, bx + y|\!) = 0 - b + b^\star + (\!|y, y|\!) = 0$. Replacing y by $bx + y$, we may assume that $(\!|y, y|\!) = 0$. Then Lemma 1.2.3.9 implies that M is the orthogonal direct sum of $M_{1,0} \oplus M_{1,0}^\vee$ and its orthogonal complement $(M_{1,0} \oplus M_{1,0}^\vee)^\perp \cong M_{1,0}^\perp / M_{1,0}$.

Note that M_1' is a totally isotropic projective $\mathcal{O}_R$-submodule of $M_{1,0}^\perp / M_{1,0}$ such that $(M_{1,0}^\perp / M_{1,0})/M_1' \cong M_{1,0}^\perp / M_1$, which is projective because $M_{1,0}^\vee \oplus M_{1,0}^\perp / M_1 \cong M/M_1$ is projective by assumption. By induction, we may write $M_{1,0}^\perp / M_{1,0}$ as the orthogonal direct sum of $M_1' \oplus (M_1')^\vee$ and the orthogonal complement of $M_1' \oplus (M_1')^\vee$ in $M_{1,0}^\perp / M_{1,0}$. Then we can conclude the proof by putting the two orthogonal direct sums together. □

Proposition 1.2.4.5. *With assumptions on k and Λ as above, let $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a finitely generated self-dual sufficiently symplectic projective $\mathcal{O}_R$-module, and let M_1 and M_2 be two **totally isotropic** projective $\mathcal{O}_R$-submodules of M, such that*

$M_1 \cong M_2$ and such that M/M_1 and M/M_2 are both projective. If $M_1 \oplus M_1^\vee$ has the same $\mathcal{O}_R$-multirank as M, then there is a symplectic automorphism of $(M, \langle\,\cdot\,,\,\cdot\,\rangle)$ that sends M_1 to M_2.

Proof. By Lemma 1.2.4.4, there exist symplectic isomorphisms

$$(M, \langle\,\cdot\,,\,\cdot\,\rangle) \xrightarrow{\sim} (M_i \oplus M_i^\vee, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}) \overset{\perp}{\oplus} ((M_i \oplus M_i^\vee)^\perp, \langle\,\cdot\,,\,\cdot\,\rangle|_{(M_i \oplus M_i^\vee)^\perp})$$

for $i = 1, 2$. By comparison of $\mathcal{O}_R$-multiranks, we have $(M_i \oplus M_i^\vee)^\perp = 0$ for $i = 1, 2$. Thus any $f_0 : M_1 \xrightarrow{\sim} M_2$ induces a symplectic isomorphism $(f_0 \oplus (f_0^\vee)^{-1}) : (M_1 \oplus M_1^\vee) \xrightarrow{\sim} (M_2 \oplus M_2^\vee)$, and the proposition follows. □

Proposition 1.2.4.6. *With assumptions on k and Λ as above, let $\tilde{R} \twoheadrightarrow R$ be a surjection of Artinian local Λ-algebras, with kernel I satisfying $I^2 = 0$. Let $(\tilde{M}, \langle\,\cdot\,,\,\cdot\,\rangle)$ be a finitely generated self-dual sufficiently symplectic projective $\mathcal{O}_{\tilde{R}}$-module, and let $(M, \langle\,\cdot\,,\,\cdot\,\rangle) := (\tilde{M}, \langle\,\cdot\,,\,\cdot\,\rangle) \underset{\tilde{R}}{\otimes} R$. Suppose M_1 is a **totally isotropic** projective $\mathcal{O}_R$-submodule of M, such that M/M_1 is projective. Then there is a totally isotropic projective $\mathcal{O}_{\tilde{R}}$-submodule $\tilde{M}_1$ of $\tilde{M}$ such that $\tilde{M} \underset{\tilde{R}}{\otimes} R = M$ and such that $\tilde{M}/\tilde{M}_1$ is projective.*

Proof. Since M_1 is finitely generated and projective, it decomposes as $M \cong M_{\tau,R}^{\oplus m_\tau}$ for some $\mathcal{O}_R$-multirank $m = (m_\tau)_\tau$. Let us proceed by induction on $|m| := \sum_\tau m_\tau$. There is nothing to prove when $|m| = 0$.

When $|m| \geq 1$, we may decompose $M_1 \cong M_{1,0} \oplus M_1'$, where $M_{1,0}$ and M_1' are finitely generated projective $\mathcal{O}_R$-modules such that $M_{1,0} \cong M_{\tau,R}$ for some τ with $m_\tau \geq 1$. By Lemma 1.2.4.4, there is an isomorphism $M \xrightarrow{\sim} (M_{1,0} \oplus M_{1,0}^\vee) \overset{\perp}{\oplus} (M_{1,0} \oplus M_{1,0}^\vee)^\perp$, so that M_1' is embedded as a projective $\mathcal{O}_R$-submodule of $(M_{1,0} \oplus M_{1,0}^\vee)^\perp \cong M_{1,0}^\perp/M_{1,0}$.

Let $(\!|\,\cdot\,,\,\cdot\,|\!)$ be the skew-Hermitian pairing associated with $\langle\,\cdot\,,\,\cdot\,\rangle$ by Lemma 1.1.4.5. Let x be any element spanning $M_{1,0}$, and let y be some element spanning $M_{1,0}^\vee$ such that $(\!|x, y|\!) = 1_{\tau\circ c}$, the identity element of $\mathcal{O}_{\tau\circ c,R}$. Let $\tilde{x}$ (resp. $\tilde{y}$) be any element in $\tilde{M}$ lifting x (resp. y). Then $(\!|\tilde{x}, \tilde{y}|\!) =: r$ is an element in $\mathcal{O}_{\tau\circ c,\tilde{R}}$ lifting $(\!|x, y|\!) = 1_{\tau\circ c} \in \mathcal{O}_{\tau\circ c,R}$. In particular, r is a unit in $\mathcal{O}_{\tau\circ c,\tilde{R}}$. Replacing $\tilde{y}$ by $r^{-1}\tilde{y}$, we may assume that $(\!|\tilde{x}, \tilde{y}|\!) = 1_{\tau\circ c} \in \mathcal{O}_{\tau\circ c,\tilde{R}}$. Let $\xi := (\!|\tilde{x}, \tilde{x}|\!) \in \mathcal{O}_{\tilde{R}}$. Since $(\!|x, x|\!) = 0$ by the assumption that $M_{1,0}$ is totally isotropic, we see that $\xi \in I \cdot \mathcal{O}_{\tilde{R}}$. By Lemma 1.2.4.3 and its proof, and by the assumption that $\langle\,\cdot\,,\,\cdot\,\rangle$ is sufficiently alternating, we see that $\xi = \eta - \eta^\star$ for some $\eta \in I \cdot \mathcal{O}_{\tilde{R}}$. Note that $\eta^\star \in I \cdot \mathcal{O}_{\tilde{R}}$ implies $\eta\eta^\star = 0$. Then $\tilde{x}' := \tilde{x} - \eta\tilde{y}$ is another lifting of x, such that $(\!|\tilde{x}', \tilde{x}'|\!) = (\!|\tilde{x}, \tilde{x}|\!) - \eta^\star(\!|\tilde{y}, \tilde{x}|\!) - \eta(\!|\tilde{x}, \tilde{y}|\!) + \eta^\star\eta(\!|\tilde{y}, \tilde{y}|\!) = \xi + \eta^\star - \eta + 0 = 0$. Similarly, we may find another lifting $\tilde{y}' := \tilde{y} - \zeta\tilde{x}$ for some $\zeta \in I \cdot \mathcal{O}_{\tilde{R}}$ such that $(\!|\tilde{y}', \tilde{y}'|\!) = 0$. For each of these choices of η and ζ, we have $(\!|\tilde{x}', \tilde{y}'|\!) = (\!|\tilde{x}, \tilde{y}|\!) - \eta(\!|\tilde{y}, \tilde{y}|\!) - \zeta(\!|\tilde{x}, \tilde{x}|\!) + \eta\zeta = 1$ because all terms but $(\!|\tilde{x}, \tilde{y}|\!)$ on the right-hand side lie in $I^2 \cdot \mathcal{O}_{\tilde{R}} = 0$. Let $\tilde{M}_{1,0}$ be the $\mathcal{O}_{\tau,\tilde{R}}$-span of $\tilde{x}'$, and let $\tilde{M}_{1,0}^\vee$ be the $\mathcal{O}_{\tau\circ c,\tilde{R}}$-span of $\tilde{y}'$. Then Lemma 1.2.3.9 shows that there is a symplectic isomorphism $\tilde{M} \xrightarrow{\sim} (\tilde{M}_{1,0} \oplus \tilde{M}_{1,0}^\vee) \overset{\perp}{\oplus} (\tilde{M}_{1,0} \oplus \tilde{M}_{1,0}^\vee)^\perp$. In particular, $\tilde{M}_{1,0}$ is a totally isotropic projective $\mathcal{O}_{\tilde{R}}$-submodule lifting $M_{1,0}$, and $(\tilde{M}_{1,0} \oplus \tilde{M}_{1,0}^\vee)^\perp \cong \tilde{M}_{1,0}^\perp/\tilde{M}_{1,0}$ is a projective self-dual symplectic $\mathcal{O}_{\tilde{R}}$-module lifting $(M_{1,0} \oplus M_{1,0}^\vee)^\perp$.

Note that M_1' is a totally isotropic projective $\mathcal{O}_R$-submodule of $M_{1,0}^\perp/M_{1,0}$ such that $(M_{1,0}^\perp/M_{1,0})/M_1' \cong M_{1,0}^\perp/M_1$ is projective. By induction, it can be lifted to a projective $\mathcal{O}_{\tilde{R}}$-submodule $\tilde{M}_1'$ of $\tilde{M}_{1,0}^\perp/\tilde{M}_{1,0}$ such that $(\tilde{M}_{1,0}^\perp/\tilde{M}_{1,0})/\tilde{M}_1' \cong \tilde{M}_{1,0}^\perp/\tilde{M}_1$ is projective. Let $\tilde{M}_1$ be the preimage of $\tilde{M}_1'$ in $\tilde{M}_{1,0}^\perp$, which is a totally isotropic $\mathcal{O}_{\tilde{R}}$-submodule of $\tilde{M}$. It is projective because it is isomorphic to $\tilde{M}_{1,0} \oplus \tilde{M}_1'$, and $\tilde{M}/\tilde{M}_1$ is $\mathcal{O}_{\tilde{R}}$-projective because it is isomorphic to $\tilde{M}_{1,0}^\vee \oplus (\tilde{M}_{1,0}^\perp/\tilde{M}_1)$. Hence it satisfies all our requirements. □

1.2.5 Reflex Fields

Let $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ be a PEL-type $\mathcal{O}$-lattice as in Definition 1.2.1.3. The natural $\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-action on $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ may differ from the $\mathbb{C}$-action given by the polarization $h : \mathbb{C} \to \mathrm{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ by the complex conjugation $1 \otimes c$. Then we can decompose

$$L \underset{\mathbb{Z}}{\otimes} \mathbb{C} = V_0 \oplus V_0^c \tag{1.2.5.1}$$

(as a direct sum of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules), where $h(z)$ acts by $1 \otimes z$ on V_0, and by $1 \otimes z^c$ on V_0^c.

Both V_0 and V_0^c are totally isotropic submodules of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$, because

$$\begin{aligned}\sqrt{-1}\langle x, y\rangle &= \langle (1 \otimes \sqrt{-1})x, y\rangle = \langle h(\sqrt{-1})x, y\rangle \\ &= \langle x, h(-\sqrt{-1})y\rangle = \langle x, (1 \otimes (-\sqrt{-1}))y\rangle = -\sqrt{-1}\langle x, y\rangle\end{aligned}$$

for all $x, y \in V_0$. (The case for V_0^c is similar.) Since $\langle\,\cdot\,,\,\cdot\,\rangle$ is nondegenerate, it induces a perfect pairing between V_0 and V_0^c, or equivalently, an isomorphism $V_0^c \xrightarrow{\sim} \mathrm{Hom}_\mathbb{C}(V_0, \mathbb{C}(1))$ of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules. This determines an isomorphism $V_0^c \xrightarrow{\sim} V_0^\vee := \mathrm{Hom}_\mathbb{C}(V_0, \mathbb{C})$ for each choice of $\sqrt{-1}$.

Let us denote the unique simple module of $B \underset{F,\tau}{\otimes} \mathbb{C}$ by W_τ. Applying Corollary 1.1.2.6 to the $B \underset{\mathbb{Q}}{\otimes} \mathbb{C}$-modules V_0 and V_0^c, we obtain decompositions

$$V_0 \cong \underset{\tau : F \to \mathbb{C}}{\oplus} W_\tau^{\oplus p_\tau}$$

and

$$V_0^c \cong \underset{\tau : F \to \mathbb{C}}{\oplus} W_\tau^{\oplus q_\tau}$$

for some (tuples of) integers $(p_\tau)_\tau$ and $(q_\tau)_\tau$.

Definition 1.2.5.2. *The integers $(p_\tau)_\tau$ and $(q_\tau)_\tau$ are called the **signatures** of V_0 and V_0^c, respectively, and the pairs of integers $(p_\tau, q_\tau)_\tau$ are called the **signatures** of $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$.*

Remark 1.2.5.3. The signatures of V_0 (resp. V_0^c) form simply the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-multirank of V_0 (resp. V_0^c) when $\Lambda = k = \mathbb{C}$ in Definition 1.1.3.5.

Since the decompositions above are determined by the action of the center F and its homomorphisms into $\mathbb{C}$ (see Corollary 1.1.2.6), we have $p_\tau = q_{\tau\circ c}$ for all τ. Suppose that $(m_{[\tau]})_{[\tau]}$ is the $\mathcal{O}$-multirank of L as in Definition 1.2.1.21, then the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-multirank $(m_\tau)_\tau$ of $L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ is given by $m_\tau = s_{[\tau]} m_{[\tau]}$ as in Lemma 1.2.1.33, and we have

$$L \underset{\mathbb{Z}}{\otimes} \mathbb{C} \cong \underset{\tau : F \to \mathbb{C}}{\oplus} W_\tau^{\oplus m_\tau}$$

with $m_\tau = p_\tau + q_\tau = p_\tau + p_{\tau\circ c}$ for all τ by (1.2.5.1).

Definition 1.2.5.4. *The* ***reflex field*** F_0 *of* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ *is the* ***field of definition*** *of* V_0 *as an* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$*-module. More precisely,* F_0 *is the fixed field of* $\mathbb{C}$ *by the elements* σ *in* $\mathrm{Aut}(\mathbb{C}/\mathbb{Q})$ *such that* V_0 *and* $V_0 \underset{\mathbb{C},\sigma}{\otimes} \mathbb{C}$ *are isomorphic as* $B \underset{\mathbb{Q}}{\otimes} \mathbb{C}$*-modules (see Definition 1.1.2.7 and Remark 1.1.2.9).*

Remark 1.2.5.5. The reflex field is by definition a subfield of $\mathbb{C}$.

Let us state the following special cases of the results in Section 1.1.2, applied with $C = B$, $E = F$, $k = \mathbb{Q}$, and $K = K^{\mathrm{sep}} = \mathbb{C}$ there.

Corollary 1.2.5.6. $F_0 = \mathbb{Q}(\mathrm{Tr}_{\mathbb{C}}(b|V_0) : b \in B) = \mathbb{Q}(\mathrm{Tr}_{\mathbb{C}}(b|V_0) : b \in \mathcal{O})$.

Proof. This is a special case of Corollary 1.1.2.16, which is applicable because $\mathrm{char}(\mathbb{Q}) = 0$. □

Corollary 1.2.5.7. *If a rational prime number* p *is unramified in* F*, then it is unramified in* F_0*.*

Proof. The discriminant of F over $\mathbb{Q}$ is the same as that of every Galois conjugate of F over $\mathbb{Q}$. Therefore, if p does not divide the discriminant of F over $\mathbb{Q}$, then it does not divide the discriminant of F^{Gal} over $\mathbb{Q}$ either. By Corollary 1.1.2.12, F_0 is contained in F^{Gal}. Hence p does not divide the discriminant of F_0 over $\mathbb{Q}$, as desired. □

Each choice of $\sqrt{-1}$ induces compatible isomorphisms $\mathbb{C}(1) \xrightarrow{\sim} \mathbb{C}$ and $V_0^c \xrightarrow{\sim} V_0^\vee$, allowing us to rewrite (1.2.5.1) as a *symplectic* isomorphism

$$(L \underset{\mathbb{Z}}{\otimes} \mathbb{C}, \langle\,\cdot\,,\,\cdot\,\rangle) \xrightarrow{\sim} (V_0 \oplus V_0^\vee, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}) \tag{1.2.5.8}$$

(see Definition 1.1.4.8 and Lemma 1.1.4.13). (The actual choice of $\sqrt{-1}$ is immaterial for our purpose.)

Lemma 1.2.5.9. *Suppose* F_0' *is any finite extension of* F_0 *in* $\mathbb{C}$ *such that* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} F_0'$ *is a product of matrix algebras over fields. Then there exists an* $\mathcal{O}_{F_0'}$*-torsion-free* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathcal{O}'_{F_0}$*-module* L_0 *such that* $L_0 \underset{\mathcal{O}'_{F_0}}{\otimes} \mathbb{C} \cong V_0$ *as* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$*-modules. We can take* F_0' *to be unramified at any prescribed finite set of rational primes unramified in* $B \cong \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$*.*

Proof. By Corollary 1.1.2.14, $p_\tau = p_{\tau'}$ for every $\tau, \tau' : F \to \mathbb{C}$ in the same $\mathrm{Aut}(\mathbb{C}/F_0')$-orbit. Since $\mathcal{O} \underset{\mathbb{Z}}{\otimes} F_0'$-modules are parameterized by multiranks labeled by $\mathrm{Gal}(F_0^{\mathrm{sep}}/F_0')$-orbits of homomorphisms $F \to F_0^{\mathrm{sep}}$, and since they decompose over F_0^{sep} (and accordingly over $\mathbb{C}$) as in (1.1.2.10) (with $s_{[\tau]} = 1$ there when $\mathcal{O} \underset{\mathbb{Z}}{\otimes} F_0'$ is a product of matrix algebras over fields), it follows that there exists an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} F_0'$-module V_{00} such that $V_{00} \underset{F_0'}{\otimes} \mathbb{C}$ has multirank $(p_\tau)_\tau$. Then we take L_0 to be the $\mathcal{O}$-span of any full $\mathcal{O}'_{F_0}$-lattice in V_{00}. The existence of F_0' unramified at any prescribed finite subset of $\Box$ follows from [107, Thm. 32.15 and 32.18] (using the Grunwald–Wang theorem). □

Lemma 1.2.5.10. *Every* L_0 *in Lemma 1.2.5.9 satisfies* $\mathrm{Det}_{\mathcal{O}|L_0} = \mathrm{Det}_{\mathcal{O}|V_0}$ *as elements in* $\mathbb{C}[\mathcal{O}^\vee] \cong \mathbb{Z}[\mathcal{O}^\vee] \underset{\mathbb{Z}}{\otimes} \mathbb{C}$*. Moreover, we can view* $\mathrm{Det}_{\mathcal{O}|V_0}$ *as an element in* $\mathcal{O}_{F_0}[\mathcal{O}^\vee] \cong \mathbb{Z}[\mathcal{O}^\vee] \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0}$*.*

Proof. For every $\sigma \in \mathrm{Aut}(\mathbb{C}/F_0)$, we have $\mathrm{Det}_{\mathcal{O}|V_0} = \sigma(\mathrm{Det}_{\mathcal{O}|V_0})$ because $V_0 \cong V_0 \underset{\mathbb{C},\sigma}{\otimes} \mathbb{C}$ as $B \underset{\mathbb{Q}}{\otimes} \mathbb{C}$-modules. Hence $\mathrm{Det}_{\mathcal{O}|V_0} \in F_0[\mathcal{O}^\vee]$. Then we have $\mathrm{Det}_{\mathcal{O}|V_0} \in \mathcal{O}_{F_0}[\mathcal{O}^\vee] = F_0[\mathcal{O}^\vee] \cap \mathcal{O}_{F_0'}[\mathcal{O}^\vee]$ by Lemma 1.2.5.9. □

Suppose we have a homomorphism $\mathcal{O}_{F_0} \to k$, and suppose k is either characteristic zero or a finite field. Suppose $p = \mathrm{char}(k)$ satisfies $p \nmid \mathrm{Disc}$. Let $\Lambda = k$ when $p = 0$, and let $\Lambda = W(k)$ when $p > 0$.

Lemma 1.2.5.11. *With assumptions as above, let* R *be a noetherian local* Λ*-algebra with residue field* k*. Let* M *and* M' *be two* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$*-modules that are finitely generated and projective as an* R*-module. Then the following are equivalent:*

1. $M \cong M'$ *as* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$*-modules.*
2. $\mathrm{Det}_{\mathcal{O}|M} = \mathrm{Det}_{\mathcal{O}|M'}$ *in* $R[\mathcal{O}^\vee] \cong \mathbb{Z}[\mathcal{O}^\vee] \underset{\mathbb{Z}}{\otimes} R$*.*
3. $M_0 := M \underset{R}{\otimes} k$ *and* $M_0' := M' \underset{R}{\otimes} k$ *satisfy* $\mathrm{Det}_{\mathcal{O}|M_0} = \mathrm{Det}_{\mathcal{O}|M_0'}$ *in* $k[\mathcal{O}^\vee] \cong \mathbb{Z}[\mathcal{O}^\vee] \underset{\mathbb{Z}}{\otimes} k$*.*

In particular, if $\mathrm{Det}_{\mathcal{O}|M_0} = \mathrm{Det}_{\mathcal{O}|V_0} = \mathrm{Det}_{\mathcal{O}|M_0'}$ *in* $k[\mathcal{O}^\vee]$*, then we have* $M \cong M'$ *as* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$*-modules. (Here we interpret* $\mathrm{Det}_{\mathcal{O}|V_0}$ *as the push-forward of an element in* $\mathcal{O}_{F_0}[\mathcal{O}^\vee]$ *by Lemma 1.2.5.10.)*

Proof. The implications from 1 to 2 and from 2 to 3 are clear. It remains to justify the implication from 3 to 1. If $\mathrm{Det}_{\mathcal{O}|M_0} = \mathrm{Det}_{\mathcal{O}|M_0'}$ in $k[\mathcal{O}^\vee] \cong \mathbb{Z}[\mathcal{O}^\vee] \underset{\mathbb{Z}}{\otimes} k$, then $M_0 \cong M_0'$ as $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$-modules by Proposition 1.1.2.20 (and by separability of $\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} k$ over k). Hence $M \cong M'$ as $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-modules by Lemma 1.1.3.1. □

Let k and Λ be as above, with $p = \mathrm{char}(k)$ such that $p \nmid \mathrm{I}_{\mathrm{bad}} \mathrm{Disc}[L^\# : L]$. Let F_0' and L_0 be chosen as in Lemma 1.2.5.9, such that F_0' is unramified at p when $p > 0$. By Corollary 1.2.3.10, and by comparing multiranks using (1.2.5.8), there exists a finite étale ring extension $\Lambda \hookrightarrow \Lambda'$, together with a homomorphism $\mathcal{O}_{F_0',(p)} \to \Lambda'$ extending $\mathcal{O}_{F_0,(\Box)} \to \Lambda$, such that we have a symplectic isomorphism

$$(L \underset{\mathbb{Z}}{\otimes} \Lambda', \langle\,\cdot\,,\,\cdot\,\rangle) \cong ((L_0 \underset{\mathcal{O}_{F_0'}}{\otimes} \Lambda') \oplus (L_0 \underset{\mathcal{O}_{F_0'}}{\otimes} \Lambda')^\vee, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}), \tag{1.2.5.12}$$

an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \Lambda'$-module analogue of (1.2.5.8). Let us fix the choice of such a Λ'.

Definition 1.2.5.13. *The subgroup functor* $\mathrm{P}_{0,\Lambda'}$ *of* $\mathrm{G}_{\Lambda'} := \mathrm{G} \underset{\mathbb{Z}}{\otimes} \Lambda'$ *is defined by setting, for each* Λ'*-algebra* R*,*

$$\mathrm{P}_{0,\Lambda'}(R) := \{g \in \mathrm{G}_{\Lambda'}(R) : g((L_0 \underset{\mathcal{O}_{F_0'}}{\otimes} R)^\vee) = (L_0 \underset{\mathcal{O}_{F_0'}}{\otimes} R)^\vee\}.$$

Lemma 1.2.5.14. *For every noetherian complete local Λ'-algebra R, the association $g \mapsto g((L_0 \underset{\mathcal{O}_{F'_0}}{\otimes} R)^\vee)$ for $g \in \mathrm{G}_{\Lambda'}(R)$ induces a **bijection** from the set $(\mathrm{G}_{\Lambda'}/\mathrm{P}_{0,\Lambda'})(R)$ of R-valued points of the quotient functor $\mathrm{G}_{\Lambda'}/\mathrm{P}_{0,\Lambda'}$ to the set of totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-submodules M of $(L \underset{\mathbb{Z}}{\otimes} R, \langle \cdot \, , \cdot \rangle)$ such that $M \cong (L_0 \underset{\mathcal{O}_{F'_0}}{\otimes} R)^\vee$ and such that $(L \underset{\mathbb{Z}}{\otimes} R)/M$ is projective as an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-module.*

Proof. By Lemma 1.2.4.2, $(L \underset{\mathbb{Z}}{\otimes} R, \langle \cdot \, , \cdot \rangle)$ is sufficiently symplectic (see Definition 1.2.4.1) because it is the pullback of a symplectic module defined over $\mathbb{Z}$. Then we can conclude by applying Proposition 1.2.4.5. □

Proposition 1.2.5.15. *The functor $\mathrm{G}_{\Lambda'}/\mathrm{P}_{0,\Lambda'}$ is formally smooth over Λ'.*

Proof. Let $\tilde{R} \twoheadrightarrow R$ be a surjection of Artinian local Λ'-algebras, with kernel I satisfying $I^2 = 0$. Suppose we have a translation $g((L_0 \underset{\mathcal{O}_{F'_0}}{\otimes} R)^\vee)$ of $L \underset{\mathbb{Z}}{\otimes} R$ by some element $g \in \mathrm{G}_{\Lambda'}(R)$. Proposition 1.2.4.6 applies because $(L \underset{\mathbb{Z}}{\otimes} \tilde{R}, \langle \cdot \, , \cdot \rangle)$ is sufficiently symplectic (as explained in the proof of Lemma 1.2.5.14 above). Hence there is a totally isotropic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \tilde{R}$-submodule M of $L \underset{\mathbb{Z}}{\otimes} \tilde{R}$ lifting $g((L_0 \underset{\mathcal{O}_{F'_0}}{\otimes} R)^\vee)$ such that $(L \underset{\mathbb{Z}}{\otimes} \tilde{R})/M$ is projective as an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-module. Now we can conclude the proof by applying Lemma 1.2.5.14. □

Proposition 1.2.5.16. *With assumptions on k and Λ' as above, the group $\mathrm{P}_{0,\Lambda'}$ is smooth.*

Proof. If $L = 0$, then $\mathrm{P}_{0,\Lambda'} = \mathrm{G}_{\Lambda'} = \mathbf{G}_{\mathrm{m},\Lambda'}$ and the proposition is clear. Hence we may assume that $L \neq 0$.

Let $\tilde{R} \twoheadrightarrow R$ be a surjection of Artinian local Λ'-algebras, with kernel I satisfying $I^2 = 0$. Let us denote by $\tilde{M} := L_0 \underset{\mathcal{O}_{F'_0}}{\otimes} \tilde{R}$, and $M := \tilde{M} \underset{\tilde{R}}{\otimes} R$. Then the $\mathcal{O} \underset{\tilde{R}}{\otimes} R$-module $\tilde{M}^\vee$ (resp. $M^\vee$) embeds as a totally isotropic submodule of $L \underset{\mathbb{Z}}{\otimes} \tilde{R}$ (resp. $L \underset{\mathbb{Z}}{\otimes} R$), and we have a canonical isomorphism $(L \underset{\mathbb{Z}}{\otimes} \tilde{R})/\tilde{M}^\vee \cong \tilde{M}$ (resp. $(L \underset{\mathbb{Z}}{\otimes} R)/M^\vee \cong M$). Let us take any isomorphism $\tilde{\psi} : (\tilde{M} \oplus \tilde{M}^\vee, \langle \cdot \, , \cdot \rangle_{\mathrm{can.}}) \xrightarrow{\sim} (L \underset{\mathbb{Z}}{\otimes} \tilde{R}, \langle \cdot \, , \cdot \rangle)$, and let $\psi := \tilde{\psi} \underset{\tilde{R}}{\otimes} R$.

Let $g \in \mathrm{P}_{0,\Lambda'}(R)$. Using ψ, we obtain three morphisms $\alpha \in \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(M)$, $\beta \in \mathrm{Hom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(M^\vee, M)$, and $\gamma \in \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(M^\vee)$ such that

$$\psi(g(\psi^{-1}(x, f))) = (\alpha(x) + \beta(f), \gamma(f))$$

for all $x \in M$ and $f \in M^\vee$. For simplicity, let us suppress ψ in the following notation. Then it is convenient to express the above relation in matrix form as $g = \begin{pmatrix} \alpha & \beta \\ & \gamma \end{pmatrix}$. In this case, the relation (viewing vectors as column vectors)

$$\begin{aligned} \langle (x_1, f_1), g(x_2, f_2) \rangle &= \langle (x_1, f_1), (\alpha x_2 + \beta f_2, \gamma f_2) \rangle \\ &= (\gamma f_2)(x_1) - f_1(\alpha x_2 + \beta f_2) = f_2(\gamma^\vee x_1) - (\alpha^\vee f_1)(x_2) - (\beta^\vee f_1)(f_2) \\ &= \langle (\gamma^\vee x_1 - \beta^\vee f_1, \alpha^\vee f_1), (x_2, f_2) \rangle = \langle g^\maltese (x_1, f_1), (x_2, f_2) \rangle \end{aligned}$$

shows that we have $g^\maltese = \begin{pmatrix} \gamma^\vee & -\beta^\vee \\ & \alpha^\vee \end{pmatrix}$, and the relation $g^\maltese g = r \in R^\times$ becomes

$$\begin{pmatrix} \alpha & \beta \\ & \gamma \end{pmatrix} \begin{pmatrix} \gamma^\vee & -\beta^\vee \\ & \alpha^\vee \end{pmatrix} = \begin{pmatrix} \alpha\gamma^\vee & -\alpha\beta^\vee + \beta\alpha^\vee \\ & \gamma\alpha^\vee \end{pmatrix} = \begin{pmatrix} r & \\ & r \end{pmatrix}.$$

Each such g can be decomposed as

$$\begin{pmatrix} \alpha & \beta \\ & \gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ & (\alpha^\vee)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}\beta \\ & 1 \end{pmatrix},$$

where $\alpha^{-1}\beta$ is symmetric in the sense that $(\alpha^{-1}\beta)^\vee = \alpha^{-1}\beta$. Then each of the three terms in the product is also an element of $\mathrm{P}_{0,\Lambda'}(R)$. Therefore it suffices to show that we can lift each of the three kinds of elements.

If $g = \left(\begin{smallmatrix} 1 & 0 \\ & r \end{smallmatrix}\right)$ for some $r \in R^\times$, then each lifting $\tilde{r} \in \tilde{R}$ of r is a unit, and hence defines a lifting $\tilde{g} := \left(\begin{smallmatrix} 1 & 0 \\ & \tilde{r} \end{smallmatrix}\right)$ of g.

If $g = \left(\begin{smallmatrix} \alpha & 0 \\ & (\alpha^\vee)^{-1} \end{smallmatrix}\right)$ for some invertible $\alpha \in \mathrm{End}_{\mathcal{O}_R}(M)$, which is a product of matrix algebras over $\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} R$, then each lifting $\tilde{\alpha}$ of α in $\mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \tilde{R}}(\tilde{M})$ is invertible by Nakayama's lemma. Hence $\tilde{g} := \left(\begin{smallmatrix} \tilde{\alpha} & 0 \\ & (\tilde{\alpha}^\vee)^{-1} \end{smallmatrix}\right)$ defines a lifting of g.

Suppose that $g = \left(\begin{smallmatrix} 1 & \beta \\ & 1 \end{smallmatrix}\right)$, where $\beta^\vee = \beta \in \mathrm{Hom}_{\mathcal{O}_R}(M^\vee, M)$. By Lemma 1.2.3.1, we may replace $\tilde{M}$ with some finitely generated projective $\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} \tilde{R}$-module $\tilde{N}$, and assume that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \tilde{R} = \mathcal{O}_F \underset{\mathbb{Z}}{\otimes} \tilde{R}$. Then elements in $\mathrm{Hom}_{\mathcal{O}_{\tilde{R}}}(\tilde{M}^\vee, \tilde{M})$ can be represented in block matrix form with entries in $\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} \tilde{R}$, so that the formation of the dual is simply $X \mapsto {}^t X^c$. (Here c is the restriction of $\star$ to $\mathcal{O}_F$.) Hence the question of lifting β is a question of lifting a matrix with the condition ${}^t X^c = X$. For an entry of the matrix above the diagonal, any lifting will do. Then they determine the liftings of the entries below the diagonal. For an entry along the diagonal, being invariant implies that they lie in $\mathcal{O}_{F^+} \underset{\mathbb{Z}}{\otimes} R$, which can also be lifted to $\mathcal{O}_{F^+} \underset{\mathbb{Z}}{\otimes} \tilde{R}$. Hence there is an element $\tilde{\beta}$ in $\mathrm{Hom}_{\mathcal{O}_{\tilde{R}}}(\tilde{M}^\vee, \tilde{M})$ such that $\tilde{\beta}^\vee = \tilde{\beta}$, and $\tilde{g} := \left(\begin{smallmatrix} 1 & \tilde{\beta} \\ & 1 \end{smallmatrix}\right)$ defines a lifting of g, as desired. □

Corollary 1.2.5.17. *With assumptions on k and Λ as above, the group $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \Lambda$ is smooth over Λ.*

Proof. It suffices to show that $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \Lambda'$ is smooth over Λ', which follows from Propositions 1.2.5.15 and 1.2.5.16. □

Remark 1.2.5.18. This is a special case of Corollary 1.2.3.12, with an alternative reasoning.

1.2.6 Filtrations

This section will not be needed until Section 5.2.2.

Let R be a (commutative) ring. Let M be an integrable $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-module. Suppose we are given an increasing filtration $\mathtt{F} = \{\mathtt{F}_{-i}\}_i$ on M indexed by nonpositive integers $-i$ consisting of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-submodules $\mathtt{F}_{-i}$.

Convention 1.2.6.1. *All filtrations $\mathtt{F} = \{\mathtt{F}_{-i}\}_i$ on M we shall consider will satisfy $\mathtt{F}_0 = M$ and $\mathtt{F}_{-i} = 0$ for sufficiently large i.*

Definition 1.2.6.2. *We say that a filtration* $\mathtt{F} = \{\mathtt{F}_{-i}\}_i$ *on* M *is* ***integrable*** *(resp.* ***projective****) if* $\mathrm{Gr}^{\mathtt{F}}_{-i}$ *is integrable (see Definition 1.2.1.23) (resp. projective) as an* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$*-module for every* i*.*

Definition 1.2.6.3. *We say that a filtration* $\mathtt{F}$ *on* M *is* ***split*** *if there exists (non-canonically) some isomorphism* $\mathrm{Gr}^{\mathtt{F}} := \bigoplus_i \mathrm{Gr}^{\mathtt{F}}_{-i} \overset{\sim}{\to} \mathtt{F}_0$ *(of* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$*-modules).*

Lemma 1.2.6.4. *Every projective filtration is split.*

Proof. Simply split the exact sequences $0 \to \mathtt{F}_{-i-1} \to \mathtt{F}_{-i} \to \mathrm{Gr}^{\mathtt{F}}_{-i} \to 0$ one by one in increasing order of i. □

Corollary 1.2.6.5. *If* $\mathcal{O}$ *is hereditary, (which is the case when* $\mathcal{O}$ *is maximal, by Proposition 1.1.1.23), then every integrable filtration* $\mathtt{F}$ *on an integrable* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$*-module* M *is split.*

Proof. If $\mathcal{O}$ is hereditary, which means that every $\mathcal{O}$-lattice is projective, then every integrable filtration $\mathtt{F}$ is projective. □

Definition 1.2.6.6. *A filtration* $\mathtt{F}$ *on an integrable* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$*-module* M *is called* ***admissible*** *if it is integrable and split (see Definitions 1.2.6.2 and 1.2.6.3).*

When $\mathcal{O}$ is maximal, this just means that the filtration is integrable.

Definition 1.2.6.7. *A surjection* $\mathtt{F} \twoheadrightarrow \mathtt{F}'$ *(resp. a submodule* $\mathtt{F}''$ *of* $\mathtt{F}$*, resp. an embedding* $\mathtt{F}'' \hookrightarrow \mathtt{F}$*) of integrable* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$*-modules is called* ***admissible*** *if* $0 \subset \ker(\mathtt{F} \twoheadrightarrow \mathtt{F}') \subset \mathtt{F}$ *(resp.* $0 \subset \mathtt{F}'' \subset \mathtt{F}$*, resp.* $0 \subset \mathrm{image}(\mathtt{F}'' \hookrightarrow \mathtt{F}) \subset \mathtt{F}$*) is an admissible filtration on* $\mathtt{F}$*.*

When M is equipped with a pairing $\langle\cdot\,,\cdot\rangle_M$ such that $(M, \langle\cdot\,,\cdot\rangle_M)$ defines a symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-module in the sense of Definition 1.1.4.7, we would also like to consider those filtrations that respect this pairing.

Definition 1.2.6.8. *Given a symplectic* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$*-module* $(M, \langle\cdot\,,\cdot\rangle_M)$ *and an integer* $m \geq 1$*, a filtration* $\mathtt{F} = \{\mathtt{F}_{-i}\}_i$ *on* $\mathtt{F}_0 = M$ *is called* m***-symplectic*** *if* $\mathtt{F}_{-m+i}$ *and* $\mathtt{F}_{-i}$ *are annihilators of each other under the pairing* $\langle\cdot\,,\cdot\rangle_M$ *on* M*. If the integer* m *is clear from the context, we shall suppress it from the notation and simply say that the filtrations are symplectic.*

Remark 1.2.6.9. Later (in Chapter 5 and onwards) m will always be 3. The symplectic filtrations we consider will be of the form $0 = \mathtt{F}_{-3} \subset \mathtt{F}_{-2} \subset \mathtt{F}_{-1} \subset \mathtt{F}_0 = M$, such that $\mathtt{F}_{-2}$ and $\mathtt{F}_{-1}$ are the annihilators of each other under $\langle\cdot\,,\cdot\rangle_M$.

1.3 Geometric Structures

1.3.1 Abelian Schemes and Quasi-Isogenies

Definition 1.3.1.1. *An* ***abelian scheme*** *is a group scheme* $A \to S$ *that is proper, smooth, and with geometrically connected fibers.*

Remark 1.3.1.2.

1. Since properness implies quasi-compactness (see [59, II, 5.4.1 and I, 6.6.3]), and since smoothness implies being locally of finite presentation (see [59, IV-2, 6.8.1]), an abelian scheme is automatically of finite presentation over its base scheme. Hence the technique of reduction to the noetherian case in [59, IV-3, §8] can be applied.
2. By [59, IV-2, 4.5.13], a fiber of a morphism $X \to S$ of schemes with a section $S \to X$ is geometrically connected as soon as it is connected. Hence, in Definition 1.3.1.1 (and similar contexts), it suffices to assume that the fibers are connected. Nevertheless, for the sake of clarity, we will often explicitly mention the geometric connectedness of fibers in our exposition.

For convenience, let us include the following important theorem:

Theorem 1.3.1.3 (see [59, IV-3, 8.2.2, 8.9.1, 8.9.5, 8.10.5, and 17.7.8])**.** *Suppose* S_0 *is a quasi-compact scheme, and* $S = \varprojlim_{i \in I} S_i$ *is a projective limit of schemes* S_i *affine over* S_0*, indexed by some directed partially ordered set* I*.*

1. *Suppose* X *is a scheme of finite presentation over* S*. Then there exists an index* $i \in I$ *and a scheme* X_i *of finite presentation over* S_i *such that* $X \cong X_i \underset{S_i}{\times} S$ *over* S*. In this case, we can define for each* $j \geq i$ *in* I *a scheme* $X_j := X_i \underset{S_i}{\times} S_j$*.*
2. *Suppose* X*,* i*, and* X_i *are as above and* $\mathscr{M}$ *is a* ***quasi-coherent sheaf*** *of modules of finite presentation over* $\mathscr{O}_X$*. Then there exists an index* $j \geq i$ *in* I *and a quasi-coherent sheaf* $\mathscr{M}_j$ *of modules of finite presentation over* $\mathscr{O}_{X_j}$ *such that* $\mathscr{M} \cong \mathscr{M}_j \underset{\mathscr{O}_{X_j}}{\otimes} \mathscr{O}_X$ *over* $\mathscr{O}_X$*.*
3. *Suppose* Y *is another scheme of finite presentation over* S*, with some index* $i \in I$ *and schemes* X_i *and* Y_i *of finite presentation such that* $X \cong X_i \underset{S_i}{\times} S$ *and* $Y \cong Y_i \underset{S_i}{\times} S$ *over* S*. Define for each* $j \geq i$ *in* I *schemes* $X_j := X_i \underset{S_i}{\times} S_j$ *and* $Y_j := Y_i \underset{S_i}{\times} S_j$*. Then the canonical map*
$$\varinjlim_{j \in I, j \geq i} \mathrm{Hom}_{S_j}(X_j, Y_j) \to \mathrm{Hom}_S(X, Y)$$
is a bijection.
4. *In the context above, suppose* $f : X \to Y$ *is a* ***morphism*** *satisfying any of the following properties:*
   - *(a) an isomorphism, a monomorphism, an immersion, an open immersion, a closed immersion, or surjective;*
   - *(b) finite, quasi-finite, or proper;*
   - *(c) projective or quasi-projective;*
   - *(d) flat (for some quasi-coherent sheaf of finite presentation); or*
   - *(e) unramified, étale, or smooth.*

   Then there exist some $j \geq i$ *and some morphism* $f_j := X_j \to Y_j$ *over* S_j *with the same property such that* $f = f_j \underset{S_j}{\times} S$*. (In 4d, the quasi-coherent sheaf is the pullback of a quasi-coherent sheaf over* X_j *for which* f_j *is flat.) We say for simplicity that the properties above are* ***of finite presentation****.*

As a result, we may reduce problems concerning schemes, modules, and morphisms of finite presentation to the case where the base scheme is locally noetherian.

The underlying schemes of abelian schemes enjoy a rather strong rigidity property, which can be described by a special case of the following:

Proposition 1.3.1.4 (rigidity lemma; cf. [96, Prop. 6.1])**.** *Suppose we are given a commutative diagram*

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"] & & Y \arrow[dl, "q"] \\ & S \arrow[ul, bend left, "e"] & \end{tikzcd}$$

of locally noetherian schemes such that the base scheme S is connected, such that $p : X \to S$ is open and closed, with a section e, such that $q : Y \to S$ is separated, and such that the canonical morphism $\mathscr{O}_S \to p_*\mathscr{O}_X$ induced by p is an isomorphism. Suppose there exists a point $s \in S$ such that $f(X_s)$ is set-theoretically a single point. Then $f = \eta \circ p$ holds for the section $\eta : S \to Y$ of q defined by $\eta = f \circ e$.

Let us include a proof for the convenience of readers. The proofs we shall present for Proposition 1.3.1.4 and for the following corollaries are originally due to Mumford in [96], with a slight rewording by us.

Proof of Proposition 1.3.1.4. First suppose that S has only one point s. Then $f = \eta \circ p = f \circ e \circ p$ hold as *topological maps* by the assumption that $f(X_s)$ is set-theoretically just one point, and we have identifications $f_*\mathscr{O}_X = f_*e_*\mathscr{O}_S$ and $f_*\mathscr{O}_X = f_*e_*p_*\mathscr{O}_X$ as the push-forwards are defined by topological morphisms. To show $f = f \circ e \circ p$ as morphisms of schemes, we need to show that $f_\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ and $(f \circ e \circ p)_\# : \mathscr{O}_Y \to f_*e_*p_*\mathscr{O}_X$ agree as morphisms of sheaves under the identification $f_*\mathscr{O}_X = f_*e_*p_*\mathscr{O}_X$. By assumption, $p_\# : \mathscr{O}_S \to p_*\mathscr{O}_X$ is an isomorphism; therefore we only need to show that $f_\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ and $(f \circ e)_\# : \mathscr{O}_Y \to f_*e_*\mathscr{O}_S$ agree under the identification $f_*\mathscr{O}_X = f_*e_*\mathscr{O}_S$. This is true by assumption, as we only need to compare the stalks at the image of e under f. Hence the proposition holds when s has only one point.

In general, let Z be the largest closed subscheme of X over which $f = \eta \circ p$. Since $q : Y \to S$ is separated, Z is the pullback of the closed diagonal of $Y \underset{S}{\times} Y$ via $(f, \eta \circ p) : X \to Y \underset{S}{\times} Y$. By assumption it contains the fiber $p^{-1}(s)$. We claim that $Z = X$. By the first part of the proof, we know that, for every Artinian local subscheme $T \subset S$ containing s, the preimage $p^{-1}(T)$ is a subscheme of Z. For every point u in $p^{-1}(s)$ and every Artinian local subscheme $W \subset X$ containing u, we have $W \subset p^{-1}(T)$ for some Artinian local subscheme T and hence $W \subset Z$. This implies that Z contains an open neighborhood of $p^{-1}(s)$ in X. Let C be the closed subset of X complementing any such open subset. Since p is a closed morphism, $p(C)$ is closed in S. By taking the complement of $p(C)$ in S, we see that there exists an open neighborhood U_0 of s such that Z contains $p^{-1}(U_0)$. Let U_1 be the maximal subscheme of S such that $p^{-1}(U_1)$ is a subscheme of Z. Then the above argument applied to an arbitrary point t of U_1 shows that U_1 is open. On the other hand, since Z is closed, and since p is open by assumption (or by its flatness when p is locally of finite presentation), we see that $p(X - Z)$ is also open. Now $p(X - Z)$ and U_1 cover the underlying topological space of S, and U_1 is nonempty. Hence by connectedness of S we know that $p(X - Z)$ is empty, and the proposition follows. □

Note that we do not need the group structure of X in this proposition.

Corollary 1.3.1.5 ([96, Cor. 6.2])**.** *Let A be an abelian scheme, and let G be a separated group scheme of finite presentation over a connected scheme S. Let f and g be two morphisms of schemes making the diagram*

$$\begin{tikzcd} A \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] \arrow[dr, "p"'] & & G \arrow[dl, "q"] \\ & S & \end{tikzcd}$$

commutative. Let $m_G : G \underset{S}{\times} G \to G$ denote the multiplication morphism of G. Suppose that, for some point $s \in S$, the morphisms f_s and g_s from A_s to G_s are equal. Then there is a section $\eta : S \to G$ such that $f = m_G \circ ((\eta \circ p) \times g)$.

Proof. By Theorem 1.3.1.3, we may assume that all A, G, and S are locally noetherian, and apply Proposition 1.3.1.4 to $m_G \circ (f \times ([-1]_G \circ g))$, where $[-1]_G : G \to G$ is the inverse morphism of the group scheme G. □

Corollary 1.3.1.6 ([96, Cor. 6.4])**.** *Let A be an abelian scheme, and let G be a separated group scheme of finite presentation over a base scheme S. If $h : A \to G$ is a morphism over S taking the identity e_A of A to the identity e_G of G, then h is a homomorphism.*

Proof. Let us denote the multiplication morphism of A by m_A, and let us define two morphisms $f, g : A \underset{S}{\times} A \to A \underset{S}{\times} G$ by setting $f(x_1, x_2) = (x_1, h(m_A(x_1, x_2)))$ and $g(x_1, x_2) = (x_1, h(x_2))$ for all functorial points x_1 and x_2 of A. Let $p : A \underset{S}{\times} A \to A$ and $q : A \underset{S}{\times} G \to A$ be the first projections. By Corollary 1.3.1.5, with f, g, p, and q as above, there exists some morphism $\eta : A \to G$ such that $h(m_A(x_1, x_2)) = m_G(\eta(x_1), h(x_2))$. By putting $x_2 = e_A$, we get $h(x_1) = \eta(x_1)$, and hence $h(m_A(x_1, x_2)) = m_G(h(x_1), h(x_2))$. This shows h is a group homomorphism. □

Corollary 1.3.1.7 ([96, Cor. 6.5])**.** *If A is an abelian scheme over a scheme S, then A is a commutative group scheme.*

Proof. Apply Corollary 1.3.1.6 to the inverse morphism $[-1]_A : A \to A$. □

Corollary 1.3.1.8 ([96, Cor. 6.6])**.** *If A is an abelian scheme over S, then there is only one structure of group scheme on A over S with the given identity $e_A : S \to A$.*

Proof. Applying Corollary 1.3.1.6 to the identity isomorphism $\mathrm{Id}_A : A \overset{\sim}{\to} A$, we see that every two group structures on A are identical to each other. □

Definition 1.3.1.9. *An **isogeny** $f : G \to G'$ of smooth group schemes over S is a group scheme homomorphism over S that is **surjective** and **with quasi-finite kernel**.*

Definition 1.3.1.10. *An **isogeny** $f : A \to A'$ of abelian schemes over S is an isogeny of smooth group schemes from A to A'.*

Lemma 1.3.1.11. *An isogeny of smooth group schemes is flat. Consequently, an isogeny of proper smooth group schemes (such as abelian schemes) is finite flat and of finite presentation.*

Proof. It suffices to show that an isogeny of smooth group schemes is flat. By Theorem 1.3.1.3, we may assume that S is noetherian. Then the lemma follows from [59, IV-3, 11.3.10 a)⇒b) and 15.4.2 e′)⇒b)]. □

Corollary 1.3.1.12. *In Definition 1.3.1.10, the kernel* $\ker(f)$ *is finite flat and of finite presentation. Hence, the rank of* $\ker(f)$ *is locally constant as a function on* S. *(In particular, it is a constant on each connected subscheme of* S.*)*

Definition 1.3.1.13. *Two abelian schemes* A *and* A' *are* ***isogenous*** *if there exists an isogeny from* A *to* A'.

In what follows, unless otherwise specified (or clear from the context), we shall consider only isogenies between abelian schemes.

Now let $\Box$ be an arbitrary set of finite rational primes.

Convention 1.3.1.14. *An integral-valued function (such as the rank of a finite flat groups scheme of finite presentation) is* ***prime-to-***$\Box$ *if its values are prime-to-*$\Box$.

Definition 1.3.1.15. *An isogeny* $f : A \to A'$ *over* S *is* ***prime-to-***$\Box$ *if the rank of* $\ker(f)$ *(as a finite flat group scheme of finite presentation) is prime-to-*$\Box$.

Definition 1.3.1.16. *A* ***quasi-isogeny*** $f : A \to A'$ *of abelian schemes over* S *is an equivalence class of triples* (B, g, h), *where* $g : B \to A$ *and* $h : B \to A'$ *are isogenies over* S, *and where two triples* (B, g, h) *and* (B', g', h') *are considered equivalent if there exist isogenies* $i : C \to B$ *and* $j : C \to B'$ *such that* $g \circ i = g' \circ j$ *and* $h \circ i = h' \circ j$.

Definition 1.3.1.17. *A quasi-isogeny* $f : A \to A'$ *of abelian schemes over* S *is* ***prime-to-***$\Box$ *if it can be represented by a triple* (B, g, h) *as in Definition 1.3.1.16 such that* g *and* h *are both prime-to-*$\Box$ *isogenies. We shall often call a prime-to-*$\Box$ *quasi-isogeny a* $\mathbb{Z}^{\times}_{(\Box)}$***-isogeny***. *(And hence, we shall call a quasi-isogeny a* $\mathbb{Q}^{\times}$*-isogeny.)*

Lemma 1.3.1.18. *The natural functor from the category of isogenies (resp. prime-to-*$\Box$ *isogenies) to the category of quasi-isogenies (resp. prime-to-*$\Box$ *quasi-isogenies), defined by sending an isogeny* $f : A \to A'$ *to the class containing the triple* (A, Id_A, f), *is fully faithful.*

Definition 1.3.1.19. *The composition of two* $\mathbb{Z}^{\times}_{(\Box)}$*-isogenies* $f : A \to A'$ *and* $f' : A' \to A''$ *represented by* (B, g, h) *and* (B', g', h'), *respectively, is represented by* $(B \underset{h,A',g'}{\times} B', g \circ \mathrm{pr}_1, h' \circ \mathrm{pr}_2)$, *where* $\mathrm{pr}_1 : B \underset{h,A',g'}{\times} B' \to B$ *and* $\mathrm{pr}_2 : B \underset{h,A',g'}{\times} B' \to B'$ *are the two projections.*

Remark 1.3.1.20. Every $\mathbb{Z}^{\times}_{(\Box)}$-isogeny is invertible: The inverse of the equivalence class of (B, g, h) is simply the equivalence class of (B, h, g).

Remark 1.3.1.21. Suppose, in Definition 1.3.1.17, that S has finitely many connected components. Then we may assume that $g = [N]$, the multiplication by N, for some integer N prime-to-$\Box$. Equivalently, this means that $f \circ [N] = [N] \circ f$ is a prime-to-$\Box$ isogeny h for some integer N prime-to-$\Box$. We shall write $f = N^{-1}h$ in this case. This makes sense because $[N]$ is invertible in the category of $\mathbb{Z}^{\times}_{(\Box)}$-isogenies. More generally we can define $[N]$ for each section N of $(\mathbb{Z}^{\times}_{(\Box)})_S$, so that we do not need to require that S has only finitely many connected components.

1.3.2 Polarizations

Definition 1.3.2.1.

1. *Let* X *be a scheme of finite presentation over a base scheme* S. *The* ***relative Picard functor*** *is defined by*
$$\underline{\mathrm{Pic}}(X/S) : T/S \mapsto \frac{\{\text{invertible sheaves } \mathcal{L} \text{ over } X \underset{S}{\times} T\}}{\left\{\begin{matrix}\text{invertible sheaves of the form } \mathrm{pr}_2^*(\mathcal{M}) \\ \text{for some } \mathcal{M} \text{ over } T\end{matrix}\right\}}.$$

2. *Let* A *be an abelian scheme over* S *with identity section* $e : S \to A$.

   (a) *For each invertible sheaf* $\mathcal{L}$ *over* A, *a* ***rigidification*** *of* $\mathcal{L}$ *is an isomorphism* $\xi : \mathscr{O}_S \xrightarrow{\sim} e^*\mathcal{L}$. *(For the convenience of language, we shall also call the inverse isomorphism* $\xi^{-1} : e^*\mathcal{L} \xrightarrow{\sim} \mathscr{O}_S$ *a rigidification.)*

   (b) *The (open) subfunctor* $\underline{\mathrm{Pic}}^0(A/S)$ *of* $\underline{\mathrm{Pic}}(A/S)$ *is defined by*
$$\underline{\mathrm{Pic}}^0(A/S) : T/S \mapsto \frac{\left\{\begin{matrix}\text{invertible sheaves } \mathcal{L} \text{ over } A_T := A \underset{S}{\times} T \\ \text{s.t. for all } t \in T, \text{ the fiber } \mathcal{L}_t \text{ is} \\ \text{algebraically equivalent to zero over } A_t\end{matrix}\right\}}{\left\{\begin{matrix}\text{invertible sheaves as above} \\ \text{of the form } \mathrm{pr}_2^*(\mathcal{M}) \text{ for some } \mathcal{M} \text{ over } T\end{matrix}\right\}}.$$

Lemma 1.3.2.2 (cf. [96, Ch. 0, §5, d)])**.** *Let* A *be an abelian scheme over* S *with identity section* $e : S \to A$.

1. *The functor* $\underline{\mathrm{Pic}}(A/S)$ *is canonically isomorphic to*
$$\underline{\mathrm{Pic}}_e(A/S) : T/S \mapsto \left\{\begin{matrix}\text{invertible sheaves } \mathcal{L} \text{ over } A_T \text{ equipped with} \\ \text{rigidifications along } e_T := e \underset{S}{\times} T\end{matrix}\right\} / \cong .$$

2. *The subfunctor* $\underline{\mathrm{Pic}}^0(A/S)$ *of* $\underline{\mathrm{Pic}}(A/S)$ *is canonically isomorphic to the (open) subfunctor*
$$\underline{\mathrm{Pic}}^0_e(A/S) : T/S \mapsto \left\{\begin{matrix}\text{invertible sheaves } \mathcal{L} \text{ over } A_T \text{ equipped with} \\ \text{rigidifications along } e_T, \text{ s.t. for all } t \in T, \\ \mathcal{L}_t \text{ is algebraically equivalent to zero over } A_t\end{matrix}\right\} / \cong$$
of $\underline{\mathrm{Pic}}_e(A/S)$.

Theorem 1.3.2.3. *For every abelian scheme* $A \to S$, *the relative Picard functors* $\underline{\mathrm{Pic}}_e(A/S)$ *and* $\underline{\mathrm{Pic}}^0_e(A/S)$ *are representable over* S. *Moreover,* $\underline{\mathrm{Pic}}^0_e(A/S)$ *is representable by an abelian scheme over* S, *which we call the* ***dual abelian scheme*** *of* A. *We shall denote the dual abelian scheme of* A *by* $A^\vee$. *(By definition, the identity section* $e_{A^\vee}$ *of* $A^\vee$ *corresponds to the trivial invertible sheaf over* A.*)*

For more details, see [42, Ch. I, §1], in which they mention the results of Artin, Raynaud, and Deligne, and explain the proof.

Remark 1.3.2.4. By reduction to the locally noetherian case by Theorem 1.3.1.3, and by the result of Hilbert schemes as in [96, Ch. 0, §5, d) and p. 117], the functors $\underline{\mathrm{Pic}}_e(A/S)$ and $\underline{\mathrm{Pic}}^0_e(A/S)$ are representable when A is locally projective over S. On the other hand, we will only consider polarized abelian schemes when we define our moduli problems later, and we know of no formulation of a polarization of an abelian scheme $A \to S$ that does not force the local projectivity of A over S (cf. Proposition 1.3.2.15 and Definition 1.3.2.16 below). If readers are unwilling to make use of a stronger result such as Theorem 1.3.2.3, they may safely add the assumption of local projectivity to all abelian schemes in what follows.

Definition 1.3.2.5. *The tautological (i.e., universal) rigidified invertible sheaf* $\mathcal{P}_A$ *over* $A \underset{S}{\times} A^\vee$ *is called the* ***Poincaré invertible sheaf*** *of* A.

Remark 1.3.2.6. The invertible sheaf $\mathcal{P}_A$ is rigidified along $(e_A, \mathrm{Id}_{A^\vee}) : A^\vee \to A \underset{S}{\times} A^\vee$ because it is the tautological object, and hence in particular, an object of $\underline{\mathrm{Pic}}^0_e(A/S)(A^\vee)$. On the other hand, it is rigidified along $(\mathrm{Id}_A, e_{A^\vee}) : A \to A \underset{S}{\times} A^\vee$ by the definition of $e_{A^\vee}$ and the universal property of $A^\vee$. That is, the pullback by

$e_{A^\vee}$ corresponds to giving the parameter of the trivial invertible sheaf. The rigidification along $(e_A, \mathrm{Id}_{A^\vee})$ is uniquely determined by the rigidification along $(e_A, \mathrm{Id}_{A^\vee})$ by the condition that the two rigidifications agree along $(e_A, e_{A^\vee}) : S \to A \underset{S}{\times} A^\vee$.

Construction 1.3.2.7. Let $m, \mathrm{pr}_1, \mathrm{pr}_2 : A \underset{S}{\times} A \to A$ denote the multiplication morphism and the two projections, respectively. For each rigidified invertible sheaf $\mathcal{L}$ over $A \to S$, define

$$\mathcal{D}_2(\mathcal{L}) := m^*\mathcal{L} \otimes \mathrm{pr}_1^* \mathcal{L}^{\otimes -1} \otimes \mathrm{pr}_2^* \mathcal{L}^{\otimes -1}.$$

Then $\mathcal{D}_2(\mathcal{L})$ is an invertible sheaf over $A \underset{S}{\times} A \to A$, which is rigidified along the identity section $(e_A, \mathrm{Id}_A) : A \to A \underset{S}{\times} A$ if we view $A \underset{S}{\times} A$ as an abelian scheme over the second factor A. By the universal property of $A^\vee$ (representing $\underline{\mathrm{Pic}}_e^0(A/S)$), we obtain a unique morphism

$$\lambda_{\mathcal{L}} : A \to A^\vee$$

of schemes over S, sending e_A to $e_{A^\vee}$. This is automatically a group scheme homomorphism by Corollary 1.3.1.6.

Lemma 1.3.2.8. *There is a canonical isomorphism $A \overset{\sim}{\to} (A^\vee)^\vee$ over S.*

Proof. By the universal property of $(A^\vee)^\vee$ applied to the Poincaré invertible sheaf $\mathcal{P}_A$ over $A \underset{S}{\times} A^\vee$, we obtain a canonical morphism $A \to (A^\vee)^\vee$ over S, which is an isomorphism because it is so over every geometric point of S, by the usual theory of abelian varieties over algebraically closed fields (see, for example, [94, §13]). □

Definition 1.3.2.9. *For each homomorphism $f : A \to A'$ between abelian schemes over S, the pullback*

$$f^* : \underline{\mathrm{Pic}}_e^0(A'/S) \to \underline{\mathrm{Pic}}_e^0(A/S)$$

induces a group scheme homomorphism

$$(A')^\vee \to A^\vee$$

over S, which we denote by $f^\vee$. This is called the ***dual isogeny*** *of f when f is an isogeny.*

Lemma 1.3.2.10. *For each homomorphism $f : A \to A'$ between abelian schemes over S, we have a canonical isomorphism*

$$(\mathrm{Id}_A \times f^\vee)^*\mathcal{P}_A \cong (f \times \mathrm{Id}_{(A')^\vee})^*\mathcal{P}_{A'}$$

over $A \underset{S}{\times} (A')^\vee$ by the universal properties of $\mathcal{P}_A$ and $\mathcal{P}_{A'}$.

Lemma 1.3.2.11. *If f is an isogeny between abelian schemes over S, then the rank of f is the same as the rank of $f^\vee$. Hence it makes sense to say that the dual of a prime-to-□ isogeny is again a prime-to-□ isogeny.*

Proof. It suffices to verify this statement over fibers over geometric points of S. This then is well known in the theory of abelian varieties (see [94, §15, Thm. 1]). (In Section 5.2.4 below, we will generalize the argument of the proof of [94, §15, Thm. 1] to abelian schemes.) □

Definition 1.3.2.12. *For each group scheme homomorphism $\lambda : A \to A^\vee$, consider the composition $A \overset{\sim}{\to} (A^\vee)^\vee \overset{\lambda^\vee}{\to} A^\vee$, where the isomorphism is the canonical one given by Lemma 1.3.2.8. By abuse of notation, we also denote this composition by $\lambda^\vee$. We say λ is* ***symmetric*** *if $\lambda = \lambda^\vee$.*

Lemma 1.3.2.13. *For every invertible sheaf $\mathcal{L}$, the homomorphism $\lambda_{\mathcal{L}}$ constructed in Construction 1.3.2.7 is symmetric.*

Proof. By Lemma 1.3.2.10, we have $(\mathrm{Id}_A \times \lambda_{\mathcal{L}}^\vee)^*\mathcal{P}_A \cong (\lambda_{\mathcal{L}} \times \mathrm{Id}_{(A^\vee)^\vee})^*\mathcal{P}_{A^\vee}$ over $A \underset{S}{\times} (A^\vee)^\vee$. By pulling back under $(\mathrm{Id}_A \times \mathrm{can.})$, we obtain $(\mathrm{Id}_A \times \lambda_{\mathcal{L}}^\vee)^*\mathcal{P}_A \cong (\lambda_{\mathcal{L}} \times \mathrm{can.})^*\mathcal{P}_{A^\vee}$ over $A \underset{S}{\times} A$. On the other hand, by the construction of the canonical $A \overset{\sim}{\to} (A^\vee)^\vee$, we have $(\mathrm{Id}_{A^\vee} \times \mathrm{can.})^*\mathcal{P}_{A^\vee} \cong s^*\mathcal{P}_A$, where $s : A^\vee \underset{S}{\times} A \to A \underset{S}{\times} A^\vee$ is the isomorphism switching the two factors. Therefore, using the fact that $\mathcal{D}_2(\mathcal{L})$ is isomorphic to its pullback under the automorphism switching the two factors of $A \underset{S}{\times} A$, we have $(\lambda_{\mathcal{L}} \times \mathrm{can.})^*\mathcal{P}_{A^\vee} \cong (\lambda_{\mathcal{L}} \times \mathrm{Id}_A)^*s^*\mathcal{P}_A \cong \mathcal{D}_2(\mathcal{L}) \cong (\mathrm{Id}_A \times \lambda_{\mathcal{L}})^*\mathcal{P}_A$. As a result, we have $(\mathrm{Id}_A \times \lambda_{\mathcal{L}}^\vee)^*\mathcal{P}_A \cong (\mathrm{Id}_A \times \lambda_{\mathcal{L}})^*\mathcal{P}_A$, and hence $\lambda_{\mathcal{L}}^\vee = \lambda_{\mathcal{L}}$ by the universal property of $\mathcal{P}_A$. □

Following [37, 1.2, 1.3, 1.4], we have the following converse:

Proposition 1.3.2.14. *Locally for the étale topology over S, every symmetric homomorphism from A to $A^\vee$ is of the form $\lambda_{\mathcal{L}}$ for some invertible sheaf $\mathcal{L}$.*

Proof. By Theorem 1.3.1.3, we may assume that S is locally noetherian. If we set $\mathcal{M} := (\mathrm{Id}_A, \lambda)^*\mathcal{P}_A$, then $\lambda_{\mathcal{M}} = \lambda + \lambda^\vee = 2\lambda$. (This follows from the universal property of $\mathcal{P}_A$ and its symmetric bilinear properties; that is, from the theorem of the cube.) Therefore the question is whether we can find some $\mathcal{L}$ such that $\mathcal{L}^{\otimes 2} \cong \mathcal{M}$.

Since $\underline{\mathrm{Hom}}^{\mathrm{sym}}(A, A^\vee)$ is an algebraic space (by [6, Cor. 6.2] and the theory of Hilbert schemes), and since it is unramified over S by rigidity (given by Corollary 1.3.1.5), it is a scheme (by [73, II, 6.16]).

By the arguments in [92, Thm. 2.3(ii)], every group extension of a commutative finite flat group scheme by $\mathbf{G}_{\mathrm{m}}$ splits fppf locally. Using the representation theory of theta group schemes, the argument in the proof of [94, §23, Thm. 3] generalizes and shows that, if $A[m] \subset \ker(\lambda_{\mathcal{L}})$ (as a closed subgroup scheme) for some integer m, then fppf locally $\mathcal{M} \cong \mathcal{L}^{\otimes m}$ for some $\mathcal{L}$. In particular, this is true for $m = 2$. This shows that the morphism $\underline{\mathrm{Pic}}_e(A/S) \to \underline{\mathrm{Hom}}^{\mathrm{sym}}(A, A^\vee) : \mathcal{L} \mapsto \lambda_{\mathcal{L}}$ (with kernel $\underline{\mathrm{Pic}}_e^0(A/S) \cong A^\vee$) over S is surjective and smooth, as we have verified these properties fppf locally. Since smooth morphisms have sections étale locally (by [22, §2.2, Prop. 14]), λ is étale locally of the form $\lambda_{\mathcal{L}}$, as desired. □

Proposition 1.3.2.15. *Let λ be a symmetric homomorphism from A to $A^\vee$ over S. The following conditions are equivalent:*

1. *Over each geometric point $\bar{s}$ of S, $\lambda_{\bar{s}}$ is of the form $\lambda_{\mathcal{L}}$ for some ample invertible sheaf $\mathcal{L}$ over $A_{\bar{s}}$.*

2. *Locally for the étale topology, λ is of the form $\lambda_{\mathcal{L}}$ for some invertible sheaf $\mathcal{L}$ over A relatively ample over S.*

3. *The invertible sheaf $(\mathrm{Id}_A, \lambda)^*\mathcal{P}_A$ over A is relatively ample over S.*

Proof. By Theorem 1.3.1.3, we may assume that S is locally noetherian. Then relative ampleness is a fiberwise condition by [59, III-1, 4.7.1]. □

Definition 1.3.2.16. *A* ***polarization*** *λ of A is a symmetric homomorphism from A to $A^\vee$ satisfying any of the conditions in Proposition 1.3.2.15. A* ***principal polarization*** *(resp.* ***prime-to-□ polarization****) is a polarization that is an isomorphism (resp. a prime-to-□ isogeny).*

Remark 1.3.2.17. A polarization is necessarily an isogeny, because $\lambda_{\mathcal{L}}$ is quasi-finite by the usual theory of abelian varieties over algebraically closed fields (see, for example, [94]).

Corollary 1.3.2.18. *An isogeny $\lambda : A \to A^\vee$ is a polarization if and only if $[N] \circ \lambda$ is a polarization for some positive integer N (or more generally for any section N of $(\mathbb{Z}_{>0})_S$).*

Motivated by this, we can extend the notion of polarizations to $\mathbb{Z}^\times_{(\Box)}$-isogenies as well:

Definition 1.3.2.19. *A $\mathbb{Z}^\times_{(\Box)}$**-polarization** of A is a $\mathbb{Z}^\times_{(\Box)}$-isogeny $\lambda : A \to A^\vee$ such that $[N] \circ \lambda$ is a positive isogeny for some section N of $(\mathbb{Z}_{>0})_S$.*

Note that we do not have to assume that N is prime-to-$\Box$ (i.e., valued in integers prime-to-$\Box$).

Definition 1.3.2.20. *The **dual** $\mathbb{Z}^\times_{(\Box)}$**-isogeny** $f^\vee : (A')^\vee \to A^\vee$ of a $\mathbb{Z}^\times_{(\Box)}$-isogeny $f : A \to A'$ represented by some triple (B, g, h) is the $\mathbb{Z}^\times_{(\Box)}$-isogeny represented by $((A')^\vee \underset{h^\vee, B^\vee, g^\vee}{\times} A^\vee, \mathrm{pr}_1, \mathrm{pr}_2)$.*

This definition makes sense because of Lemma 1.3.2.11.

Corollary 1.3.2.21. *If $\lambda : A \to A^\vee$ is a $\mathbb{Z}^\times_{(\Box)}$-polarization, then $f^\vee \circ \lambda \circ f : A' \to (A')^\vee$ is a $\mathbb{Z}^\times_{(\Box)}$-polarization for each $\mathbb{Z}^\times_{(\Box)}$-isogeny $f : A' \to A$. Moreover, $\lambda^{-1} : A^\vee \to A$ is also a $\mathbb{Z}^\times_{(\Box)}$-polarization.*

Proof. To show this, we use 3 of Proposition 1.3.2.15, Lemma 1.3.2.10, and the fact that the pullback of an invertible sheaf under an isogeny is relatively ample if and only if the original invertible sheaf is relatively ample. ☐

Definition 1.3.2.22. *A $\mathbb{Z}^\times_{(\Box)}$**-isogeny** $f : (A, \lambda) \to (A', \lambda')$ is a $\mathbb{Z}^\times_{(\Box)}$-isogeny $f : A \to A'$ such that $\lambda = f^\vee \circ \lambda' \circ f$.*

1.3.3 Endomorphism Structures

Assume as in Section 1.2.1 that $\mathcal{O}$ is an order in a finite-dimensional semisimple algebra B over $\mathbb{Q}$ with a positive involution $\star$.

Definition 1.3.3.1. *Let A be an abelian scheme with a $\mathbb{Z}^\times_{(\Box)}$-polarization λ over S. Recall that the λ**-Rosati involution** of $\mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_S$ is defined by sending each $\mathbb{Q}^\times$-isogeny $f : A \to A$ to the composition $A \xrightarrow{\lambda} A^\vee \xrightarrow{f^\vee} A^\vee \xrightarrow{\lambda^{-1}} A$ of $\mathbb{Q}^\times$-isogenies (cf. [94, p. 189]).*

*Let R be any $\mathbb{Z}$-subalgebra of $\mathbb{Q}$. An $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$**-endomorphism structure** (or simply an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$**-structure**) of (A, λ) is a ring homomorphism*

$$i : \mathcal{O} \underset{\mathbb{Z}}{\otimes} R \to \mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} R_S \qquad (1.3.3.2)$$

satisfying the following conditions:

1. *The image of the composition of (1.3.3.2) with the canonical morphism $\mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} R_S \to \mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_S$ is preserved by the λ-Rosati involution of $\mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}_S$.*

2. *The restriction of the λ-Rosati involution to this image of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$ agrees with the one induced by the involution $\star$ of $\mathcal{O}$.*

*We say that i satisfies the **Rosati condition** if it satisfies the above two conditions. Concretely, this means the equality $i(b)^\vee \circ \lambda = \lambda \circ i(b^\star)$ holds for every $b \in \mathcal{O} \underset{\mathbb{Z}}{\otimes} R$.*

Remark 1.3.3.3. If $R = \mathbb{Z}$, then we are given a ring homomorphism $i : \mathcal{O} \to \mathrm{End}_S(A)$ called an *$\mathcal{O}$-structure* by definition. In this case, we shall think of A as a *left* $\mathcal{O}$-module via i.

Remark 1.3.3.4. Let R be any $\mathbb{Z}$-subalgebra of $\mathbb{Q}$. If i is an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-structure of (A, λ), then i is an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-structure of $(A, r\lambda)$ for all $r \in \mathbb{Z}^\times_{(\Box), >0}$.

Definition 1.3.3.5. *Let i (resp. i') be an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-structure of (A, λ) (resp. (A', λ')). A $\mathbb{Z}^\times_{(\Box)}$-isogeny $f : (A, \lambda) \to (A', \lambda')$ is **$\mathcal{O}$-equivariant** if $f \circ i(b) = i'(b) \circ f$ for all $b \in \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$. We say in this case that we have a $\mathbb{Z}^\times_{(\Box)}$-isogeny $f : (A, \lambda, i) \to (A', \lambda', i')$.*

Remark 1.3.3.6. If f and (A, λ, i) are prescribed in a $\mathbb{Z}^\times_{(\Box)}$-isogeny $f : (A, \lambda, i) \to (A', \lambda', i')$, then λ' and i' are determined by $\lambda' = (f^\vee)^{-1} \circ \lambda \circ f^{-1}$ and $i'(b) = f \circ i(b) \circ f^{-1}$ for all $b \in \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$.

Proposition 1.3.3.7. *Let A be an abelian scheme over S, and let $\lambda : A \to A^\vee$ be a polarization of A. Then the functor that assigns to each scheme T over S the set of $\mathcal{O}$-endomorphism structures $i : \mathcal{O} \to \mathrm{End}_T(A \underset{S}{\times} T)$ of $(A \underset{S}{\times} T, \lambda \underset{S}{\times} T)$ (as in Definition 1.3.3.1) is representable by a scheme finite over S.*

Proof. Since A is polarized (see Proposition 1.3.2.15 and Definition 1.3.2.16), and since $\mathcal{O}$ is finitely generated over $\mathbb{Z}$, by reduction to the locally noetherian case by Theorem 1.3.1.3, and by replacing S with its noetherian open subschemes, we may assume that A is projective over S. By the result of Hilbert schemes as in [96, Ch. 0, §5, d) and p. 117], we know that this functor is representable by a disjoint union S' of schemes that are projective over S.

It remains to show that S' is quasi-finite (and hence finite) over S. For this purpose, we may replace S with its geometric points and assume that $S = \mathrm{Spec}(k)$ for some algebraically closed field k. Let $\mathcal{O}' = \mathrm{End}_S(A)$ and let $\star'$ be the λ-Rosati involution (see Definition 1.3.3.1) of $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. Let $\mathcal{O}'_l$ (resp. $\mathcal{O}'_r$) be the order in $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ (containing $\mathcal{O}'$ and) consisting of elements of $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ whose left (resp. right) action on $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ stabilizes $\mathcal{O}'$. Let $\mathcal{I}$ be the set of ring homomorphisms $i : \mathcal{O} \to \mathcal{O}'_l$ such that $i(b^\star) = (i(b))^{\star'}$ for all $b \in \mathcal{O}$. It suffices to show that $\mathcal{I}$ is a finite set, because $S'(\mathrm{Spec}(k))$ can be identified with a subset of $\mathcal{I}$.

First we claim that there are only finitely many ring homomorphisms $i : \mathcal{O} \to \mathcal{O}'_l$ (not necessarily in $\mathcal{I}$) up to conjugation by elements in $(\mathcal{O}'_l)^\times$. (This is an analogue of the Noether–Skolem theorem; see, for example, [63, Thm. 4.3.1].) Let $(\mathcal{O}'_r)^{\mathrm{op}}$ denote the opposite ring of $\mathcal{O}'_r$, which has a canonical left action on $\mathcal{O}'$ (induced by the canonical right action of $\mathcal{O}'_r$) commuting with that of $\mathcal{O}'_l$. Hence there is a canonical (left) $\mathcal{O}'_l \underset{\mathbb{Z}}{\otimes} (\mathcal{O}'_r)^{\mathrm{op}}$-lattice structure on $\mathcal{O}'$. By composition with the canonical action of $\mathcal{O}'_l$ on $\mathcal{O}'$, each ring homomorphism $i : \mathcal{O} \to \mathcal{O}'_l$ defines an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} (\mathcal{O}'_r)^{\mathrm{op}}$-lattice structure on $\mathcal{O}'$. If two ring homomorphisms $i_1, i_2 : \mathcal{O} \to \mathcal{O}'_l$ define isomorphic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} (\mathcal{O}'_r)^{\mathrm{op}}$-lattice structures on $\mathcal{O}'$, then there exists some $f \in (\mathrm{End}_{\mathbb{Z}}(\mathcal{O}'))^\times$ such that $(i_2(b) \otimes b') = f \circ (i_1(b) \otimes b') \circ f^{-1}$ (as elements of $\mathrm{End}_{\mathbb{Z}}(\mathcal{O}')$) for all $b \in \mathcal{O}$ and $b' \in (\mathcal{O}'_r)^{\mathrm{op}}$. By taking $b = 1$, we have $(1 \otimes b') = f \circ (1 \otimes b') \circ f^{-1}$ for all $b' \in (\mathcal{O}'_r)^{\mathrm{op}}$. This shows that the image of f under the canonical embedding $\mathrm{End}_{\mathbb{Z}}(\mathcal{O}') \hookrightarrow \mathrm{End}_{\mathbb{Q}}(\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ commutes with the right action of $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ on $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$; therefore f can be identified with an element of $(\mathcal{O}'_l)^\times$. By taking $b' = 1$, we have $i_2(b) = f \circ i_1(b) \circ f^{-1}$ in $\mathcal{O}'_l$ for all $b \in \mathcal{O}$; that is, i_1 and i_2 are the same up to conjugation by $f \in (\mathcal{O}'_l)^\times$. Thus the claim follows from the Jordan–Zassenhaus theorem (which asserts, in the special case here, that there are only finitely many isomorphism classes of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} (\mathcal{O}'_r)^{\mathrm{op}}$-lattices; see, for example, [107, Thm. 26.4]).

Let i_0 be an element of $\mathcal{I}$ and let $\mathcal{C}$ be the centralizer of $i_0(\mathcal{O})$ in $\mathcal{O}'_l$. Let $\mathcal{F} := \{f \in (\mathcal{O}'_l)^\times : \exists\, i_f \in \mathcal{I} \text{ such that } i_f(b) = f \circ i_0(b) \circ f^{-1} \text{ for all } b \in \mathcal{O}\}$. We would like to show that $\overline{\mathcal{F}} := \mathcal{F}/\mathcal{C}^\times$ is a finite set. Since i_0 is arbitrary, the finiteness of $\mathcal{I}$ (and the proposition) will follow, because (by the previous paragraph) $\mathcal{I}$ can be (noncanonically) identified with a finite disjoint union of sets like $\overline{\mathcal{F}}$.

Suppose $f \in \mathcal{F}$, and suppose $i_f \in \mathcal{I}$ is as in the above definition of $\mathcal{F}$. Then $i_f(b^\star) = (i_f(b))^{\star'}$ and $i_0(b^\star) = (i_0(b))^{\star'} = f^{\star'} \circ i_f(b^\star) \circ (f^{-1})^{\star'} = (f^{\star'} f) \circ i_f(b) \circ (f^{\star'} f)^{-1}$ for all $b \in \mathcal{O}$; hence $f^{\star'} f \in \mathcal{C}^\times$. By the positivity of the λ-Rosati involution $\star'$ and by the classification of real positive involutions (see [94, §21] and [76, §2]), $\mathcal{K} := \{g \in \mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{R} : g^{\star'} g = \mathrm{Id}_{\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{R}}\}$ is a compact subset of $(\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{R})^\times$ (with the standard topology induced by that of $\mathbb{R}$), and there exists $c \in (\mathcal{C} \underset{\mathbb{Z}}{\otimes} \mathbb{R})^\times$ such that $f^{\star'} f = c^{\star'} c$. This shows that $\mathcal{F} \subset \mathcal{K} \cdot (\mathcal{C} \underset{\mathbb{Z}}{\otimes} \mathbb{R})^\times$ in $(\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{R})^\times$. Since the image of the composition of canonical maps $(\mathcal{O}'_l)^\times \hookrightarrow (\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{R})^\times \twoheadrightarrow (\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{R})^\times / (\mathcal{C} \underset{\mathbb{Z}}{\otimes} \mathbb{R})^\times$ is a discrete subset (because $\mathrm{Hom}_{\mathrm{ring}}(\mathcal{O}, \mathcal{O}'_l)$ is a discrete subset of $\mathrm{Hom}_{\mathrm{ring}}(\mathcal{O}, \mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{R})$), the image $\overline{\mathcal{F}}$ of the composition of canonical maps $\mathcal{F} \hookrightarrow \mathcal{K} \cdot (\mathcal{C} \underset{\mathbb{Z}}{\otimes} \mathbb{R})^\times \twoheadrightarrow \mathcal{K}/(\mathcal{K} \cap (\mathcal{C} \underset{\mathbb{Z}}{\otimes} \mathbb{R})^\times)$ is also a discrete subset. Since $\mathcal{K}/(\mathcal{K} \cap (\mathcal{C} \underset{\mathbb{Z}}{\otimes} \mathbb{R})^\times)$ is compact (because $\mathcal{K}$ is), we see that $\overline{\mathcal{F}}$ is finite, as desired. $\square$

1.3.4 Conditions on Lie Algebras

We will use the polarized symplectic vector space $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h)$ to define a condition for the Lie algebras of the abelian schemes we consider.

Recall that (in Sections 1.2.1 and 1.2.5) we have a decomposition $L \underset{\mathbb{Z}}{\otimes} \mathbb{C} = V_0 \oplus V_0^c$, where $h(z)$ acts by $1 \otimes z$ on V_0, and by $1 \otimes z^c$ on V_0^c. Moreover, both V_0 and V_0^c are totally isotropic under the pairing $\langle \cdot, \cdot \rangle$.

The reason to consider V_0 is that, according to the Hodge decomposition for abelian varieties over $\mathbb{C}$, it is natural to compare V_0 with the Lie algebra of an abelian variety.

By Definition 1.1.2.18, the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-module V_0 defines an element $\mathrm{Det}_{\mathcal{O}|V_0}$ in $\mathbb{C}[\mathcal{O}^\vee]$, which lies in $\mathcal{O}_{F_0}[\mathcal{O}^\vee]$ by Lemma 1.2.5.10.

On the other hand, suppose $A \to S$ is an abelian scheme over $\mathcal{O}_{F_0,(\Box)}$, together with a $\mathbb{Z}_{(\Box)}^\times$-polarization λ and an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-endomorphism structure $i : \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \to \mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} (\mathbb{Z}_{(\Box)})_S$ giving an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-action on $\underline{\mathrm{Lie}}_{A/S}$. Since the Lie algebra $\underline{\mathrm{Lie}}_{A/S}$ is a locally free $\mathscr{O}_S$-module with $\mathcal{O}$-action, it defines an element $\mathrm{Det}_{\mathcal{O}|\underline{\mathrm{Lie}}_{A/S}}$ in $\mathscr{O}_S[\mathcal{O}^\vee]$ (see Definition 1.1.2.21).

Definition 1.3.4.1. *The* ***(Kottwitz) determinantal condition*** *defined by* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h)$ *on* $\underline{\mathrm{Lie}}_{A/S}$ *is that* $\mathrm{Det}_{\mathcal{O}|\underline{\mathrm{Lie}}_{A/S}}$ *agrees with the image of* $\mathrm{Det}_{\mathcal{O}|V_0}$ *under the structural homomorphism from* $\mathcal{O}_{F_0,(\Box)}$ *to* $\mathscr{O}_S$*. The* ***Lie algebra condition*** *defined by* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h)$ *on* (A, λ, i) *is this determinantal condition defined by* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h)$ *on* $\underline{\mathrm{Lie}}_{A/S}$.

Remark 1.3.4.2. Suppose two polarizations h and h' of $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle)$ differ by conjugation by an element in $\mathrm{G}(\mathbb{R})$ (or rather $\mathrm{G}^+(\mathbb{R})$, the subgroup of elements in $\mathrm{G}(\mathbb{R})$ with positive similitudes). Then we have an isomorphism $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h) \xrightarrow{\sim} (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h')$ of polarized symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-lattices (see Definition 1.2.1.2), and therefore these two triples define the same conditions in Definition 1.2.5.4 and 1.3.4.1.

Although we can define this condition for all $\mathcal{O}_{F_0,(\Box)}$-schemes, the module V_0 is defined by objects over $\mathbb{C}$. Therefore one may certainly doubt whether this condition is also good for fields or complete local rings over $\mathcal{O}_{F_0,(\Box)}$ when the residue characteristic is positive. We shall see in the proof of Proposition 2.2.2.9 that this is indeed the case if we assume that $\Box \nmid \mathrm{Disc}$.

1.3.5 Tate Modules

Definition 1.3.5.1. *Let G be a commutative group. Let Δ be a semisubgroup of $\mathbb{Z}_{>0}$, the multiplicative semigroup of positive integers. Suppose that G is* ***Δ-divisible*** *in the sense that the multiplication by N in G is* ***surjective*** *for every $N \in \Delta$.*

1. *We define $\mathrm{V}_\Delta(G)$ to be the group of sequences $(\alpha_i)_{i \in \Delta}$ such that*
   (a) *$N\alpha_{Ni} = \alpha_i$ for all $i, N \in \Delta$;*
   (b) *for each $i \in \Delta$, there exists $N \in \Delta$ such that $N\alpha_i = 0$.*

   If $1 \in \Delta$, $\mathrm{T}_\Delta(G)$ is defined to be the subgroup of $\mathrm{V}_\Delta(G)$ with $\alpha_1 = 0$.
2. *We define $\mathrm{V}(G)$ to be $\mathrm{V}_{\mathbb{Z}_{>0}}(G)$, and $\mathrm{T}(G)$ to be $\mathrm{T}_{\mathbb{Z}_{>0}}(G)$.*
3. *For each prime number $l > 0$, we set $\Delta_l := l^{\mathbb{Z}_{\geq 0}}$ and define $\mathrm{V}_l(G)$ (resp. $\mathrm{T}_l(G)$) to be $\mathrm{V}_{\Delta_l}(G)$ (resp. $\mathrm{T}_{\Delta_l}(G)$).*

4. *For each set of prime numbers* $\Box$, *we set* $\Delta^\Box := \mathbb{Z}^\times_{(\Box)} \cap \mathbb{Z}_{>0}$ *and define* $\mathrm{V}^\Box(G)$ *(resp.* $\mathrm{T}^\Box(G)$*) to be* $\mathrm{V}_{\Delta^\Box}(G)$ *(resp.* $\mathrm{T}_{\Delta^\Box}(G)$*).*

Let us denote by G_{tors} the torsion subgroup of G, and denote by $G^\Box_{\mathrm{tors}}$ the prime-to-$\Box$ part of the torsion subgroup of G_{tors}. Then we have the canonical exact sequences $0 \to \mathrm{T}(G) \to \mathrm{V}(G) \to G_{\mathrm{tors}} \to 0$ and $0 \to \mathrm{T}^\Box(G) \to \mathrm{V}^\Box(G) \to G^\Box_{\mathrm{tors}} \to 0$. Note that the surjectivity of $\mathrm{V}(G) \to G_{\mathrm{tors}}$ (resp. $\mathrm{V}^\Box(G) \to G^\Box_{\mathrm{tors}}$) requires the assumption that multiplication by N in G is *surjective* for every $N \in \mathbb{Z}_{>0}$ (resp. every $N \in \mathbb{Z}^\times_{(\Box)} \cap \mathbb{Z}_{>0}$).

Let A be any abelian variety over an algebraic closed field k. Consider $G := A(k)$, the k-points of A. We shall denote $\mathrm{V}(G)$, $\mathrm{V}_l(G)$, $\mathrm{V}^\Box(G)$, $\mathrm{T}(G)$, $\mathrm{T}_l(G)$, and $\mathrm{T}^\Box(G)$ by $\mathrm{V}\,A$, $\mathrm{V}_l\,A$, $\mathrm{V}^\Box\,A$, $\mathrm{T}\,A$, $\mathrm{T}_l\,A$, and $\mathrm{T}^\Box\,A$, respectively.

Let $p := \mathrm{char}(k)$. Assume that $\Box$ contains $p = \mathrm{char}(k)$ if $p > 0$. Let $A^\Box_{\mathrm{tors}}$ denote the subgroup of all prime-to-$\Box$ torsion points of A. Then we have the exact sequence $0 \to \mathrm{T}^\Box\,A \to \mathrm{V}^\Box\,A \to A^\Box_{\mathrm{tors}} \to 0$.

Each group scheme homomorphism $f : A \to A'$ sends A_{tors} to A'_{tors} and induces a homomorphism $\mathrm{V}^\Box(f) : \mathrm{V}^\Box\,A \to \mathrm{V}^\Box\,A'$. The homomorphism $\mathrm{V}^\Box(f)$ is an isomorphism when f is an isogeny, and we can extend the definition of $\mathrm{V}^\Box(f)$ to the case that f is a $\mathbb{Z}^\times_{(\Box)}$-isogeny by setting $\mathrm{V}^\Box(f) = \mathrm{V}^\Box(g)^{-1} \circ \mathrm{V}^\Box(h)$ if f is represented by some triple (B, g, h) as in Definition 1.3.1.17. In particular, for each $\mathbb{Z}^\times_{(\Box)}$-isogeny of the form $N^{-1}f$ where N is an integer prime-to-$\Box$ and f is an isogeny from A to A', we can define $\mathrm{V}^\Box(N^{-1}f)$ by setting $\mathrm{V}^\Box(N^{-1}f)((\alpha_i)) = (f(\alpha_{Ni}))$ for each $\alpha = (\alpha_i) \in \mathrm{V}^\Box\,A$.

Lemma 1.3.5.2. *Fix a triple* (A, λ, i)*, where* A *is an abelian variety over an algebraically closed field* k*, where* λ *is a* $\mathbb{Z}^\times_{(\Box)}$*-polarization of* A*, and where* i *is an* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$*-structure of* (A, λ)*. Then there is a one–one correspondence*

$$\left\{ \begin{matrix} \text{equivalence classes of } \mathbb{Z}^\times_{(\Box)}\text{-isogenies} \\ f : (A, \lambda, i) \to (A', \lambda', i') \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{open compact} \\ \text{subgroups of } \mathrm{V}^\Box\,A \end{matrix} \right\}$$

given by sending a $\mathbb{Z}^\times_{(\Box)}$*-isogeny* $f : (A, \lambda, i) \to (A', \lambda', i')$ *to* $\mathrm{V}(f)^{-1}(\mathrm{T}^\Box\,A')$*. The* $\mathcal{O}$*-invariant open compact subgroups of* $\mathrm{V}^\Box\,A$ *correspond to* $\mathbb{Z}^\times_{(\Box)}$*-isogenies* $f : (A, \lambda, i) \to (A', \lambda', i')$ *such that* i' *has its image in* $\mathrm{End}_S(A')$*.*

Here two $\mathbb{Z}^\times_{(\Box)}$*-isogenies* $f_1 : (A, \lambda, i) \to (A_1, \lambda_1, i_1)$ *and* $f_2 : (A, \lambda, i) \to (A_2, \lambda_2, i_2)$ *are equivalent if there exists an isomorphism* $h : A_1 \xrightarrow{\sim} A_2$ *(of abelian varieties over* k*) such that* $h \circ f_1 = f_2$*.*

Since prime-to-$\Box$ isogenies are characterized by their kernels, which are commutative group schemes finite étale over the base field, it is useful to have the following:

Proposition 1.3.5.3 (cf. [56, V, §7] or [44, A I.7])**.** *Let* S *be a connected locally noetherian scheme, and* $\bar{s}$ *any fixed geometric point on* S*. Then there is an equivalence of categories between*

$$\left\{ \begin{matrix} \text{commutative group schemes} \\ \text{finite étale over } S \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{finite continuous} \\ \pi_1(S, \bar{s})\text{-modules} \end{matrix} \right\}$$

given by sending a group scheme H *over* S *to its geometric fiber* $H_{\bar{s}}$ *over* $\bar{s}$*.*

Combining the above two propositions, we have the following corollary:

Corollary 1.3.5.4. *Let* S *be a connected locally noetherian scheme, with residual characteristics either* 0 *or a prime number in* $\Box$*. Let* $\bar{s}$ *be any fixed geometric point of* S*.*

Fix a triple (A, λ, i)*, where* A *is an abelian scheme over* S*, where* λ *is a* $\mathbb{Z}^\times_{(\Box)}$*-polarization of* A*, and where* i *is an* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$*-structure of* (A, λ)*. Then there is a one–one correspondence*

$$\left\{ \begin{matrix} \text{equivalence classes of} \\ \mathbb{Z}^\times_{(\Box)}\text{-isogenies} \\ f : (A, \lambda, i) \to (A', \lambda', i') \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \pi_1(S, \bar{s})\text{-invariant} \\ \text{open compact} \\ \text{subgroups of } \mathrm{V}^\Box\,A_{\bar{s}} \end{matrix} \right\}$$

given by sending a $\mathbb{Z}^\times_{(\Box)}$*-isogeny* $f : (A, \lambda, i) \to (A', \lambda', i')$ *to* $\mathrm{V}(f_{\bar{s}})^{-1}(\mathrm{T}^\Box\,A'_{\bar{s}})$*. Here two* $\mathbb{Z}^\times_{(\Box)}$*-isogenies* $f_1 : (A, \lambda, i) \to (A_1, \lambda_1, i_1)$ *and* $f_2 : (A, \lambda, i) \to (A_2, \lambda_2, i_2)$ *are equivalent if there exists an isomorphism* $h : A_1 \xrightarrow{\sim} A_2$ *(of group schemes over* S*) such that* $h \circ f_1 = f_2$*.*

The $\pi_1(S, \bar{s})$*-invariant* $\mathcal{O}$*-invariant open compact subgroups of* $\mathrm{V}^\Box\,A_{\bar{s}}$ *correspond to* $\mathbb{Z}^\times_{(\Box)}$*-isogenies* $f : (A, \lambda, i) \to (A', \lambda', i')$ *such that* i' *defines an* $\mathcal{O}$*-structure (see Definition 1.3.3.1).*

Remark 1.3.5.5. In Corollary 1.3.5.4, if (A, λ, i) is a triple such that λ is not prime-to-$\Box$, then there is no triple (A', λ', i') in the equivalence class of (A, λ, i) such that λ' is a prime-to-$\Box$ polarization.

1.3.6 Principal Level Structures

Let $(L, \langle \cdot\,, \cdot \rangle, h)$ be a PEL-type $\mathcal{O}$-lattice as in Definition 1.2.1.3. Let $\Box$ be a set of rational prime numbers, and let $n \geq 1$ be an integer prime-to-$\Box$.

Let us begin with a naive candidate for a level structure on (A, λ, i):

Definition 1.3.6.1. *Let* (A, λ, i) *be a triple such that*

1. A *is an abelian scheme over a scheme* S *over* $\mathrm{Spec}(\mathbb{Z}_{(\Box)})$*;*
2. $\lambda : A \to A^\vee$ *is a prime-to-*$\Box$ *polarization of* A*;*
3. $i : \mathcal{O} \to \mathrm{End}_S(A)$ *defines an* $\mathcal{O}$*-structure of* (A, λ)*.*

An $\mathcal{O}$*-equivariant* ***symplectic isomorphism*** *from* $(L/nL)_S$ *to* $A[n]$ *(cf. Definition 1.1.4.8) consists of the following data:*

1. *An* $\mathcal{O}$*-equivariant isomorphism* $\alpha_n : (L/nL)_S \xrightarrow{\sim} A[n]$ *of group schemes over* S*.*
2. *An isomorphism* $\nu_n : ((\mathbb{Z}/n\mathbb{Z})(1))_S \xrightarrow{\sim} \boldsymbol{\mu}_{n,S}$ *of group schemes over* S *making the diagram*

$$\begin{tikzcd} (L/nL)_S \underset{S}{\times} (L/nL)_S \arrow[r, "\langle \cdot\,,\cdot \rangle"] \arrow[d, "\alpha_n \times \alpha_n"', "\wr"] & ((\mathbb{Z}/n\mathbb{Z})(1))_S \arrow[d, dotted, "\nu_n", "\wr"'] \\ A[n] \underset{S}{\times} A[n] \arrow[r, "\mathrm{e}^\lambda"'] & \boldsymbol{\mu}_{n,S} \end{tikzcd}$$

commutative, where e^λ *is the* λ*-Weil pairing.*

By abuse of notation, we often denote such a symplectic isomorphism by

$$(\alpha_n, \nu_n) : (L/nL)_S \xrightarrow{\sim} A[n],$$

or simply by

$$\alpha_n : (L/nL)_S \xrightarrow{\sim} A[n].$$

This candidate works perfectly well for the moduli of *principally polarized* abelian schemes as in [42].

However, since the lattice L is not necessarily self-dual, the pairing on L/nL induced by $\langle\,\cdot\,,\,\cdot\,\rangle$ may be *degenerate*, or even trivial. Moreover, the $\mathcal{O}$-equivariance might not be detectable modulo n. Therefore, for an arbitrary geometric point $\bar{s}$ of S, the symplectic isomorphism $(\alpha_n, \nu_n) : (L/nL)_S \xrightarrow{\sim} A[n]$ may not be *liftable* to an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$-equivariant symplectic isomorphism $(\hat{\alpha}, \hat{\nu}) : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square \xrightarrow{\sim} \mathrm{T}^\square A_{\bar{s}}$ (in the obvious sense), making the definition of Hecke actions of $\mathrm{G}(\mathbb{A}^{\infty,\square})$ an unreasonable task. Thus, for the pair (α_n, ν_n) to qualify as a level structure, we need to take the issue of liftability into account.

Definition 1.3.6.2. *Let (A, λ, i) be a triple over S as in Definition 1.3.6.1. An* ***(integral) principal level-n structure of*** *(A, λ, i)* ***of type*** *$(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square, \langle\,\cdot\,,\,\cdot\,\rangle)$ is an $\mathcal{O}$-equivariant symplectic isomorphism*

$$(\alpha_n, \nu_n) : (L/nL)_S \xrightarrow{\sim} A[n]$$

that is a ***symplectic-liftable isomorphism*** *in the following sense:*

There exists (noncanonically) a tower $(S_m \twoheadrightarrow S)_{n|m,\square\nmid m}$ of finite étale coverings such that

1. *$S_n = S$;*
2. *for each l such that $n|l$ and $l|m$, there is a finite étale covering $S_m \twoheadrightarrow S_l$ whose composition with $S_l \to S$ is the finite étale covering $S_m \to S$;*
3. *over each S_m, there is an $\mathcal{O}$-equivariant symplectic isomorphism*
$$(\alpha_{m,S_m}, \nu_{m,S_m}) : (L/mL)_{S_m} \xrightarrow{\sim} A[m]_{S_m};$$
4. *for each l such that $n|l$ and $l|m$, the pullback of $(\alpha_{l,S_l}, \nu_{l,S_l})$ to S_m is the reduction modulo l of $(\alpha_{m,S_m}, \nu_{m,S_m})$.*

If $L \neq 0$, then ν_n is uniquely determined by α_n. If $L = 0$, then ν_n is the essential nontrivial information. By abuse of notation, we shall often suppress the datum ν_n and denote it by $\nu(\alpha_n)$ (as if ν_n were always determined by α_n), and denote level structures simply by $\alpha_n : (L/nL)_S \xrightarrow{\sim} A[n]$. (We shall adopt similar conventions for other symplectic isomorphisms.)

Remark 1.3.6.3. The symplectic-liftability condition is nontrivial even when $n = 1$. Moreover, it forces the kernel of the prime-to-$\square$ polarization λ to be isomorphic to $((L^\#\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square)/(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square))_S$.

Remark 1.3.6.4. For simplicity, when the context is clear, we shall speak of a level structure without the term *integral* and other modifiers.

Lemma 1.3.6.5. *Let (A, λ, i) be a triple over S as in Definition 1.3.6.1. Let $\bar{s}$ be any geometric point of S. Recall that an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$-equivariant* ***symplectic isomorphism***

$$(\hat{\alpha}, \hat{\nu}) : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square \xrightarrow{\sim} \mathrm{T}^\square A_{\bar{s}}$$

(see Definition 1.1.4.8) is an isomorphism

$$\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square \xrightarrow{\sim} \mathrm{T}^\square A_{\bar{s}}$$

of the underlying $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$-modules, together with an isomorphism

$$\hat{\nu} : \hat{\mathbb{Z}}^\square(1) \xrightarrow{\sim} \mathrm{T}^\square\,\mathbf{G}_{\mathrm{m},\bar{s}}$$

(of $\hat{\mathbb{Z}}^\square$-modules) making the diagram

$$\begin{tikzcd}
(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square) \times (L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square) \arrow[r, "\langle\,\cdot\,,\,\cdot\,\rangle"] \arrow[d, "\hat{\alpha}\times\hat{\alpha}"', "\wr"] & \hat{\mathbb{Z}}^\square(1) \arrow[d, dotted, "\hat{\alpha}", "\wr"'] \\
\mathrm{T}^\square A_{\bar{s}} \times \mathrm{T}^\square A_{\bar{s}} \arrow[r, "\mathrm{e}^\lambda"'] & \mathrm{T}^\square\,\mathbf{G}_{\mathrm{m},\bar{s}}
\end{tikzcd}$$

commutative. Let $(\alpha_n, \nu_n) : (L/nL)_S \xrightarrow{\sim} A[n]$ be an (integral) principal level-n structure of (A, λ, i) of type $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square, \langle\,\cdot\,,\,\cdot\,\rangle)$. Then (α_n, ν_n) is ***symplectic-liftable at*** *$\bar{s}$ in the sense that there exists (noncanonically) an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$-equivariant symplectic isomorphism $(\hat{\alpha}, \hat{\nu}) : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square \xrightarrow{\sim} \mathrm{T}^\square A_{\bar{s}}$ lifting $(\alpha_{n,\bar{s}}, \nu_{n,\bar{s}}) : L/nL \xrightarrow{\sim} A[n]_{\bar{s}}$ as a reduction modulo n of $(\hat{\alpha}, \hat{\nu})$.*

Proof. The pullback of the compatible tower $(\alpha_{m,S_m})_{n|m,\square\nmid m}$ defined over $(S_m \to S)_{n|m,\square\nmid m}$ to the geometric point $\bar{s}$ of S (with a compatible choice of liftings to each $S_m \to S$) allows us to choose a compatible tower $((\alpha_{m,\bar{s}}, \nu_{m,\bar{s}}) : L/mL \xrightarrow{\sim} A_{\bar{s}}[m])_{n|m,\square\nmid m}$ of $\mathcal{O}$-equivariant symplectic isomorphisms, which is equivalent to the desired $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$-equivariant symplectic isomorphism $(\hat{\alpha}, \hat{\nu}) : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square \xrightarrow{\sim} \mathrm{T}^\square A_{\bar{s}}$. $\square$

By abuse of notation, we shall often suppress the datum $\hat{\nu}$ and denote it by $\nu(\hat{\alpha})$ (as if $\nu(\hat{\alpha})$ were always determined by $\hat{\alpha}$), and denote $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$-equivariant symplectic isomorphisms simply by $\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square \xrightarrow{\sim} \mathrm{T}^\square A_{\bar{s}}$.

Lemma 1.3.6.6. *Let (A, λ, i) be a triple over S as in Definition 1.3.6.1. Suppose moreover that S is* ***locally noetherian****. Then an $\mathcal{O}$-equivariant symplectic isomorphism $\alpha_n : (L/nL)_S \xrightarrow{\sim} A[n]$ is symplectic-liftable (in the sense of Definition 1.3.6.2) if and only if it is symplectic-liftable at every geometric point $\bar{s}$ of S (in the sense of Lemma 1.3.6.5).*

Proof. If α_n is symplectic-liftable, then the existence of $\hat{\alpha}_{\bar{s}}$ is given by Lemma 1.3.6.5 without the locally noetherian hypothesis.

Conversely, since S is locally noetherian, it is the disjoint union of its connected components; therefore, to show that α_n is symplectic-liftable in the sense of Definition 1.3.6.2, we may replace S with each of its connected components and assume that S is *connected*. Let $\bar{s}$ be any geometry point of S, and let $\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square \xrightarrow{\sim} \mathrm{T}^\square A_{\bar{s}}$ be any $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$-equivariant symplectic isomorphism $\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square \xrightarrow{\sim} \mathrm{T}^\square A_{\bar{s}}$ whose reduction modulo n is $\alpha_{n,\bar{s}} : L/nL \xrightarrow{\sim} A[n]_{\bar{s}}$. Note that the $\mathcal{U}^\square(n)$-orbit of this lifting is unique, and (by Proposition 1.3.5.3) this orbit is invariant under the action of $\pi_1(S, \bar{s})$ because $A[n]$ and $\boldsymbol{\mu}_{n,S}$ are locally constant étale sheaves over S. The $\pi_1(S, \bar{s})$-invariance of the $\mathcal{U}^\square(n)$-orbit $[\hat{\alpha}]_n$ of $\hat{\alpha}$ expresses the fact that we have a continuous homomorphism $\pi_1(S, \bar{s}) \to \mathcal{U}^\square(n)$. By taking the preimage of $\mathcal{U}^\square(m)$ under this homomorphism, we obtain an open compact subgroup $\pi_1(S_m, \bar{s})$ of $\pi_1(S, \bar{s})$ corresponding to some finite étale covering $S_m \to S$ and some lifting $\bar{s} \to S_m$ of $\bar{s} \to S$. The $\mathcal{U}^\square(m)$-orbit of $\hat{\alpha}$ is therefore invariant under $\pi_1(S_m, \bar{s})$ when we pass to the finite étale covering $S_m \to S$, and we obtain a $\pi_1(S_m, \bar{s})$-equivariant symplectic isomorphism $L/mL \xrightarrow{\sim} A[m]_{\bar{s}}$. By Proposition 1.3.5.3, this is equivalent to an $\mathcal{O}$-equivariant symplectic isomorphism $\alpha_{m,S_m} : (L/mL)_{S_m} \xrightarrow{\sim} A[m]_{S_m}$. The compatibility between different m and l follows from the natural containment relation between different $\mathcal{U}^{\Box}(m) \subset \mathcal{U}^{\Box}(n)$ and $\mathcal{U}^{\Box}(l) \subset \mathcal{U}^{\Box}(n)$. Thus we obtain the symplectic-liftability of α_n as in Definition 1.3.6.2. $\Box$

Corollary 1.3.6.7 (of the proof of Lemma 1.3.6.6)**.** *Let (A, λ, i) be a triple over S as in Definition 1.3.6.1. Suppose moreover that S is locally noetherian. Then an $\mathcal{O}$-equivariant symplectic isomorphism $\alpha_n : (L/nL)_S \xrightarrow{\sim} A[n]$ is symplectic-liftable at a geometric point $\bar{s}$ of S if and only if it is symplectic-liftable at every geometric point $\bar{s}'$ on the same connected component of S.*

1.3.7 General Level Structures

Let us continue with the setting in Section 1.3.6 and introduce level structures other than principal level structures.

Definition 1.3.7.1. *For each $\hat{\mathbb{Z}}^{\Box}$-algebra R, set $\mathrm{G}^{\mathrm{ess}}(R) := \mathrm{image}(\mathrm{G}(\hat{\mathbb{Z}}^{\Box}) \to \mathrm{G}(R))$.*

Lemma 1.3.7.2. *Let $n \geq 1$ be an integer such that $\Box \nmid n$. With the setting in Definition 1.3.6.2, assume moreover that the base S is connected. Then each two level-n structures $\alpha_n, \alpha'_n : (L/nL)_S \xrightarrow{\sim} A[n]$ (see Definition 1.3.6.2) are related by $(\alpha'_n, \nu(\alpha'_n)) = (\alpha_n \circ g_n, \nu(\alpha_n) \circ \nu(g_n))$ for a unique $g_n \in \mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$.*

In what follows, we shall often suppress the expression $\nu(\alpha'_n) = \nu(\alpha_n) \circ \nu(g_n)$ from the context, although it is an essential ingredient when we use the expression $\alpha'_n = \alpha_n \circ g_n$ to mean we are relating two level structures.

Definition 1.3.7.3. *Let $n \geq 1$ be an integer such that $\Box \nmid n$. Let (A, λ, i) and S be as in Definition 1.3.6.2. Let H_n be a subgroup of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$. By an **$H_n$-orbit of étale-locally-defined level-n structures**, we mean a subscheme α_{H_n} of*

$$\underline{\mathrm{Isom}}_S((L/nL)_S, A[n]) \underset{S}{\times} \underline{\mathrm{Isom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S})$$

over S that becomes the disjoint union of all elements in some H_n-orbit of level-n structures after a finite étale surjective base change in S. (We need to include the second factor $\underline{\mathrm{Isom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S})$ because we do not exclude the possibility that $L = 0$.) In this case, we denote by $\nu(\alpha_{H_n})$ the projection of α_{H_n} to $\underline{\mathrm{Isom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S})$, which is a $\nu(H_n)$-orbit of étale-locally-defined isomorphisms with its natural interpretation.

Remark 1.3.7.4. In Definition 1.3.7.3, if S is locally noetherian, then (by Lemma 1.3.6.6) the finite étale surjective base change (in S) can be replaced with an étale surjective base change (without finiteness).

Lemma 1.3.7.5. *Let (A, λ, i) and S be as in Definition 1.3.6.2. Let $\mathcal{H}$ be any open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. For each integer $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, set $H_n := \mathcal{H}/\mathcal{U}^{\Box}(n)$, the image of $\mathcal{H}$ under $\mathrm{G}(\hat{\mathbb{Z}}^{\Box}) \twoheadrightarrow \mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z}) = \mathrm{G}(\hat{\mathbb{Z}}^{\Box})/\mathcal{U}^{\Box}(n)$ (see Remark 1.2.1.9 for the definition of $\mathcal{U}^{\Box}(n)$). Then there is a canonical bijection from the set of H_m-orbits of étale-locally-defined level-m structures for (A, λ, i) to the set of H_n-orbits of étale-locally-defined level-n structures for (A, λ, i), induced (étale locally over S) by taking the reduction modulo n of an level-m structure.*

Proof. Taking reduction modulo n induces a surjection because of the symplectic-liftability condition in Definition 1.3.6.2. It induces an injection because the canonical isomorphism $(\mathrm{G}(\hat{\mathbb{Z}}^{\Box})/\mathcal{U}^{\Box}(m))/(\mathcal{U}^{\Box}(n)/\mathcal{U}^{\Box}(m)) \cong \mathrm{G}(\hat{\mathbb{Z}}^{\Box})/\mathcal{U}^{\Box}(n)$ identifies $H_m/(\mathcal{U}^{\Box}(n)/\mathcal{U}^{\Box}(m))$ with H_n. $\Box$

Definition 1.3.7.6. *Let (A, λ, i) and S be as in Definition 1.3.6.2. Let $\mathcal{H}$ be any open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. For each integer $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$, set $H_n := \mathcal{H}/\mathcal{U}^{\Box}(n)$ as in Lemma 1.3.7.5. Then an **(integral) level-$\mathcal{H}$ structure of** (A, λ, i) **of type** $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle \cdot , \cdot \rangle)$ is a collection $\alpha_{\mathcal{H}} = \{\alpha_{H_n}\}_n$ labeled by integers $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$, with elements α_{H_n} described as follows:*

1. *For each index n, the element α_{H_n} is an H_n-orbit of étale-locally-defined level-n structures as in Definition 1.3.7.3.*
2. *For all indices n and m such that $n|m$, the H_m-orbit α_{H_m} corresponds to the H_n-orbit α_{H_n} under Lemma 1.3.7.5.*

Remark 1.3.7.7. According to Lemma 1.3.7.5, the collection $\alpha_{\mathcal{H}} = \{\alpha_{H_n}\}_n$ is determined by any element α_{H_n} in it.

1.3.8 Rational Level Structures

Definition 1.3.8.1. *Let S be a locally noetherian scheme over $\mathrm{Spec}(\mathbb{Z}_{(\Box)})$, and let $\bar{s}$ be a geometric point of S. Let (A, λ, i) be a triple such that*

1. *A is an abelian scheme over S;*
2. *$\lambda : A \to A^{\vee}$ is a $\mathbb{Z}^{\times}_{(\Box)}$-polarization of A;*
3. *$i : \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \to \mathrm{End}_S(A) \underset{\mathbb{Z}}{\otimes} (\mathbb{Z}_{(\Box)})_S$ defines an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-structure of (A, λ).*

*An $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}$-equivariant **symplectic isomorphism***

$$\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} A_{\bar{s}}$$

is an isomorphism of the underlying $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}$-modules together with an isomorphism

$$\nu(\hat{\alpha}) : \mathbb{A}^{\infty,\Box}(1) \xrightarrow{\sim} \mathrm{V}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{s}}$$

(of $\mathbb{A}^{\infty,\Box}$-modules) making the diagram

$$\begin{tikzcd}
(L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}) \times (L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}) \arrow[r, "\langle \cdot , \cdot \rangle"] \arrow[d, "\hat{\alpha} \times \hat{\alpha}"', "\wr"] & \mathbb{A}^{\infty,\Box}(1) \arrow[d, dotted, "\nu(\hat{\alpha})", "\wr"'] \\
\mathrm{V}^{\Box} A_{\bar{s}} \times \mathrm{V}^{\Box} A_{\bar{s}} \arrow[r, "\mathrm{e}^{\lambda}"'] & \mathrm{V}^{\Box}\, \mathbf{G}_{\mathrm{m},\bar{s}}
\end{tikzcd}$$

commutative. The group $\mathrm{G}(\mathbb{A}^{\infty,\Box})$ has a natural right action on the set of such symplectic isomorphisms, defined by the composition $(\hat{\alpha}, \nu(\hat{\alpha})) \mapsto (\hat{\alpha} \circ g, \nu(\hat{\alpha}) \circ \nu(g))$ for each $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$.

For $\mathbb{Z}^{\times}_{(\Box)}$-isogeny classes of similar triples over *locally noetherian bases*, a better notion of level structures is given as follows:

Definition 1.3.8.2. *Let $\mathcal{H}$ be an open compact subgroup of $\mathrm{G}(\mathbb{A}^{\infty,\Box})$. Let (A,λ,i), S, and $\bar{s}$ be as in Definition 1.3.8.1. A* ***rational level-$\mathcal{H}$ structure of*** *(A,λ,i)* ***of type*** *$(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box},\langle\cdot,\cdot\rangle)$* ***based at*** *$\bar{s}$ is a $\pi_1(S,\bar{s})$-invariant $\mathcal{H}$-orbit $[\hat{\alpha}]_{\mathcal{H}}$ of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}$-equivariant symplectic isomorphisms $\hat{\alpha}: L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}\overset{\sim}{\to}\mathrm{V}^{\Box}A_{\bar{s}}$ (as in Definition 1.3.8.1).*

Remark 1.3.8.3. When the context is clear, we shall abbreviate a *rational principal level-$\mathcal{H}$ structure of (A,λ,i) of type $(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box},\langle\cdot,\cdot\rangle)$ based at $\bar{s}$* as a *rational level-$\mathcal{H}$ structure based at $\bar{s}$*.

Construction 1.3.8.4. With the same setting as in Definition 1.3.7.6, assume moreover that S is locally noetherian. Let $\alpha_{\mathcal{H}}=\{\alpha_{H_n}\}_n$ be any level-$\mathcal{H}$ structure of (A,λ,i) of type $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box},\langle\cdot,\cdot\rangle)$. Let $\bar{s}\to S$ be any geometric point of S. Let us choose an integer $n\geq 1$ such that $\Box\nmid n$ and $\mathcal{U}^{\Box}(n)\subset\mathcal{H}$. Let $\tilde{S}\twoheadrightarrow S$ be a finite étale covering such that the pullback of α_{H_n} to $\tilde{S}$ is the disjoint union of all elements in some H_n-orbit of some level-n structure $\alpha_n:(L/nL)_{\tilde{S}}\overset{\sim}{\to}A[n]_{\tilde{S}}$ over $\tilde{S}$. Let us lift $\bar{s}\to S$ to some $\bar{s}\to\tilde{S}$ and view $\bar{s}$ as a geometric point of $\tilde{S}$ by this particular lifting. Then we obtain by Lemma 1.3.6.5 some $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$-equivariant symplectic isomorphism $\hat{\alpha}:L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}\overset{\sim}{\to}\mathrm{T}^{\Box}A_{\bar{s}}$ lifting α_n. The $\mathcal{H}$-orbit $[\hat{\alpha}]_{\mathcal{H}}$ of $\hat{\alpha}$ is independent of the choice of n, $\tilde{S}$, and α_n, because it is the H_n-orbit of the $\mathcal{U}^{\Box}(n)$-orbit of $\hat{\alpha}$. Moreover, $[\hat{\alpha}]_{\mathcal{H}}$ is invariant under $\pi_1(\tilde{S},\bar{s})$ for every choice of liftings $\bar{s}\to\tilde{S}$, which means that it is invariant under $\pi_1(S,\bar{s})$. As a result, we obtain a well-defined rational level-$\mathcal{H}$ structure $[\hat{\alpha}]_{\mathcal{H}}$ of (A,λ,i) of type $(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box},\langle\cdot,\cdot\rangle)$ such that every $\hat{\alpha}$ in $[\hat{\alpha}]_{\mathcal{H}}$ sends $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box}A_{\bar{s}}$.

Lemma 1.3.8.5. *With the same setting as in Definition 1.3.7.6, assume moreover that S is locally noetherian and connected. Let $\bar{s}$ be a geometric point of S. Let $\mathcal{H}$ be any open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. Then a rational level-$\mathcal{H}$ structure $[\hat{\alpha}]_{\mathcal{H}}$ of (A,λ,i) based at $\bar{s}$ comes from a (necessarily unique) integral level-$\mathcal{H}$ structure $\alpha_{\mathcal{H}}$ as in Construction 1.3.8.4 if and only if the following condition is satisfied: Each isomorphism $\hat{\alpha}:L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}\overset{\sim}{\to}\mathrm{V}^{\Box}A_{\bar{s}}$ in the $\pi_1(S,\bar{s})$-invariant $\mathcal{U}^{\Box}(n)$-orbit $[\hat{\alpha}]_{\mathcal{H}}$ induces an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$-equivariant symplectic isomorphism $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}\overset{\sim}{\to}\mathrm{T}^{\Box}A_{\bar{s}}$ (such that $\nu(\hat{\alpha}):\mathbb{A}^{\infty,\Box}(1)\overset{\sim}{\to}\mathrm{V}^{\Box}\mathbf{G}_{\mathrm{m},\bar{s}}$ induces an isomorphism $\hat{\mathbb{Z}}^{\Box}(1)\overset{\sim}{\to}\mathrm{T}^{\Box}\mathbf{G}_{\mathrm{m},\bar{s}}$).*

Proof. Let us take any integer $n\geq 1$ such that $\Box\nmid n$ and $\mathcal{U}^{\Box}(n)\subset\mathcal{H}$. Under the assumption, we recover a $\pi_1(S,\bar{s})$-equivariant H_n-orbit of some $\mathcal{O}$-invariant isomorphisms $\alpha_{n,\bar{s}}:L/nL\overset{\sim}{\to}A[n]_{\bar{s}}$. Let $\tilde{S}\twoheadrightarrow S$ be the finite étale covering corresponding to the open compact subgroup of $\pi_1(S,\bar{s})$ that leaves $\alpha_{n,\bar{s}}$ invariant. By Proposition 1.3.5.3, $\alpha_{n,\bar{s}}$ is the specialization of an $\mathcal{O}$-equivariant isomorphism $\alpha_n:(L/nL)_{\tilde{S}}\overset{\sim}{\to}A[n]_{\tilde{S}}$ of group schemes over $\tilde{S}$. Moreover, by Lemma 1.3.6.6 and its proof (cf. Corollary 1.3.6.7), the symplectic-liftability of α_n at a single point $\bar{s}$ (in the sense of Lemma 1.3.6.5) implies that α_n is a level-n structure over $\tilde{S}$. The $\pi_1(S,\bar{s})$-invariance of the H_n-orbit of $\alpha_{n,\bar{s}}$ implies that the disjoint union of the elements in the H_n-orbit of α_n is the pullback of a subscheme α_{H_n} of $\underline{\mathrm{Isom}}_S((L/nL)_S,A[n])\times\underline{\mathrm{Isom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S,\boldsymbol{\mu}_{n,S})$ over S. As a result, α_{H_n} is an H_n-orbit of étale-locally-defined (integral) level-n structures (see Definition 1.3.7.3), which defines an integral level-$\mathcal{H}$ structure $\alpha_{\mathcal{H}}$ as in Definition 1.3.7.6, as desired. $\Box$

Now we are ready to show that the choice of the base point $\bar{s}$ is immaterial in practice, because we have the following:

Lemma 1.3.8.6. *Let S be a locally noetherian scheme over $\mathrm{Spec}(\mathbb{Z}_{(\Box)})$, and let $\bar{s}$ and $\bar{s}'$ be two geometric points of S lying on the same connected component. Let (A,λ,i) be a triple as in Definition 1.3.8.1. Let $\mathcal{H}$ be any open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. Then the rational level-$\mathcal{H}$ structures of (A,λ,i) of type $(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box},\langle\cdot,\cdot\rangle)$ based at $\bar{s}$ are canonically in bijection with the rational level-$\mathcal{H}$ structures based at $\bar{s}'$. Under this bijection, those rational level-$\mathcal{H}$ structures $[\hat{\alpha}]_{\mathcal{H}}$ based at $\bar{s}$ that are represented by $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}$-equivariant symplectic isomorphisms $\hat{\alpha}:L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}\overset{\sim}{\to}\mathrm{V}^{\Box}A_{\bar{s}}$ sending $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box}A_{\bar{s}}$ correspond to rational level-$\mathcal{H}$ structures $[\hat{\alpha}']_{\mathcal{H}}$ based at $\bar{s}$ that are represented by $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}$-equivariant symplectic isomorphisms $\hat{\alpha}':L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}\overset{\sim}{\to}\mathrm{V}^{\Box}A_{\bar{s}'}$ sending $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box}A_{\bar{s}'}$.*

Proof. Let us first describe how the bijection is constructed. Start with a rational level-$\mathcal{H}$ structure $[\hat{\alpha}]_{\mathcal{H}}$ based at $\bar{s}$, and let $\hat{\alpha}:L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}\overset{\sim}{\to}\mathrm{V}^{\Box}A_{\bar{s}}$ be a representative of $[\hat{\alpha}]_{\mathcal{H}}$. By Corollary 1.3.5.4, there is a $\mathbb{Z}_{(\Box)}^{\times}$-isogeny $f:(A,\lambda,i)\to(A_1,\lambda_1,i_1)$ such that the composition $\hat{\alpha}_1:=\mathrm{V}^{\Box}(f)\circ\hat{\alpha}$ sends $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box}A_{1,\bar{s}}$. Then the $\mathcal{H}$-orbit of $\hat{\alpha}_1$ gives a well-defined rational level-$\mathcal{H}$ structure $[\hat{\alpha}_1]_{\mathcal{H}}$ of (A_1,λ_1,i_1) based at $\bar{s}$. By Lemma 1.3.8.5, $[\hat{\alpha}_1]_{\mathcal{H}}$ comes from a (necessarily unique) integral level-$\mathcal{H}$ structure $\alpha_{\mathcal{H}}$ of (A_1,λ_1,i_1) under Construction 1.3.8.4. Applying Construction 1.3.8.4 to $\alpha_{\mathcal{H}}$ with a different base point $\bar{s}'$, we obtain a rational level-$\mathcal{H}$ structure $[\hat{\alpha}_1']_{\mathcal{H}}$ of (A_1,λ_1,i_1) based at $\bar{s}'$. Let $\hat{\alpha}_1':L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}\overset{\sim}{\to}\mathrm{V}^{\Box}A_{1,\bar{s}'}$ be any representative of $[\hat{\alpha}_1']_{\mathcal{H}}$. Then the $\mathcal{H}$-orbit of $\hat{\alpha}':=\mathrm{V}^{\Box}(f)^{-1}\circ\hat{\alpha}_1':L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}\overset{\sim}{\to}\mathrm{V}^{\Box}A_{\bar{s}'}$ gives a well-defined rational level-$\mathcal{H}$ structure $[\hat{\alpha}']_{\mathcal{H}}$ of (A,λ,i) based at $\bar{s}'$. This procedure gives a bijection because it is reversible (by switching the roles of $\bar{s}$ and $\bar{s}'$). It is clear from the construction that it satisfies the remaining properties described in the lemma. $\Box$

Definition 1.3.8.7. *Let $\mathcal{H}$, (A,λ,i), and S be as in Definition 1.3.8.1. A* ***rational level-$\mathcal{H}$ structure of*** *(A,λ,i)* ***of type*** *$(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box},\langle\cdot,\cdot\rangle)$ is an assignment to each geometric $\bar{s}$ on S a rational level-$\mathcal{H}$ structure of (A,λ,i) of type $(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box},\langle\cdot,\cdot\rangle)$ based at $\bar{s}$, such that the assignments to two geometric points of S lying on the same connected component correspond to each other under the canonical bijection in Lemma 1.3.8.6.*

Convention 1.3.8.8. *By abuse of notation, we shall still denote the rational level-$\mathcal{H}$ structures of (A,λ,i) of type $(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box},\langle\cdot,\cdot\rangle)$ by the same notation $[\hat{\alpha}]_{\mathcal{H}}$ that we use for the structures based at a particular geometric point $\bar{s}$ of S. This is reasonable because we have to take a particular choice of a geometric point $\bar{s}$ of S only when we take a representative $\hat{\alpha}$ of $[\hat{\alpha}]_{\mathcal{H}}$.*

Remark 1.3.8.9. By Lemma 1.3.8.6, Construction 1.3.8.4 determines a well-defined rational level-$\mathcal{H}$ structure $[\hat{\alpha}]_{\mathcal{H}}$ whose choice is independent of the geometric point $\bar{s}$ at which it is based.

1.4 Definitions of Moduli Problems

Assume as in Section 1.2.1 that B is a finite-dimensional semisimple algebra over $\mathbb{Q}$ with a positive involution $\star$, and $\mathcal{O}$ is a $\mathbb{Z}$-order invariant under $\star$. Let Disc be the discriminant of $\mathcal{O}$ over $\mathbb{Z}$ (see Definition 1.1.1.6; see also Proposition 1.1.1.16). Closely related to Disc is the invariant $\mathrm{I}_{\mathrm{bad}}$ for $\mathcal{O}$ defined in Definition 1.2.1.18, which is either 1 or 2.

1.4.1 Definition by Isomorphism Classes

Let us fix a choice of a PEL-type $\mathcal{O}$-lattice $(L, \langle \cdot , \cdot \rangle, h)$ (see Definition 1.2.1.3) and an integer $n \geq 1$. Let $L^{\#}$ be the dual lattice of L with respect to $\langle \cdot , \cdot \rangle$ (see Definition 1.1.4.11).

Definition 1.4.1.1. *We say that a prime number p is **bad** if $p | n \, \mathrm{I}_{\mathrm{bad}} \, \mathrm{Disc}[L^{\#} : L]$. We say a prime number p is **good** if it is not bad. We say that $\Box$ is a set of good primes if it does not contain any bad primes.*

Let us fix the choice of a set $\Box$ of good primes. Let $\mathsf{S}_0 := \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ and let $(\mathrm{Sch}/\mathsf{S}_0)$ be the category of schemes over S_0. We shall define our moduli problem in the language of categories fibered in groupoids (see Appendix A).

Definition 1.4.1.2. *The moduli problem M_n is defined as the category fibered in groupoids over $(\mathrm{Sch}/\mathsf{S}_0)$ whose fiber over each scheme S is the groupoid $\mathsf{M}_n(S)$ described as follows: The objects of $\mathsf{M}_n(S)$ are tuples $(A, \lambda, i, \alpha_n)$, where*

1. *A is an abelian scheme over S;*
2. *$\lambda : A \to A^\vee$ is a prime-to-$\Box$ polarization of A;*
3. *$i : \mathcal{O} \to \mathrm{End}_S(A)$ defines an $\mathcal{O}$-structure of (A, λ);*
4. *$\underline{\mathrm{Lie}}_{A/S}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-module structure given naturally by i satisfies the determinantal condition in Definition 1.3.4.1 given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle, h)$;*
5. *$\alpha_n : (L/nL)_S \xrightarrow{\sim} A[n]$ is an (integral) principal level-n structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle \cdot , \cdot \rangle)$ as in Definition 1.3.6.2.*

The isomorphisms
$$(A, \lambda, i, \alpha_n) \sim_{\mathrm{isom.}} (A', \lambda', i', \alpha'_n)$$
of $\mathsf{M}_n(S)$ are given by isomorphisms $f : A \xrightarrow{\sim} A'$ (of abelian schemes over S) such that

1. *$\lambda = f^\vee \circ \lambda' \circ f$;*
2. *$f \circ i(b) = i'(b) \circ f$ for all $b \in \mathcal{O}$;*
3. *$f|_{A[n]} : A[n] \xrightarrow{\sim} A'[n]$ satisfies $\alpha'_n = (f|_{A[n]}) \circ \alpha_n$.*

Definition 1.4.1.3. *If we have two tuples $(A, \lambda, i, \alpha_n) \sim_{\mathrm{isom.}} (A', \lambda', i', \alpha'_n)$ as in Definition 1.4.1.2 under an isomorphism $f : A \xrightarrow{\sim} A'$, then we say in this case that we have an isomorphism $f : (A, \lambda, i, \alpha_n) \xrightarrow{\sim} (A', \lambda', i', \alpha'_n)$.*

The definition for general level structures is as follows:

Definition 1.4.1.4. *Let $\mathcal{H}$ be an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. The moduli problem $\mathsf{M}_{\mathcal{H}}$ is defined as the category fibered in groupoids over $(\mathrm{Sch}/\mathsf{S}_0)$ whose fiber over each scheme S is the groupoid $\mathsf{M}_{\mathcal{H}}(S)$ described as follows: The objects of $\mathsf{M}_{\mathcal{H}}(S)$ are tuples $(A, \lambda, i, \alpha_{\mathcal{H}})$, where*

1. *A is an abelian scheme over S;*
2. *$\lambda : A \to A^\vee$ is a prime-to-$\Box$ polarization of A;*
3. *$i : \mathcal{O} \to \mathrm{End}_S(A)$ defines an $\mathcal{O}$-structure of (A, λ);*
4. *$\underline{\mathrm{Lie}}_{A/S}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-module structure given naturally by i satisfies the determinantal condition in Definition 1.3.4.1 given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot , \cdot \rangle, h)$;*
5. *$\alpha_{\mathcal{H}}$ is an (integral) level-$\mathcal{H}$ structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle \cdot , \cdot \rangle)$ as in Definition 1.3.7.6.*

The isomorphisms
$$(A, \lambda, i, \alpha_{\mathcal{H}}) \sim_{\mathrm{isom.}} (A', \lambda', i', \alpha'_{\mathcal{H}})$$
of $\mathsf{M}_{\mathcal{H}}(S)$ are given by isomorphisms $f : A \xrightarrow{\sim} A'$ (of abelian schemes over S) such that

1. *$\lambda = f^\vee \circ \lambda' \circ f$;*
2. *$f \circ i(b) = i'(b) \circ f$ for all $b \in \mathcal{O}$;*
3. *we have the symbolical relation $f \circ \alpha_{\mathcal{H}} = \alpha'_{\mathcal{H}}$ defined in the following sense: For each integer $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$, let*
$$\alpha_{H_n} \subset \underline{\mathrm{Hom}}_S((L/nL)_S, A[n]) \underset{S}{\times} \underline{\mathrm{Hom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S})$$
and
$$\alpha'_{H_n} \subset \underline{\mathrm{Hom}}_S((L/nL)_S, A'[n]) \underset{S}{\times} \underline{\mathrm{Hom}}_S(((\mathbb{Z}/n\mathbb{Z})(1))_S, \boldsymbol{\mu}_{n,S})$$
be the subschemes defining $\alpha_{\mathcal{H}}$ and $\alpha'_{\mathcal{H}}$, respectively, as in Definition 1.3.7.6. Then α_{H_n} is the pullback of α'_{H_n} under the morphism $f|_{A[n]} \times \mathrm{Id}$. (It suffices to verify this condition for one n.)

Definition 1.4.1.5. *If we have two tuples $(A, \lambda, i, \alpha_{\mathcal{H}}) \sim_{\mathrm{isom.}} (A', \lambda', i', \alpha'_{\mathcal{H}})$ as in Definition 1.4.1.4 under an isomorphism $f : A \xrightarrow{\sim} A'$, then we say in this case that we have an isomorphism $f : (A, \lambda, i, \alpha_{\mathcal{H}}) \xrightarrow{\sim} (A', \lambda', i', \alpha'_{\mathcal{H}})$.*

Remark 1.4.1.6. We have a canonical identification $\mathsf{M}_n \cong \mathsf{M}_{\mathcal{U}^{\Box}(n)}$ because (integral) level-$\mathcal{H}$ structures are just (integral) principal level-n structures when $\mathcal{H} = \mathcal{U}^{\Box}(n)$. When $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$, the $H_n = \mathcal{H}/\mathcal{U}^{\Box}(n)$-orbit α_{H_n} of α_n defines a canonical finite étale morphism $\mathsf{M}_n \to \mathsf{M}_{\mathcal{H}}$, making $\mathsf{M}_{\mathcal{H}}$ a quotient of M_n by H_n.

Remark 1.4.1.7. Over $\mathbb{C}$, each abelian variety has an associated lattice that determines itself up to homothety. However, we cannot talk about lattices over arbitrary bases. The close approximations we have are the (co)homologies such as the de Rham and ℓ-adic homologies, with rigidifications given by the Lie algebra conditions and level structures. Therefore, we should view both of them as "*level structures*", although the terminology has been reserved for the ℓ-adic versions only. Practically, we often discuss the Lie algebra conditions when we discuss the endomorphism structures. But one should keep in mind that this is because of its formulation, not because of its meaning.

Following Pink [102, 0.6], we define the neatness of open compact subgroups $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ as follows: Let us view $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ as a *subgroup* of $\mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}}(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}) \times \mathbf{G}_{\mathrm{m}}(\hat{\mathbb{Z}}^{\Box})$ (as in Definition 1.2.1.6). (Or we may use any faithful linear algebraic representation of G.) Then, for each rational prime $p > 0$ not in $\Box$, it makes sense to talk about *eigenvalues* of elements g_p in $\mathrm{G}(\mathbb{Z}_p)$, which are elements in $\bar{\mathbb{Q}}_p^{\times}$. Let $g = (g_p) \in \mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, with p running through rational primes such that $\Box \nmid p$. For each such p, let Γ_{g_p} be the subgroup of $\bar{\mathbb{Q}}_p^{\times}$ generated by eigenvalues of g_p. For any embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, consider the subgroup $(\bar{\mathbb{Q}}^{\times} \cap \Gamma_{g_p})_{\mathrm{tors}}$ of torsion elements of $\bar{\mathbb{Q}}^{\times} \cap \Gamma_{g_p}$, which is independent of the choice of the embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$.

Definition 1.4.1.8. *We say that $g = (g_p)$ is **neat** if $\underset{p \notin \Box}{\cap} (\bar{\mathbb{Q}}^{\times} \cap \Gamma_{g_p})_{\mathrm{tors}} = \{1\}$. We say that an open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ is **neat** if all its elements are neat.*

Remark 1.4.1.9. The usual version of Serre's lemma, which asserts that no nontrivial root of unity can be congruent to 1 modulo n if $n \geq 3$, shows that $\mathcal{H}$ is neat if $\mathcal{H} \subset \mathcal{U}^{\Box}(n)$ for some $n \geq 3$ such that $\Box \nmid n$.

Lemma 1.4.1.10. *The moduli problem $\mathsf{M}_{\mathcal{H}}$ is locally of finite presentation as a category fibered in groupoids (see Definition A.5.9).*

Proof. This is because objects of $\mathsf{M}_{\mathcal{H}}$ are defined by schemes and morphisms of finite presentation (see Remark 1.3.1.2 and Theorem 1.3.1.3). $\Box$

Theorem 1.4.1.11. *Let $\mathcal{H}$ be an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. The moduli problem $\mathsf{M}_{\mathcal{H}}$ is an algebraic stack separated, smooth, and of finite type over S_0. It is representable by an algebraic space if the objects it parameterizes have no nontrivial automorphism, which is in particular, the case when $\mathcal{H}$ is neat (see Definition 1.4.1.8).*

As a special case,

Corollary 1.4.1.12. *The moduli problem M_n is an algebraic stack separated, smooth, and of finite type over S_0. It is representable by an algebraic space if $n \geq 3$ (see Remark 1.4.1.9).*

The proof of Theorem 1.4.1.11 will be carried out in Chapter 2. We shall denote the algebraic stack or algebraic space representing $\mathsf{M}_{\mathcal{H}}$ (resp. M_n) by the same notation, $\mathsf{M}_{\mathcal{H}}$ (resp. M_n).

Remark 1.4.1.13. We shall see in Corollary 7.2.3.10, which is a by-product of an intermediate construction in the proof of Theorem 7.2.4.1, that $\mathsf{M}_{\mathcal{H}}$ is quasi-projective over S_0 when $\mathcal{H}$ is neat. Therefore it is not necessary to argue that it is a scheme at this moment.

1.4.2 Definition by $\mathbb{Z}_{(\Box)}^{\times}$-Isogeny Classes

Let $V := L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. Then we may write $(V\underset{\mathbb{Q}}{\otimes}\mathbb{R}, \langle\cdot,\cdot\rangle, h)$ (resp. $(V\underset{\mathbb{Q}}{\otimes}\mathbb{A}^{\infty,\Box}, \langle\cdot,\cdot\rangle)$) in place of $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \langle\cdot,\cdot\rangle, h)$ (resp. $(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}, \langle\cdot,\cdot\rangle)$).

Let $(\mathrm{LNSch}/\mathsf{S}_0)$ be the full subcategory of $(\mathrm{Sch}/\mathsf{S}_0)$ whose objects are *locally noetherian schemes* over S_0.

Definition 1.4.2.1. *Let $\mathcal{H}$ be an open compact subgroup of $\mathrm{G}(\mathbb{A}^{\infty,\Box})$. The moduli problem $\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}$ is defined as the category fibered in groupoids over $(\mathrm{LNSch}/\mathsf{S}_0)$ whose fiber over each scheme S is the groupoid $\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}(S)$ described as follows: The objects of $\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}(S)$ are tuples $(A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}})$, where*

1. *A is an abelian scheme over S;*
2. *$\lambda : A \to A^{\vee}$ is a $\mathbb{Z}_{(\Box)}^{\times}$-polarization of A;*
3. *$i : \mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)} \to \mathrm{End}_S(A)\underset{\mathbb{Z}}{\otimes}(\mathbb{Z}_{(\Box)})_S$ defines an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$-structure of (A, λ);*
4. *$\underline{\mathrm{Lie}}_{A/S}$ with its $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$-module structure given by i satisfies the determinantal condition as in Definition 1.3.4.1 given by $(V\underset{\mathbb{Q}}{\otimes}\mathbb{R}, \langle\cdot,\cdot\rangle, h)$;*
5. *$[\hat{\alpha}]_{\mathcal{H}}$ is a rational principal level-$\mathcal{H}$ structure of (A, λ, i) of type $(V\underset{\mathbb{Q}}{\otimes}\mathbb{A}^{\infty,\Box}, \langle\cdot,\cdot\rangle)$ as in Definition 1.3.8.7.*

The isomorphisms

$$(A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}}) \sim_{\mathbb{Z}_{(\Box)}^{\times}\text{-isog.}} (A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$$

of $\mathsf{M}_{\mathcal{H}}^{\mathrm{rat}}(S)$ are given by $\mathbb{Z}_{(\Box)}^{\times}$-isogenies $f : A \to A'$ such that

1. *over each connected component S, we have $\lambda = r f^{\vee} \circ \lambda' \circ f$ for some $r \in \mathbb{Z}_{(\Box),>0}^{\times}$;*
2. *$f \circ i(b) = i'(b) \circ f$ for all $b \in \mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$;*
3. *for each geometric point $\bar{s}$ of S, the morphism $\mathrm{V}^{\Box}(f) : \mathrm{V}^{\Box} A_{\bar{s}} \xrightarrow{\sim} \mathrm{V}^{\Box} A'_{\bar{s}}$ induced by f satisfies the condition that, for all representatives $\hat{\alpha}$ and $\hat{\alpha}'$ representing $[\hat{\alpha}]_{\mathcal{H}}$ and $[\hat{\alpha}']_{\mathcal{H}}$ at $\bar{s}$, respectively (see Convention 1.3.8.8), $(\hat{\alpha}')^{-1} \circ \mathrm{V}^{\Box}(f) \circ \hat{\alpha}$ lies in the $\mathcal{H}$-orbit of the identity on $V\underset{\mathbb{Q}}{\otimes}\mathbb{A}^{\infty,\Box}$, and $\nu(\hat{\alpha}')^{-1} \circ \nu(\hat{\alpha})$ lies in the $\nu(\mathcal{H})$-orbit of the $r \in \mathbb{Z}_{(\Box),>0}^{\times}$ such that $\lambda = r f^{\vee} \circ \lambda' \circ f$ at $\bar{s}$.*

Definition 1.4.2.2. *If we have two tuples $(A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}}) \sim_{\mathbb{Z}_{(\Box)}^{\times}\text{-isog.}} (A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$ as in Definition 1.4.2.1 under a $\mathbb{Z}_{(\Box)}^{\times}$-isogeny $f : A \xrightarrow{\sim} A'$, then we say in this case that we have a $\mathbb{Z}_{(\Box)}^{\times}$-isogeny $f : (A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}}) \xrightarrow{\sim} (A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$.*

Remark 1.4.2.3. Suppose $L \neq 0$. Let $\bar{s}$ be any geometric point of S. Then the $r \in \mathbb{Z}_{(\Box),>0}^{\times}$ above (in the definition of a $\mathbb{Z}_{(\Box)}^{\times}$-isogeny $f : (A, \lambda, i, [\hat{\alpha}]_{\mathcal{H}}) \xrightarrow{\sim} (A', \lambda', i', [\hat{\alpha}']_{\mathcal{H}})$) such that $\lambda = r f^{\vee} \circ \lambda' \circ f$ at $\bar{s}$ implies that

$$\mathrm{e}^{\lambda'}(\mathrm{V}^{\Box}(f)(x), \mathrm{V}^{\Box}(f)(y)) = \mathrm{e}^{\lambda}(x, r^{-1} y)$$

for all $x, y \in \mathrm{V}^\square A_{\bar{s}}$. In this case, we may interpret r^{-1} as some *similitude factor* for $\mathrm{V}^\square(f)$. Then the condition that $(\hat{\alpha}')^{-1}\circ \mathrm{V}^\square(f)\circ\hat{\alpha}$ lies in the $\mathcal{H}$-orbit of the identity on $V \underset{\mathbb{Q}}{\otimes} \mathbb{A}^{\infty,\square}$ forces $\nu(\hat{\alpha}')^{-1}\circ r^{-1}\circ\nu(\hat{\alpha})$ to lie in the $\nu(\mathcal{H})$-orbit of the identity on $\mathbb{A}^{\infty,\square}$, and hence the equivalent condition that $\nu(\hat\alpha')^{-1}\circ\nu(\hat\alpha)$ lies in the $\nu(\mathcal{H})$-orbit of the $r \in \mathbb{Z}^\times_{(\square),>0}$.

Remark 1.4.2.4. Definition 1.4.2.1 uses only the existence of some $\mathbb{Z}$-order $\mathcal{O}$ in B (with positive involution) and some $\mathcal{O}$-lattice $(L, \langle\cdot\,,\cdot\rangle)$ inside $(V\underset{\mathbb{Q}}{\otimes}\mathbb{R}, \langle\cdot\,,\cdot\rangle)$ (with some polarization h) and $(V\underset{\mathbb{Q}}{\otimes}\mathbb{A}^{\infty,\square}, \langle\cdot\,,\cdot\rangle)$, or rather $(V\underset{\mathbb{Q}}{\otimes}\mathbb{A}^{\square}, \langle\cdot\,,\cdot\rangle)$, such that $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\square)}$ is maximal as a $\mathbb{Z}_{(\square)}$-order, and such that $(L\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\square)}, \langle\cdot\,,\cdot\rangle)$ is self-dual. We could have started (as in [76]) with a polarized symplectic vector space $(V, \langle\cdot\,,\cdot\rangle, h)$ (over $B \cong \mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$), together with the existence of a maximal $\mathbb{Z}_{(\square)}$-order and the existence of some self-dual $\mathbb{Z}_{(\square)}$-lattice over this order. (The definitions of the subgroups $\mathrm{G}(\hat{\mathbb{Z}}^\square)$ and $\mathcal{U}^\square(n)$ depend nevertheless on the choice of $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$.)

1.4.3 Comparison between Two Definitions

Let $\mathcal{H}$ be a fixed choice of an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^\square)$. Let us denote by $\mathsf{M}^{\mathrm{LN}}_{\mathcal{H}}$ the pullback of the category $\mathsf{M}_{\mathcal{H}}$ fibered in groupoids over $(\mathrm{Sch}/\mathsf{S}_0)$ to $(\mathrm{LNSch}/\mathsf{S}_0)$ (see Definition A.5.5). A consequence of Theorem 1.4.1.11 is:

Corollary 1.4.3.1. *Suppose* $S \in \mathrm{Ob}(\mathrm{Sch}/\mathsf{S}_0)$ *and* $S \cong \varprojlim_{i\in I} S_i$, *where* $S_i \in \mathrm{Ob}(\mathrm{LNSch}/\mathsf{S}_0)$ *are* ***affine schemes*** *for all* $i \in I$. *Then*
$$\mathsf{M}_{\mathcal{H}}(S) \cong \varinjlim_{i\in I} \mathsf{M}_{\mathcal{H}}(S_i).$$
In particular, $\mathsf{M}_{\mathcal{H}}$ *is uniquely determined by* $\mathsf{M}^{\mathrm{LN}}_{\mathcal{H}}$.

Proof. First note that $\mathsf{M}_{\mathcal{H}}$ is uniquely determined (up to isomorphism) by its fibers over *affine schemes*. Since algebraic stacks of finite type over S_0 are, in particular, locally of finite presentation, their fibers over affine schemes are uniquely determined (up to isomorphism) by their pullbacks to $(\mathrm{LNSch}/\mathsf{S}_0)$, by the explicit formula in the statement of the corollary. ☐

Construction 1.4.3.2. We can define a canonical morphism
$$\mathsf{M}^{\mathrm{LN}}_{\mathcal{H}} \to \mathsf{M}^{\mathrm{rat}}_{\mathcal{H}} \tag{1.4.3.3}$$
(over $(\mathrm{LNSch}/\mathsf{S}_0)$) as follows: Over each scheme S in $(\mathrm{LNSch}/\mathsf{S}_0)$, we associate with each object $(A, \lambda, i, \alpha_n)$ in $\mathsf{M}^{\mathrm{LN}}_{\mathcal{H}}(S) \cong \mathsf{M}_{\mathcal{H}}(S)$ the object $(A, \lambda, i, [\hat\alpha]_{\mathcal{H}})$ in $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}(S)$, where $[\hat\alpha]_{\mathcal{H}}$ is associated with $\alpha_{\mathcal{H}}$ as in Construction 1.3.8.4 and Remark 1.3.8.9.

Proposition 1.4.3.4. *The morphism* (*1.4.3.3*) *is an isomorphism* (*i.e., equivalence of categories*).

Remark 1.4.3.5. This is in the sense of 1-isomorphisms between 2-categories. In particular, this only requires the morphism (1.4.3.3) to induce *equivalences of categories* $\mathsf{M}^{\mathrm{LN}}_{\mathcal{H}}(S) \cong \mathsf{M}_{\mathcal{H}}(S) \to \mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}(S)$ for each locally noetherian scheme S (see Remark A.1.2.7 and Definition A.5.3).

Proof of Proposition 1.4.3.4. Without loss of generality, we may assume that S is locally noetherian and connected, and fix a particular choice $\bar{s}$ of a geometric point on S. All rational level-$\mathcal{H}$ structures we consider will be based at $\bar{s}$, without further explanation (see Remark 1.3.8.9).

Suppose $(A, \lambda, i, [\hat\alpha]_{\mathcal{H}})$ (resp. $(A', \lambda', i', [\hat\alpha']_{\mathcal{H}})$) is associated with $(A, \lambda, i, \alpha_{\mathcal{H}})$ (resp. $(A', \lambda', i', \alpha'_{\mathcal{H}})$) as in Construction 1.4.3.2. Let us take any choice of $\hat\alpha$ (resp. $\hat\alpha'$) that represents $[\hat\alpha]_{\mathcal{H}}$ (resp. $[\hat\alpha']_{\mathcal{H}}$).

Suppose $(A, \lambda, i, [\hat\alpha]_{\mathcal{H}}) \sim_{\mathbb{Z}^\times_{(\square)}\text{-isog.}} (A', \lambda', i', [\hat\alpha']_{\mathcal{H}})$. By definition, this means there is a $\mathbb{Z}^\times_{(\square)}$-isogeny $f : (A, \lambda, i) \to (A', r\lambda', i')$ for some $r \in \mathbb{Z}^\times_{(\square),>0}$, such that $(\hat\alpha')^{-1}\circ\mathrm{V}^\square(f)\circ\hat\alpha$ lies in the $\mathcal{H}$-orbit of the identity on $L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\square}$, and such that $\nu(\hat\alpha')^{-1}\circ\nu(\hat\alpha)$ lies in the $\nu(\mathcal{H})$-orbit of r.

By Construction 1.4.3.2, we have $\mathrm{T}^\square A_{\bar s} = \hat\alpha(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square)$, and $\mathrm{T}^\square A'_{\bar s} = \hat\alpha'(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square)$. Therefore we have $\mathrm{V}^\square(f)(\mathrm{T}^\square A_{\bar s}) = \mathrm{T}^\square A'_{\bar s}$, which by Corollary 1.3.5.4 implies that $f : A \to A'$ is an isomorphism of abelian schemes. Since $\mathrm{T}^\square \mathbf{G}_{\mathrm{m},\bar s} = \nu(\hat\alpha)(\hat{\mathbb{Z}}^\square(1)) = \nu(\hat\alpha')(\hat{\mathbb{Z}}^\square(1))$ by construction, $\nu(\hat\alpha')^{-1}\circ\nu(\hat\alpha)$ lies in $\hat{\mathbb{Z}}^{\square,\times}$, which by assumption has to contain the $\nu(\mathcal{H})$-orbit of r. Since $\nu(\mathcal{H}) \subset \hat{\mathbb{Z}}^{\square,\times}$ (because $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^\square)$), the approximation $\mathbb{A}^{\infty,\square,\times} = \mathbb{Z}^\times_{(\square),>0}\cdot\hat{\mathbb{Z}}^{\square,\times}$ forces $r = 1$. Furthermore, it is clear that we have $f\circ\alpha_{\mathcal{H}} = \alpha'_{\mathcal{H}}$ in the sense of Definition 1.4.1.4. Therefore $(A, \lambda, i, \alpha_{\mathcal{H}}) \sim_{\text{isom.}} (A', \lambda', i', \alpha'_{\mathcal{H}})$, and we can conclude the injectivity of (1.4.3.3).

On the other hand, suppose $(A, \lambda, i, [\hat\alpha]_{\mathcal{H}})$ is any object in $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}(S)$. We must show that there exists an object $(A', \lambda', i', [\hat\alpha']_{\mathcal{H}})$ in $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}(S)$, satisfying $(A, \lambda, i, [\hat\alpha]_{\mathcal{H}}) \sim_{\mathbb{Z}^\times_{(\square)}\text{-isog.}} (A', \lambda', i', [\hat\alpha']_{\mathcal{H}})$, such that $(A', \lambda', i', [\hat\alpha']_{\mathcal{H}})$ comes from an object in $\mathsf{M}_{\mathcal{H}}(S)$, or equivalently has the following properties:

1. λ' is a polarization (instead of merely a $\mathbb{Z}^\times_{(\square)}$-polarization).

2. i' defines an $\mathcal{O}$-structure (mapping $\mathcal{O}$ to $\mathrm{End}_S(A')$ rather than $\mathrm{End}_S(A')\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\square)}$).

3. Let $\hat\alpha'$ be any representative of $[\hat\alpha']_{\mathcal{H}}$. Then $\hat\alpha'$ induces (by Lemma 1.3.8.5) an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$-equivariant symplectic isomorphism $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square \xrightarrow{\sim} \mathrm{T}^\square A'_{\bar s}$, and $\nu(\hat\alpha')$ induces an isomorphism $\hat{\mathbb{Z}}^\square(1) \xrightarrow{\sim} \mathrm{T}^\square \mathbf{G}_{\mathrm{m},\bar s}$ (of $\hat{\mathbb{Z}}^\square$-modules) making the diagram
$$\begin{tikzcd} (L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square)\times(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square) \arrow[r, "\langle\cdot\,,\cdot\rangle"] \arrow[d, "\hat\alpha'\times\hat\alpha'"', "\wr"] & \hat{\mathbb{Z}}^\square(1) \arrow[d, dashed, "\nu(\hat\alpha')", "\wr"'] \\ \mathrm{T}^\square A'_{\bar s}\times\mathrm{T}^\square A'_{\bar s} \arrow[r, "e^{\lambda'}"'] & \mathrm{T}^\square \mathbf{G}_{\mathrm{m},\bar s} \end{tikzcd} \tag{1.4.3.6}$$
commutative. (This is also a condition for λ': If we replace λ' with a positive multiple different from itself, then this will not hold.)

Let $\hat\alpha$ be a representative of $[\hat\alpha]_{\mathcal{H}}$. By Corollary 1.3.5.4, the $\mathcal{O}$-invariant open compact subgroup $\hat\alpha(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square)$ of $\mathrm{V}^\square A_{\bar s}$ corresponds to a $\mathbb{Z}^\times_{(\square)}$-isogeny $f : (A, \lambda, i) \to (A', \lambda'', i')$ such that $\mathrm{V}^\square(f)^{-1}(\mathrm{T}^\square A'_{\bar s}) = \hat\alpha(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square)$, such that i' has its image in

$\mathrm{End}_S(A')$, and such that the diagram

$$\begin{tikzcd}
(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})\times(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}) \arrow[r, "\langle\,\cdot\,,\,\cdot\,\rangle"] \arrow[d, "\hat{\alpha}'\times\hat{\alpha}'"'] & \mathbb{A}^{\infty,\Box}(1) \arrow[d, dotted, "\wr\ r\nu(\hat{\alpha})"] \\
\mathrm{V}^\Box A'_{\bar{s}}\times\mathrm{V}^\Box A'_{\bar{s}} \arrow[r, "\mathrm{e}^{r\lambda''}"'] & \mathrm{V}^\Box\,\mathbf{G}_{\mathrm{m},\bar{s}}
\end{tikzcd}$$

is commutative for every $r\in\mathbb{Z}^\times_{(\Box),>0}$.

By the approximation $\mathbb{A}^{\infty,\Box,\times}=\mathbb{Z}^\times_{(\Box),>0}\cdot\hat{\mathbb{Z}}^{\Box,\times}$, there exists a unique $r\in\mathbb{Z}^\times_{(\Box),>0}$ such that $(r\nu(\hat{\alpha}))(\hat{\mathbb{Z}}^\Box(1))=\mathrm{T}^\Box\,\mathbf{G}_{\mathrm{m},\bar{s}}$. Let $\lambda':=r\lambda''$, let $\hat{\alpha}':=\mathrm{V}^\Box(f)\circ\hat{\alpha}$, let $\nu(\hat{\alpha}'):=r\nu(\hat{\alpha})$, and let $[\hat{\alpha}']_{\mathcal{H}}$ be the $\mathcal{H}$-orbit of $\hat{\alpha}'$. Then $\hat{\alpha}'(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)=\mathrm{V}^\Box(f)(\hat{\alpha}(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box))=\mathrm{T}^\Box A'_{\bar{s}}$, and we have $(A,\lambda,i,[\hat{\alpha}]_{\mathcal{H}})\sim_{\mathbb{Z}^\times_{(\Box)}\text{-isog.}}(A',\lambda',i',[\hat{\alpha}']_{\mathcal{H}})$.

Moreover, since $\hat{\alpha}'(\hat{\mathbb{Z}}^\Box(1))=\mathrm{T}^\Box\,\mathbf{G}_{\mathrm{m},\bar{s}}$, the inclusion $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box\subset L^\#\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$ corresponds under $\hat{\alpha}'$ to the inclusion $\mathrm{V}^\Box(\lambda')(\mathrm{T}^\Box A'_{\bar{s}})\subset\mathrm{T}^\Box((A'_{\bar{s}})^\vee)$ in $\mathrm{V}^\Box A'_{\bar{s}}$, showing that λ' is a polarization (instead of merely a $\mathbb{Z}^\times_{(\Box)}$-polarization). This concludes the proof of the surjectivity of (1.4.3.3). $\Box$

As a trivial consequence,

Corollary 1.4.3.7 (of Proposition 1.4.3.4 and Theorem 1.4.1.11)**.** *The moduli problem $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$ is an algebraic stack separated, smooth, and of finite type over S_0 in the sense that $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$ is isomorphic to the pullback* (*as a category fibered in groupoids over* $(\mathrm{Sch}/\mathsf{S}_0)$) *to* $(\mathrm{LNSch}/\mathsf{S}_0)$ *of an algebraic stack separated, smooth, and of finite type over S_0. It is representable by an algebraic space* (*in a similar sense*) *if the objects it parameterizes have no nontrivial automorphism, which is in particular, the case when $\mathcal{H}$ is neat* (*see Definition 1.4.1.8*)*.*

Moreover, we obtain the following *exotic* isomorphism between moduli problems defined by reasonably different choices of PEL-type $\mathcal{O}$-lattices:

Corollary 1.4.3.8. *Let $\mathcal{O}$* (*resp. $\mathcal{O}'$*) *be a $\mathbb{Z}$-order invariant under the involution $\star$ of B, and let $(L,\langle\,\cdot\,,\,\cdot\,\rangle,h)$* (*resp. $(L',\langle\,\cdot\,,\,\cdot\,\rangle',h')$*) *be a PEL-type $\mathcal{O}$-lattice* (*resp. a PEL-type $\mathcal{O}'$-lattice*)*. Suppose $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}=\mathcal{O}'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$* (*both canonically embedded as subalgebras of B*)*, and $(L\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)},\langle\,\cdot\,,\,\cdot\,\rangle,h)\cong(L'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)},\langle\,\cdot\,,\,\cdot\,\rangle',h')$* (*as polarized symplectic modules over $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}=\mathcal{O}'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$*)*, such that $\Box$ is a set of good primes for both of them. Then the two moduli problems $\mathsf{M}_{\mathcal{H}}$ and $\mathsf{M}'_{\mathcal{H}}$ over $\mathsf{S}_0=\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ defined respectively by them are isomorphic to each other* (*over S_0*)*.*

Proof. This follows from Corollary 1.4.3.1, Proposition 1.4.3.4, and Remark 1.4.2.4. $\Box$

Remark 1.4.3.9. Let $\mathcal{O}'$ be any maximal order in $B\cong\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ containing $\mathcal{O}$. (We do not assume that $\mathcal{O}'$ is invariant under the involution $\star$ of B.) Let L' denote the $\mathcal{O}'$-span of L in $V=L\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. Then L' is an $\mathcal{O}$-lattice because it is still $\mathbb{Z}$-torsion-free and finitely generated over $\mathcal{O}$. By Proposition 1.1.1.21, we have $\mathcal{O}'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}=\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$ (as orders in B), and hence $(L')\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}=L\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$ (as lattices in V) remains self-dual. Therefore, by Corollary 1.4.3.8, the isomorphism class of the moduli problem $\mathsf{M}_{\mathcal{H}}$ remains unchanged if we retain $\mathcal{O}$ but replace L with L'. (Certainly, the meaning of $\mathcal{U}^\Box(n)$ for $n\geq 1$ has to be modified accordingly, because we have modified $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$. This is one reason that it is more natural to work with general level structures.)

Condition 1.4.3.10. *The PEL-type $\mathcal{O}$-lattice $(L,\langle\,\cdot\,,\,\cdot\,\rangle,h)$ is chosen such that the action of $\mathcal{O}$ on L extends to an action of some maximal order $\mathcal{O}'$ in B containing $\mathcal{O}$.*

As explained in Remark 1.4.3.9, this is harmless for our purpose of compactifications. We will need this technical condition only when we study the degeneration of objects in M_n (see Lemma 5.2.2.4 below).

Remark 1.4.3.11. If we form the tower $\mathsf{M}^\Box:=\varprojlim\limits_{\mathcal{H},\mathcal{H}\subset\mathrm{G}(\hat{\mathbb{Z}}^\Box)}\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$, then the objects of this tower can be represented by tuples of the form $(A,\lambda,i,[\hat{\alpha}])$ in the obvious sense, and there is a natural right action of elements $g\in\mathrm{G}(\mathbb{A}^{\infty,\Box})$ on $\mathsf{M}^\Box$ defined by sending a representative $(A,\lambda,i,\hat{\alpha})$ to $(A,\lambda,i,\hat{\alpha}\circ g)$. At finite levels, the action can be defined more precisely by sending $(A,\lambda,i,[\hat{\alpha}]_{\mathcal{H}'})$ at level $\mathcal{H}'$ to $(A,\lambda,i,[\hat{\alpha}\circ g]_{\mathcal{H}})$ at level $\mathcal{H}$, if $\mathcal{H}'\subset\mathcal{H}\cap(g\mathcal{H}g^{-1})$. (We use $\mathcal{H}\cap(g\mathcal{H}g^{-1})$ rather than $\mathcal{H}\cap(g^{-1}\mathcal{H}g)$, because we are using a right action.) We will elaborate more on this idea in Sections 5.4.3 and 6.4.3.

Remark 1.4.3.12. The characteristic zero fiber of $\mathsf{M}_{\mathcal{H}}$ might contain unnecessary components other than the canonical model of the Shimura variety we want: By first identifying $\mathsf{M}^{\mathrm{LN}}_{\mathcal{H}}$ with $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$ by Proposition 1.4.3.4, we see from Definition 1.4.2.1 that the definition involves only $(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^\Box,\langle\,\cdot\,,\,\cdot\,\rangle,h)$, but not $(L,\langle\,\cdot\,,\,\cdot\,\rangle,h)$. Therefore, two nonisomorphic PEL-type $\mathcal{O}$-lattices $(L_1,\langle\,\cdot\,,\,\cdot\,\rangle_1,h_1)$ and $(L_2,\langle\,\cdot\,,\,\cdot\,\rangle_2,h_2)$ define the same moduli problem $\mathsf{M}_{\mathcal{H}}$ if they become isomorphic after tensoring with $\mathbb{A}^\Box$. This is the issue of the so-called *failure of Hasse's principle*. When B is a simple algebra, and when $\Box$ has only one element, the characteristic zero fiber of $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$ can be identified with the one defined by Kottwitz in [76, §5]. Moreover, when B is not of type D, it is explained in [76, §8] that the canonical models of Shimura varieties appearing in the characteristic zero fiber of $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$ are all isomorphic to each other (even as canonical models). Therefore, the failure of Hasse's principle in the definition of our moduli problems is harmless in such cases.

Remark 1.4.3.13. Even if the failure of Hasse's principle does not occur, the algebraic stack (or algebraic space) $\mathsf{M}_{\mathcal{H}}$ (or $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$) is not geometrically connected in general.

Remark 1.4.3.14. Although Definition 1.4.2.1 uses only $(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^\Box,\langle\,\cdot\,,\,\cdot\,\rangle,h)$ but not $(L,\langle\,\cdot\,,\,\cdot\,\rangle,h)$, the existence of the lattice $(L,\langle\,\cdot\,,\,\cdot\,\rangle,h)$ is indispensable. Suppose we have defined the moduli problem $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$ using only some adelic object $(M_{\mathbb{A}^\Box},\langle\,\cdot\,,\,\cdot\,\rangle,h)$ (without assuring that the pairings come from some particular integral object $(L,\langle\,\cdot\,,\,\cdot\,\rangle,h)$, or some rational analogue). Then, by redefining everything over the categories $(\mathrm{Sch}/\mathsf{S}_0)$ or $(\mathrm{LNSch}/\mathsf{S}_0)$, the proofs of Theorem 1.4.1.11 and Proposition 1.4.3.4 still work, and they show that $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$ is smooth over S_0. However, it is not clear that it is nonempty! In fact, the existence of any geometric point will force the existence of some complex point by smoothness, and the H_1 (with pairing) of the corresponding polarized complex abelian variety will force the existence of some PEL-type $\mathcal{O}$-lattice $(L,\langle\,\cdot\,,\,\cdot\,\rangle,h)$ inducing $(M_{\mathbb{A}^\Box},\langle\,\cdot\,,\,\cdot\,\rangle,h)$. Conversely, if we have some PEL-type $\mathcal{O}$-lattice $(L,\langle\,\cdot\,,\,\cdot\,\rangle,h)$, then we can define a complex abelian variety (with additional PEL structures) by taking the real torus $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})/L$

with complex structure given by $h : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$. In particular, the moduli problem defined by $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ is nonempty. This justifies our use of PEL-type $\mathcal{O}$-lattices (integral or rational versions, rather than adelic versions) in the definition of moduli problems.

1.4.4 Definition by Different Sets of Primes

Let $\Box$ and $\Box'$ be any two sets of good primes (see Definition 1.4.1.1) such that $\Box \subset \Box'$. Let $\mathcal{U}_{\Box'-\Box} := \prod_{p \in \Box'-\Box} \mathrm{G}(\mathbb{Z}_p)$. Let $\mathcal{H}'$ be an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box'})$, and let $\mathcal{H}$ be the open compact subgroup $\mathcal{H}' \times \mathcal{U}_{\Box'-\Box}$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$. Let $\mathsf{M}_{\mathcal{H}}$ and $\mathsf{M}_{\mathcal{H}'}$ be defined over $\mathsf{S}_0 := \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ and $\mathsf{S}'_0 := \mathrm{Spec}(\mathcal{O}_{F_0,(\Box')})$, respectively, as in Definition 1.4.1.2. There is a canonical *forgetful functor*

$$\mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}'} \underset{\mathsf{S}'_0}{\times} \mathsf{S}_0 \tag{1.4.4.1}$$

defined by viewing a level-$\mathcal{H}$ structure as a level-$\mathcal{H}'$ structure. (This makes sense because level-$\mathcal{H}'$ structures require a weaker liftability condition than level-$\mathcal{H}$ structures. See Definition 1.3.7.6.)

Lemma 1.4.4.2. *The forgetful functor (1.4.4.1) is representable by an open and closed immersion over* S_0.

Proof. By Lemma 1.4.1.10, it suffices to show that, for each morphism $S' \to \mathsf{M}_{\mathcal{H}'} \underset{\mathsf{S}'_0}{\times} \mathsf{S}_0$ from a locally noetherian scheme S', the pullback $S \to S'$ of (1.4.4.1) is representable by an open and closed immersion. The definitions of the two moduli problems $\mathsf{M}_{\mathcal{H}}$ and $\mathsf{M}_{\mathcal{H}'}$ are the same except that the level structures (see Definition 1.3.7.6) of objects of $\mathsf{M}_{\mathcal{H}}$ require the symplectic-liftability conditions at those primes p in $\Box' - \Box$. Since level structures are defined by isomorphisms between finite étale group schemes, the pullback $S \to S'$ of (1.4.4.1) is representable by a finite étale morphism. By [59, IV-4, 17.9.1 and IV-2, 6.15.3], to show that $S \to S'$ is an open and closed immersion, it suffices to show that, for each geometric point $\bar{s}$ of S', there is at most one way to lift $\bar{s}$ to a geometric point of S. Let $(A_{\bar{s}}, \lambda_{\bar{s}}, i_{\bar{s}}, \alpha_{\mathcal{H}',\bar{s}})$ be the object of $\mathsf{M}_{\mathcal{H}'}(\bar{s})$ defined by composing $\bar{s} \to S'$ with $S' \to \mathsf{M}_{\mathcal{H}'} \underset{\mathsf{S}'_0}{\times} \mathsf{S}_0$. Since the degree of $\lambda_{\bar{s}}$ is prime-to-$\Box'$, at each $p \in \Box' - \Box$, the $\lambda_{\bar{s}}$-Weil pairing $\mathrm{e}^{\lambda_{\bar{s}}}$ defines a *perfect* alternating pairing on $\mathrm{T}_p A_{\bar{s}}$ valued in $\mathrm{T}_p \mathbf{G}_{\mathrm{m},\bar{s}}$. If there exists an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-equivariant symplectic isomorphism from $(L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p, \langle\,\cdot\,,\,\cdot\,\rangle, \mathbb{Z}_p(1))$ to $(\mathrm{T}_p \mathbf{G}_{\mathrm{m},\bar{s}}, \mathrm{e}^{\lambda_{\bar{s}}}, \mathrm{T}_p \mathbf{G}_{\mathrm{m},\bar{s}})$, then its $\mathrm{G}(\mathbb{Z}_p)$-orbit is necessarily unique, because $\mathrm{G}(\mathbb{Z}_p)$ is by definition the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-equivariant symplectic automorphism of $(L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p, \langle\,\cdot\,,\,\cdot\,\rangle, \mathbb{Z}_p(1))$ (see Definition 1.2.1.6). Hence there is at most one way to lift $\bar{s}$ to a geometric point of S, as desired. $\Box$

A more interesting question is to find sufficient conditions for (1.4.4.1) to be an isomorphism.

Proposition 1.4.4.3. *With assumptions as above, suppose there is a unique isomorphism class of self-dual $\mathbb{Z}_p(1)$-valued integrable $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-lattices of each $\mathcal{O}$-multirank for each $p \in \Box' - \Box$. Then the forgetful functor (1.4.4.1) is an isomorphism.*

Proof. Let us continue with the setting in the proof of Lemma 1.4.4.2. It suffices to show that the pullback $S \to S'$ of (1.4.4.1) is surjective. Since p is different from the residue characteristic of $\bar{s}$, there exists some (noncanonical) isomorphism $\mathbb{Z}_p(1) \xrightarrow{\sim} \mathrm{T}_p \mathbf{G}_{\mathrm{m},\bar{s}}$ of $\mathbb{Z}_p$-modules which allows us to consider $(\mathrm{T}_p \mathbf{G}_{\mathrm{m},\bar{s}}, \mathrm{e}^{\lambda}, \mathrm{T}_p \mathbf{G}_{\mathrm{m},\bar{s}})$ as $\mathbb{Z}_p(1)$-valued. By assumption, there exists an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-equivariant symplectic isomorphism from $(L \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p, \langle\,\cdot\,,\,\cdot\,\rangle, \mathbb{Z}_p(1))$ to $(\mathrm{T}_p \mathbf{G}_{\mathrm{m},\bar{s}}, \mathrm{e}^{\lambda}, \mathrm{T}_p \mathbf{G}_{\mathrm{m},\bar{s}})$, because they have the same $\mathcal{O}$-multirank. This shows that $S \to S'$ is surjective, as desired. $\Box$

Remark 1.4.4.4. According to [118, Lem. 3.4] and [76, Lem. 7.2], or according to Proposition 1.2.3.7 and Corollary 1.2.3.10, the condition that there is a unique symplectic isomorphism class of each $\mathcal{O}$-multirank over the good primes holds except when the semisimple algebra B involves some simple factor of type D (see Definition 1.2.1.15).

Remark 1.4.4.5. When the semisimple algebra B involves any simple factor of type D, the question is delicate already in the case that B is simple, that $\Box = \emptyset$, and that $\Box' = \{p\}$ for some good prime $p > 0$. See [76, §8].

Chapter 2

Representability of Moduli Problems

In this chapter, let us assume the same setting as in Section 1.4. Let us fix a choice of an open compact subgroup $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^\Box)$.

Our main objective is to prove Theorem 1.4.1.11, with Proposition 2.3.5.2 as a by-product. Technical results worth noting are Proposition 2.1.6.8 and Corollary 2.2.4.12. The proof of Theorem 1.4.1.11 is carried out by verifying Artin's criterion in Section 2.3.4 (see, in particular, Theorems B.3.7, B.3.9, and B.3.11). For readers who might have wondered, let us make it clear that we will not need Condition 1.4.3.10 in this chapter.

Let us outline the strategy of our proof before we begin. (Those readers who are willing to believe the representability statement as explained in [76, §5] should feel free to skip this chapter.)

There exist at least two different methods for showing that the moduli problem $\mathsf{M}_{\mathcal{H}}$ defined in Definition 1.4.1.4 is an algebraic stack.

The first one is given in [96, Ch. 7] using geometric invariant theory. The advantage of this method is that it is then clear that $\mathsf{M}_{\mathcal{H}}$ is a scheme when the objects it parameterizes have no nontrivial automorphism (because it always works in the category of schemes). Indeed, there is always a morphism from $\mathsf{M}_{\mathcal{H}}$ to the Siegel moduli (of some polarization degree possibly greater than one), which is relatively representable by a scheme of finite type over its image. The image is closed in the Siegel moduli schemes, as the existence of additional structures is described by closed conditions. Therefore the general result using geometric invariant theory in [96, Ch. 7] for the Siegel moduli implies that $\mathsf{M}_{\mathcal{H}}$ is of finite type over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, and is actually a scheme when the objects it parameterizes have no nontrivial automorphism. However, this (sketchy) argument gives no information about the local moduli, let alone the smoothness of $\mathsf{M}_{\mathcal{H}}$ over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$.

The second method is Artin's criterion (for algebraic spaces or algebraic stacks), which has the advantage that it requires little more than showing the prorepresentability of local moduli. Note that to prove the claim of smoothness in Theorem 1.4.1.11 we have to understand the local moduli anyway. Therefore it seems justified to us that our point of view should be biased toward the second method: Following the well-explained arguments in [109] and [99, §2] we will show that the local moduli is prorepresentable and formally smooth at each point of finite type over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. The endomorphism structure, the Lie algebra condition, and the level structures require some explanation, but they do not incur any essential difficulty. Since the moduli problem can be shown to be of finite type by the theory of Hilbert schemes, we conclude from Artin's criterion that $\mathsf{M}_{\mathcal{H}}$ is an algebraic stack separated, smooth, and of finite type over the base scheme (see Appendices A and B for more details). When $\mathcal{H}$ is neat, $\mathsf{M}_{\mathcal{H}}$ is representable by an algebraic space, because the existence of a level-$\mathcal{H}$ structure forces all automorphisms of objects of $\mathsf{M}_{\mathcal{H}}$ to be trivial.

As already mentioned in Remark 1.4.1.13, the fact that $\mathsf{M}_{\mathcal{H}}$ is actually a scheme when $\mathcal{H}$ is neat will be a by-product of our later work, and therefore can be suppressed at this moment.

We will not need the Serre–Tate theory of local moduli (as in [85] or [65]), and hence will not need Barsotti–Tate groups nor any kind of Cartier–Dieudonné theory. They would be important for the study of integral models of Shimura varieties that are not smooth. However, as explained in the introduction, we have decided not to discuss them in such generality.

2.1 Theory of Obstructions for Smooth Schemes

Let us introduce some basic terminology for the deformation of smooth schemes. Unless otherwise specified, all schemes in this section will be assumed to be *noetherian* and *separated.* (Readers might want to take a look at Section B.1 before reading this section.)

The idea of deforming smooth objects originated from the fundamental works of Kodaira and Spencer in [74] and [75], and the algebraic version dates back to Grothendieck's fundamental works in [50] and [56]. Some of the formulations we adopt here follow closely the presentation of [99, §2].

Remark 2.1.1. When the structural morphisms are clear, we shall denote by $\underline{H}^0$ and $\underline{H}^i$ the push-forwards and higher direct images, respectively, which are analogues of global sections and its derived functors in the relative setting.

Remark 2.1.2. For simplicity, the index sets of open coverings in this section (and in subsequent sections based on arguments here) will be omitted.

2.1.1 Preliminaries

Lemma 2.1.1.1 (cf. [56, III, Lem. 4.2] or [99, Lem. 2.2.2])**.** *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of nilpotent ideals $\mathscr{I}$, and let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be a morphism of schemes over $\tilde{S}$ such that $f := \tilde{f} \underset{\tilde{S}}{\times} S$ is an isomorphism. Suppose $\tilde{X}$ is flat over $\tilde{S}$. Then $\tilde{f}$ is an isomorphism.*

Proof. Since $\tilde{f}$ induces a homeomorphism on the underlying topological spaces, it suffices to treat the following affine case: Let $I := \ker(\tilde{R} \twoheadrightarrow R)$ be a nilpotent ideal in $\tilde{R}$. Let $u : N \to M$ be a morphism of $\tilde{R}$-modules such that M is flat over $\tilde{R}$, and such that $u \underset{\tilde{R}}{\otimes} R : N/(I \cdot N) \to M/(I \cdot M)$ is an isomorphism. Then u is an isomorphism.

To show this, let $K := \ker(u)$ and $Q := M/u(N)$. By assumption, we have $Q/(I \cdot Q) = 0$, and so $Q = I \cdot Q = I^2 \cdot Q = \cdots = I^n \cdot Q = 0$ for some n, because I is nilpotent. Thus

$$0 = \mathrm{Tor}_1^{\tilde{R}}(M, \tilde{R}/I) \to K/(I \cdot K) \to N/(I \cdot N) \to M/(I \cdot M) \to 0,$$

because M is $\tilde{R}$-flat. Hence $K/(I \cdot K) = 0$, and $K = 0$ as before. □

Lemma 2.1.1.2 (cf. [56, III, §5]). *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let $\tilde{X} \to \tilde{S}$ and $\tilde{Y} \to \tilde{S}$ be schemes over $\tilde{S}$, $X := \tilde{X} \underset{\tilde{S}}{\times} S$, $Y := \tilde{Y} \underset{\tilde{S}}{\times} S$, and denote by $\underline{\mathrm{Der}}_{Y/S}$ the sheaf of germs of $\mathscr{O}_S$-derivations from $\mathscr{O}_Y$ into itself. Let $f : X \to Y$ be a morphism of schemes over S. Let us denote by $\mathrm{Mor}_{\tilde{S}}(\tilde{X}, \tilde{Y}, f)$ the set of morphisms $\tilde{f} : \tilde{X} \to \tilde{Y}$ over $\tilde{S}$ such that $\tilde{f} \underset{\tilde{S}}{\times} S = f$. Suppose moreover that $\tilde{X}$ is flat over $\tilde{S}$, and that $\tilde{Y}$ is smooth over $\tilde{S}$. Then $\mathrm{Mor}_{\tilde{S}}(\tilde{X}, \tilde{Y}, f)$ is either empty or a torsor under $H^0(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$.*

Proof. It suffices to treat the affine case because the morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ on the underlying topology spaces is already determined by $f : X \to Y$. Let us assume $\tilde{S} = \mathrm{Spec}(\tilde{R})$, $S = \mathrm{Spec}(R)$, with $S \hookrightarrow \tilde{S}$ given by $\tilde{R} \twoheadrightarrow R$ with kernel I such that $I^2 = 0$. Let $\tilde{X} = \mathrm{Spec}(\tilde{M})$, $\tilde{Y} = \mathrm{Spec}(\tilde{N})$. Let us denote $\tilde{M}/(I \cdot \tilde{M})$ (resp. $\tilde{N}/(I \cdot \tilde{N})$) by M (resp. N), and let $u : N \to M$ denote the morphism given by $f : X \to Y$. Suppose we have a morphism $\tilde{u} : \tilde{N} \to \tilde{M}$ lifting u.

If $\tilde{u}' : \tilde{N} \to \tilde{M}$ is any other lifting, then $D := \tilde{u}' - \tilde{u}$ maps $\tilde{N}$ to $I \cdot \tilde{M}$. By [88, Thm. 7.7], the flatness of $\tilde{M}$ implies that the canonical morphism $\tilde{M} \underset{\tilde{R}}{\otimes} I \to \tilde{M} \underset{\tilde{R}}{\otimes} \tilde{R} \cong \tilde{M}$ is injective. Hence $I \cdot \tilde{M} \cong \tilde{M} \underset{\tilde{R}}{\otimes} I \cong M \underset{R}{\otimes} I$ because $I^2 = 0$. Moreover, the kernel of D contains $I \cdot \tilde{N}$. Therefore we may identify D as an R-module morphism $D : N \to M \underset{R}{\otimes} I$.

Let n_1 and n_2 be elements in $\tilde{N}$. Then the comparison between $u'(n_1 n_2) - u(n_1 n_2) = D(n_1 n_2)$ and $u'(n_1)u'(n_2) - u(n_1)u(n_2) = (u'(n_1) - u(n_1))u'(n_2) + (u'(n_2) - u(n_2))u'(n_1) = D(n_1)u(n_2) + D(n_2)u(n_1)$ shows that $u'(n_1 n_2) = u'(n_1)u'(n_2)$ if and only if $D(n_1 n_2) = D(n_1)u(n_2) + D(n_2)u(n_1)$. Combining this with other more trivial relations, we see that u' is an algebra homomorphism if and only if D is an R-derivative from N to $M \underset{R}{\otimes} I$, where the N-module structure of M is given by $u : N \to M$. Note that we have canonical isomorphisms

$$\mathrm{Der}_R(N, M \underset{R}{\otimes} I) \cong \mathrm{Hom}_N(\Omega^1_{N/R}, M \underset{R}{\otimes} I) \cong \mathrm{Hom}_M(\Omega^1_{N/R} \underset{N,u}{\otimes} M, M \underset{R}{\otimes} I).$$

Written globally, this is the group of global sections of

$$\underline{\mathrm{Der}}_{\mathscr{O}_S}(\mathscr{O}_Y, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \cong \underline{\mathrm{Hom}}_{\mathscr{O}_X}(f^*\Omega^1_{Y/S}, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I}),$$

which is isomorphic to the group $H^0(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ in the statement of the lemma in this affine case. □

Corollary 2.1.1.3 (cf. [99, Lem. 2.2.3]). *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let $\tilde{Z} \to \tilde{S}$ be a smooth morphism, $Z = \tilde{Z} \underset{\tilde{S}}{\times} S$, and denote by $\underline{\mathrm{Der}}_{Z/S}$ the sheaf of germs of $\mathscr{O}_S$-derivations from $\mathscr{O}_Z$ into itself. Let us denote by $\mathrm{Aut}_{\tilde{S}}(\tilde{Z}, S)$ the set of automorphisms of $\tilde{Z}$ over $\tilde{S}$ inducing the identity on Z. Then there is a canonical isomorphism $\mathrm{Aut}_{\tilde{S}}(\tilde{Z}, S) \xrightarrow{\sim} H^0(Z, \underline{\mathrm{Der}}_{Z/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$.*

Proof. Take $\tilde{X} = \tilde{Y} = \tilde{Z}$ in Lemma 2.1.1.2, and take $f : Z \to Z$ to be the identity morphism. Since the identity isomorphism $\tilde{Z} \to \tilde{Z}$ lifts f, by Lemma 2.1.1.1 and the flatness of $\tilde{Z}$ over $\tilde{S}$, we obtain a composition of canonical isomorphisms

$$\mathrm{Aut}_{\tilde{S}}(\tilde{Z}, S) \cong \mathrm{Mor}_{\tilde{S}}(\tilde{Z}, \tilde{Z}, f) \xrightarrow{\sim} H^0(Z, \underline{\mathrm{Der}}_{Z/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}),$$

as desired. □

Corollary 2.1.1.4. *With the setting as in Lemma 2.1.1.2, consider the natural homomorphisms*

$$df : H^0(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \to H^0(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

and

$$f^* : H^0(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \to H^0(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I}),$$

which define actions of the sources on the targets by addition. Then, by Lemma 2.1.1.2 and Corollary 2.1.1.3, these actions are compatible with the natural actions of $\mathrm{Aut}_{\tilde{S}}(\tilde{X}, S)$ and $\mathrm{Aut}_{\tilde{S}}(\tilde{Y}, S)$ on $\mathrm{Mor}_{\tilde{S}}(\tilde{X}, \tilde{Y}, f)$ given by pre- and postcompositions.

Proof. Recall that the proof of Lemma 2.1.1.2 is achieved by identifying $\mathrm{Aut}_{\tilde{S}}(\tilde{X}, \tilde{Y}, f)$ in the affine case as a torsor under global sections of $\underline{\mathrm{Der}}_{\mathscr{O}_S}(\mathscr{O}_X, \mathscr{O}_Y \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \cong \underline{\mathrm{Hom}}_{\mathscr{O}_X}(f^*\Omega^1_{Y/S}, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. The homomorphisms df and f^* we see in the statement of the corollary are induced locally by $df^* : \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\Omega^1_{X/S}, \mathscr{O}_X) \to \underline{\mathrm{Hom}}_{\mathscr{O}_X}(f^*\Omega^1_{Y/S}, \mathscr{O}_X)$ and $f^* : f^*\underline{\mathrm{Hom}}_{\mathscr{O}_Y}(\Omega^1_{Y/S}, \mathscr{O}_Y) \to \underline{\mathrm{Hom}}_{\mathscr{O}_X}(f^*\Omega^1_{Y/S}, \mathscr{O}_X)$. Now it suffices to observe that, following the proof of Lemma 2.1.1.2, the additions of the images of these morphisms are compatible with pre- and postcompositions of automorphisms. □

Lemma 2.1.1.5 (cf. [59, IV-4, 17.11.4] or [22, §2.2, Prop. 11]). *A morphism $\tilde{Y} \to \tilde{S}$ is smooth at $y \in \tilde{Y}$ if and only if there exists an open neighborhood $\tilde{U} \subset \tilde{Y}$ of y, an integer r, and an étale morphism $\tilde{U} \to \mathbb{A}^r_{\tilde{S}}$ over $\tilde{S}$, where $\mathbb{A}^r_{\tilde{S}}$ is the affine r-space over $\tilde{S}$.*

Lemma 2.1.1.6 ([59, IV-4, 18.1.2]). *For every closed immersion $S \hookrightarrow \tilde{S}$ defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$, as in the case of Lemma 2.1.1.7, the functor $\tilde{Y} \mapsto \tilde{Y} \underset{\tilde{S}}{\times} S$ is an equivalence of categories between schemes étale over the respective bases $\tilde{S}$ and S.*

Combining Lemmas 2.1.1.5 and 2.1.1.6, we obtain:

Lemma 2.1.1.7 ([56, III, Thm. 4.1] or [99, Lem. 2.2.4]). *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let $X \to S$ be a smooth scheme. For every $x \in X$, there exists an affine open neighborhood $U \subset X$ of x, and* *a smooth morphism $\tilde{U} \to \tilde{S}$, such that $\tilde{U} \underset{\tilde{S}}{\times} S \cong U$ over S. Moreover, suppose V is another such affine open neighborhood, with $\tilde{V} \to \tilde{S}$ smooth and $\tilde{V} \underset{\tilde{S}}{\times} S \cong V$ over S. Then for every affine neighborhood W of x in $U \cap V$, there exists an isomorphism $\tilde{U}|_W \xrightarrow{\sim} \tilde{V}|_W$ over $\tilde{S}$ whose pullback to S is the identity isomorphism on W.*

Remark 2.1.1.8. Here $\tilde{U}|_W$ has a meaning because $\tilde{U}$ is a scheme defined over the underlying topological space of U, and the underlying topological space of W is an open subset.

2.1.2 Deformation of Smooth Schemes

Definition 2.1.2.1. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let X be a scheme smooth over S. Then we denote by $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ the set of isomorphism classes of pairs $(\tilde{X}, \varphi)$ such that $\tilde{X} \to \tilde{S}$ is smooth and such that $\varphi : \tilde{X} \underset{\tilde{S}}{\times} S \to X$ is an isomorphism over S.*

Proposition 2.1.2.2 (cf. [50] or [56, III, Thm. 6.3, Prop. 5.1] or [99, Prop. 2.2.5])**.** *Suppose we have the same setting as in Definition 2.1.2.1. Then the following are true:*

1. *There exists a unique element*
$$\mathsf{o}(X; S \hookrightarrow \tilde{S}) \in H^2(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}),$$
called the ***obstruction*** *to $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$, such that*
$$\mathsf{o}(X; S \hookrightarrow \tilde{S}) = 0$$
if and only if
$$\mathsf{Lift}(X; S \hookrightarrow \tilde{S}) \neq \emptyset.$$

2. *If $\mathsf{o}(X; S \hookrightarrow \tilde{S}) = 0$ then $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ is a torsor under the group $H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$.*

3. *Let $f : X \xrightarrow{\sim} Y$ be any isomorphism of schemes smooth over S. Then the two natural isomorphisms*
$$df : H^2(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \xrightarrow{\sim} H^2(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$
and
$$f^* : H^2(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \xrightarrow{\sim} H^2(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$
induce the identification
$$df(\mathsf{o}(X; S \hookrightarrow \tilde{S})) = f^*(\mathsf{o}(Y; S \hookrightarrow \tilde{S})).$$

Proof. By Lemma 2.1.1.7, there is an affine open covering $\{U_\alpha\}_\alpha$ of X such that each U_α can be lifted to an affine scheme $\tilde{U}_\alpha$ smooth over $\tilde{S}$, with an isomorphism $\varphi_\alpha : \tilde{U}_\alpha \underset{\tilde{S}}{\times} S \xrightarrow{\sim} U_\alpha$ over S. Let us write $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and write similarly when there are more indices. Since $X \to S$ is separated, we know that each $U_{\alpha\beta}$ is affine, and hence there exists a morphism
$$\xi_{\alpha\beta} : \tilde{U}_\alpha|_{U_{\alpha\beta}} \to \tilde{U}_\beta|_{U_{\alpha\beta}}$$
over $\tilde{S}$ inducing $\varphi_\beta^{-1} \circ \varphi_\alpha$ over $U_{\alpha\beta}$. By Lemma 2.1.1.1, $\xi_{\alpha\beta}$ is an isomorphism.

Let us denote the restrictions of $\xi_{\alpha\beta}$ to $U_{\alpha\beta\gamma}$ by the same notation. For these $\tilde{U}_\alpha$ to glue together and form a scheme $\tilde{X}$ lifting X over $\tilde{S}$, these morphisms $\xi_{\alpha\beta}$ have to satisfy the so-called cocycle condition
$$\xi_{\alpha\gamma} = \xi_{\beta\gamma} \circ \xi_{\alpha\beta} \tag{2.1.2.3}$$
over each $U_{\alpha\beta\gamma}$. Let us measure the failure of this by defining
$$c_{\alpha\beta\gamma} := \xi_{\alpha\gamma}^{-1} \circ \xi_{\beta\gamma} \circ \xi_{\alpha\beta} \in \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}).$$
(We do not need to know if $\xi_{\alpha\gamma}^{-1} = \xi_{\gamma\alpha}$.) Since $c_{\alpha\beta\gamma} \underset{\tilde{S}}{\times} S$ is the identity on $U_{\alpha\beta\gamma}$, by Corollary 2.1.1.3, the automorphism $c_{\alpha\beta\gamma}$ defines an element of $H^0(U_{\alpha\beta\gamma}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$, which we also denote by $c_{\alpha\beta\gamma}$. By Corollary 2.1.1.3, we obtain an isomorphism
$$\mathrm{Aut}_{\tilde{S}}(\tilde{U}_\beta|_{U_{\alpha\beta}}, S) \xrightarrow{\sim} \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta}}, S) \tag{2.1.2.4}$$
by sending a to $\xi_{\alpha\beta}^{-1} \circ a \circ \xi_{\alpha\beta}$. Since the group $\mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha, S)$ is commutative for every $\tilde{U}_\alpha$ (and their intersections), the isomorphism (2.1.2.4) does not depend on the isomorphism $\xi_{\alpha\beta}$ we choose.

We claim that $c = \{c_{\alpha\beta\gamma}\}_{\alpha\beta\gamma}$ is a 2-cocycle with respect to the open covering $\{U_\alpha\}_\alpha$. By definition, its coboundary is given by
$$(\partial c)_{\alpha\beta\gamma\delta} := c_{\beta\gamma\delta} \circ c_{\alpha\gamma\delta}^{-1} \circ c_{\alpha\beta\delta} \circ c_{\alpha\beta\gamma}^{-1}. \tag{2.1.2.5}$$
We would like to represent all four elements on the right-hand side by elements of $\mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta\gamma\delta}}, S)$, via (2.1.2.4) if necessary. Since this group is commutative, we may switch the order of elements on the right-hand side of (2.1.2.5), and therefore
$$\begin{aligned}(\partial c)_{\alpha\beta\gamma\delta} &= c_{\alpha\gamma\delta}^{-1} \circ c_{\alpha\beta\delta} \circ c_{\beta\gamma\delta} \circ c_{\alpha\beta\gamma}^{-1} = [\xi_{\alpha\gamma}^{-1} \circ \xi_{\gamma\delta}^{-1} \circ \xi_{\alpha\delta}] \circ [\xi_{\alpha\delta}^{-1} \circ \xi_{\beta\delta} \circ \xi_{\alpha\beta}] \\ &\quad \circ [\xi_{\alpha\beta}^{-1} \circ (\xi_{\beta\delta}^{-1} \circ \xi_{\gamma\delta} \circ \xi_{\beta\gamma}) \circ \xi_{\alpha\beta}] \circ [\xi_{\alpha\beta}^{-1} \circ \xi_{\beta\gamma}^{-1} \circ \xi_{\alpha\gamma}] \\ &= \xi_{\alpha\gamma}^{-1} \circ [\xi_{\gamma\delta}^{-1} \circ [\xi_{\alpha\delta} \circ \xi_{\alpha\delta}^{-1}] \circ [\xi_{\beta\delta} \circ [\xi_{\alpha\beta} \circ \xi_{\alpha\beta}^{-1}] \circ \xi_{\beta\delta}^{-1}] \circ \xi_{\gamma\delta}] \\ &\quad \circ [\xi_{\beta\gamma} \circ [\xi_{\alpha\beta} \circ \xi_{\alpha\beta}^{-1}] \circ \xi_{\beta\gamma}^{-1}] \circ \xi_{\alpha\gamma} = \mathrm{Id}_{\tilde{U}_\alpha|_{U_{\alpha\beta\gamma\delta}}}.\end{aligned}$$
Suppose we have chosen a collection $\{\xi'_{\alpha\beta} : \tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}\}_{\alpha\beta}$ that differs from $\{\xi_{\alpha\beta}\}_{\alpha\beta}$ by
$$\xi'_{\alpha\beta} = \xi_{\alpha\beta} \circ \eta_{\alpha\beta}$$
for some $\eta_{\alpha\beta} \in \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta}}, S)$. Then we may identify $\eta = \{\eta_{\alpha\beta}\}_{\alpha\beta}$ with a 1-cochain in $C^1(\{U_\alpha\}_\alpha, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. Its coboundary is given by
$$(\partial\eta)_{\alpha\beta\gamma} := \eta_{\alpha\gamma}^{-1} \circ (\xi_{\alpha\beta}^{-1} \circ \eta_{\beta\gamma} \circ \xi_{\alpha\beta}) \circ \eta_{\alpha\beta}$$
in $\mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}, S)$, where we have used (2.1.2.4) again. Then $c_{\alpha\beta\gamma}$ becomes
$$\begin{aligned}c'_{\alpha\beta\gamma} &:= [\eta_{\alpha\gamma}^{-1} \circ (\xi_{\alpha\gamma})^{-1}] \circ [\xi_{\beta\gamma} \circ \eta_{\beta\gamma}] \circ [\xi_{\alpha\beta} \circ \eta_{\alpha\beta}] \\ &= [\xi_{\alpha\gamma}^{-1} \circ \xi_{\beta\gamma} \circ \xi_{\alpha\beta}] \circ [\eta_{\alpha\gamma}^{-1} \circ (\xi_{\alpha\beta}^{-1} \circ \eta_{\beta\gamma} \circ \xi_{\alpha\beta}) \circ \eta_{\alpha\beta}] = c_{\alpha\beta\gamma} \circ (\partial\eta)_{\alpha\beta\gamma}\end{aligned}$$
over $U_{\alpha\beta\gamma}$, where we can move $\eta_{\alpha\gamma}^{-1}$ because $\mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta}} S)$ is commutative. Hence we obtain a class $[c]$ in $H^2(\{U_\alpha\}_\alpha, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \cong H^2(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ that does not depend on the choices of $\xi_{\alpha\beta}$.

To show that this is independent of the choices of $\tilde{U}_\alpha$ over U_α, note that any different choice of $\tilde{U}'_\alpha$ over a particular U_α is (noncanonically) isomorphic to $\tilde{U}_\alpha$ by Lemma 2.1.1.7. Hence the class $[c]$ does not depend on the choice of $\tilde{U}_\alpha$ over each U_α. For the same reason, the class $[c]$ defined in $H^2(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ by a particular open covering $\{U_\alpha\}_\alpha$ remains unchanged if we refine the open covering. Thus we have shown that the definition of $[c]$ is independent of all choices.

Now, if $[c]$ is trivial, then it means there exist particular choices of $\{(U_\alpha, \tilde{U}_\alpha, \varphi_\alpha)\}_\alpha$ and $\{\xi_{\alpha\beta}\}_{\alpha\beta}$ such that, up to modification of $\{\xi_{\alpha\beta}\}_{\alpha\beta}$ by some $\{\eta_{\alpha\beta}\}_{\alpha\beta}$ as above, the cocycle condition (2.1.2.3) can be satisfied. By gluing the $\{\tilde{U}_\alpha\}_\alpha$ together using the modified $\{\xi_{\alpha\beta}\}_{\alpha\beta}$, we obtain a scheme $\tilde{X}$ smooth over $\tilde{S}$, together with an isomorphism $\varphi : \tilde{X} \underset{\tilde{S}}{\times} S \xrightarrow{\sim} X$ over S, as desired. Conversely, if any such smooth $\tilde{X} \to \tilde{S}$ exists, then there exists an affine open covering $\{U_\alpha\}_\alpha$ such that each $\tilde{U}_\alpha := \tilde{X}|_{U_\alpha}$ is smooth over $\tilde{S}$ and affine. Then $[c]$ is necessarily trivial because we can compute it by this choice of $\{\tilde{U}_\alpha\}_\alpha$, and the isomorphisms in $\{\xi_{\alpha\beta} : \tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}\}_{\alpha\beta}$ coming from the identity morphisms of schemes in $\{\tilde{X}|_{U_{\alpha\beta}}\}_{\alpha\beta}$ are certainly compatible with each other. This proves the first statement of the proposition if we set $\mathsf{o}(X; S \hookrightarrow \tilde{S}) := [c]$.

For the second statement, suppose there exists an element $(\tilde{X}, \varphi)$ in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$. Let $\{U_\alpha\}_\alpha$ be an affine open covering of X such that each $\tilde{U}_\alpha := \tilde{X}|_{U_\alpha}$ is affine.

Suppose we are given a 1-cocycle $d = \{d_{\alpha\beta}\}_{\alpha\beta}$ in $C^1(\{U_\alpha\}_\alpha, \underline{\mathrm{Der}}_{X/S})$, where $d_{\alpha\beta} \in H^0(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \cong \mathrm{Aut}_{\tilde{S}}(\tilde{X}|_{U_{\alpha\beta}}, S)$. We can interpret each $d_{\alpha\beta}$ as an isomorphism $\tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}$ over $\tilde{S}$ as both the source and target are canonically identified with $\tilde{X}|_{U_{\alpha\beta}}$. Since d is a 1-cocycle, these isomorphisms glue the affine schemes $\tilde{U}_\alpha$ together and define an object $(\tilde{X}^d, \varphi^d)$ in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$. Suppose we take another 1-cocycle d' that differs from d by a 1-coboundary. This means there exists a 0-cochain $e = \{e_\alpha\}_\alpha$ with $e_\alpha \in H^0(U_\alpha, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \cong \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha, S)$. As in (2.1.2.4), its coboundary is given by

$$(\partial e)_{\alpha\beta} = (d_{\alpha\beta}^{-1} \circ e_\beta^{-1} \circ d_{\alpha\beta}) \circ e_\alpha$$

in $\mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta}}, S)$, and therefore

$$d'_{\alpha\beta} = d_{\alpha\beta} \circ (\partial e)_{\alpha\beta} = d_{\alpha\beta} \circ [(d_{\alpha\beta}^{-1} \circ e_\beta^{-1} \circ d_{\alpha\beta}) \circ e_\alpha] = e_\beta^{-1} \circ d_{\alpha\beta} \circ e_\alpha$$

implies the relations $d_{\alpha\beta} = e_\beta \circ d'_{\alpha\beta} \circ e_\alpha^{-1}$ that glue together the collection $\{e_\alpha : \tilde{U}_\alpha \xrightarrow{\sim} \tilde{U}_\alpha\}_\alpha$ into an isomorphism $\tilde{X}^{d'} \xrightarrow{\sim} \tilde{X}^d$. Hence there is a well-defined map sending the $[d]$ in $H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ to the isomorphism class of $(\tilde{X}^d, \varphi^d)$ in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$. This map is injective because every isomorphism $\tilde{X}^{d'} \xrightarrow{\sim} \tilde{X}^d$ defines by restriction a collection $\{e_\alpha : \tilde{U}_\alpha \xrightarrow{\sim} \tilde{U}_\alpha\}_\alpha$ that necessarily defines a 1-coboundary giving the difference between d' and d.

Let us show that it is also surjective. Suppose there is any other element $(\tilde{X}', \varphi')$ in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$. By smoothness of $\tilde{X}' \to S$, for each α, the morphism $U_\alpha \to \tilde{U}'_\alpha := \tilde{X}'|_{U_\alpha}$ over S can be lifted to a morphism $\tilde{U}_\alpha \to \tilde{U}'_\alpha$ over $\tilde{S}$, which is necessarily an isomorphism by Lemma 2.1.1.1. For each α and β, the isomorphism $\xi'_{\alpha\beta} : \tilde{U}'_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}'_\beta|_{U_{\alpha\beta}}$ coming from the identity morphism on $\tilde{X}'|_{U_{\alpha\beta}}$ pulls back to an isomorphism $\tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}$ over $\tilde{S}$ that differs from $\xi_{\alpha\beta}$ by an automorphism $d_{\alpha\beta}$ of $\tilde{U}_\alpha|_{U_{\alpha\beta}}$, which we identify as an element in $H^0(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. The cochain $d = \{d_{\alpha\beta}\}_{\alpha\beta}$ necessarily satisfies the cocycle condition, as both $\{\xi_{\alpha\beta}\}_{\alpha\beta}$ and $\{\xi'_{\alpha\beta}\}_{\alpha\beta}$ do (in a slightly different context). This gives a class $[d]$ of d in $H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ and shows the surjectivity.

Finally, let us explain the third statement. By abuse of notation, for each open subscheme U of Y, and for each smooth morphism $\tilde{U} \to \tilde{S}$ with an isomorphism $\tilde{U} \underset{\tilde{S}}{\times} S \xrightarrow{\sim} U$ (inducing an isomorphism between the underlying topological spaces of $\tilde{U}$ and U), let $\tilde{f}^{-1}(\tilde{U}) := \tilde{U} \underset{U,f}{\times} f^{-1}(U)$ be the pullback of $\tilde{U}$ under $f : X \xrightarrow{\sim} Y$, and let $\tilde{f} : \tilde{f}^{-1}(\tilde{U}) \xrightarrow{\sim} \tilde{U}$ denote the induced isomorphism (between ringed spaces). By Corollary 2.1.1.4, the restriction of the composition $(df)^{-1} \circ f^*$ to U is nothing but the isomorphism

$$H^0(U, \underline{\mathrm{Der}}_{U/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \xrightarrow{\sim} H^0(f^{-1}(U), \underline{\mathrm{Der}}_{f^{-1}(U)/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

corresponding to

$$\mathrm{Aut}_{\tilde{S}}(\tilde{U}, S) \xrightarrow{\sim} \mathrm{Aut}_{\tilde{S}}(\tilde{f}^{-1}(\tilde{U}), S) : a \mapsto \tilde{f}^{-1} \circ a \circ \tilde{f}$$

under Corollary 2.1.1.3. Suppose $\{U_\alpha\}_\alpha$ is an affine open covering of Y that is lifted to some collection $\{\tilde{U}_\alpha\}_\alpha$ of schemes smooth over $\tilde{S}$, defining elements $c_{\alpha\beta\gamma} \in \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}, S) \cong H^0(U_{\alpha\beta\gamma}, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ representing the class $\mathsf{o}(Y; S \hookrightarrow \tilde{S}) \in H^2(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. Then $\{f^{-1}(U_\alpha)\}_\alpha$ is an affine open covering of X that is lifted to the collection $\{\tilde{f}^{-1}(\tilde{U}_\alpha)\}_\alpha$ (defined as above), defining the elements $((df)^{-1} \circ f^*)(c_{\alpha\beta\gamma})$ representing the class $\mathsf{o}(X; S \hookrightarrow \tilde{S}) \in H^2(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$, as desired. □

2.1.3 Deformation of Morphisms

Definition 2.1.3.1. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let X be a smooth scheme over S. Suppose $f : X \to Y$ is a morphism between smooth schemes over S such that X and Y lift respectively to smooth schemes $\tilde{X}$ and $\tilde{Y}$ over $\tilde{S}$. Then we denote by $\mathsf{Lift}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S})$ the set of morphisms $\tilde{f} : \tilde{X} \to \tilde{Y}$ such that $\tilde{f} \underset{\tilde{S}}{\times} S = f$.*

Proposition 2.1.3.2. *Suppose that we have the same setting as in Definition 2.1.3.1. Then the following are true:*

1. *There exists a unique element*
$$\mathsf{o}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S}) \in H^1(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I}),$$
*called the **obstruction** to $\mathsf{Lift}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S})$, such that*
$$\mathsf{o}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S}) = 0$$
if and only if
$$\mathsf{Lift}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S}) \neq \emptyset.$$

2. *If $\mathsf{o}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S}) = 0$ then $\mathsf{Lift}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S})$ is a torsor under the group $H^0(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$.*

3. *By Proposition 2.1.2.2, the set $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ (resp. $\mathsf{Lift}(Y; S \hookrightarrow \tilde{S})$) is a torsor under the group $H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ (resp. $H^1(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$). Hence*

it makes sense to write

$$\tilde{X}' = \mathsf{m}_{\tilde{X}} + \tilde{X}$$

and

$$\tilde{Y}' = \mathsf{m}_{\tilde{Y}} + \tilde{Y}$$

for elements $\mathsf{m}_{\tilde{X}} \in H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ *and* $\mathsf{m}_{\tilde{Y}} \in H^1(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. *Now consider the natural morphisms*

$$df : H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \to H^1(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$$

and

$$f^* : H^1(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \to H^1(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I}).$$

Then we have the relation

$$\mathsf{o}(f; \mathsf{m}_{\tilde{X}} + \tilde{X}, \mathsf{m}_{\tilde{Y}} + \tilde{Y}, S \hookrightarrow \tilde{S}) = \mathsf{o}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S}) - df(\mathsf{m}_{\tilde{X}}) + f^*(\mathsf{m}_{\tilde{Y}}).$$

Proof. Let $\{U_\alpha\}_\alpha$ and $\{V_\alpha\}_\alpha$ be affine open coverings of X and Y, respectively, indexed by the same set, such that $f(U_\alpha) \subset V_\alpha$ for each index α. Let $\tilde{U}_\alpha := \tilde{X}|_{U_\alpha}$ and $\tilde{V}_\alpha := \tilde{Y}|_{U_\alpha}$. Then $\{\tilde{U}_\alpha\}_\alpha$ and $\{\tilde{V}_\alpha\}_\alpha$ are open coverings of $\tilde{X}$ and $\tilde{Y}$, respectively. By smoothness of $\tilde{Y}$, for each α, the morphism $f|_{U_\alpha} : U_\alpha \to V_\alpha$ over S can be lifted to a morphism $\tilde{f}_\alpha : \tilde{U}_\alpha \to \tilde{V}_\alpha$ over $\tilde{S}$. By Lemma 2.1.1.2, such liftings over each open subscheme U of U_α form a torsor under the group $H^0(U, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. Let us write the group action additively. Comparing the restrictions to $U_{\alpha\beta}$, there exist elements $c_{\alpha\beta} \in H^0(U_{\alpha\beta}, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ such that

$$\tilde{f}_\alpha|_{U_{\alpha\beta}} = c_{\alpha\beta} + \tilde{f}_\beta|_{U_{\alpha\beta}}. \tag{2.1.3.3}$$

Comparing the relations over $U_{\alpha\beta\gamma}$, we obtain the cocycle relation $c_{\alpha\gamma} = c_{\beta\gamma} + c_{\alpha\beta}$. If we replace each choice of $\tilde{f}_\alpha$ with $\tilde{f}'_\alpha = \tilde{f}_\alpha + e_\alpha$ for some $e_\alpha \in H^0(\tilde{U}_\alpha, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$, then we obtain $c'_{\alpha\beta} = c_{\alpha\beta} + (\partial e)_{\alpha\beta}$, where $(\partial e)_{\alpha\beta} := -e_\beta + e_\alpha$. Thus there is a well-defined class $[c]$ for $c = \{c_{\alpha\beta}\}_{\alpha\beta}$ in $H^1(\{U_\alpha\}_\alpha, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \cong H^1(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ that is independent of the choice of $\{\tilde{f}_\alpha\}_\alpha$. The class $[c]$ in $H^1(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ is independent of the choices of $\{U_\alpha\}_\alpha$ and $\{V_\alpha\}_\alpha$ because we can always replace them with refinements.

If $[c]$ is trivial, then there exists $e = \{e_\alpha\}_\alpha$ as above such that $c_{\alpha\beta} = e_\beta - e_\alpha$. Hence $\{e_\alpha + \tilde{f}_\alpha = e_\beta + \tilde{f}_\beta\}_\alpha$ defines a global morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$. Conversely, the existence of any global morphism forces $[c]$ to be trivial. Hence we can conclude the proof of the first statement by setting $\mathsf{o}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S}) = [c]$.

For the second statement, note that the existence of any global lifting $\tilde{f}$ gives a choice of the $\tilde{f}_\alpha$ as above over each U_α, and hence any other global choice $\tilde{f}'$ must differ by some $\{e_\alpha\}_\alpha$ that patches together to some e describing the difference between $\tilde{f}$ and $\tilde{f}'$.

For the third statement, note that in the proof of Proposition 2.1.2.2, the group action of $H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ (resp. $H^1(Y, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$) is given by modifying the gluing isomorphisms for $\{\tilde{U}_\alpha\}_\alpha$ (resp. $\{\tilde{V}_\alpha\}_\alpha$). Suppose $\mathsf{m}_{\tilde{X}}$ (resp. $\mathsf{m}_{\tilde{Y}}$) is represented by some 1-cochain $m_{\tilde{X}} = \{m_{\tilde{X},\alpha\beta}\}_{\alpha\beta}$ (resp. $m_{\tilde{Y}} = \{m_{\tilde{Y},\alpha\beta}\}_{\alpha\beta}$), where $m_{\tilde{X},\alpha\beta} \in H^0(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ (resp. $m_{\tilde{Y},\alpha\beta} \in H^0(V_{\alpha\beta}, \underline{\mathrm{Der}}_{Y/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$). By Corollary 2.1.1.4, the relation (2.1.3.3) corresponds to $df(m_{\tilde{X},\alpha\beta}) + \tilde{f}_\alpha|_{U_{\alpha\beta}} = c'_{\alpha\beta} + f^*(m_{\tilde{Y},\alpha\beta}) + \tilde{f}_\beta|_{U_{\alpha\beta}}$, which implies that $c'_{\alpha\beta} = c_{\alpha\beta} + df(m_{\tilde{X},\alpha\beta}) - f^*(m_{\tilde{Y},\alpha\beta})$, as desired. □

2.1.4 Base Change

The arguments used in the proofs in Sections 2.1.2 and 2.1.3 above have functorial implications in the following situation: Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$, and let $T \hookrightarrow \tilde{T}$ be a closed immersion defined by a sheaf of ideals $\mathscr{J}$ such that $\mathscr{J}^2 = 0$. Suppose we are given a commutative diagram

$$\begin{tikzcd} S \arrow[r, hook] \arrow[d, hook] & \tilde{S} \arrow[d, hook, "\iota"] \\ T \arrow[r, hook] & \tilde{T} \end{tikzcd}$$

of closed embeddings. (We do not assume that this is Cartesian.) The commutativity shows that $\mathscr{J}$ is mapped to $\mathscr{I}$ by the pullback $\iota^* : \mathscr{O}_{\tilde{T}} \to \mathscr{O}_{\tilde{S}}$. We shall denote the induced morphism simply by $\iota^* : \mathscr{J} \to \mathscr{I}$.

If X is any scheme smooth over T, then $X \underset{T}{\times} S$ is a scheme smooth over S. Therefore it makes sense to compare the two sets $\mathsf{Lift}(X; T \hookrightarrow \tilde{T})$ and $\mathsf{Lift}(X \underset{T}{\times} S; S \hookrightarrow \tilde{S})$. Moreover, suppose we have a morphism $f : X \to Y$ between schemes smooth over T such that the source X and the target Y are lifted respectively to schemes $\tilde{X}$ and $\tilde{Y}$ smooth over $\tilde{T}$. Then $f \underset{T}{\times} S$ is a morphism from $X \underset{T}{\times} S$ to $Y \underset{T}{\times} S$ over S, and it also makes sense to compare the two sets $\mathsf{Lift}(f; \tilde{X}, \tilde{Y}, T \hookrightarrow \tilde{T})$ and $\mathsf{Lift}(f \underset{T}{\times} S; \tilde{X} \underset{\tilde{T}}{\times} \tilde{S}, \tilde{Y} \underset{\tilde{T}}{\times} \tilde{S}, S \hookrightarrow \tilde{S})$.

If $\tilde{U}$ is any scheme smooth over $\tilde{T}$ lifting an affine open subscheme U of X, then $\tilde{U} \underset{\tilde{T}}{\times} \tilde{S}$ is a lifting of the affine open subscheme $U \underset{T}{\times} S$ of $X \underset{T}{\times} S$ over $\tilde{S}$. Moreover, if $\tilde{V}$ is any scheme smooth over $\tilde{T}$ lifting an affine open subscheme V of Y such that $f(U) \subset V$, then $\tilde{V} \underset{\tilde{T}}{\times} \tilde{S}$ is a lifting over $\tilde{S}$ of the affine open subscheme $V \underset{T}{\times} S$ of $Y \underset{T}{\times} S$ such that $(f \underset{T}{\times} S)(U \underset{T}{\times} S) \subset V \underset{T}{\times} S$. Therefore, if $\tilde{f}_{\tilde{U}} : \tilde{U} \to \tilde{V}$ is a morphism over $\tilde{T}$ lifting $f|_U : U \to V$, then $\tilde{f}_U \underset{\tilde{T}}{\times} \tilde{S} : \tilde{U} \underset{\tilde{T}}{\times} \tilde{S} \to \tilde{V} \underset{\tilde{T}}{\times} \tilde{S}$ is a morphism over $\tilde{S}$ lifting $(f \underset{T}{\times} S)|_{U \underset{T}{\times} S} : U \underset{T}{\times} S \to V \underset{T}{\times} S$.

Lemma 2.1.4.1. *The diagram*

$$\begin{tikzcd} \mathrm{Mor}_{\tilde{T}}(\tilde{U}, \tilde{V}, f|_U) \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\underset{\tilde{T}}{\times} \tilde{S}"'] & H^0(U, f^*(\underline{\mathrm{Der}}_{Y/T}) \underset{\mathscr{O}_T}{\otimes} \mathscr{J}) \arrow[d, "\iota^*"] \\ \mathrm{Mor}_{\tilde{S}}(\tilde{U} \underset{\tilde{T}}{\times} \tilde{S}, \tilde{V} \underset{\tilde{T}}{\times} \tilde{S}, f|_{U \underset{T}{\times} S}) \arrow[r, "\sim", "\text{can.}"'] & H^0(U \underset{T}{\times} S, (f \underset{T}{\times} S)^*(\underline{\mathrm{Der}}_{Y \underset{T}{\times} S/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \end{tikzcd}$$

with horizontal isomorphisms given by Lemma 2.1.1.2 is commutative.

Proof. This follows if we note that the proof of Lemma 2.1.1.2 is compatible with base change. □

Corollary 2.1.4.2. *The diagram*

$$\begin{tikzcd} \mathrm{Aut}_{\tilde{T}}(\tilde{U}, T) \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\times_{\tilde{T}} \tilde{S}"'] & H^0(U, \underline{\mathrm{Der}}_{X/T} \otimes_{\mathscr{O}_T} \mathscr{I}) \arrow[d, "\iota^*"] \\ \mathrm{Aut}_{\tilde{S}}(\tilde{U} \times_{\tilde{T}} \tilde{S}, S) \arrow[r, "\sim", "\text{can.}"'] & H^0(U \times_T S, \underline{\mathrm{Der}}_{X \times_T S/S} \otimes_{\mathscr{O}_S} \mathscr{I}) \end{tikzcd}$$

with horizontal isomorphisms given by Corollary 2.1.1.3 is commutative.

Corollary 2.1.4.3. 1. *The two obstructions* $\mathsf{o}(X; T \hookrightarrow \tilde{T})$ *and* $\mathsf{o}(X \times_T S; S \hookrightarrow \tilde{S})$ *are related by*

$$\iota^*(\mathsf{o}(X; T \hookrightarrow \tilde{T})) = \mathsf{o}(X \times_T S; S \hookrightarrow \tilde{S})$$

under the morphism

$$\iota^* : H^2(X, \underline{\mathrm{Der}}_{X/T} \otimes_{\mathscr{O}_T} \mathscr{I}) \to H^2(X \times_T S, \underline{\mathrm{Der}}_{X \times_T S/S} \otimes_{\mathscr{O}_S} \mathscr{I}).$$

2. *If* $\mathsf{o}(X; T \hookrightarrow \tilde{T}) = 0$ *and* $\mathsf{o}(X \times_T S; S \hookrightarrow \tilde{S}) = 0$*, then the morphism*

$$\cdot \times_{\tilde{T}} \tilde{S} : \mathsf{Lift}(X; T \hookrightarrow \tilde{T}) \to \mathsf{Lift}(X \times_T S; S \hookrightarrow \tilde{S})$$

of torsors is equivariant under the morphism

$$\iota^* : H^1(X, \underline{\mathrm{Der}}_{X/T} \otimes_{\mathscr{O}_T} \mathscr{I}) \to H^1(X \times_T S, \underline{\mathrm{Der}}_{X \times_T S/S} \otimes_{\mathscr{O}_S} \mathscr{I}).$$

Proof. The statements follow from the proof of Proposition 2.1.2.2, as each affine open covering $\{U_\alpha\}_\alpha$ of X that defines the obstructions and torsor structures also defines the corresponding objects for $X \times_T S$ by the operation $\cdot \times_{\tilde{T}} \tilde{S}$ corresponding to the tensor operation $\cdot \otimes_{\mathscr{O}_{\tilde{T}}} \mathscr{O}_{\tilde{S}}$ on sheaves. □

Similarly, following the proof of Proposition 2.1.3.2, we obtain the following corollary:

Corollary 2.1.4.4. *The two obstructions* $\mathsf{o}(f; \tilde{X}, \tilde{Y}, T \hookrightarrow \tilde{T})$ *and* $\mathsf{o}(f \times_T S; \tilde{X} \times_{\tilde{T}} \tilde{S}, \tilde{Y} \times_{\tilde{T}} \tilde{S}, S \hookrightarrow \tilde{S})$ *are related by*

$$\iota^*(\mathsf{o}(f; \tilde{X}, \tilde{Y}, T \hookrightarrow \tilde{T})) = \mathsf{o}(f \times_T S; \tilde{X} \times_{\tilde{T}} \tilde{S}, \tilde{Y} \times_{\tilde{T}} \tilde{S}, S \hookrightarrow \tilde{S})$$

under the morphism

$$\iota^* : H^1(X, f^*\underline{\mathrm{Der}}_{X/T} \otimes_{\mathscr{O}_T} \mathscr{I}) \to H^1(X \times_T S, (f \times_T S)^*\underline{\mathrm{Der}}_{X \times_T S/S} \otimes_{\mathscr{O}_S} \mathscr{I}).$$

2.1.5 Deformation of Invertible Sheaves

Let us review the definition of *cup products* (and, in particular, the sign convention we use) before stating the results in this section. Suppose we have a scheme Z over S, and two invertible sheaves $\mathscr{F}$ and $\mathscr{G}$ over Z. We shall define a morphism

$$\cup : H^i(Z, \mathscr{F}) \otimes H^j(Z, \mathscr{G}) \to H^{i+j}(Z, \mathscr{F} \otimes_{\mathscr{O}_Z} \mathscr{G}) \quad (2.1.5.1)$$

as follows. Suppose there is an affine open covering $\{Z_\alpha\}_\alpha$ of Z, with the convention that $Z_{\alpha\beta} = Z_\alpha \cap Z_\beta$ etc. as before. Given an i-cochain $a = \{a_{\alpha_0 \ldots \alpha_i}\}_{\alpha_0 \ldots \alpha_i} \in C^i(\{Z_{\alpha_0 \ldots \alpha_i}\}_{\alpha_0 \ldots \alpha_i}, \mathscr{F})$ and a j-cochain $b = \{b_{\alpha_0 \ldots \alpha_j}\}_{\alpha_0 \ldots \alpha_j} \in C^j(\{Z_{\alpha_0 \ldots \alpha_j}\}_{\alpha_0 \ldots \alpha_j}, \mathscr{G})$, we define an $(i+j)$-cochain $a \cup b$ by setting

$$(a \cup b)_{\alpha_0 \ldots \alpha_{i+j}} := a_{\alpha_0 \ldots \alpha_i} b_{\alpha_i \ldots \alpha_{i+j}},$$

where the notation $a_{\alpha_0 \ldots \alpha_i} b_{\alpha_i \ldots \alpha_{i+j}}$ means the image of $a_{\alpha_0 \ldots \alpha_i} \otimes b_{\alpha_i \ldots \alpha_{i+j}}$ under the canonical morphism

$$H^0(Z_{\alpha_0 \ldots \alpha_{i+j}}, \mathscr{F}) \otimes_{\mathscr{O}_S} H^0(Z_{\alpha_0 \ldots \alpha_{i+j}}, \mathscr{G}) \overset{\text{can.}}{\to} H^0(Z_{\alpha_0 \ldots \alpha_{i+j}}, \mathscr{F} \otimes_{\mathscr{O}_Z} \mathscr{G}).$$

Since $(\partial a)_{\alpha_0 \ldots \alpha_{i+1}} := \sum_{k=0}^{i+1} (-1)^k a_{\alpha_0 \ldots \hat{\alpha}_k \ldots \alpha_{i+1}}$ and $(\partial b)_{\alpha_0 \ldots \alpha_{j+1}} := \sum_{k=0}^{j+1} (-1)^k b_{\alpha_0 \ldots \hat{\alpha}_k \ldots \alpha_{i+1}}$, one verifies easily that $\partial(a \cup b) = (\partial a) \cup b + (-1)^i a \cup (\partial b)$. This shows that the operation $([a], [b]) \mapsto [a] \cup [b] := [a \cup b]$ on Čech cohomology classes is well defined, inducing the desired morphism (2.1.5.1) above.

Definition 2.1.5.2. *Let* $S \hookrightarrow \tilde{S}$ *be a closed immersion defined by a sheaf of ideals* $\mathscr{I}$ *such that* $\mathscr{I}^2 = 0$*. Let* X *be a scheme smooth over* S *that is lifted to some scheme* $\tilde{X}$ *smooth over* $\tilde{S}$*. Suppose* $\mathcal{L}$ *is an invertible sheaf over* X*. Then we denote by* $\mathsf{Lift}(\mathcal{L}; \tilde{X}, S \hookrightarrow \tilde{S})$ *the set of isomorphism classes of pairs* $(\tilde{\mathcal{L}}, \psi)$ *such that* $\tilde{\mathcal{L}}$ *is an invertible sheaf over* $\tilde{X}$ *and such that* $\psi : \tilde{\mathcal{L}} \otimes_{\mathscr{O}_{\tilde{S}}} \mathscr{O}_S \to \mathcal{L}$ *is an isomorphism over* X.

Proposition 2.1.5.3. *Suppose that we have the same setting as in Definition 2.1.5.2. Then the following are true:*

1. *There exists a unique element*

$$\mathsf{o}(\mathcal{L}; \tilde{X}, S \hookrightarrow \tilde{S}) \in H^2(X, \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{I}),$$

called the ***obstruction*** *to* $\mathsf{Lift}(\mathcal{L}; \tilde{X}, S \hookrightarrow \tilde{S})$*, such that*

$$\mathsf{o}(\mathcal{L}; \tilde{X}, S \hookrightarrow \tilde{S}) = 0$$

if and only if

$$\mathsf{Lift}(\mathcal{L}; \tilde{X}, S \hookrightarrow \tilde{S}) \neq \emptyset.$$

2. *If* $\mathsf{o}(\mathcal{L}; \tilde{X}, S \hookrightarrow \tilde{S}) = 0$ *and the canonical morphism*

$$H^0(\tilde{X}, \mathscr{O}_{\tilde{X}}^\times) \to H^0(X, \mathscr{O}_X^\times) \quad (2.1.5.4)$$

is surjective, then $\mathsf{Lift}(\mathcal{L}; \tilde{X}, S \hookrightarrow \tilde{S})$ *is a torsor under the group* $H^1(X, \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{I})$.

3. *By Proposition 2.1.2.2, the set* $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ *is a torsor under the group* $H^1(X, \underline{\mathrm{Der}}_{X/S} \otimes_{\mathscr{O}_S} \mathscr{I})$*. Hence it makes sense to write*

$$\tilde{X}' = \mathsf{m}_{\tilde{X}} + \tilde{X}$$

for every element $\mathsf{m}_{\tilde{X}} \in H^1(X, \underline{\mathrm{Der}}_{X/S} \otimes_{\mathscr{O}_S} \mathscr{I})$*. Let*

$$d\log : \mathrm{Pic}(X) \cong H^1(X, \mathscr{O}_X^\times) \to H^1(X, \Omega^1_{X/S})$$

be the morphism induced by

$$d\log : \mathscr{O}_X^\times \to \Omega^1_{X/S} : a \mapsto d\log(a) := a^{-1}da.$$

Then the cup product with $d\log(\mathcal{L})$ *defines a natural morphism*

$$\begin{aligned} \mathsf{d}_{\mathcal{L}} :\ & H^i(X, \underline{\mathrm{Der}}_{X/S} \otimes_{\mathscr{O}_S} \mathscr{I}) \\ & \to H^{i+1}(X, \underline{\mathrm{Der}}_{X/S} \otimes_{\mathscr{O}_X} \Omega^1_{X/S} \otimes_{\mathscr{O}_S} \mathscr{I}) \overset{\text{can.}}{\to} H^{i+1}(X, \mathscr{O}_X \otimes_{\mathscr{O}_S} \mathscr{I}), \end{aligned} \quad (2.1.5.5)$$

which in the case $i = 1$ makes the following identity hold:
$$\mathsf{o}(\mathcal{L}; \mathsf{m}_{\tilde{X}} + \tilde{X}, S \hookrightarrow S) = \mathsf{o}(\mathcal{L}; \tilde{X}, S \hookrightarrow S) + \mathsf{d}_{\mathcal{L}}(\mathsf{m}_{\tilde{X}}).$$

Proof. First let us take any smooth affine open covering $\{U_\alpha\}_\alpha$ of X such that $\mathcal{L}$ is given by a cohomology class $[l] \in H^1(X, \mathscr{O}_X^\times)$ represented by some $l = \{l_{\alpha\beta} \in \mathscr{O}_{U_{\alpha\beta}}^\times\}_{\alpha\beta}$. Note that we have the cocycle condition
$$l_{\alpha\gamma}^{-1} \cdot l_{\beta\gamma} \cdot l_{\alpha\beta} = 1 \tag{2.1.5.6}$$
over $U_{\alpha\beta\gamma}$. Let $\tilde{l}_{\alpha\beta}$ be any element in $\mathscr{O}_{\tilde{U}_\alpha|_{U_{\alpha\beta}}}^\times$ lifting $l_{\alpha\beta}$. Let $\xi_{\alpha\beta} : \tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}$ denote the isomorphism giving the gluing of the lifting $\tilde{X}$ of X. If $\tilde{l} = \{\tilde{l}_{\alpha\beta}\}_{\alpha\beta}$ comes from some invertible sheaf $\tilde{\mathcal{L}}$ over X lifting $\mathcal{L}$, then we have
$$\tilde{l}_{\alpha\gamma}^{-1} \cdot [\xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma})] \cdot \tilde{l}_{\alpha\beta} = 1. \tag{2.1.5.7}$$
In general, let us measure the failure of liftability by
$$h_{\alpha\beta\gamma} := \tilde{l}_{\alpha\gamma}^{-1} \cdot [\xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma})] \cdot \tilde{l}_{\alpha\beta} - 1 \in \mathscr{O}_{\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}}.$$
By (2.1.5.6), $h_{\alpha\beta\gamma} \in \mathscr{I} \cdot \mathscr{O}_{\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}} \cong \mathscr{O}_{U_{\alpha\beta\gamma}} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}$. Moreover, $h = \{h_{\alpha\beta\gamma}\}_{\alpha\beta\gamma}$ is a 2-cocycle because $(1+h_{\beta\gamma\delta})(1-h_{\alpha\gamma\delta})(1+h_{\alpha\beta\delta})(1-h_{\alpha\beta\gamma}) = 1+(\partial h)_{\alpha\beta\gamma\delta} = 1$. If we replace $\tilde{l}$ with another lifting of l, then we arrive at a 2-cocycle that differs from h by a coboundary. This shows that $[h]$ defines a cohomology class in $H^2(X, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ independent of the choice of $\tilde{l}$. Moreover, $[h]$ is trivial if and only if we can find some choice $\tilde{l}$ such that the cocycle condition (2.1.5.7) is satisfied by $\tilde{l}$. This shows that we can define $\mathsf{o}(\mathcal{L}; \tilde{X}, S \hookrightarrow \tilde{S})$ to be $[h]$. Note that this is simply the image of the class of $\mathcal{L} \in \mathrm{Pic}(X) \cong H^1(X, \mathscr{O}_X^\times)$ under the connecting morphism in the long exact sequence
$$H^1(X, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \to H^1(\tilde{X}, \mathscr{O}_{\tilde{X}}^\times) \to H^1(X, \mathscr{O}_X^\times) \to H^2(X, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \to \cdots \tag{2.1.5.8}$$
associated with $0 \to \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \mathscr{I} \to \mathscr{O}_{\tilde{X}}^\times \to \mathscr{O}_X^\times \to 0$. This proves the first statement of the proposition. The second statement then follows because the first morphism in (2.1.5.8) is injective when (2.1.5.4) is surjective.

To prove the third statement, let us investigate what happens when we replace the $\{\xi_{\alpha\beta}\}_{\alpha\beta}$ defining $\tilde{X}$ in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ with some different element defining $\mathsf{m}_{\tilde{X}} + \tilde{X}$ with $\mathsf{m}_{\tilde{X}} \in H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. By refining the open covering $\{U_\alpha\}_\alpha$ if necessary, let us suppose $\mathsf{m}_{\tilde{X}}$ is defined by some $\{\eta_{\alpha\beta}\}_{\alpha\beta}$ with $\eta_{\alpha\beta} \in \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta}}, S) \cong H^0(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. Then $\{\xi'_{\alpha\beta} := \xi_{\alpha\beta} \circ \eta_{\alpha\beta}\}_{\alpha\beta}$ defines $\tilde{X}' = \mathsf{m}_{\tilde{X}} + \tilde{X}$ which gives a possibly different lifting of X, and we have to replace each $h_{\alpha\beta\gamma}$ accordingly with $h'_{\alpha\beta\gamma} := \tilde{l}_{\alpha\gamma}^{-1} \cdot [(\xi'_{\alpha\beta})^*(\tilde{l}_{\beta\gamma})] \cdot \tilde{l}_{\alpha\beta} - 1 \in \mathscr{O}_{\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}}$. Let us write $\eta_{\alpha\beta}^* : \mathscr{O}_{\tilde{U}_\alpha|_{U_{\alpha\beta}}} \to \mathscr{O}_{\tilde{U}_\alpha|_{U_{\alpha\beta}}}$ as
$$\eta_{\alpha\beta}^* = \mathrm{Id} + T_{\alpha\beta} \circ d$$
for $T_{\alpha\beta} \circ d \in H^0(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$, as in Lemma 2.1.1.2 and Corollary 2.1.1.3. We use the notation $T_{\alpha\beta} \circ d$ to signify the fact that it is a composition of the universal differentiation $d : \mathscr{O}_{U_{\alpha\beta}/S} \to \Omega^1_{U_{\alpha\beta}/S}$ and some morphism $T_{\alpha\beta} \in \mathrm{Hom}_{\mathscr{O}_{U_{\alpha\beta}}}(\Omega^1_{U_{\alpha\beta}/S}, \mathscr{O}_{U_{\alpha\beta}} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. Note that $\{\eta_{\alpha\beta}\}_{\alpha\beta}$ and $\{T_{\alpha\beta}\}_{\alpha\beta}$ are simply different ways of representing the same class $\mathsf{m}_{\tilde{X}}$. Both $\xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma})$ and $(\xi'_{\alpha\beta})^*(\tilde{l}_{\beta\gamma})$ become the same $l_{\beta\gamma}$ modulo $\mathscr{I}$. Since $\eta_{\alpha\beta}^* = (\xi_{\alpha\beta}^{-1} \circ \xi'_{\alpha\beta})^* = (\xi'_{\alpha\beta})^* \circ (\xi_{\alpha\beta}^*)^{-1}$, we have $(\xi'_{\alpha\beta})^*(\tilde{l}_{\beta\gamma}) = \eta_{\alpha\beta}^* \xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma}) = (\mathrm{Id} + T_{\alpha\beta} \circ d)(\xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma})) = \xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma}) + (T_{\alpha\beta} \circ d)(l_{\beta\gamma}) = (\xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma}))(1 + (\xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma}))^{-1} T_{\alpha\beta}(dl_{\beta\gamma})) = (\xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma}))(1 + T_{\alpha\beta}(d\log(l_{\beta\gamma})))$, where we have used the usual convention of log differentiation. As a result, we have $h'_{\alpha\beta\gamma} - h_{\alpha\beta\gamma} = \tilde{l}_{\alpha\gamma}^{-1}[(\xi'_{\alpha\beta})^*(\tilde{l}_{\beta\gamma}) - \xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma})]\tilde{l}_{\beta\gamma} = (\tilde{l}_{\alpha\gamma}^{-1} \cdot [\xi_{\alpha\beta}^*(\tilde{l}_{\beta\gamma})] \cdot \tilde{l}_{\beta\gamma}) \cdot T_{\alpha\beta}(d\log(l_{\beta\gamma})) = (1 + h_{\alpha\beta\gamma}) \cdot T_{\alpha\beta}(d\log(l_{\beta\gamma})) = T_{\alpha\beta}(d\log(l_{\beta\gamma}))$. This is just the cup product of the class $\mathsf{m}_{\tilde{X}}$ represented by $T = \{T_{\alpha\beta}\}_{\alpha\beta}$ and the class $d\log(\mathcal{L})$ represented by $\{d\log(l_{\alpha\beta})\}_{\alpha\beta}$. This proves the third statement. □

Corollary 2.1.5.9. *Suppose X is a scheme smooth over S. Then we have a canonical isomorphism*
$$\underline{H}^1(X, \mathscr{O}_X) \cong \underline{\mathrm{Lie}}_{\underline{\mathrm{Pic}}(X/S)/S}. \tag{2.1.5.10}$$

Proof. It suffices to verify $H^1(X, \mathscr{O}_X) \cong \mathrm{Lie}_{\underline{\mathrm{Pic}}(X/S)/S}$ when S is affine. By definition, $\underline{\mathrm{Lie}}_{\underline{\mathrm{Pic}}(X/S)/S}$ is the set of liftings of the trivial invertible sheaf $\mathscr{O}_X$ over X. Set $\tilde{S} := \underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathscr{O}_S[\varepsilon]/(\varepsilon^2))$. By abuse of notation, $\mathscr{O}_{\tilde{S}} \twoheadrightarrow \mathscr{O}_S$ has kernel $\mathscr{I} := \varepsilon\mathscr{O}_S$ satisfying $\mathscr{I}^2 = 0$. As $\mathscr{O}_S$-modules, we have $\mathscr{I} \cong \mathscr{O}_S$ because it is generated by the single element ε. The surjection $\mathscr{O}_{\tilde{S}} \twoheadrightarrow \mathscr{O}_S$ has a canonical section given by $a \mapsto a \in \mathscr{O}_S[\varepsilon]/(\varepsilon^2)$ for all $a \in \mathscr{O}_S$. Therefore we may pullback X to a family $\tilde{X}$ over $\tilde{S}$, together with the trivial invertible sheaf lifting the trivial invertible sheaf over X. This forces the obstruction to vanish, and the second statement of Proposition 2.1.5.3 applies. The torsor $\mathrm{Lie}_{\underline{\mathrm{Pic}}(X/S)/S}$ under $H^1(X, \mathscr{O}_X)$ can be canonically trivialized by the section of the surjection above. □

Lemma 2.1.5.11. *Suppose $f : X \to S$ is a smooth group scheme. Then the canonical morphism $\underline{\mathrm{Der}}_{X/S} \to f^*\underline{\mathrm{Lie}}_{X/S}$ (induced by adjunction of the evaluation along the identity section) is an isomorphism, and therefore there is a canonical isomorphism*
$$\underline{H}^0(X, \underline{\mathrm{Der}}_{X/S}) \cong \underline{H}^0(X, \mathscr{O}_X) \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{X/S} \tag{2.1.5.12}$$
by the projection formula [59, 0_I, 5.4.10.1]. The analogous statement is true if we replace $\underline{\mathrm{Der}}_{X/S}$ (resp. $\underline{\mathrm{Lie}}_{X/S}$) with $\Omega^1_{X/S}$ (resp. $\underline{\mathrm{Lie}}^\vee_{X/S}$)

Proof. The morphism $\underline{\mathrm{Der}}_{X/S} \to f^*\underline{\mathrm{Lie}}_{X/S}$ is an isomorphism by [22, §4.2, Prop. 2]. The remaining statements are clear. □

If X is an abelian scheme over S, then $\underline{\mathrm{Lie}}_{\underline{\mathrm{Pic}}(X/S)/S} \cong \underline{\mathrm{Lie}}_{X^\vee/S}$ by definition. By the two identifications (2.1.5.12) and (2.1.5.10), we can interpret (2.1.5.5) as a morphism
$$\mathsf{d}_{\mathcal{L}} : \underline{\mathrm{Lie}}_{X/S} \to \underline{\mathrm{Lie}}_{X^\vee/S}.$$

Proposition 2.1.5.13. *This morphism $\mathsf{d}_{\mathcal{L}}$ agrees with the differential $d\lambda_{\mathcal{L}}$ of the homomorphism $\lambda_{\mathcal{L}} : X \to X^\vee$ defined by $\mathcal{L}$ (in Construction 1.3.2.7).*

Proof. Again we may assume that S is affine.

Each section of $\mathrm{Lie}_{X/S}$ can be realized as a morphism $T : \tilde{S} := \underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathscr{O}_S[\varepsilon]/(\varepsilon^2)) \to X$ extending the identity section $e_X : S \to X$. It can be identified with a section of $H^0(X, \underline{\mathrm{Der}}_{X/S})$ as follows: If a function f on X is evaluated as some $a + b\varepsilon$ under T, for $a, b \in \mathscr{O}_S$, then $b = T(df)$. Once this identification is made, we may also regard T as a differentiation.

On the other hand, the pullback $(\mathrm{Id}_X \underset{S}{\times} T)^*\mathcal{D}_2(\mathcal{L})$ of $\mathcal{D}_2(\mathcal{L})$ under $(\mathrm{Id}_X \underset{S}{\times} T) : X \underset{S}{\times} \tilde{S} \to X \underset{S}{\times} X$ gives a deformation of $\mathcal{D}_2(\mathcal{L})|_{X \underset{S}{\times} eX} \cong \mathscr{O}_X$ over $\tilde{S}$. By the universal property of the Poincaré invertible sheaf $\mathcal{P}_X$, this invertible sheaf $(\mathrm{Id}_X \underset{S}{\times} T)^*\mathcal{D}_2(\mathcal{L})$ is the pullback of $\mathcal{P}_X$ under some unique morphism $(\mathrm{Id}_X \times T') : X \underset{S}{\times} \tilde{S} \to X \underset{S}{\times} X^\vee$. Since $\lambda_{\mathcal{L}}$ is by definition the unique homomorphism such that $\mathcal{D}_2(\mathcal{L})$ is the pullback of $\mathcal{P}_X$ under $(\mathrm{Id}_X \times \lambda_{\mathcal{L}})$, the morphism $T' : \tilde{S} \to X^\vee$ is nothing but the section $d\lambda_{\mathcal{L}}(T)$ of $\mathrm{Lie}_{X^\vee/S}$.

Let us interpret T' as a deformation of $\mathscr{O}_X$. Let $\mathcal{L}$ be defined by some cocycle represented by some $\{l_{\alpha\beta}\}_{\alpha\beta}$ in $H^1(X, \mathscr{O}_X^\times)$. If we interpret $X \underset{S}{\times} \tilde{S}$ as an abelian scheme over $\tilde{S}$ lifting X over S, then $(\mathrm{Id}_X \underset{S}{\times} T)^*\mathcal{D}_2(\mathcal{L})$ is an invertible sheaf lifting the trivial invertible sheaf $\mathscr{O}_X$ over S. The cocycle for $(\mathrm{Id}_X \underset{S}{\times} T)^*\mathcal{D}_2(\mathcal{L})$ can be given explicitly by $m_{\alpha\beta} := [l_{\alpha\beta,0} + T(dl_{\alpha\beta,0})\varepsilon]l_{\alpha\beta,0}^{-1} = 1 + T(d\log(l_{\alpha\beta,0}))\varepsilon$. By reading the coefficient of ε, we see that the deformation $(\mathrm{Id}_X \underset{S}{\times} T)^*\mathcal{D}_2(\mathcal{L})$ corresponds to $\{T(d\log(l_{\alpha\beta}))\}_{\alpha\beta}$ in $H^1(X, \mathscr{O}_X)$ under the isomorphism (2.1.5.10) given by Corollary 2.1.5.9. This is exactly the morphism $\mathsf{d}_{\mathcal{L}}$ defined by the cup product with $d\log(\mathcal{L})$. □

Proposition 2.1.5.14. *If X is an abelian scheme over S, then the cup product morphisms induce an isomorphism $\wedge^i \underline{H}^1(X, \mathscr{O}_X) \cong \underline{H}^i(X, \mathscr{O}_X)$ of locally free sheaves for all $i \geq 0$, making $\underline{H}^\bullet(X, \mathscr{O}_X)$ an exterior algebra.*

See [18, Prop. 2.5.2] for a proof.

Corollary 2.1.5.15. *Let X be an abelian scheme over S. Then the diagram*

$$\begin{tikzcd}
\underline{\mathrm{Lie}}_{X^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{X/S} \arrow[rr, "\mathrm{Id}_{X^\vee} \otimes d\lambda_{\mathcal{L}}"] \arrow[d, "\mathrm{can.} \otimes \mathrm{can.}"', "\wr"] & & \underline{\mathrm{Lie}}_{X^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{X^\vee/S} \arrow[d, "\wr"', "\mathrm{can.} \otimes \mathrm{can.}"] \\
\underline{H}^1(X, \mathscr{O}_X) \underset{\mathscr{O}_S}{\otimes} \underline{H}^0(X, \underline{\mathrm{Der}}_{X/S}) \arrow[rr, "\mathrm{Id} \otimes \mathsf{d}_{\mathcal{L}}"] \arrow[d, "\mathrm{can.}"', "\wr"] & & \underline{H}^1(X, \mathscr{O}_X) \underset{\mathscr{O}_S}{\otimes} \underline{H}^1(X, \mathscr{O}_X) \arrow[d, "\cup"] \\
\underline{H}^1(X, \underline{\mathrm{Der}}_{X/S}) \arrow[rr, "\mathsf{d}_{\mathcal{L}}"'] & & \underline{H}^2(X, \mathscr{O}_X)
\end{tikzcd}$$

is commutative.

2.1.6 De Rham Cohomology

Let $\pi : X \to S$ be a morphism of schemes. Let $\Omega^\bullet_{X/S}$ be the complex over X whose differentials $d : \Omega^i_{X/S} \to \Omega^{i+1}_{X/S}$ are induced by the canonical $d : \mathscr{O}_{X/S} \to \Omega^1_{X/S}$ of the Kähler differentials. Then one can define the (*relative*) *de Rham cohomology* $\underline{H}^i_{\mathrm{dR}}(X/S)$ to be the higher direct image $R^i\pi_*\Omega^\bullet_{X/S}$ relative to S.

Let us suppose from now on that X is smooth. By Lemma 2.1.1.5, we may find an affine open covering $\{U_\alpha\}_\alpha$ of X such that each U_α is étale over some affine r-space $\mathbb{A}^r_S$ over S. In this case, locally over the base scheme S, the sheaf of differentials over U_α has a basis $dx_1, \ldots, dx_r$ given by the coordinates $x_1, \ldots, x_r$ of $\mathbb{A}^r_S$. In particular, $R^i\pi_*\Omega^q_{U_\alpha/S}$ is trivial for all $i > 0$ and all q.

This allows us to compute the de Rham cohomology of X explicitly: Let $\underline{C}^{\bullet,\bullet}$ be the double complex of sheaves with terms $\underline{C}^{p,q} := \underline{C}^p(\{U_\alpha\}_\alpha, \Omega^q_{X/S})$ and differentials

$$\partial : \underline{C}^p(\{U_\alpha\}_\alpha, \Omega^q_{X/S}) \to \underline{C}^{p+1}(\{U_\alpha\}_\alpha, \Omega^q_{X/S}) :$$

$$(x_{\alpha_0\ldots\alpha_p}) \mapsto ((\partial x)_{\alpha_0\ldots\alpha_{p+1}}) := \Big(\sum_{k=0}^{p+1}(-1)^k x_{\alpha_0\ldots\hat{\alpha}_k\ldots\alpha_{p+1}}\Big)$$

and

$$d : \underline{C}^p(\{U_\alpha\}_\alpha, \Omega^q_{X/S}) \to \underline{C}^p(\{U_\alpha\}_\alpha, \Omega^{q+1}_{X/S}) : (x_{\alpha_0\ldots\alpha_p}) \mapsto (dx_{\alpha_0\ldots\alpha_p}).$$

Let $\underline{C}^\bullet$ be the total complex of $\underline{C}^{\bullet,\bullet}$ with terms $\underline{C}^n := \underset{p+q=n}{\oplus} \underline{C}^{p,q}$ and differentials

$$D = D_{p,q} = \partial \oplus (-1)^p d : \underline{C}^{p,q} \to \underline{C}^{p+1,q} \oplus \underline{C}^{p,q+1}.$$

Note that $\partial d = d\partial$, and hence $D^2 = 0$ by the sign twist we have specified. The differentials of $\underline{C}^\bullet$ are $\pi^{-1}(\mathscr{O}_S)$-linear, and the formation of $\underline{C}^\bullet$ is compatible with localizations over S. This is called the *Čech complex* associated with $\Omega^\bullet_{X/S}$.

Let us define a morphism $\Omega^\bullet_{X/S} \to \underline{C}^\bullet$ by assigning over each open subscheme U of X the morphism $\Omega^n_{X/S}(U) \to \underline{C}^{0,n}(U) \subset \underline{C}^n(U)$ sending a differential over U to its restrictions over $U \cap U_\alpha$. Under the assumption that $R^i\pi_*\Omega^q_{U_\alpha/S}$ is trivial for all $i > 0$ and all q, it is well known that the higher direct image of $\Omega^\bullet_{X/S}$ can be calculated using (*relative*) *Čech cohomology*:

Proposition 2.1.6.1. *The morphism $\Omega^\bullet_{X/S} \to \underline{C}^\bullet$ induces an isomorphism $R^i\pi_*\Omega^\bullet_{X/S} \overset{\sim}{\to} R^i\pi_*\underline{C}^\bullet$. (In fact, the morphism $\Omega^\bullet_{X/S} \to \underline{C}^\bullet$ is a quasi-isomorphism.)*

Proof. Since the question is local over S, we may assume that S is affine. Then the proposition follows from [48, Ch. II, §5]. (See also [59, III-2, 6.2.2] and the remark in [67, Sec. 3, p. 206].) □

Now suppose $S \hookrightarrow \tilde{S}$ is a closed immersion defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let $\tilde{X}$ be a scheme smooth over $\tilde{S}$, and let $X := \tilde{X} \underset{\tilde{S}}{\times} S$. By Proposition 2.1.3.2 and its proof, we know that $\tilde{X}$ is glued from liftings $\{\tilde{U}_\alpha\}_\alpha$ of affine open smooth subschemes $\{U_\alpha\}_\alpha$ forming an open covering of X. By refining the open covering if necessary, we may assume that each of the affine open subscheme $\tilde{U}_\alpha$ is étale over some affine r-space $\mathbb{A}^r_{\tilde{S}}$ as above. Then we know that the de Rham cohomology $\underline{H}^i_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ can be computed as the relative cohomology (i.e., higher direct image) of the total complex $\underline{C}^\bullet$ of $\underline{C}^{p,q} = \underline{C}^p(\{\tilde{U}_\alpha\}_\alpha, \Omega^q_{\tilde{X}/\tilde{S}})$. Note that the sheaves $\underline{C}^{p,q}$ are defined only using the information on each $\tilde{U}_\alpha$. Namely, we just need to know the sheaves $\underline{H}^0(U_{\alpha_0\ldots\alpha_p}, \Omega^q_{\tilde{U}_{\alpha_1\ldots\alpha_p}/S})$. On the other hand, the differential D of the complex $\underline{C}^\bullet$ does depend on the gluing isomorphisms $\xi_{\alpha\beta} : \tilde{U}_\alpha|_{U_{\alpha\beta}} \overset{\sim}{\to} \tilde{U}_\beta|_{U_{\alpha\beta}}$. Nevertheless,

Proposition 2.1.6.2. *Let $\mathsf{m}_{\tilde{X}} \in H^1(X, \underline{\mathrm{Der}} X/S \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$, and let $\tilde{X}' := \mathsf{m}_{\tilde{X}} + \tilde{X} \in \mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ denote the object given by $\mathsf{m}_{\tilde{X}}$ and $\tilde{X}$ under the action of*

$H^1(X, \underline{\mathrm{Der}} X/S \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ *on* $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$. *Then there is a canonical isomorphism* $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S}) \cong \underline{H}^1_{\mathrm{dR}}(\tilde{X}'/\tilde{S})$, *lifting the identity isomorphism on* $\underline{H}^1_{\mathrm{dR}}(X/S)$.

Although this is quite well known, we would like to give a proof here to show its relation to the theory of obstruction that we have studied so far.

Proof of Proposition 2.1.6.2. Take an affine open covering $\{\tilde{U}_\alpha\}_\alpha$ of $\tilde{X}$ such that, if we set $U_\alpha := \tilde{U}_\alpha \underset{\tilde{S}}{\times} S$ and set $\xi_{\alpha\beta} : \tilde{U}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \tilde{U}_\beta|_{U_{\alpha\beta}}$ to be the isomorphism over $\tilde{S}$ identifying the open subscheme $\tilde{U}_{\alpha\beta}$ with itself, then $\tilde{X}'$ is obtain by replacing this gluing isomorphism $\xi_{\alpha\beta}$ with $\xi'_{\alpha\beta} \circ \eta_{\alpha\beta}$, where $\eta_{\alpha\beta} \in \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta}}, S) \cong H^0(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ represents the class of $\mathsf{m}_{\tilde{X}} \in H^1(U_{\alpha\beta}, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. As in the proof of Proposition 2.1.5.3, we identify $\eta^*_{\alpha\beta} = \mathrm{Id} + T_{\alpha\beta} \circ d$ with $T_{\alpha\beta} \in \mathrm{Hom}_{\mathscr{O}_{U_{\alpha\beta}}}(\Omega^1_{U_{\alpha\beta}/S}, \mathscr{O}_{U_{\alpha\beta}} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. Then $(\xi'_{\alpha\beta})* = (\eta_{\alpha\beta})^*(\xi_{\alpha\beta})^* = (\mathrm{Id} + T_{\alpha\beta})(\xi_{\alpha\beta})^*$ and $T = \{T_{\alpha\beta}\}_{\alpha\beta}$ defines a class of $H^1(X, \Omega^1_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$.

Let $\underline{C}^\bullet$ and $(\underline{C}')^\bullet$ be the respective complexes computing $\underline{H}^i_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ and $\underline{H}^i_{\mathrm{dR}}(\tilde{X}'/\tilde{S})$ as explained above, with the same affine open subschemes $\{\tilde{U}_\alpha\}_\alpha$ covering both $\tilde{X}$ and $\tilde{X}'$ (with different gluing isomorphisms along the overlaps). More precisely, we have $\underline{C}^n = \underset{p+q=n}{\oplus} \underline{C}^{p,q}$, $(\underline{C}')^n = \underset{p+q=n}{\oplus} (\underline{C}')^{p,q}$, $\underline{C}^{p,q} = \underline{H}^0(\tilde{U}_{\alpha_0}|_{U_{\alpha_0 \dots \alpha_p}}, \Omega^q_{\tilde{X}/\tilde{S}})$, and $(\underline{C}')^{p,q} = \underline{H}^0(\tilde{U}_{\alpha_0}|_{U_{\alpha_0 \dots \alpha_p}}, \Omega^q_{\tilde{X}'/\tilde{S}})$, with natural identifications $\underline{C}^{p,q} \cong (\underline{C}')^{p,q}$ which we shall assume in what follows. Each local section of $\underline{C}^n$ is represented by a tuple $(x^{(p,q)})_{p+q=n}$ with $x^{(p,q)} = \{x^{(p,q)}_{\alpha_0 \dots \alpha_p}\}_{\alpha_0 \dots \alpha_p}$ representing an element in $\underline{C}^p(\{\tilde{U}_\alpha\}_\alpha, \Omega^q_{\tilde{X}/\tilde{S}}) = \underline{C}^p(\{\tilde{U}_\alpha\}_\alpha, \Omega^q_{\tilde{X}'/\tilde{S}})$.

The cup product with $T = \{T_{\alpha\beta}\}_{\alpha\beta}$ defines a morphism $\underline{C}^{p,q} \to \underline{C}^{p+1,q-1} : x^{(p,q)} \mapsto (T \cup x^{(p,q)})$ by $(T \cup x^{(p,q)})_{\alpha_0 \dots \alpha_{p+1}} := T_{\alpha_0\alpha_1} x^{(p,q)}_{\alpha_1 \dots \alpha_{p+1}}$. Naturally, $\partial(T \cup x^{(p,q)}) = (\partial T \cup x^{(p,q)}) + (-1)^1 (T \cup \partial x^{(p,q)})$, with $\partial T = 0$ (as T is a cocycle). The differential $\partial' : (\underline{C}')^{p,q} \to (\underline{C}')^{p+1,q} : x^{(p,q)} \mapsto \partial' x^{(p,q)}$ is defined by $(\partial' x^{(p,q)})_{\alpha_0 \dots \alpha_{p+1}} := (\xi'_{\alpha_0\alpha_1})^*(x^{(p,q)}_{\alpha_1 \dots \alpha_{p+1}}) + \sum_{k=1}^{p+1} (-1)^k x^{(p,q)}_{\alpha_0 \dots \hat{\alpha}_k \dots \alpha_{p+1}} = (\xi_{\alpha_0\alpha_1})^*(x^{(p,q)}_{\alpha_1 \dots \alpha_{p+1}}) + (T_{\alpha_0\alpha_1} \circ d)((\xi_{\alpha_0\alpha_1})^*(x^{(p,q)}_{\alpha_1 \dots \alpha_{p+1}})) + \sum_{k=1}^{p+1} (-1)^k x^{(p,q)}_{\alpha_0 \dots \hat{\alpha}_k \dots \alpha_{p+1}} = (\partial x^{(p,q)})_{\alpha_0 \dots \alpha_{p+1}} + (T_{\alpha_0\alpha_1} \circ d)(x^{(p,q)}_{\alpha_1 \dots \alpha_{p+1}})$. Thus we simply have $\partial' = \partial + (T \cup d)$.

On the other hand, although the morphism $d : \mathscr{O}_{\tilde{U}_\alpha} \to \Omega^1_{\tilde{U}_\alpha/S}$ coming from the restriction of $d : \mathscr{O}_{\tilde{X}} \to \Omega^1_{\tilde{X}/\tilde{S}}$ is unique up to canonical isomorphism, it does not mean that it is the *same* as the morphism d' coming from $\Omega^1_{\tilde{X}'/\tilde{S}}$. Let us measure this difference over $\tilde{U}_\alpha$ by $d' = (\mathrm{Id} + E_\alpha) \circ d$, for some morphism $E_\alpha : \Omega^1_{\tilde{U}_\alpha/\tilde{S}} \to \mathscr{I} \cdot \Omega^1_{\tilde{U}_\alpha/\tilde{S}}$, or rather a morphism $E_\alpha : \Omega^1_{U_\alpha/S} \to \Omega^1_{U_\alpha/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}$. This E is canonical because of the universal properties of d and d'. Note that we need $(E_\alpha \circ d + d \circ E_\alpha) \circ d = 0$ in order to make $(d')^2 = 0$. Then we have $d \circ E_\alpha \circ d = 0 = E_\alpha \circ d \circ d$, which means that $d \circ E_\alpha = 0 = E_\alpha \circ d$ as everything in $\Omega^1_{\tilde{U}_\alpha/\tilde{S}}$ is in the image of $d : \mathscr{O}_{\tilde{U}_\alpha} \to \Omega^1_{\tilde{U}_\alpha/\tilde{S}}$. Moreover, we need to glue d (resp. d') as well using $\xi_{\alpha\beta}$ (resp. $\xi'_{\alpha\beta}$). Therefore, we need both the relations $d(\xi^*_{\alpha\beta}(x)) = \xi^*_{\alpha\beta}(dx)$ and $d'((\xi'_{\alpha\beta})^*(x)) = (\xi'_{\alpha\beta})^*(dx)$. If we expand all the terms in the second relation and substitute the first relation into it, then we get $E_\alpha(dx) + dT_{\alpha\beta}(dx) = \xi^*_{\alpha\beta}(E_\beta)(dx)$, and $dT_{\alpha\beta} = \xi^*_{\alpha\beta}(E_\beta) - E_\alpha =: -(\partial E)_{\alpha\beta}$, or simply $dT = -\partial E$.

Now let us specialize to the case $n = 1$ and consider the morphism

$$\underline{C}^1 \to (\underline{C}')^1 : x = (x^{(1,0)}, x^{(0,1)}) \mapsto x' = ((x')^{(1,0)}, (x')^{(0,1)})$$

given explicitly by

$$x' := (x^{(1,0)} + (T \cup x^{(0,1)}), x^{(0,1)} + Ex^{(0,1)}). \tag{2.1.6.3}$$

The differential D on $\underline{C}^1$ sends

$$x = (x^{(1,0)}, x^{(0,1)}) \mapsto Dx = (\partial x^{(1,0)}, -dx^{(1,0)} + \partial x^{(0,1)}, dx^{(0,1)}).$$

On the other hand, we have a similar formula for D' on x', whose three components are given by

1. $\partial'(x')^{(1,0)} = (\partial + (T \cup d))(x^{(1,0)} + (T \cup x^{(0,1)})) = \partial x^{(1,0)} + (T \cup dx^{(1,0)}) + \partial(T \cup \partial x^{(0,1)}) = \partial x^{(1,0)} - (T \cup (-dx^{(1,0)} + \partial x^{(0,1)}))$;

2. $-d'(x')^{(1,0)} + \partial'(x')^{(0,1)} = -(d + Ed)(x^{(1,0)} + (T \cup x^{(0,1)})) + (\partial + (T \cup d))(x^{(0,1)} + Ex^{(0,1)}) = (-dx^{(1,0)} + \partial x^{(0,1)}) - Edx^{(1,0)} - d(T \cup x^{(0,1)}) + T \cup dx^{(0,1)} - \partial(Ex^{(0,1)}) = (-dx^{(1,0)} + \partial x^{(0,1)}) - Edx^{(1,0)} - dT \cup x^{(0,1)} - (\partial E) \cup x^{(0,1)} - E\partial x^{(0,1)} = (\mathrm{Id} + E)(-dx^{(1,0)} + \partial x^{(0,1)})$;

3. $d'(x')^{(0,1)} = (d + Ed)(x^{(0,1)} + Ex^{(0,1)}) = dx^{(0,1)} + Edx^{(0,1)} + dEx^{(0,1)} = dx^{(0,1)}$.

We have used $dT = -\partial E$ in the second one. Since $\mathrm{Id} + E$ is an automorphism, we see that $D'x' = 0$ if and only if $Dx = 0$. On the other hand, if $x^{(1,0)} = \partial x^{(0,0)}$ and $x^{(0,1)} = dx^{(0,0)}$ for some $(x^{(0,0)}) \in \underline{C}^0$, then

$$x^{(1,0)} + (T \cup x^{(0,0)}) = \partial x^{(0,0)} + T \cup dx^{(0,0)} = \partial' x^{(0,0)}$$

and

$$x^{(0,1)} + Ex^{(0,1)} = dx^{(0,0)} + Edx^{(0,0)} = d'x^{(0,0)}.$$

As a result, there is a unique way to associate with each representative of $\mathsf{m}_X \in H^1(X, \underline{\mathrm{Der}} X/S \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ an isomorphism from the first relative cohomology of $(\underline{C}^\bullet, D)$ to the one of $((\underline{C}')^\bullet, D')$. If we modify the representative $T = \{T_{\alpha\beta}\}_{\alpha\beta}$ of $\mathsf{m}_{\tilde{X}}$ by a coboundary, then all the E_α are also modified in a uniquely determined way. This shows that we have constructed a canonical isomorphism $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S}) \cong \underline{H}^1_{\mathrm{dR}}(\tilde{X}'/\tilde{S})$, as desired. □

Now suppose that $\tilde{Y}$ is a scheme over $\tilde{S}$ such that there is a morphism f from X to $Y := \tilde{Y} \underset{\tilde{S}}{\times} S$. Then,

Proposition 2.1.6.4. *There is a canonical morphism*

$$\tilde{f}^* : \underline{H}^1_{\mathrm{dR}}(\tilde{Y}/\tilde{S}) \to \underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$$

lifting the canonical morphism $f^* : \underline{H}^1_{\mathrm{dR}}(Y/S) \to \underline{H}^1_{\mathrm{dR}}(X/S)$ *induced by* f.

Proof. We shall denote by $d_{\tilde{X}} : \mathscr{O}_{\tilde{X}} \to \Omega^1_{\tilde{X}/\tilde{S}}$ and $d_{\tilde{Y}} : \mathscr{O}_{\tilde{Y}} \to \Omega^1_{\tilde{Y}/\tilde{S}}$ the respective morphisms of universal differentials. Take affine open coverings $\{\tilde{U}_\alpha\}_\alpha$ of $\tilde{X}$ and $\{\tilde{V}_\alpha\}_\alpha$ of $\tilde{Y}$ as in the proof of Proposition 2.1.3.2 such that $f(U_\alpha) \subset V_\alpha$. For each α, let $\tilde{f}_\alpha$ be any morphism over $\tilde{S}$ lifting the restriction of f to U_α. The composition $d_{\tilde{X}} \circ \tilde{f}^*_\alpha : \mathscr{O}_{\tilde{V}_\alpha} \to \Omega^1_{\tilde{U}_\alpha/\tilde{S}}$ induces, by the universal property of $\Omega^1_{\tilde{V}_\alpha/\tilde{S}}$, a unique morphism $\Omega^1_{\tilde{V}_\alpha/\tilde{S}} \to \Omega^1_{\tilde{U}_\alpha/\tilde{S}}$, which we again denote by $\tilde{f}^*_\alpha$, such that $\tilde{f}^*_\alpha \circ d_{\tilde{Y}} = d_{\tilde{X}} \circ \tilde{f}^*_\alpha$.

By Proposition 2.1.3.2, the obstruction of lifting f globally to some morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ over $\tilde{S}$ is a cohomology class in $H^1(X, f^*(\underline{\mathrm{Der}}_{Y/S}) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ represented by some $T \circ d_{\tilde{Y}} = \{T_{\alpha\beta} \circ d_{\tilde{Y}}\}_{\alpha\beta}$, where the notation $T_{\alpha\beta} \circ d_{\tilde{Y}}$ means it is the composition of $d_{\tilde{Y}}$ with $T_{\alpha\beta} \in \mathrm{Hom}_{\mathscr{O}_{U_{\alpha\beta}}}(f^*\Omega^1_{V_{\alpha\beta}/S}, \mathscr{O}_{U_{\alpha\beta}} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$. By (2.1.3.3) in the proof of Proposition 2.1.3.2, we may assume that $T_{\alpha\beta} \circ d_{\tilde{Y}} = \tilde{f}^*_\alpha - \tilde{f}^*_\beta$, which implies that $d_{\tilde{X}} \circ T_{\alpha\beta} = \tilde{f}^*_\alpha - \tilde{f}^*_\beta$.

Let $\underline{C}^\bullet_{\tilde{X}}$ (resp. $\underline{C}^\bullet_{\tilde{Y}}$) be the complex computing $\underline{H}^i_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ (resp. $\underline{H}^i_{\mathrm{dR}}(\tilde{Y}/\tilde{S})$) as above, with the affine open subschemes $\tilde{U}_\alpha$ (resp. $\tilde{V}_\alpha$) covering $\tilde{X}$ (resp. $\tilde{Y}$). Let us define a morphism $\underline{C}^1_{\tilde{Y}} \to \underline{C}^1_{\tilde{X}} : y = (y^{(1,0)}, y^{(0,1)}) \mapsto x = (x^{(1,0)}, x^{(0,1)})$ by $x^{(1,0)}_{\alpha\beta} := \tilde{f}^*_\alpha(y^{(1,0)}_{\alpha\beta}) + T_{\alpha\beta}(y^{(0,1)}_\beta)$ and $x^{(0,1)}_\alpha := \tilde{f}^*_\alpha(y^{(0,1)}_\alpha)$.

The differential $D_{\tilde{Y}}$ on $\underline{C}^1_{\tilde{Y}}$ sends a section $y = (y^{(1,0)}, y^{(0,1)})$ to $D_{\tilde{Y}} y = (\partial_{\tilde{Y}} y^{(1,0)}, -d_{\tilde{Y}} y^{(1,0)} + \partial_{\tilde{Y}} y^{(0,1)}, d_{\tilde{Y}} y^{(0,1)})$. On the other hand, we have a similar formula for $D_{\tilde{X}}$ on sections x of $\underline{C}^1_{\tilde{X}}$, with three components given by

1. $(\partial_{\tilde{X}} x^{(1,0)})_{\alpha\beta\gamma} = [\tilde{f}^*_\beta(y^{(1,0)}_{\beta\gamma}) + T_{\beta\gamma}(y^{(0,1)}_\gamma)] - [\tilde{f}^*_\alpha(y^{(1,0)}_{\alpha\gamma}) + T_{\alpha\gamma}(y^{(0,1)}_\gamma)] + [\tilde{f}^*_\alpha(y^{(1,0)}_{\alpha\beta}) + T_{\alpha\beta}(y^{(0,1)}_\beta)] = \tilde{f}^*_\alpha((\partial_{\tilde{Y}} y^{(1,0)})_{\alpha\beta\gamma}) + (\tilde{f}^*_\beta - \tilde{f}^*_\alpha)(y^{(1,0)}_{\beta\gamma}) + (\partial T)_{\alpha\beta\gamma}(y^{(0,1)}_\gamma) + T_{\alpha\beta}((\partial y^{(0,1)})_{\beta\gamma}) = \tilde{f}^*_\alpha((\partial_{\tilde{Y}} y^{(1,0)})_{\alpha\beta\gamma}) + T_{\alpha\beta}(-d_{\tilde{Y}} y^{(1,0)}_{\beta\gamma} + (\partial_{\tilde{Y}} y^{(0,1)})_{\beta\gamma})$;
2. $(-d_{\tilde{X}} x^{(1,0)} + \partial_{\tilde{X}} x^{(0,1)})_{\alpha\beta} = -d_{\tilde{X}}(\tilde{f}^*_\alpha(y^{(1,0)}_{\alpha\beta})) - (d_{\tilde{X}} \circ T_{\alpha\beta})(y^{(0,1)}_\beta) - \tilde{f}^*_\beta(y^{(0,1)}_\beta) + \tilde{f}^*_\alpha(y^{(0,1)}_\alpha) = -d_{\tilde{X}}(\tilde{f}^*_\alpha(y^{(1,0)}_{\alpha\beta})) - (\tilde{f}^*_\alpha - \tilde{f}^*_\beta)(y^{(0,1)}_\beta) - \tilde{f}^*_\beta(y^{(0,1)}_\beta) + \tilde{f}^*_\alpha(y^{(0,1)}_\alpha) = \tilde{f}^*_\alpha(-d_{\tilde{Y}} y^{(1,0)}_{\alpha\beta} + (-y^{(0,1)}_\beta + y^{(0,1)}_\alpha)) = \tilde{f}^*_\alpha(-d_{\tilde{Y}} y^{(1,0)}_{\alpha\beta} + (\partial_{\tilde{Y}} y^{(0,1)})_{\alpha\beta})$;
3. $d_{\tilde{X}} x^{(0,1)}_\alpha = \tilde{f}^*_\alpha(d_{\tilde{Y}} y^{(0,0)}_\alpha) = d_{\tilde{X}}(\tilde{f}^*_\alpha(y^{(0,0)}_\alpha))$.

Therefore, if $D_{\tilde{Y}} y = 0$ then $D_{\tilde{X}} x = 0$ as well. On the other hand, if $y^{(1,0)} = \partial_{\tilde{Y}} y^{(0,0)}$ and $y^{(0,1)} = d_{\tilde{Y}} y^{(0,0)}$ for some $(y^{(0,0)}) \in \underline{C}^0_{\tilde{Y}}$, then

$$\begin{aligned} x^{(1,0)}_{\alpha\beta} &= \tilde{f}^*_\alpha(y^{(1,0)}_{\alpha\beta}) + T_{\alpha\beta}(y^{(0,1)}_\beta) = \tilde{f}^*_\alpha(-y^{(0,0)}_\beta + y^{(0,0)}_\alpha) + T_{\alpha\beta}(d_{\tilde{Y}} y^{(0,0)}_\beta) \\ &= \tilde{f}^*_\alpha(-y^{(0,0)}_\beta + y^{(0,0)}_\alpha) + (\tilde{f}^*_\alpha - \tilde{f}^*_\beta)(y^{(0,0)}_\beta) = \tilde{f}^*_\alpha(y^{(0,0)}_\alpha) - \tilde{f}^*_\beta(y^{(0,0)}_\beta) \end{aligned}$$

and

$$x^{(0,1)}_\alpha = \tilde{f}^*_\alpha(d_{\tilde{Y}} y^{(0,0)}_\alpha) = d_{\tilde{X}}(\tilde{f}^*_\alpha(y^{(0,0)}_\alpha)).$$

This shows that $x = (x^{(1,0)}, x^{(0,1)}) = D_{\tilde{X}}(x^{(0,0)})$ for $x^{(0,0)} = (x^{(0,0)}_\alpha) := (\tilde{f}^*_\alpha(y^{(0,0)}_\alpha))$. As a result, there is a unique way to associate with each representative of $\mathsf{o}(f; \tilde{X}, \tilde{Y}, S \hookrightarrow \tilde{S}) \in H^1(X, f^*(\underline{\mathrm{Der}} Y/S) \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ an isomorphism from the first relative cohomology of $(\underline{C}^\bullet_{\tilde{Y}}, D_{\tilde{Y}})$ to the one of $(\underline{C}^\bullet_{\tilde{X}}, D_{\tilde{X}})$. If we modify $T = \{T_{\alpha\beta}\}_{\alpha\beta}$ by a coboundary then all the morphisms are also modified in a uniquely determined way that does not affect the result. This shows that we have constructed the desired morphism $\underline{H}^1_{\mathrm{dR}}(\tilde{Y}/\tilde{S}) \to \underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$. □

By construction of $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ using $\underline{C}^\bullet$, there is a morphism from $\underline{H}^0(\tilde{X}, \Omega^1_{\tilde{X}/\tilde{S}})$ to $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ and a morphism from $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ to $\underline{H}^1(\tilde{X}, \mathscr{O}_{\tilde{X}})$. Indeed, they correspond respectively to $\underline{C}^{0,1}$ and $\underline{C}^{1,0}$ in $\underline{C}^1$. Alternatively, consider the truncated subcomplex $\Omega^{\bullet \geq 1}_{\tilde{X}/\tilde{S}}$ of $\Omega^\bullet_{\tilde{X}/\tilde{S}}$ and the exact sequence $0 \to \Omega^{\bullet\geq 1}_{\tilde{X}/\tilde{S}} \to \Omega^\bullet_{\tilde{X}/\tilde{S}} \to \mathscr{O}_{\tilde{X}} \to 0$, where $\mathscr{O}_{\tilde{X}}$ is considered a complex with only one nonzero term of degree 0. Then taking (relative) first hypercohomology gives an exact sequence

$$0 \to \underline{H}^0(\tilde{X}, \Omega^1_{\tilde{X}/\tilde{S}}) \to \underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S}) \to \underline{H}^1(\tilde{X}, \mathscr{O}_{\tilde{X}}). \tag{2.1.6.5}$$

When $\tilde{X}$ is an abelian scheme over $\tilde{S}$, it is known that $\underline{H}^0(\tilde{X}, \Omega^1_{\tilde{X}/\tilde{S}}) \cong e^*_{\tilde{X}} \Omega^1_{\tilde{X}/\tilde{S}} =: \underline{\mathrm{Lie}}^\vee_{\tilde{X}/\tilde{S}} \cong (\underline{\mathrm{Lie}}_{\tilde{X}/\tilde{S}})^\vee$, that $\underline{H}^1(\tilde{X}, \mathscr{O}_{\tilde{X}}) \cong \underline{\mathrm{Lie}}_{\tilde{X}^\vee/\tilde{S}}$, and that these are all locally free $\mathscr{O}_{\tilde{S}}$-modules. Moreover, by [18, Lem. 2.5.3], the last morphism in (2.1.6.5) is actually surjective, and we obtain the exact sequence

$$0 \to \underline{\mathrm{Lie}}^\vee_{\tilde{X}/\tilde{S}} \to \underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S}) \to \underline{\mathrm{Lie}}_{\tilde{X}^\vee/\tilde{S}} \to 0. \tag{2.1.6.6}$$

If we dualize this exact sequence (2.1.6.6), then we obtain the exact sequence

$$0 \to \underline{\mathrm{Lie}}^\vee_{\tilde{X}^\vee/\tilde{S}} \to \underline{H}^{\mathrm{dR}}_1(\tilde{X}/\tilde{S}) \to \underline{\mathrm{Lie}}_{\tilde{X}/\tilde{S}} \to 0. \tag{2.1.6.7}$$

Here $\underline{H}^{\mathrm{dR}}_1(\tilde{X}/\tilde{S})$ is the dual of $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$, formally defined to be the *first de Rham homology* of $\tilde{X}$.

If $\tilde{X}'$ is a different lifting in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$, then we have a similar exact sequence for $\tilde{X}'$. Note that the dual of the canonical isomorphism $\underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S}) \cong \underline{H}^1_{\mathrm{dR}}(\tilde{X}'/\tilde{S})$ does not map $\underline{\mathrm{Lie}}^\vee_{\tilde{X}^\vee/\tilde{S}}$ to $\underline{\mathrm{Lie}}^\vee_{(\tilde{X}')^\vee/\tilde{S}}$: We saw in the proof of Proposition 2.1.6.2, in particular in the explicit morphism (2.1.6.3), that if $\tilde{X}'$ is a different lifting in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$, then the part mapping onto $\underline{\mathrm{Lie}}_{\tilde{X}^\vee/\tilde{S}}$ is mapped, under the map $\underline{C}^1 \to (\underline{C}')^1$ defining the canonical isomorphism, to a submodule different from the part mapping onto $\underline{\mathrm{Lie}}_{(\tilde{X}')^\vee/\tilde{S}}$. On the other hand, since $\tilde{X}$ and $\tilde{X}'$ are both liftings of X, all their corresponding objects are identical after pullback from $\tilde{S}$ to S. Therefore, we have two subsheaves $\underline{\mathrm{Lie}}^\vee_{\tilde{X}^\vee/\tilde{S}}$ and $\underline{\mathrm{Lie}}^\vee_{(\tilde{X}')^\vee/\tilde{S}}$ in $\underline{H}^1_{\mathrm{dR}}(\tilde{X}'/\tilde{S}) \cong \underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ such that

$$\underline{\mathrm{Lie}}^\vee_{\tilde{X}^\vee/\tilde{S}} \underset{\mathscr{O}_{\tilde{S}}}{\otimes} \mathscr{O}_S = \underline{\mathrm{Lie}}^\vee_{(\tilde{X}')^\vee/\tilde{S}} \underset{\mathscr{O}_{\tilde{S}}}{\otimes} \mathscr{O}_S = \underline{\mathrm{Lie}}^\vee_{X^\vee/S}$$

in the *same* space $\underline{H}^{\mathrm{dR}}_1(X/S)$. We say in this case that these two subsheaves are the same modulo $\mathscr{I}$. Let us consider the projective $\mathscr{O}_{\tilde{S}}$-submodules $\mathscr{M}$ in $\underline{H}^{\mathrm{dR}}_1(\tilde{X}/\tilde{S})$ such that $\mathscr{M}$ become the same as $\underline{\mathrm{Lie}}^\vee_{X^\vee/S}$ modulo $\mathscr{I}$ and such that $\underline{H}^{\mathrm{dR}}_1(\tilde{X}/\tilde{S})/\mathscr{M}$ are projective. Using the exact sequence (2.1.6.7), we see that such $\mathscr{O}_{\tilde{S}}$-submodules are parameterized by the global sections of

$$\begin{aligned} \underline{\mathrm{Hom}}_{\mathscr{O}_{\tilde{S}}}(\underline{\mathrm{Lie}}^\vee_{\tilde{X}^\vee/\tilde{S}}, \mathscr{I} \cdot \underline{\mathrm{Lie}}_{\tilde{X}/\tilde{S}}) &\cong \underline{\mathrm{Hom}}_{\mathscr{O}_S}(\underline{\mathrm{Lie}}^\vee_{X^\vee/S}, \underline{\mathrm{Lie}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}) \\ &\cong \underline{\mathrm{Lie}}_{X^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I} \cong \underline{H}^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I}). \end{aligned}$$

When S is *affine*, this is the same set $H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ that parameterizes different liftings in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$. Thus they must coincide. Now we can conclude with the following analogue of a weaker form of the main theorem in *Grothendieck–Messing theory* (cf. [90] and [58]):

Proposition 2.1.6.8. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion of* ***affine schemes*** *defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let $\tilde{X}$ be an abelian scheme over $\tilde{S}$. Consider the exact sequence (2.1.6.7) associated with $\tilde{X}$. Then the objects in $\mathsf{Lift}(X; S \hookrightarrow \tilde{S})$ are in bijection with $\mathscr{O}_{\tilde{S}}$-submodules $\mathscr{M}$ in exact sequences*

$$0 \to \mathscr{M} \to \underline{H}^{\mathrm{dR}}_1(\tilde{X}/\tilde{S}) \to \mathscr{N} \to 0$$

*of **projective** $\mathscr{O}_{\tilde{S}}$**-modules** such that $\mathscr{M} \underset{\mathscr{O}_{\tilde{S}}}{\otimes} \mathscr{O}_S = \underline{\mathrm{Lie}}^\vee_{X^\vee/S}$ in $\underline{H}^{\mathrm{dR}}_1(X/S) = \underline{H}^{\mathrm{dR}}_1(\tilde{X}/\tilde{S}) \underset{\mathscr{O}_{\tilde{S}}}{\otimes} \mathscr{O}_S$.*

Similar analysis for the case of lifting morphisms (following the explicit construction in the proof of Proposition 2.1.6.4) shows the following:

Proposition 2.1.6.9. *With the setting as above, if $\tilde{Y}$ is an abelian scheme over $\tilde{S}$ and $f : X := \tilde{X} \underset{\tilde{S}}{\times} S \to Y := \tilde{Y} \underset{\tilde{S}}{\times} S$ is a morphism of underlying schemes defined over S, then f can be lifted to a morphism of schemes $\tilde{f} : \tilde{X} \to \tilde{Y}$ over $\tilde{S}$ if and only if $\underline{\mathrm{Lie}}^\vee_{\tilde{X}^\vee/\tilde{S}}$ is mapped to $\underline{\mathrm{Lie}}^\vee_{\tilde{Y}^\vee/\tilde{S}}$ under the dual of the canonical morphism $\underline{H}^1_{\mathrm{dR}}(\tilde{Y}/\tilde{S}) \to \underline{H}^1_{\mathrm{dR}}(\tilde{X}/\tilde{S})$ (whose reduction modulo $\mathscr{I}$ maps $\underline{\mathrm{Lie}}^\vee_{X^\vee/S}$ to $\underline{\mathrm{Lie}}^\vee_{Y^\vee/S}$).*

That is, the bijection in Proposition 2.1.6.8 is *functorial* in nature.

2.1.7 Kodaira–Spencer Morphisms

Let S be a scheme over some fixed choice of universal base scheme U. Let $\tilde{S}$ be the first infinitesimal neighborhood of the image of the closed immersion $\Delta : S \hookrightarrow S \underset{\mathsf{U}}{\times} S$. (We are using the running assumption that the scheme S is separated here.) Concretely, if $\mathscr{I}$ is the ideal defining the image of the closed immersion $\Delta : S \hookrightarrow S \underset{\mathsf{U}}{\times} S$, then $\tilde{S}$ is the subscheme of $S \underset{\mathsf{U}}{\times} S$ defined by $\mathscr{I}^2$. By abuse of notation, we shall denote the pullback of $\mathscr{I}$ to $\tilde{S}$ by the same notation $\mathscr{I}$. Then the closed immersion $S \hookrightarrow \tilde{S}$ is defined by the ideal sheaf $\mathscr{I}$ satisfying $\mathscr{I}^2 = 0$, a typical situation we have studied so far.

The two projections of $S \underset{\mathsf{U}}{\times} S \to S$ induce two canonical sections $\mathrm{pr}_1, \mathrm{pr}_2 : \tilde{S} \to S$ of $S \hookrightarrow \tilde{S}$. By definition, $\Omega^1_{S/\mathsf{U}} := \Delta^*(\mathscr{I}/\mathscr{I}^2)$, and there is a universal differential $d : \mathscr{O}_S \to \Omega^1_{S/\mathsf{U}}$ given by $a \mapsto a \otimes 1 - 1 \otimes a = \mathrm{pr}_1^*(a) - \mathrm{pr}_2^*(a)$ for all $a \in \mathscr{O}_S$. In other words, $\Omega^1_{S/\mathsf{U}}$ is simply the sheaf of ideals $\mathscr{I}$ over $\mathscr{O}_{\tilde{S}}$ considered as a sheaf of ideals over $\mathscr{O}_S$.

Now suppose X is a scheme smooth over S. Then $\tilde{X}_1 := \mathrm{pr}_1^*(X)$ and $\tilde{X}_2 := \mathrm{pr}_2^*(X)$ are two elements of $\mathsf{Lift}(X, S \hookrightarrow \tilde{S})$. By Proposition 2.1.3.2, there is an element $\mathsf{m} \in H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{I})$ such that $\tilde{X}_1 = \mathsf{m} + \tilde{X}_2$ by the torsor structure of $\mathsf{Lift}(X, S \hookrightarrow \tilde{S})$. Since $\mathscr{I}$ is identified with $\Omega^1_{X/S}$ in this situation, we have obtained an element m in $H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}})$ describing the difference between $\tilde{X}_1$ and $\tilde{X}_2$.

Definition 2.1.7.1. *This element m is called the **Kodaira–Spencer class** of X over S (over the universal base scheme U). We shall denote m by the symbol $\mathsf{KS}_{X/S/\mathsf{U}}$ to signify its meaning as a Kodaira–Spencer class.*

Let f denote the structural morphism $X \to S$, which is smooth by our assumption. Then the *first exact sequence* for $X \to S \to \mathsf{U}$ is of the form

$$0 \to f^*(\Omega^1_{S/\mathsf{U}}) \to \Omega^1_{X/\mathsf{U}} \to \Omega^1_{X/S} \to 0. \tag{2.1.7.2}$$

By [59, IV-4, 17.2.3, 17.3.1, 17.5.2], smoothness of f implies that (2.1.7.2) is exact and locally split. By splitting this exact sequence locally over affine open subschemes, the extension class of this exact sequence in $\mathrm{Ext}^1_{\mathscr{O}_X}(\Omega^1_{X/S}, f^*(\Omega^1_{S/\mathsf{U}}))$ is described by a cohomology class of

$$\begin{aligned} H^1(X, \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\Omega^1_{X/S}, f^*\Omega^1_{S/\mathsf{U}})) &\cong H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_X}{\otimes} f^*(\Omega^1_{S/\mathsf{U}})) \\ &\cong H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}). \end{aligned}$$

Proposition 2.1.7.3. *The extension class of (2.1.7.2), when represented in $H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}})$, is (up to a sign convention) the Kodaira–Spencer class $\mathsf{KS}_{X/S/\mathsf{U}}$ defined in Definition 2.1.7.1.*

Proof. Take an affine open covering $\{U_\alpha\}_\alpha$ of X such that each U_α is étale over the affine r-space $\mathbb{A}^r_S$ over S for some integer $r \geq 0$. Via the two projections from $\tilde{S}$ to S splitting $S \hookrightarrow \tilde{S}$, this open covering can be lifted to open coverings of $\tilde{X}_i := \mathrm{pr}_i^*(X)$ over $\tilde{S}$, for $i = 1, 2$. Therefore, it suffices to compare the gluing isomorphisms for $\tilde{X}_1$ and $\tilde{X}_2$. As in the proof of Proposition 2.1.3.2, the comparison is given by morphisms $T_{\alpha\beta} : \Omega^1_{U_{\alpha\beta}/S} \to \mathscr{O}_{U_{\alpha\beta}} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}$, and $\mathsf{KS}_{X/S/\mathsf{U}}$ is represented by the 1-cocycle formed by these $T_{\alpha\beta}$.

Suppose $\Omega^1_{U_\alpha/S}$ has $\mathscr{O}_S$-basis elements $dx_1, \ldots, dx_r$ given by the coordinates of $\mathbb{A}^r_S$, and suppose $\Omega^1_{U_\beta/S}$ has $\mathscr{O}_S$-basis elements $dy_1, \ldots, dy_r$ given by the coordinates of $\mathbb{A}^r_S$ (for the same r), then there is a change of coordinates $dx_i = \sum\limits_{1 \leq j \leq r} a_{ij} dy_j$ for some $a_{ij} \in \mathscr{O}_S$ as both induce bases for $\Omega^1_{U_{\alpha\beta}/S}$ by restriction. This form an invertible matrix $a = (a_{ij})_{1 \leq i \leq r, 1 \leq j \leq r}$ over $\mathscr{O}_S$, and for convenience let us denote its inverse matrix by $a^{-1} = (a^{ij})_{1 \leq i \leq r, 1 \leq j \leq r}$. Note that the comparison of two pullbacks by projections over $\tilde{S}$ gives the universal differentiation $d : \mathscr{O}_S \to \Omega^1_{S/\mathsf{U}}$ for $\mathscr{O}_S$. Therefore, we have two morphisms $dx_i \mapsto \sum\limits_{1 \leq j \leq r} \mathrm{pr}_1^*(a_{ij}) dy_j$ and $dx_i \mapsto \sum\limits_{1 \leq j \leq r} \mathrm{pr}_2^*(a_{ij}) dy_j$, the first given by multiplying the corresponding matrix entries of $\mathrm{Id} + (da)a^{-1}$ by the second one, or more explicitly by multiplying $\mathrm{Id} + \sum\limits_{1 \leq k \leq r} da_{ik} a^{kj}$ by $\sum\limits_{1 \leq j \leq r} a_{ij} dx_j$. This shows that (up to a sign convention) $T_{\alpha\beta}$ is given by the matrix $(da)a^{-1}$.

On the other hand, the statement that U_α is étale over $\mathbb{A}^r_S$ with $\mathscr{O}_S$ coordinates $x_1, \ldots, x_r$ also shows that we may split (2.1.7.2) over U_α by taking the basis elements $dx_1, \ldots, dx_r$ of $\Omega^1_{U_\alpha/S}$ as part of a basis of $\Omega^1_{U_\alpha/\mathsf{U}}$. Similarly, if we split (2.1.7.2) over U_β by the basis elements $dy_1, \ldots, dy_r$ of $\Omega^1_{U_\beta/S}$, then the difference of the two splittings is again measured by $(da)a^{-1}$. This gives exactly the 1-cocycle representing the extension class of (2.1.7.2) in $H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}})$. $\square$

Remark 2.1.7.4. In the context of Section 2.1.6, there is a canonical isomorphism between the first de Rham cohomologies of the two liftings $\tilde{X}_1$ and $\tilde{X}_2$ of X given by the Kodaira–Spencer class $\mathsf{KS}_{X/S/\mathsf{U}}$. On the other hand, since the two liftings come from pullback by projections, they are naturally isomorphic if we identify the two projections by a flipping isomorphism. The difference of the two canonical isomorphisms gives a morphism

$$\underline{H}^1_{\mathrm{dR}}(\tilde{X}_1/\tilde{S}) \to \mathscr{I} \cdot \underline{H}^1_{\mathrm{dR}}(\tilde{X}_2/\tilde{S}), \tag{2.1.7.5}$$

which induces the morphism

$$\underline{H}^0(X, \Omega^1_{X/S}) \to \underline{H}^1(X, \mathscr{O}_X \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}) \tag{2.1.7.6}$$

on the graded pieces. This is nothing but the morphism defined by the cup product with the Kodaira–Spencer class $\mathsf{KS}_{X/S/\mathsf{U}}$. When $\Omega^1_{S/\mathsf{U}}$ is locally free over $\mathscr{O}_S$ of finite rank (which is, in particular, the case when S is smooth over U), we can rewrite the morphism (2.1.7.6) as

$$\underline{H}^0(X, \Omega^1_{X/S}) \to \underline{H}^1(X, \mathscr{O}_X) \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}} \tag{2.1.7.7}$$

(by the projection formula [59, 0_{I}, 5.4.10.1]). Hence we can rewrite the morphism (2.1.7.5) as

$$\underline{H}^1_{\mathrm{dR}}(X/S) \to \underline{H}^1_{\mathrm{dR}}(X/S) \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}},$$

which is the so-called *Gauss–Manin connection* of $\underline{H}^1_{\mathrm{dR}}(X/S)$.

Now let us assume that $\Omega^1_{S/\mathsf{U}}$ is locally free of finite rank over $\mathscr{O}_S$. Suppose X is an abelian scheme over S. Then we have canonical identifications $\underline{H}^0(X, \Omega^1_{X/S}) \cong \underline{H}^0(X, \mathscr{O}_X) \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{X/S} \cong \underline{\mathrm{Lie}}^\vee_{X/S} \cong (\underline{\mathrm{Lie}}_{X/S})^\vee$ and $\underline{H}^1(X, \mathscr{O}_X) \cong \underline{\mathrm{Lie}}_{X^\vee/S}$ (given by Lemma 2.1.5.11 and Corollary 2.1.5.9). By the canonical identification $\underline{\mathrm{Lie}}^\vee_{X^\vee/S} \cong (\underline{\mathrm{Lie}}_{X^\vee/S})^\vee$ as well, we may interpret the morphism (2.1.7.7) as

$$\mathrm{KS}_{X/S/\mathsf{U}} : \underline{\mathrm{Lie}}^\vee_{X/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{X^\vee/S} \to \Omega^1_{S/\mathsf{U}}. \tag{2.1.7.8}$$

Definition 2.1.7.9. *The morphism* $\mathrm{KS}_{X/S/\mathsf{U}}$ *in* (*2.1.7.8*) *is called the* ***Kodaira–Spencer morphism*** *for the abelian scheme* X *over* S (*over the base scheme* U).

2.2 Formal Theory

2.2.1 Local Moduli Functors and Schlessinger's Criterion

Let us make precise the meaning of local moduli problems, or rather infinitesimal deformations, associated with $\mathsf{M}_{\mathcal{H}}$.

Let $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ and let s be a point of finite type over S_0. Let k be a finite field extension of $k(s)$. Let $p = \mathrm{char}(k)$. If $p > 0$, then $p \in \Box$ and $k(s)$ and k are necessarily finite fields of characteristic p. By assumption, p is unramified in F, and by Corollary 1.2.5.7, p is unramified in F_0. Therefore the completion $\hat{\mathscr{O}}_{\mathsf{S}_0,s}$ is simply the Witt vectors $W(k(s))$ (by Theorem B.1.1.9). If $p = 0$, then $\Box$ is necessarily a finite set, and k is a field extension of F_0.

Following Lemma B.1.1.11, set $\Lambda := k$ when $p = 0$ and set $\Lambda := W(k)$ when $p > 0$. We denote by C_Λ the category of Artinian local Λ-algebras with residue field k and by $\hat{\mathsf{C}}_\Lambda$ the category of noetherian complete local Λ-algebras with residue field k. (These are the same definitions as in Notation B.1.1.) Then we see that $\hat{\mathsf{C}}_\Lambda$ can be viewed as the infinitesimal neighborhoods of $\mathrm{Spec}(k) \to \mathsf{S}_0$, and C_Λ can be viewed as those in which $\mathrm{Spec}(k)$ is defined by nilpotent ideals. For simplicity, let us abbreviate C_Λ by C and $\hat{\mathsf{C}}_\Lambda$ by $\hat{\mathsf{C}}$. Note that the notation Λ here is consistent with the one in Section 1.1.3 (with $R_0 = \mathbb{Z}$).

Let us denote by $\xi_0 : \mathrm{Spec}(k) \to \mathsf{M}_{\mathcal{H}}$ a point of $\mathsf{M}_{\mathcal{H}}$ corresponding to an object $\xi_0 = (A_0, \lambda_0, i_0, \alpha_{n,0})$ in $\mathsf{M}_{\mathcal{H}}(\mathrm{Spec}(k))$. Let us denote by Def_{ξ_0} the functor from $\hat{\mathsf{C}}$ to (Sets) defined by the assignment

$$R \mapsto \{\text{isomorphism classes of pairs } (\xi, f_0)\ \},$$

where $\xi = (A, \lambda, i, \alpha_{\mathcal{H}})$ is an object in $\mathsf{M}_{\mathcal{H}}(\mathrm{Spec}(R))$, and where

$$f_0 : \xi \underset{R}{\otimes} k := \mathsf{M}_{\mathcal{H}}(\mathrm{Spec}(R) \to \mathrm{Spec}(k))(\xi) \xrightarrow{\sim} \xi_0$$

is an isomorphism (in the sense of Definition 1.4.1.3) identifying $\xi \underset{R}{\otimes} k$ with ξ_0 (see Appendix B, especially Theorem B.3.9 for the reasoning behind this definition).

In the remainder of this chapter, our first main objective is to show that Def_{ξ_0} is effectively prorepresentable and formally smooth, and to show that Theorems B.3.7, B.3.9, and B.3.11 can be applied. (Note that, without the effectiveness, prorepresentability is a condition for $\mathsf{Def}_{\xi_0}|_{\mathsf{C}}$ only.)

To achieve this, it is helpful to introduce some other functors deforming fewer structures, and hence easier to understand than $\mathsf{Def}_{\xi_0} = \mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$:

1. Let us denote by Def_{A_0} the functor from $\hat{\mathsf{C}}$ to (Sets) defined by
$$R \mapsto \{\text{isomorphism classes of pairs } (A, f_0) \text{ over } R\},$$
where
   (a) A is an abelian scheme over R;
   (b) $f_0 : A \underset{R}{\otimes} k \xrightarrow{\sim} A_0$ is an isomorphism.

2. Let us denote by $\mathsf{Def}_{(A_0,\lambda_0)}$ the functor from $\hat{\mathsf{C}}$ to (Sets) defined by
$$R \mapsto \{\text{isomorphism classes of triples } (A, \lambda, f_0) \text{ over } R\},$$
where
   (a) A is an abelian scheme over R;
   (b) $\lambda : A \to A^\vee$ is a polarization of A;
   (c) $f_0 : A \underset{R}{\otimes} k \xrightarrow{\sim} A_0$ is an isomorphism that pulls λ_0 back to $\lambda \underset{R}{\otimes} k$.

3. Let us denote by $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$ the functor from $\hat{\mathsf{C}}$ to (Sets) defined by
$$R \mapsto \{\text{isomorphism classes of tuples } (A, \lambda, i, f_0) \text{ over } R\},$$
where
   (a) A is an abelian scheme over R;
   (b) $\lambda : A \to A^\vee$ is a polarization of A;
   (c) $i : \mathcal{O} \to \mathrm{End}_R(A)$ is an endomorphism structure;
   (d) $\underline{\mathrm{Lie}}_{A/\mathrm{Spec}(R)}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-module structure given naturally by i satisfies the determinantal condition in Definition 1.3.4.1 given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h)$;
   (e) $f_0 : A \underset{R}{\otimes} k \xrightarrow{\sim} A_0$ is an isomorphism that pulls λ_0 back to $\lambda \underset{R}{\otimes} k$ and pulls i_0 back to $i \underset{R}{\otimes} k$.

We will show the prorepresentability and formal smoothness separately.

Let us record Schlessinger's general criterion of prorepresentability before we proceed. Following Schlessinger's fundamental paper [109],

Definition 2.2.1.1. *A surjection* $p : \tilde{R} \twoheadrightarrow R$ *in* C *is a* ***small surjection*** *if the kernel of* p *is an ideal* I *such that* $I \cdot \mathfrak{m}_{\tilde{R}} = 0$, *where* $\mathfrak{m}_{\tilde{R}}$ *is the maximal ideal of* $\tilde{R}$.

Remark 2.2.1.2. Every surjection in C is the composition of a finite number of small surjections.

If a functor $F : \mathsf{C} \to (\text{Sets})$ is prorepresentable, then the following two conditions necessarily hold:

1. $F(k)$ has exactly one element. (Here k is understood as the final object of C.)
2. For all surjections $\tilde{R} \twoheadrightarrow R$ and $Q \twoheadrightarrow R$ in C, the functoriality of F gives an isomorphism
$$F(Q \underset{R}{\times} \tilde{R}) \xrightarrow{\sim} F(Q) \underset{F(R)}{\times} F(\tilde{R}). \tag{2.2.1.3}$$

We may ask how much of the converse is true. The answer is provided by the following theorem of Schlessinger's:

Theorem 2.2.1.4. *A covariant functor $F : \mathsf{C} \to (\text{Sets})$ such that $F(k)$ has exactly one element is prorepresentable if and only if F satisfies (2.2.1.3) for all surjections $\tilde{R} \twoheadrightarrow R$ and $Q \twoheadrightarrow R$, and $\dim_k(F(k[\varepsilon]/(\varepsilon^2))) < \infty$. It suffices to check (2.2.1.3) in the case that the surjection $\tilde{R} \twoheadrightarrow R$ is a small surjection (as in Definition 2.2.1.1).*

Moreover, suppose F is prorepresentable by some $R^{\text{univ}} \in \hat{\mathsf{C}}$ that is formally smooth (namely, $F(\tilde{R}) \to F(R)$ is surjective for every surjection $\tilde{R} \twoheadrightarrow R$ in C) and satisfies $\dim_k(F(k[\varepsilon]/(\varepsilon^2))) = m$. Then there is an isomorphism $R^{\text{univ}} \cong \Lambda[[t_1, \ldots, t_m]]$.

Proof. The first half is a weakened form of [109, Thm. 2.11]. The second half is just [109, Prop. 2.5(i)]. □

Remark 2.2.1.5. For ease of notation, when $\tilde{S} = \operatorname{Spec}(\tilde{R})$ and $S = \operatorname{Spec}(R)$, we shall write $\mathsf{o}(X; \tilde{R} \twoheadrightarrow R)$ etc. in place of $\mathsf{o}(X; S \hookrightarrow \tilde{S})$ etc. in Section 2.1. If $\tilde{R} \twoheadrightarrow R$ is a *small surjection* with kernel I (as in Definition 2.2.1.1), then $\tilde{M} \underset{\tilde{R}}{\otimes} I \cong (\tilde{M}/\mathfrak{m}_{\tilde{R}} \cdot \tilde{M}) \underset{k}{\otimes} I$ for every $\tilde{R}$-module $\tilde{M}$, because $I \cdot \mathfrak{m}_{\tilde{R}} = 0$. Hence we shall define $S_0 := \operatorname{Spec}(k)$, $X_0 := X \underset{R}{\otimes} k$, etc., and write $H^1(X_0, \underline{\mathrm{Der}}_{X_0/S_0}) \underset{k}{\otimes} I$ in place of $H^1(X, \underline{\mathrm{Der}}_{X/S} \underset{R}{\otimes} I)$, as it is more precise. Note that we are allowed to write $H^1(X_0, \underline{\mathrm{Der}}_{X_0/S_0}) \underset{k}{\otimes} I$ instead of $H^1(X_0, \underline{\mathrm{Der}}_{X_0/S_0} \underset{k}{\otimes} I)$ (by the projection formula [59, 0_{I}, 5.4.10.1]) because I is free of finite rank over the residue field $k = \tilde{R}/\mathfrak{m}_{\tilde{R}}$.

2.2.2 Rigidity of Structures

Let us retain the notation in Section 2.2.1 and the conventions mentioned in Remark 2.2.1.5. Apart from Proposition 2.2.2.9 and consequently Corollary 2.2.2.10, where we do need more refined assumptions on k and hence on Λ (to make Proposition 1.1.2.20 work), the remaining results in this section work for arbitrary choices of k and Λ as at the beginning of Section 2.2.1 (or in Section B.1).

Let us first show that Def_{A_0} can be understood by the deformation of the underlying smooth scheme structures. The rigidity of abelian schemes has the following implication:

Lemma 2.2.2.1. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of nilpotent ideals $\mathscr{I}$. Let $\tilde{A}$ and $\tilde{A}'$ be two abelian schemes over $\tilde{S}$, and let $A := \tilde{A} \underset{\tilde{S}}{\times} S$ and $A' := \tilde{A}' \underset{\tilde{S}}{\times} S$. Then the restriction map from the set of group homomorphisms $\operatorname{Hom}_{\tilde{S}}(\tilde{A}, \tilde{A}')$ to $\operatorname{Hom}_S(A, A')$ is **injective**. Similarly, the restriction map from the set of group isomorphisms $\operatorname{Isom}_{\tilde{S}}(\tilde{A}, \tilde{A}')$ to $\operatorname{Isom}_S(A, A')$ is **injective**.*

Proof. Suppose f and g are two group homomorphisms from $\tilde{A}$ to $\tilde{A}'$ such that $f \underset{\tilde{S}}{\times} S = g \underset{\tilde{S}}{\times} S$. Then, by Corollary 1.3.1.5, there exists a section $\eta : \tilde{S} \to \tilde{A}'$ such that $g = f + \eta$. Since f and g are group homomorphisms, both of them send the identity section $e_{\tilde{A}}$ of $\tilde{A}$ to the identity section $e_{\tilde{A}'}$ of $\tilde{A}'$. This forces η to be the identity section $e_{\tilde{A}'}$, and hence $f = g$. The argument for group isomorphisms is identical. □

In particular, there are no *infinitesimal automorphisms* for A as an *abelian scheme* over S. Note that we might have infinitesimal automorphisms if we only consider A as a *scheme* smooth over S. For example, when $R = k[\varepsilon]/(\varepsilon^2)$, we know they are parameterized by $H^0(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \cong \mathrm{Lie}_{A_0/S_0}$. The essential extra freedom is controlled by the choice of identity sections:

Corollary 2.2.2.2. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of nilpotent ideals $\mathscr{I}$. Suppose $\tilde{A}$ and $\tilde{A}'$ are abelian schemes over $\tilde{S}$, $A := \tilde{A} \underset{\tilde{S}}{\times} S$, $A' := \tilde{A}' \underset{\tilde{S}}{\times} S$, and $f : A \to A'$ is a (group scheme) homomorphism that is lifted to some morphism $\tilde{f} : \tilde{A} \to \tilde{A}'$ (not necessarily a homomorphism) over $\tilde{S}$ in the sense that $\tilde{f} \underset{\tilde{S}}{\times} S = f$. Then, by replacing $\tilde{f}$ with $\tilde{f} - \tilde{f}(e_{\tilde{A}})$, we obtain the unique homomorphism lifting f.*

Proof. The replacement works because of Corollary 1.3.1.6. The claim of uniqueness is just Lemma 2.2.2.1. □

Now we can state the following important fact:

Proposition 2.2.2.3 ([99, Prop. 2.27])**.** *Suppose $p : \tilde{R} \twoheadrightarrow R$ is a small surjection in $\hat{\mathsf{C}}$ with kernel I, and suppose (A, f_0) represents an object of $\mathsf{Def}_{A_0}(R)$. Let*
$$\mathsf{Def}_{A_0}(p)^{-1}([(A, f_0)])$$
be the set of isomorphism classes of objects $(\tilde{A}, \tilde{f}_0)$ in $\mathsf{Def}_{A_0}(\tilde{R})$ that are mapped to the class of (A, f_0). Then forgetting the structure of abelian schemes induces a well-defined bijection
$$\mathsf{Def}_{A_0}(p)^{-1}([(A, f_0)]) \to \mathsf{Lift}(A; \tilde{R} \twoheadrightarrow R). \tag{2.2.2.4}$$

Proof. Let us first show that the map (2.2.2.4) is well defined. If $\mathsf{Def}_{A_0}(p)$ maps the class of $(\tilde{A}, \tilde{f}_0)$ to (A, f_0), then we have some group isomorphism $\psi : \tilde{A} \underset{\tilde{R}}{\otimes} R \xrightarrow{\sim} A$ reducing to $f_0^{-1} \circ \tilde{f}_0 : \tilde{A} \underset{\tilde{R}}{\otimes} k \xrightarrow{\sim} A \underset{R}{\otimes} k$. Suppose $(\tilde{A}', \tilde{f}_0')$ is also a representative of some object in $\mathsf{Def}_{A_0}(p)^{-1}([(A, f_0)])$. Let $(\tilde{A}', \psi')$ be associated with $(\tilde{A}', \tilde{f}_0')$ by the above recipe. If there is an isomorphism $h : \tilde{A} \xrightarrow{\sim} \tilde{A}'$ over $\tilde{R}$ such that $h \underset{\tilde{R}}{\otimes} k = (\tilde{f}_0')^{-1} \circ \tilde{f}_0$ defines an isomorphism $(\tilde{A}, \tilde{f}_0) \xrightarrow{\sim} (\tilde{A}', \tilde{f}_0')$, then both ψ and $\psi' \circ (h \underset{\tilde{R}}{\otimes} R)$ are isomorphisms from $\tilde{A} \underset{\tilde{R}}{\otimes} R$ to A reducing to $\tilde{f}_0 : \tilde{A} \underset{\tilde{R}}{\otimes} k \xrightarrow{\sim} A_0$. Therefore, by Lemma 2.2.2.1, we have $\psi = \psi' \circ (h \underset{\tilde{R}}{\otimes} R)$. This shows that the isomorphism h

defines an isomorphism $(\tilde{A}, \psi) \xrightarrow{\sim} (\tilde{A}', \psi')$ and that the map (2.2.2.4) sending the isomorphism class of $(\tilde{A}, f_0)$ in $\mathsf{Def}_{A_0}(p)^{-1}([(A, f_0)])$ to the isomorphism class of $(\tilde{A}, \psi)$ in $\mathsf{Lift}(A; \tilde{R} \twoheadrightarrow R)$ is well defined.

Now suppose we have two pairs $(\tilde{A}, \tilde{f}_0)$ and $(\tilde{A}', \tilde{f}'_0)$ defining classes in $\mathsf{Def}_{A_0}(p)^{-1}([(A, f_0)])$. Let $(\tilde{A}, \psi)$ and $(\tilde{A}', \psi')$ be any two choices of pairs associated respectively with $(\tilde{A}, \tilde{f}_0)$ and $(\tilde{A}', \tilde{f}'_0)$ as above. Note that ψ and ψ' are chosen to be group isomorphisms. Suppose that there is an isomorphism $h : \tilde{A} \to \tilde{A}'$ of the underlying schemes over $\tilde{R}$ that induces an isomorphism $(\tilde{A}, \psi) \xrightarrow{\sim} (\tilde{A}', \psi')$. This means that $\psi = \psi' \circ (h \underset{\tilde{R}}{\otimes} R)$, and implies that $h \underset{\tilde{R}}{\otimes} R$ is a group isomorphism, as both ψ and ψ' are. Then, by Corollary 2.2.2.2, we may assume that h is a group isomorphism by replacing h with $h - h(e_{\tilde{A}})$. Since $\psi \underset{R}{\otimes} k = f_0^{-1} \circ \tilde{f}_0$ and $\psi' \underset{R}{\otimes} k = f_0^{-1} \circ \tilde{f}'_0$ by construction, we have $h \underset{\tilde{R}}{\otimes} k = (\psi' \underset{R}{\otimes} k)^{-1} \circ (\psi \underset{R}{\otimes} k) = (\tilde{f}'_0)^{-1} \circ \tilde{f}_0$, and hence h defines an isomorphism $(\tilde{A}, \tilde{f}_0) \xrightarrow{\sim} (\tilde{A}', \tilde{f}'_0)$. This shows the injectivity of (2.2.2.4).

Finally, the surjectivity of (2.2.2.4) follows from Proposition 2.2.2.5 below, by the automatic existence of identity sections by smoothness. $\square$

Proposition 2.2.2.5 (cf. [96, Prop. 6.15])**.** *Let $p : \tilde{R} \twoheadrightarrow R$ be a small surjection in C with kernel I. Let $\tilde{S} := \mathrm{Spec}(\tilde{R})$ and $S := \mathrm{Spec}(R)$. Let $\tilde{\pi} : \tilde{A} \to \tilde{S}$ be a proper smooth morphism with a section $\tilde{e} : \tilde{S} \to \tilde{A}$. Suppose $A := \tilde{A} \underset{\tilde{S}}{\times} S \to S$ is an abelian scheme with identity section $e := \tilde{e} \underset{\tilde{S}}{\times} S$. Then $\tilde{A} \to \tilde{S}$ is an abelian scheme with identity section $\tilde{e}$.*

Proof. Let $g : A \underset{S}{\times} A \to A$ be the morphism $g(x, y) = x - y$ defined for all functorial points x and y of A. To show that $\tilde{A}$ is an abelian scheme, our first task is to lift g to some morphism $\tilde{g} : \tilde{A} \underset{\tilde{S}}{\times} \tilde{A} \to \tilde{A}$ over $\tilde{S}$. Let $S_0 := \mathrm{Spec}(k)$, $A_0 := \tilde{A} \underset{\tilde{S}}{\times} S_0$, and $g_0 := \tilde{g} \underset{\tilde{S}}{\times} S_0$. By Proposition 2.1.3.2, there is an element

$$\mathsf{o}_0 := \mathsf{o}(g; \tilde{A} \underset{\tilde{S}}{\times} \tilde{A}, \tilde{A}, \tilde{R} \twoheadrightarrow R) \in H^1(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I$$

whose vanishing is equivalent to the existence of a morphism $\tilde{g}$ lifting g.

Let $j_1, j_2 : \tilde{A} \to \tilde{A} \underset{\tilde{S}}{\times} \tilde{A}$ be the morphisms defined by $j_1(x) = (x, e)$ and $j_2(x) = (x, x)$, respectively, for all functorial points x of $\tilde{A}$, and let $\mathrm{pr}_1, \mathrm{pr}_2 : \tilde{A} \underset{\tilde{S}}{\times} \tilde{A} \to \tilde{A}$ be the two projections. By abuse of notation, we shall use the same symbols for their pullbacks to S and S_0. Then, by repeating the arguments in the proof of Proposition 2.1.3.2 if necessary, the obstructions

$$\mathsf{o}_i := \mathsf{o}(g \circ j_i; \tilde{A}, \tilde{A}, \tilde{R} \twoheadrightarrow R) \in H^1(A_0, j_i^* g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I$$

to lifting $g \circ j_i$, for $i = 1, 2$, are related to the obstruction of lifting g by $\mathsf{o}_i = j_i^*(\mathsf{o}_0)$ under the canonical homomorphisms

$$j_i^* : H^1(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I \to H^1(A_0, j_i^* g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I.$$

Since $\mathrm{Id}_{\tilde{A}}$ and $\tilde{e} \circ \tilde{\pi}$ do lift the morphisms $g \circ j_1 = \mathrm{Id}_A$ and $g \circ j_2 = e \circ \pi$, we must have $\mathsf{o}_i = 0$ for $i = 1, 2$.

On the other hand, as A_0 is an abelian variety, the structure of $H^1(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0}))$ can be completely understood by the Künneth formula (see [59, III-2, 6.7.8]). Moreover, by Lemma 2.1.5.11, $\underline{\mathrm{Der}}_{A_0/S_0}$ is canonically isomorphic to the pullback of $\underline{\mathrm{Lie}}_{A_0/S_0}$, the latter of which is constant over $S_0 = \mathrm{Spec}(k)$ with values in the k-vector space Lie_{A_0/S_0}. Explicitly,

$$\begin{aligned} H^1(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I &\cong H^1(A_0 \underset{S_0}{\times} A_0, \mathscr{O}_{A_0 \underset{S_0}{\times} A_0}) \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I \\ &\cong [\mathrm{pr}_1^* H^1(A_0, \mathscr{O}_{A_0}) \oplus \mathrm{pr}_2^* H^1(A_0, \mathscr{O}_{A_0})] \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I. \end{aligned}$$

As a result, every element in $H^1(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I$ can be described as a sum of elements of the two factors $[\mathrm{pr}_i^* H^1(A_0, \mathscr{O}_{A_0})] \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I$, where $i = 1, 2$. As $j_1^*(\mathsf{o}_0) = \mathsf{o}_1 = 0$ and $\mathrm{pr}_1 \circ j_1 = \mathrm{Id}_{\tilde{A}}$, we see that the first factor of o_0 is trivial. On the other hand, as $j_2^*(\mathsf{o}_0) = \mathsf{o}_2 = 0$ and $\mathrm{pr}_2 \circ j_2 = \mathrm{Id}_{\tilde{A}}$, we see that the second factor of o_0 is trivial as well. Hence we must have $\mathsf{o}_0 = 0$, and the existence of some morphism $\tilde{g} : \tilde{A} \underset{\tilde{S}}{\times} \tilde{A} \to \tilde{A}$ lifting g.

By Proposition 2.1.3.2, $\mathsf{Lift}(g; \tilde{A} \underset{\tilde{S}}{\times} \tilde{A}, \tilde{A}, \tilde{R} \twoheadrightarrow R)$ is a torsor under the group $H^0(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I$, which is canonically isomorphic to $\mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I$ by Lemma 2.1.5.11. Similarly, the restrictions of the liftings $\tilde{g}$ to $(\tilde{e}, \tilde{e}) : \tilde{S} \to \tilde{A} \underset{\tilde{S}}{\times} \tilde{A}$ form a torsor under the group $H^0(S_0, (g_0|_{(e_{A_0}, e_{A_0})})^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I$, which is also canonically isomorphic to $\mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I$. As the restriction to $(\tilde{e}, \tilde{e})$ defines a natural morphism between the two torsors equivariant under the same group, we see that there exists a unique lifting $\tilde{g}$ of g that sends $(\tilde{e}, \tilde{e})$ to $\tilde{e}$.

It remains to prove that $\tilde{g}$ determines a group structure of $\tilde{A}$ with identity section $\tilde{e}$. The existence of $\tilde{g}$ gives formal definitions of the inverse and multiplication morphisms, and it only remains to check the various compatibility relations given by morphisms of the form $h : \tilde{A} \underset{\tilde{S}}{\times} \cdots \underset{\tilde{S}}{\times} \tilde{A} \to \tilde{A}$, which sends $(\tilde{e}, \ldots, \tilde{e})$ to $\tilde{e}$, and sends everything to e over S. Certainly, the condition to check is that h sends everything to $\tilde{e}$ over $\tilde{S}$. By Proposition 1.3.1.4, there is necessarily a section $\eta : \tilde{S} \to \tilde{A}$ such that h is the composition of the structural projection $\tilde{A} \underset{\tilde{S}}{\times} \cdots \underset{\tilde{S}}{\times} \tilde{A} \to \tilde{S}$ with η. Since h sends $(\tilde{e}, \ldots, \tilde{e})$ to $\tilde{e}$, this η must be the section $\tilde{e}$, as desired. $\square$

Let us state similar rigidity results for some other structures as corollaries of Lemma 2.2.2.1:

Corollary 2.2.2.6. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of nilpotent ideals $\mathscr{I}$. Let $\tilde{A}$ be an abelian scheme over $\tilde{S}$, $A := \tilde{A} \underset{\tilde{S}}{\times} S$, and $\lambda : A \to A^\vee$ some **polarization** of A (see Definition 1.3.2.16). Suppose $\tilde{\lambda} : \tilde{A} \to \tilde{A}^\vee$ is any homomorphism such that $\tilde{\lambda} \underset{\tilde{S}}{\times} S = \lambda$. Then $\tilde{\lambda}$ is necessarily a polarization of $\tilde{A}$.*

Proof. Note that both $\tilde{\lambda}$ and $\tilde{\lambda}^\vee$ lift $\lambda = \lambda^\vee$, where the symmetry follows because λ is a polarization. Hence $\tilde{\lambda} = \tilde{\lambda}^\vee$ by Lemma 2.2.2.1. Now that $\tilde{\lambda}$ is symmetric, by Proposition 1.3.2.15, it suffices to know that $\tilde{\lambda}$ is a polarization over each geometric point of $\tilde{S}$, which is true because λ is a polarization over each geometric point of S. $\square$

Corollary 2.2.2.7. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of nilpotent ideals $\mathscr{I}$. Let $\tilde{A}$ be an abelian scheme over $\tilde{S}$ and $A := \tilde{A} \underset{\tilde{S}}{\times} S$. With the setting as in Section 1.3.3, there is at most one way to lift an $\mathcal{O}$-endomorphism structure from A to $\tilde{A}$. Moreover, to check the existence of liftings, we do not have to check the Rosati condition.*

Proof. It is clear from Lemma 2.2.2.1 that liftings of morphisms are unique (if they exist). Since the Rosati condition is defined by relations of group homomorphisms that are already verified over S, it is automatic over $\tilde{S}$ by Corollary 1.3.1.5. $\square$

Corollary 2.2.2.8. *Let $S \hookrightarrow \tilde{S}$ be a closed immersion defined by a sheaf of nilpotent ideals $\mathscr{I}$. Let $(\tilde{A}, \tilde{\lambda}, \tilde{i})$ be a polarized abelian scheme with endomorphism structures over $\tilde{S}$, and let $(A, \lambda, i) := (\tilde{A}, \tilde{\lambda}, \tilde{i}) \underset{\tilde{S}}{\times} S$. With the setting as in Section 1.3.7, each level structure over A is uniquely liftable to $\tilde{A}$.*

Proof. By definition, level structures are orbits of étale-locally-defined morphisms between étale group schemes that are successively liftable to symplectic isomorphisms of higher levels after making étale base changes. Then the corollary is true because morphisms between schemes étale over S are uniquely liftable to $\tilde{S}$ by Lemma 2.1.1.6. $\square$

Proposition 2.2.2.9. *With assumptions as above, let $\tilde{R} \twoheadrightarrow R$ be a surjection in $\hat{\mathsf{C}}$. Suppose that $\tilde{A} \to \tilde{S}$ is an abelian scheme, that $A := \tilde{A} \underset{\tilde{S}}{\times} S$, and that both of them admit compatibly the necessary polarizations and endomorphism structures such that the determinantal condition in Definition 1.3.4.1 is defined. Then $\underline{\mathrm{Lie}}_{\tilde{A}/\tilde{S}}$ satisfies the condition if and only if $\underline{\mathrm{Lie}}_{A/S}$ does.*

Proof. It suffices to treat the case $R = k$. Let $A_0 := \tilde{A} \underset{\tilde{S}}{\times} S_0$, so that $(\mathrm{Lie}_{\tilde{A}/\tilde{S}}) \underset{\tilde{R}}{\otimes} k \cong \mathrm{Lie}_{A_0/S_0}$ as $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$-modules. Let F_0' and L_0 be chosen as in Lemma 1.2.5.9, with F_0' unramified at p when $p > 0$. Since our purpose is to verify whether an equation is satisfied, we may replace k with a finite field extension and assume that the homomorphism $\mathcal{O}_{F_0,(\Box)} \to \Lambda$ extends to $\mathcal{O}_{F_0',(p)} \to \Lambda$. Then we have $\mathrm{Det}_{\mathcal{O}|V_0} = \mathrm{Det}_{\mathcal{O}|L_0} = \mathrm{Det}_{\mathcal{O}|L_0 \underset{\mathcal{O}_{F_0'}}{\otimes} \tilde{R}}$ in $k[\mathcal{O}^\vee]$, and it follows from Lemma 1.2.5.11 that $\mathrm{Det}_{\mathcal{O}|\underline{\mathrm{Lie}}_{\tilde{A}/\tilde{S}}} = \mathrm{Det}_{\mathcal{O}|V_0}$ in $k[\mathcal{O}^\vee]$ if and only if $\mathrm{Det}_{\mathcal{O}|\underline{\mathrm{Lie}}_{A_0/S_0}} = \mathrm{Det}_{\mathcal{O}|V_0}$ in $k[\mathcal{O}^\vee]$. $\square$

Combining the results above,

Corollary 2.2.2.10. *The series of forgetful functors*
$$\mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})} \to \mathsf{Def}_{(A_0,\lambda_0,i_0)} \to \mathsf{Def}_{(A_0,\lambda_0)} \to \mathsf{Def}_{A_0}$$
induces the series of equivalence or embeddings
$$\mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})} \cong \mathsf{Def}_{(A_0,\lambda_0,i_0)} \hookrightarrow \mathsf{Def}_{(A_0,\lambda_0)} \hookrightarrow \mathsf{Def}_{A_0},$$
realizing each category as a full subcategory of the next one. Moreover, we may ignore the Rosati condition and the Lie algebra condition when studying $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$.

2.2.3 Prorepresentability

Consider a Cartesian diagram
$$\begin{tikzcd} \tilde{Q} \arrow[r, two heads, "\tilde{\pi}"] \arrow[d, two heads, "q"'] & \tilde{R} \arrow[d, two heads, "r"] \\ Q \arrow[r, two heads, "\pi"'] & R \end{tikzcd} \tag{2.2.3.1}$$
of surjections in which r and q are small surjections with kernels I and J, respectively. In this case $\tilde{\pi} : \tilde{Q} \twoheadrightarrow \tilde{R}$ induces an isomorphism $J \xrightarrow{\sim} I$, which we again denote by $\tilde{\pi}$. If we define $\tilde{S} := \mathrm{Spec}(\tilde{R})$, $S := \mathrm{Spec}(R)$, $\tilde{T} := \mathrm{Spec}(\tilde{Q})$, and $T := \mathrm{Spec}(Q)$, then we arrive at the setting of Section 2.1.4, together with the isomorphism $\tilde{\pi} : J \xrightarrow{\sim} I$ which makes the results there more powerful.

According to Schlessinger's criterion (stated as Theorem 2.2.1.4), a covariant functor $F : \mathsf{C} \to (\mathrm{Sets})$ such that $F(k)$ has exactly one element is prorepresentable if and only if the following two conditions are satisfied:

1. The natural map
$$F(\tilde{Q}) \to F(Q) \underset{F(R)}{\times} F(\tilde{R}) \tag{2.2.3.2}$$
is a bijection for each Cartesian diagram (2.2.3.1).
2. Let $k[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers over k, then
$$\dim_k(F(k[\varepsilon]/(\varepsilon^2))) < \infty. \tag{2.2.3.3}$$

We shall check these conditions one by one for Def_{A_0}, $\mathsf{Def}_{(A_0,\lambda_0)}$, $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$, and $\mathsf{Def}_{\xi_0} = \mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$.

Proposition 2.2.3.4. *The functor Def_{A_0} is prorepresentable.*

Proof. Let us first check the bijectivity of the map
$$\mathsf{Def}_{A_0}(\tilde{Q}) \to \mathsf{Def}_{A_0}(Q) \underset{\mathsf{Def}_{A_0}(R)}{\times} \mathsf{Def}_{A_0}(\tilde{R}) \tag{2.2.3.5}$$
for each diagram (2.2.3.1). If $\mathsf{Def}_{A_0}(Q) \underset{\mathsf{Def}_{A_0}(R)}{\times} \mathsf{Def}_{A_0}(\tilde{R})$ is empty, then $\mathsf{Def}_{A_0}(\tilde{Q})$ is also empty and (2.2.3.5) is bijective. Therefore we may assume that it is nonempty. Let $((A_Q, f_{0,Q}), (A_{\tilde{R}}, f_{0,\tilde{R}}))$ be a representative of any object in $\mathsf{Def}_{A_0}(Q) \underset{\mathsf{Def}_{A_0}(R)}{\times} \mathsf{Def}_{A_0}(\tilde{R})$. Then we have
$$\mathsf{Def}_{A_0}(\pi)([(A_Q, f_{0,Q})]) = \mathsf{Def}_{A_0}(r)([(A_{\tilde{R}}, f_{0,\tilde{R}})]) = [(A_R, f_{0,R})]$$
for some object in $\mathsf{Def}_{A_0}(R)$ represented by some $(A_R, f_{0,R})$. By Proposition 2.2.2.3 and Corollary 2.1.4.3, we have a commutative diagram
$$\begin{tikzcd} \mathsf{Def}_{A_0}(r)^{-1}([(A_R, f_{0,R})]) \arrow[r, "\sim"] & \mathsf{Lift}(A_R; \tilde{R} \twoheadrightarrow R) \\ \mathsf{Def}_{A_0}(q)^{-1}([(A_Q, f_{0,Q})]) \arrow[r, "\sim"'] \arrow[u, dotted, "\wr"] & \mathsf{Lift}(A_Q; \tilde{Q} \twoheadrightarrow Q) \arrow[u, "\wr"', "\otimes_{\tilde{Q}} \tilde{R}"] \end{tikzcd} \tag{2.2.3.6}$$
of isomorphisms (with the dotted arrow induced by the other arrows), which shows that there must be an isomorphism class $[(A_{\tilde{Q}}, f_{0,\tilde{Q}})] \in \mathsf{Def}_{A_0}(q)^{-1}([(A_Q, f_{0,Q})])$ corresponding to the given $[(A_{\tilde{R}}, f_{0,\tilde{R}})] \in \mathsf{Def}_{A_0}(r)^{-1}([(A_R, f_{0,R})])$. Moreover, for each $(A_{\tilde{Q}}, f_{0,\tilde{Q}})$ representing the class, there is an isomorphism $A_{\tilde{Q}} \underset{\tilde{Q}}{\otimes} \tilde{R} \cong A_{\tilde{R}}$ over

$\tilde{R}$, as this is how the solid vertical arrow in (2.2.3.6) is defined. By Corollary 2.2.2.2, the existence of such an isomorphism implies the unique existence of an isomorphism of abelian schemes lifting the isomorphism $f_{0,\tilde{R}} \circ f_{0,\tilde{Q}}^{-1} : A_{\tilde{Q}} \underset{\tilde{R}}{\otimes} k \xrightarrow{\sim} A_{\tilde{R}} \underset{\tilde{R}}{\otimes} k$ between the special fibers. Hence we also have

$$\mathsf{Def}_{A_0}(\tilde{\pi})([(A_{\tilde{Q}}, f_{0,\tilde{Q}})]) = [(A_{\tilde{R}}, f_{0,\tilde{R}})],$$

which shows the surjectivity of (2.2.3.5). Note that this argument shows that the dotted arrow in (2.2.3.6) can be identified with $\mathsf{Def}_{A_0}(\tilde{\pi})$. As a result, no two distinct elements in $\mathsf{Def}_{A_0}(q)^{-1}([(A_Q, f_{0,Q})])$ can be mapped to the same element in $\mathsf{Def}_{A_0}(r)^{-1}([(A_R, f_{0,R})])$ by $\mathsf{Def}_{A_0}(\tilde{\pi})$, which shows the injectivity of (2.2.3.6) as well.

Now let us compute the dimension of $\mathsf{Def}_{A_0}(k[\varepsilon]/(\varepsilon^2))$. Let $S_0 := \mathrm{Spec}(k)$ and let $t : k[\varepsilon]/(\varepsilon^2) \twoheadrightarrow k$ denote the canonical surjection. Since $\mathsf{Def}_{A_0}(k)$ has only one object $[(A_0, \mathrm{Id}_{A_0})]$ and since t has a section forcing $\mathsf{o}(A_0; t) = 0$, we have

$$\begin{aligned}\mathsf{Def}_{A_0}(k[\varepsilon]/(\varepsilon^2)) &= \mathsf{Def}_{A_0}(t)^{-1}([(A_0, \mathrm{Id}_{A_0})]) \\ &\cong \mathsf{Lift}(A_0; t) \cong H^1(A_0, \underline{\mathrm{Der}}_{A_0/S_0})\end{aligned}$$

by Propositions 2.2.2.3 and 2.1.2.2. Since A_0 is an abelian scheme over S_0, by Lemma 2.1.5.11, $\underline{\mathrm{Der}}_{A_0/S_0}$ is canonically isomorphic to the pullback of $\underline{\mathrm{Lie}}_{A_0/S_0}$, the latter of which is constant over $S_0 = \mathrm{Spec}(k)$ with values in the k-vector space Lie_{A_0/S_0}. By this fact and Corollary 2.1.5.9, we see that

$$H^1(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \cong H^1(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \cong \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0},$$

where $\mathrm{Lie}_{A_0^\vee/S_0} = \mathrm{Lie}_{\mathrm{Pic}(A_0)/S_0}$. The dimensions of Lie_{A_0/S_0} and $\mathrm{Lie}_{A_0^\vee/S_0}$ are both finite and equal to the dimension of A_0 over k. Hence

$$\dim_k \mathsf{Def}_{A_0}(k[\varepsilon]/(\varepsilon^2)) = (\dim_k A_0)^2 < \infty,$$

as desired. □

Proposition 2.2.3.7. *The functor* $\mathsf{Def}_{(A_0,\lambda_0)}$ *is prorepresentable.*

Proof. By Corollary 2.2.2.10, $\mathsf{Def}_{(A_0,\lambda_0)}$ is a subfunctor of Def_{A_0}. Hence the map

$$\mathsf{Def}_{(A_0,\lambda_0)}(\tilde{Q}) \to \mathsf{Def}_{(A_0,\lambda_0)}(Q) \underset{\mathsf{Def}_{(A_0,\lambda_0)}(R)}{\times} \mathsf{Def}_{(A_0,\lambda_0)}(\tilde{R}) \tag{2.2.3.8}$$

is always injective because (2.2.3.5) is so. It suffices to show that it is also surjective. Suppose $((A_Q, \lambda_Q, f_{0,Q}), (A_{\tilde{R}}, \lambda_{\tilde{R}}, f_{0,\tilde{R}}))$ is a representative of any object on the right-hand side of (2.2.3.8). Then we have

$$\begin{aligned}\mathsf{Def}_{(A_0,\lambda_0)}(\pi)([(A_Q, \lambda_Q, f_{0,Q})]) &= \mathsf{Def}_{A_0}(p)([(A_{\tilde{R}}, \lambda_{\tilde{R}}, f_{0,\tilde{R}})]) \\ &= [(A_R, \lambda_R, f_{0,R})]\end{aligned}$$

for some object in $\mathsf{Def}_{(A_0,\lambda_0)}(R)$ represented by some $(A_R, \lambda_R, f_{0,R})$. In particular, because of the existence of the polarization $\lambda_{\tilde{R}} : A_{\tilde{R}} \to A_{\tilde{R}}^\vee$ as a morphism between schemes, $\mathsf{Lift}(\lambda_R; A_{\tilde{R}}, A_{\tilde{R}}^\vee, \tilde{R} \twoheadrightarrow R)$ is nonempty, and

$$\mathsf{o}(\lambda_Q; A_{\tilde{Q}}, A_{\tilde{Q}}^\vee, \tilde{Q} \twoheadrightarrow Q) = \mathsf{o}(\lambda_R; A_{\tilde{R}}, A_{\tilde{R}}^\vee, \tilde{R} \twoheadrightarrow R) = 0$$

by Corollary 2.1.4.4. Therefore there is some morphism $\lambda_{\tilde{Q},0} : A_{\tilde{Q}} \to A_{\tilde{Q}}^\vee$ lifting $\lambda_Q : A_Q \to A_Q^\vee$, which might not be a homomorphism. But this is sufficient because Corollaries 2.2.2.2 and 2.2.2.6 then imply the unique existence of a polarization $\lambda_{\tilde{Q}} : A_{\tilde{Q}} \to A_{\tilde{Q}}^\vee$ lifting λ_Q.

Finally, note that

$$\dim_k \mathsf{Def}_{(A_0,\lambda_0)}(k[\varepsilon]/(\varepsilon^2)) \leq \dim_k \mathsf{Def}_{A_0}(k[\varepsilon]/(\varepsilon^2)) < \infty,$$

again because $\mathsf{Def}_{(A_0,\lambda_0)}$ is a subfunctor of Def_{A_0}. □

Proposition 2.2.3.9. *The functor* $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$ *is prorepresentable.*

Proof. By Corollary 2.2.2.10, $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$ is a subfunctor of $\mathsf{Def}_{(A_0,\lambda_0)}$. Moreover, we may ignore the Rosati condition and the Lie algebra condition when studying $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$. Therefore the prorepresentability is just a question about lifting morphisms of schemes, which can be shown by exactly the same argument as in the proof of Proposition 2.2.3.7. □

Finally,

Theorem 2.2.3.10. *The functor* $\mathsf{Def}_{\xi_0} = \mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$ *is prorepresentable.*

Proof. Simply combine Corollary 2.2.2.10 and Proposition 2.2.3.9. □

2.2.4 Formal Smoothness

Proposition 2.2.4.1. *The functor* Def_{A_0} *is formally smooth.*

Proof. Let $S_0 := \mathrm{Spec}(k)$. By Proposition 2.2.2.3, this will follow if we can show that, for every small surjection $\tilde{R} \twoheadrightarrow R$ in C with kernel I, and every (A, f_0) defining an object of $\mathsf{Def}_{A_0}(R)$, the obstruction

$$\mathsf{o} := \mathsf{o}(A; \tilde{R} \twoheadrightarrow R) \in H^2(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \underset{k}{\otimes} I$$

to $\mathsf{Lift}(A; \tilde{R} \twoheadrightarrow R)$ vanishes. Let $\tilde{S} := \mathrm{Spec}(\tilde{R})$ and $S := \mathrm{Spec}(R)$ as usual. Let us also look at the obstruction

$$\mathsf{o}_2 := \mathsf{o}(A \underset{S}{\times} A; \tilde{R} \twoheadrightarrow R) \in H^2(A_0 \underset{S_0}{\times} A_0, \underline{\mathrm{Der}}_{A_0 \underset{S_0}{\times} A_0/S_0}) \underset{k}{\otimes} I$$

to $\mathsf{Lift}(A \underset{S}{\times} A; \tilde{R} \twoheadrightarrow R)$. According to the proof of Proposition 2.1.2.2, the cohomology class o can be calculated by forming an affine open covering $\{U_\alpha\}_\alpha$ of A over S such that each U_α is lifted to a scheme $\tilde{U}_\alpha$ smooth over $\tilde{S}$, and by forming $c = \{c_{\alpha\beta\gamma}\}_{\alpha\beta\gamma}$ with

$$c_{\alpha\beta\gamma} \in \mathrm{Aut}_{\tilde{S}}(\tilde{U}_\alpha|_{U_{\alpha\beta\gamma}}, S) \cong \Gamma((U_{\alpha\beta\gamma})_0, \underline{\mathrm{Der}}_{A_0/S_0}) \underset{k}{\otimes} I$$

defining the class $\mathsf{o} = [c]$ in $H^2(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \underset{k}{\otimes} I$. Then we have an affine open covering $\{U_\alpha \underset{S}{\times} U_{\alpha'}\}_{\alpha\alpha'}$ of $A \underset{S}{\times} A$ enjoying the same smooth lifting properties, and we can calculate o_2 by forming the class of $c_2 := \{\mathrm{pr}_1^*(c_{\alpha\beta\gamma}) + \mathrm{pr}_2^*(c_{\alpha'\beta'\gamma'})\}_{\alpha\alpha'\beta\beta'\gamma\gamma'}$ in its natural sense.

Let $j_1, j_2 : A_0 \to A_0 \underset{S_0}{\times} A_0$ be the morphisms defined by $j_1(x) = (x, e)$ and $j_2(x) = (e, x)$, respectively, for all functorial points x of A_0, and let $\mathrm{pr}_1, \mathrm{pr}_2 : A_0 \underset{S_0}{\times} A_0 \to A_0$ be the two projections. As in the proof of Proposition 2.2.2.5, by the Künneth formula (see [59, III-2, 6.7.8]) and the fact that $\underline{\mathrm{Der}}_{A_0 \underset{S_0}{\times} A_0/S_0}$ is canonically isomorphic to the pullback of the constant sheaf $\underline{\mathrm{Lie}}_{A_0 \underset{S_0}{\times} A_0/S_0}$, we know that there is a canonical isomorphism

$$\begin{aligned}&H^2(A_0 \underset{S_0}{\times} A_0, \underline{\mathrm{Der}}_{A_0 \underset{S_0}{\times} A_0/S_0}) \underset{k}{\otimes} I \\ &\cong H^2(A_0 \underset{S_0}{\times} A_0, \mathscr{O}_{A_0 \underset{S_0}{\times} A_0}) \underset{k}{\otimes} \mathrm{Lie}_{A_0 \underset{S_0}{\times} A_0/S_0} \underset{k}{\otimes} I \\ &\cong [\mathrm{pr}_1^* H^2(A_0, \mathscr{O}_{A_0}) \oplus (\mathrm{pr}_1^* H^1(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} \mathrm{pr}_2^* H^1(A_0, \mathscr{O}_{A_0})) \\ &\qquad \oplus \mathrm{pr}_2^* H^2(A_0, \mathscr{O}_{A_0})] \underset{k}{\otimes} \mathrm{Lie}_{A_0 \underset{S_0}{\times} A_0/S_0} \underset{k}{\otimes} I,\end{aligned}$$

which decomposes $H^2(A_0 \underset{S_0}{\times} A_0, g_0^*(\underline{\mathrm{Der}}_{A_0/S_0})) \underset{k}{\otimes} I$ into three factors. By compatibly decomposing

$$\underline{\mathrm{Der}}_{A_0 \underset{S_0}{\times} A_0/S_0} \cong \mathrm{pr}_1^* \underline{\mathrm{Der}}_{A_0/S_0} \oplus \mathrm{pr}_2^* \underline{\mathrm{Der}}_{A_0/S_0}$$

and

$$\mathrm{Lie}_{A_0 \underset{S_0}{\times} A_0/S_0} \cong \mathrm{Lie}_{A_0/S_0} \oplus \mathrm{Lie}_{A_0/S_0}$$

as well (using j_1 and j_2), we obtain a projection

$$\begin{aligned} &H^2(A_0 \underset{S_0}{\times} A_0, \underline{\mathrm{Der}}_{A_0 \underset{S_0}{\times} A_0/S_0}) \underset{k}{\otimes} I \\ &\to \left(\mathrm{pr}_1^*[H^2(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0}] \oplus \mathrm{pr}_2^*[H^2(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0}]\right) \underset{k}{\otimes} I \qquad (2.2.4.2) \\ &\to \left(\mathrm{pr}_1^*[H^2(A_0, \underline{\mathrm{Der}}_{A_0/S_0})] \oplus \mathrm{pr}_2^*[H^2(A_0, \underline{\mathrm{Der}}_{A_0/S_0})]\right) \underset{k}{\otimes} I, \end{aligned}$$

which is determined by the two pullbacks under j_1^* and j_2^*.

From the above explicit construction, it is clear that $\mathsf{o}_2 = \mathrm{pr}_1^*(\mathsf{o}) + \mathrm{pr}_2^*(\mathsf{o})$. On the other hand, by Proposition 2.1.2.2, o_2 is preserved under the automorphism of $A \underset{S}{\times} A$ defined by $\alpha : (x, y) \mapsto (x + y, y)$ for all functorial points x and y of A. Therefore we also have

$$\mathsf{o}_2 = (\mathrm{pr}_1 \circ \alpha)^*(\mathsf{o}) + (\mathrm{pr}_2 \circ \alpha)^*(\mathsf{o}) = m^*(\mathsf{o}) + \mathrm{pr}_2^*(\mathsf{o}),$$

where $m : A_0 \underset{S_0}{\times} A_0 \to A_0$ is the multiplication morphism. Since $m \circ j_1 = m \circ j_2 = \mathrm{Id}_X$, we see that $m^*(\mathsf{o})$ and $\mathrm{pr}_1^*(\mathsf{o}) + \mathrm{pr}_2^*(\mathsf{o})$ have the same projection under (2.2.4.2). As a result, o_2 and $\mathrm{pr}_1^*(\mathsf{o}) + 2\,\mathrm{pr}_2^*(\mathsf{o})$ have the same projection under (2.2.4.2). Since $\mathsf{o}_2 = \mathrm{pr}_1^*(\mathsf{o}) + \mathrm{pr}_2^*(\mathsf{o})$, this implies $\mathrm{pr}_2^*(\mathsf{o}) = 0$ and hence $\mathsf{o} = 0$ as pr_2^* is injective. □

Corollary 2.2.4.3. *The functor Def_{A_0} is (noncanonically) prorepresented by the formally smooth algebra $\Lambda[[x_1, \ldots, x_{g^2}]]$ over Λ, where $g = \dim_k A_0$.*

Proof. By Propositions 2.2.3.4 and 2.2.4.1, and Theorem 2.2.1.4, it suffices to show that $\dim_k \mathsf{Def}_{A_0}(k[\varepsilon]/(\varepsilon^2)) = g^2$. But this has already been seen in the proof of Proposition 2.2.3.4. □

Proposition 2.2.4.4. *The functor $\mathsf{Def}_{(A_0,\lambda_0)}$ is formally smooth.*

Proof. For each small surjection $\tilde{R} \twoheadrightarrow R$ in C with kernel I, and each (A, λ, f_0) defining an object of $\mathsf{Def}_{(A_0,\lambda_0)}(R)$, we know from Proposition 2.2.4.1 that there always exists some abelian scheme $(\tilde{A}, \tilde{f}_0)$ lifting (A, f_0). By Proposition 1.3.2.15, we know that, after making an étale surjective base change if necessary, we may suppose that $\lambda = \lambda_{\mathcal{L}}$ for some ample invertible sheaf $\mathcal{L}$, where $\lambda_{\mathcal{L}}$ is associated with $\mathcal{L}$ by Construction 1.3.2.7. Then the question becomes whether we can lift $\mathcal{L}$ to some invertible sheaf $\tilde{\mathcal{L}}$ over $\tilde{A}$. Or, if not, the question becomes whether there exists a different lifting $\tilde{A}'$ of A such that $\mathcal{L}$ can be lifted to an invertible sheaf $\tilde{\mathcal{L}}$ on $\tilde{A}'$.

Let $\mathcal{L}_0 := \mathcal{L} \underset{\tilde{R}}{\otimes} k$, so that $\lambda_0 = \lambda_{\mathcal{L}_0}$. By Proposition 2.1.5.3, we know that there is an element

$$\mathsf{o}(\mathcal{L}; \tilde{A}, \tilde{R} \twoheadrightarrow R) \in H^2(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} I$$

such that we can lift $\mathcal{L}$ to some $\tilde{\mathcal{L}}$ if and only if $\mathsf{o}(\mathcal{L}; \tilde{A}, \tilde{R} \twoheadrightarrow R) = 0$. Let $S_0 := \mathrm{Spec}(k)$. If we replace $\tilde{A}$ with $\mathsf{m} + \tilde{A}$ for some $\mathsf{m} \in H^1(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \underset{k}{\otimes} I$, then there is a relation

$$\mathsf{o}(\mathcal{L}; \mathsf{m} + \tilde{A}, \tilde{R} \twoheadrightarrow R) = \mathsf{o}(\mathcal{L}; \tilde{A}, \tilde{R} \twoheadrightarrow R) + \mathsf{d}_{\mathcal{L}_0}(\mathsf{m}).$$

By Corollary 2.1.5.15, the morphism

$$\mathsf{d}_{\mathcal{L}_0} : H^1(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \underset{k}{\otimes} I \to H^2(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} I$$

is surjective when

$$(\mathrm{Id}_{A_0^\vee} \otimes d\lambda_0 \otimes \mathrm{Id}) : \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I \to \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} I$$

is surjective. By assumption, λ_0 is prime-to-$\Box$ and hence *separable*. Therefore $d\lambda_0 : \mathrm{Lie}_{A_0/S_0} \to \mathrm{Lie}_{A_0^\vee/S_0}$ is an isomorphism, which is in particular, surjective. This shows the surjectivity of $\mathsf{d}_{\mathcal{L}_0}$, and hence the existence of some element m such that $\mathsf{o}(\mathcal{L}; \mathsf{m} + \tilde{A}, R \twoheadrightarrow R) = 0$, as desired.

Note that the elements m making $\mathsf{o}(\mathcal{L}; \mathsf{m} + \tilde{A}, R \twoheadrightarrow R) = 0$ form a torsor under the *symmetric elements* in

$$H^1(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \underset{k}{\otimes} I \cong \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} I,$$

namely, the elements that are mapped (under $\mathsf{d}_{\mathcal{L}_0}$) to zero in

$$H^2(A_0, \mathscr{O}_{A_0}) \underset{k}{\otimes} I \cong [\wedge^2 H^1(A_0, \mathscr{O}_{A_0})] \underset{k}{\otimes} I \cong [\wedge^2 \mathrm{Lie}_{A_0^\vee/S_0}] \underset{k}{\otimes} I$$

(see Proposition 2.1.5.14 for the first isomorphism). □

Corollary 2.2.4.5. *Let $\tilde{R} \twoheadrightarrow R$ be a small surjection in C with kernel I, and let (A, λ, f_0) define an object in $\mathsf{Def}_{(A_0,\lambda_0)}(R)$. Let $\mathsf{Lift}(A, \lambda; \tilde{R} \twoheadrightarrow R)$ denote the subset of $\mathsf{Lift}(A; \tilde{R} \twoheadrightarrow R)$ consisting of liftings $\tilde{A}$ of A that admit liftings $\tilde{\lambda} : \tilde{A} \to \tilde{A}^\vee$ of $\lambda : A \to A^\vee$. Note that $\mathsf{Lift}(A; \tilde{R} \twoheadrightarrow R)$ is a torsor under the group $H^1(A_0, \underline{\mathrm{Der}}_{A_0/S_0}) \underset{k}{\otimes} I \cong \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I$. Then $\mathsf{Lift}(A, \lambda; \tilde{R} \twoheadrightarrow R)$ is a torsor under the group of* ***symmetric elements*** *in*

$$\mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I.$$

Proof. Once the we know that this set is nonempty, the statement follows from either Proposition 2.1.3.2 or simply the observation at the very end of the proof of Proposition 2.2.4.4. □

Corollary 2.2.4.6. *Under the assumption that λ_0 is separable, the functor $\mathsf{Def}_{(A_0,\lambda_0)}$ is (noncanonically) prorepresented by the formally smooth algebra $\Lambda[[x_1, \ldots, x_{\frac{1}{2}g(g+1)}]]$ over Λ, where $g = \dim_k A_0$.*

Proof. By applying Corollary 2.2.4.5 to the small surjection $k[\varepsilon]/(\varepsilon^2) \twoheadrightarrow k$, we see that $\dim_k \mathsf{Def}_{(A_0,\lambda_0)}(k[\varepsilon]/(\varepsilon^2))$ is the same as the dimension of the subspace of symmetric elements in $\mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0^\vee/S_0}$, which is $\frac{1}{2}g(g+1)$. Then we can conclude the proof by applying Propositions 2.2.3.7 and 2.2.4.4, and Theorem 2.2.1.4. □

Let us formulate Proposition 2.2.4.4 and Corollary 2.2.4.5 in the context of Proposition 2.1.6.8. Each polarization λ of an abelian scheme A over S defines a canonical pairing $\langle \cdot, \cdot \rangle_\lambda$ on $\underline{H}_1^{\mathrm{dR}}(A/S)$ as follows: As in [37, 1.5], the first Chern class of the Poincaré invertible sheaf $\mathcal{P}_A$ over $A \underset{S}{\times} A^\vee$ induces an alternating pairing between $\underline{H}_1^{\mathrm{dR}}(A/S)$ and $\underline{H}_1^{\mathrm{dR}}(A^\vee/S)$, which is a perfect duality by [18, 5.1]. In particular, there is a canonical isomorphism $\underline{H}_1^{\mathrm{dR}}(A/S) \cong \underline{H}_{\mathrm{dR}}^1(A^\vee/S)$. Thus, each polarization $\lambda : A \to A^\vee$ canonically induces a morphism $\lambda^* : \underline{H}_{\mathrm{dR}}^1(A^\vee/S) \to \underline{H}_{\mathrm{dR}}^1(A/S)$, and hence a morphism $\underline{H}_1^{\mathrm{dR}}(A/S) \to \underline{H}_{\mathrm{dR}}^1(A/S)$ giving a pairing $\langle \cdot, \cdot \rangle_\lambda$ on $\underline{H}_1^{\mathrm{dR}}(A/S)$. Under this pairing, the $\underline{\mathrm{Lie}}_{A^\vee/S}^\vee$ in the exact sequence

$$0 \to \underline{\mathrm{Lie}}_{A^\vee/S}^\vee \to \underline{H}_1^{\mathrm{dR}}(A/S) \to \underline{\mathrm{Lie}}_{A/S} \to 0 \qquad (2.2.4.7)$$

is a totally isotropic submodule of $\underline{H}_1^{\mathrm{dR}}(A/S)$. Moreover, when λ is separable, the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ is a perfect pairing, and the embedding $\underline{\mathrm{Lie}}_{A^\vee/S}^\vee \hookrightarrow \underline{H}_1^{\mathrm{dR}}(A/S)$ induces an isomorphism $\underline{\mathrm{Lie}}_{A/S} \to \underline{\mathrm{Lie}}_{A^\vee/S}$, which is nothing but the differential $d\lambda$ of the separable isogeny $\lambda : A \to A^\vee$.

Suppose we have a small surjection $\tilde{R} \twoheadrightarrow R$ in C with kernel I, and suppose that we have an abelian scheme $\tilde{A}$ over $\tilde{S} := \mathrm{Spec}(\tilde{R})$. Let $A := \tilde{A} \underset{\tilde{R}}{\otimes} R$, and let $\lambda : A \to A^\vee$ be a polarization. We claim that we can define a canonical pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ on $\underline{H}_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$ without knowing the existence of some polarization $\tilde{\lambda} : \tilde{A} \to \tilde{A}^\vee$ lifting λ. Indeed, the existence of λ gives a canonical morphism $\lambda_* : \underline{H}_1^{\mathrm{dR}}(\tilde{A}/\tilde{S}) \to \underline{H}_1^{\mathrm{dR}}(\tilde{A}^\vee/\tilde{S})$ by dualizing Proposition 2.1.6.4, which necessarily maps $\underline{\mathrm{Lie}}_{(\tilde{A}')^\vee/\tilde{S}}^\vee$ to $\underline{\mathrm{Lie}}_{\tilde{A}'/\tilde{S}}^\vee$ for every lifting $\tilde{A}'$ of A that does admit a lifting $\tilde{\lambda}'$ of λ, as can be seen in the proofs of Propositions 2.1.6.2 and 2.1.6.4. This morphism is necessarily an isomorphism, because it induces an isomorphism $\underline{H}_1^{\mathrm{dR}}(A/S) \xrightarrow{\sim} \underline{H}_1^{\mathrm{dR}}(A^\vee/S)$ modulo I (see Lemma 2.1.1.1). Therefore it defines a canonical pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ on $\underline{H}_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$, which agrees with $\langle\,\cdot\,,\,\cdot\,\rangle_{\tilde{\lambda}}$ whenever there does exist a lifting $\tilde{\lambda}$ of λ. Moreover, suppose f_1 and f_2 are two endomorphisms of A satisfying $\lambda \circ f_1 = f_2^\vee \circ \lambda$. (For example, suppose $f_1 = i(b^\star)$ and $f_2 = i(b)$ for some endomorphism structure $i : \mathcal{O} \to \mathrm{End}_S(A)$.) Then the canonical morphisms $\underline{H}_1^{\mathrm{dR}}(\tilde{A}/\tilde{S}) \to \underline{H}_1^{\mathrm{dR}}(\tilde{A}^\vee/\tilde{S})$ defined by $\lambda \circ f_1$ and by $f_2^\vee \circ \lambda$ have to agree. This shows that we can have $\langle f_{1,*}(x), y\rangle_\lambda = \langle x, f_{2,*}(y)\rangle_\lambda$ even without lifting the endomorphisms f_1 and f_2 to $\tilde{A}$.

Let us return to the situation that $\tilde{A}$ is a lifting of A that admits a lifting $\tilde{\lambda}$ of the polarization λ. Let us work with modules rather than sheaves, as our base scheme is now affine. Starting with the totally isotropic submodule $\mathrm{Lie}_{\tilde{A}/\tilde{S}}^\vee$ of $H_1^{\mathrm{dR}}(\tilde{A}^\vee/\tilde{S})$, the $\tilde{R}$-submodules M as in Proposition 2.1.6.8 that become the same as $\mathrm{Lie}_{A^\vee/S}^\vee$ modulo I and are moreover *totally isotropic* under the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\tilde{\lambda}}$ are parameterized by the subgroup of *symmetric elements* in $\mathrm{Hom}_{\tilde{R}}(\mathrm{Lie}_{\tilde{A}^\vee/\tilde{S}}^\vee, I \cdot \mathrm{Lie}_{\tilde{A}/\tilde{S}}) \cong \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I \overset{\mathrm{Id}\otimes d\lambda}{\underset{\sim}{\to}} \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} I$. According to Corollary 2.2.4.5, this is the same set that parameterizes the liftings of abelian schemes that admit liftings of the polarization λ of A.

Corollary 2.2.4.8. *Let $\tilde{R} \twoheadrightarrow R$ be a small surjection in C with kernel I, and let $\tilde{A}$ be an abelian scheme over $\tilde{S} = \mathrm{Spec}(\tilde{R})$. As explained in Proposition 2.1.6.8, the objects in $\mathsf{Lift}(\tilde{A}; \tilde{R} \twoheadrightarrow R)$ (which are necessarily abelian schemes by Proposition 2.2.2.3) are in bijection with modules M in exact sequences*

$$0 \to M \to H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S}) \to N \to 0$$

*of **projective** $\tilde{R}$-**modules** such that $M \underset{\tilde{R}}{\otimes} R = \mathrm{Lie}_{A^\vee/S}^\vee$ in $H_1^{\mathrm{dR}}(A/S) = H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S}) \underset{\tilde{R}}{\otimes} R$. Let us denote the abelian scheme corresponding to a submodule M as above by $\tilde{A}_M$. Suppose, moreover, that $A := \tilde{A} \underset{\tilde{R}}{\otimes} R$ has a separable polarization $\lambda : A \to A^\vee$, which defines a perfect pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ on $H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$ as explained above. Then the following are true:*

1. *The polarization λ can be lifted to some polarization $\tilde{\lambda}_M : \tilde{A}_M \to \tilde{A}_M^\vee$ if and only if M is a totally isotropic submodule of $H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$ under $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ (see Corollaries 2.2.2.2 and 2.2.2.6).*

2. *Suppose that A has a collection of endomorphisms $f_i : A \to A$ (of abelian schemes). Then these endomorphisms f_i can be lifted to endomorphisms $\tilde{f}_{i,M} : \tilde{A}_M \to \tilde{A}_M$ if and only if M is invariant under the actions of $f_{i,*}$ on $H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$ (see Proposition 2.1.6.4).*

3. *Suppose that $i : \mathcal{O} \to \mathrm{End}_S(A)$ is an $\mathcal{O}$-endomorphism structure for (A, λ). Then i can be lifted to an $\mathcal{O}$-endomorphism structure $\tilde{i} : \mathcal{O} \to \mathrm{End}_{\tilde{S}}(\tilde{A}_M)$ for some $\tilde{A}_M$ that admits a lifting $\tilde{\lambda}_M$ of λ if and only if M is both totally isotropic under $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ and invariant under the actions on $H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$ defined by $i(b)_*$ for all $b \in \mathcal{O}$.*

Proof. This is just a combination of what we have explained, together with Corollary 2.2.2.7, with the following additional remark: The compatibility between different morphisms (including the Rosati condition) is given by relations defined by group homomorphisms that are trivial over S. Therefore the usual trick using Corollary 1.3.1.5 is applicable. □

Proposition 2.2.4.9. *The functor $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$ is formally smooth.*

Proof. For each small surjection $\tilde{R} \twoheadrightarrow R$ in C with kernel I, and each (A, λ, i, f_0) defining an object of $\mathsf{Def}_{(A_0,\lambda_0,i_0)}(R)$, we know from Proposition 2.2.4.1 that there always exists some triple $(\tilde{A}, \tilde{\lambda}, \tilde{f}_0)$ lifting (A, λ, f_0). The question is whether we can also lift i to an endomorphism structure of $(\tilde{A}, \tilde{\lambda}, \tilde{f}_0)$. If so, then $(\tilde{A}, \tilde{\lambda}, \tilde{i})$ will automatically satisfy the determinantal condition (see Definition 1.3.4.1) by Proposition 2.2.2.9.

It suffices to show the existence of a projective submodule M of $\underline{H}_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$ lifting $\mathrm{Lie}_{A^\vee/S}^\vee$, as in Corollary 2.2.4.8, that is both totally isotropic under the perfect pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\tilde{\lambda}}$ defined by the separable isogeny $\tilde{\lambda}$ and invariant under the action defined by i. As explained before, $\langle\,\cdot\,,\,\cdot\,\rangle_{\tilde{\lambda}}$ is determined by λ and has the Hermitian property given by the Rosati condition satisfied by λ and i. That is, $(H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S}), \langle\,\cdot\,,\,\cdot\,\rangle_{\tilde{\lambda}})$ is a self-dual symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \tilde{R}$-module as in Definition 1.1.4.10.

Our assumptions on k and Λ include the hypotheses that there is a homomorphism $\mathcal{O}_{F_0,(\Box)} \to \Lambda$ whose composition with $\Lambda \to k$ is of finite type, that $\Lambda = k$ when $\mathrm{char}(k) = 0$, and that $\Lambda = W(k)$ when k is a finite field of $\mathrm{char}(k) = p > 0$. Let F_0' and L_0 be chosen as in Lemma 1.2.5.9, with F_0' unramified at p when $p > 0$. For the purpose of showing formal smoothness of $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$, we may (and we shall) replace k with a finite extension such that $\mathcal{O}_{F_0,(\Box)} \to \Lambda$ extends to a homomorphism $\mathcal{O}_{F_0',(p)} \to \Lambda$. By the assumption that (A_0, λ_0, i_0) satisfies the Lie algebra condition, and by Lemma 1.2.5.11, we know as in the proof of Proposition 2.2.2.9 that there is an isomorphism $\mathrm{Lie}_{A/S} \cong L_0 \underset{\mathcal{O}_{F_0'}}{\otimes} R$ of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R$-modules. By replacing R with a ring R' finite étale over R if necessary, we may assume that R is an Λ'-algebra, where Λ' is the ring chosen in the decomposition (1.2.5.12), such that $(L_0 \underset{\mathcal{O}_{F_0'}}{\otimes} R)^\vee$ is a maximal totally isotropic submodule of $L \underset{\mathbb{Z}}{\otimes} R$; we may also assume that there is a symplectic isomorphism $f : (H_1^{\mathrm{dR}}(A/S), \langle\,\cdot\,,\,\cdot\,\rangle_\lambda) \xrightarrow{\sim} (L \underset{\mathbb{Z}}{\otimes} R, \langle\,\cdot\,,\,\cdot\,\rangle)$

sending $\mathrm{Lie}^\vee_{A^\vee/S}$ to $(L_0 \underset{\mathcal{O}_{F'_0}}{\otimes} R)^\vee$, by Proposition 1.2.3.7, Corollary 1.2.3.10, Lemma 1.2.4.4, and Proposition 1.2.4.5. Moreover, by the fact that the choices of modules of standard type (see Definition 1.2.3.6) are discrete in nature, we see that there is also a symplectic isomorphism $\tilde{f} : (H_1^{\mathrm{dR}}(\tilde{A}/\tilde{S}), \langle\,\cdot\,,\,\cdot\,\rangle_{\tilde{\lambda}}) \xrightarrow{\sim} (L \underset{\mathbb{Z}}{\otimes} \tilde{R}, \langle\,\cdot\,,\,\cdot\,\rangle)$. Since f maps $\mathrm{Lie}^\vee_{A^\vee/S}$ to $(L_0 \underset{\mathcal{O}_{F'_0}}{\otimes} R)^\vee$ in $(L \underset{\mathbb{Z}}{\otimes} R, \langle\,\cdot\,,\,\cdot\,\rangle)$, we obtain a point of $(\mathrm{G}_{\Lambda'}/\mathrm{P}_{0,\Lambda'})(R)$. Then the liftability of $\underline{\mathrm{Lie}}^\vee_{A^\vee/S}$ to a projective submodule M of $\underline{H}_1^{\mathrm{dR}}(\tilde{A}/\tilde{S})$ corresponds to the liftability of the point of $(\mathrm{G}_{\Lambda'}/\mathrm{P}_{0,\Lambda'})(R)$ to a point in $(\mathrm{G}_{\Lambda'}/\mathrm{P}_{0,\Lambda'})(\tilde{R})$, which follows from Proposition 1.2.5.15. □

Corollary 2.2.4.10. *Let $\tilde{R} \twoheadrightarrow R$ be a small surjection in C with kernel I, and let (A, λ, i, f_0) define an object in $\mathsf{Def}_{(A_0,\lambda_0,i_0)}(R)$. Let $\mathsf{Lift}(A, \lambda, i; \tilde{R} \twoheadrightarrow R)$ denote the subset of $\mathsf{Lift}(A; \tilde{R} \twoheadrightarrow R)$ consisting of liftings $\tilde{A}$ of A that admit liftings $\tilde{\lambda} : \tilde{A} \to \tilde{A}^\vee$ of $\lambda : A \to A^\vee$ and liftings $\tilde{i} : \mathcal{O} \to \mathrm{End}_{\tilde{S}}(\tilde{A})$ of the $\mathcal{O}$-endomorphism structure $i : \mathcal{O} \to \mathrm{End}_S(A)$. Then $\mathsf{Lift}(A, \lambda, i; \tilde{R} \twoheadrightarrow R)$ is a torsor under the group of symmetric elements in $\mathrm{Lie}_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}_{A_0/S_0} \underset{k}{\otimes} I$ that are annihilated by the endomorphisms $(d(i(b)^\vee) \otimes \mathrm{Id} \otimes \mathrm{Id}) - (\mathrm{Id} \otimes d(i(b)) \otimes \mathrm{Id})$ for all $b \in \mathcal{O}$, or equivalently a torsor under the group of k-linear homomorphisms*

$$\mathrm{Lie}^\vee_{A_0^\vee/S_0} \underset{k}{\otimes} \mathrm{Lie}^\vee_{A_0/S_0} / \begin{pmatrix} x \otimes \lambda_0^*(x') - x' \otimes \lambda_0^*(x) \\ (bx) \otimes y - x \otimes (b^\star y) \end{pmatrix}_{\substack{x,x' \in \underline{\mathrm{Lie}}^\vee_{A_0^\vee/S_0}, \\ y \in \underline{\mathrm{Lie}}^\vee_{A_0/S_0}, b \in \mathcal{O}}} \to I, \qquad (2.2.4.11)$$

where bx and $b^\star y$ mean $(i(b)^\vee)^*(x)$ and $i(b)^*(y)$, respectively.

Proof. Once we know that the set of liftings is nonempty, the statement follows from Proposition 2.1.3.2. □

Corollary 2.2.4.12. *The functor $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$ is (noncanonically) prorepresented by the formally smooth algebra $\Lambda[[x_1, \ldots, x_r]]$ over Λ, where r is an integer that can be calculated as follows: Let V_0 be the complex vector space defined in Section 1.3.4. Then $r = \dim_{\mathbb{C}} \mathbf{Sym}_\varrho(V_0)$, where*

$$\mathbf{Sym}_\varrho(V_0) := (V_0 \underset{\mathbb{C}}{\otimes} V_0) / \begin{pmatrix} x \otimes y - y \otimes x \\ (bx) \otimes z - x \otimes (b^\star z) \end{pmatrix}_{x,y,z \in V_0, b \in \mathcal{O}}.$$

Proof. By applying Corollary 2.2.4.10 to the small surjection $k[\varepsilon]/(\varepsilon^2) \twoheadrightarrow k$, we see that $r := \dim_k \mathsf{Def}_{(A_0,\lambda_0,i_0)}(k[\varepsilon]/(\varepsilon^2))$ is the same as the k-vector-space dimension of the domain in Corollary 2.2.4.10. By replacing k with a finite extension as in the proof of Proposition 2.2.4.9, we may assume that $\mathrm{Lie}_{A_0^\vee/S_0} \cong L_0 \underset{\mathcal{O}_{F'_0}}{\otimes} k$ as $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$-modules. Let $L_1 := L_0 \underset{\mathcal{O}_{F'_0}}{\otimes} \Lambda$ and let $L_1^\vee := \mathrm{Hom}_\Lambda(L_1, \Lambda)$ (with left $\mathcal{O}$-module structure given by composing the canonical right $\mathcal{O}$-module structure with ${}^\star : \mathcal{O} \xrightarrow{\sim} \mathcal{O}$). Then, under identifications induced by λ_0, we have $\mathrm{Lie}_{A_0/S_0} \cong \mathrm{Lie}_{A_0^\vee/S_0} \cong L_1 \underset{\Lambda}{\otimes} k$ and $\underline{\mathrm{Lie}}^\vee_{A_0^\vee/S_0} \cong \underline{\mathrm{Lie}}^\vee_{A_0/S_0} \cong L_1^\vee \underset{\Lambda}{\otimes} k$ as projective $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k$-modules. By Proposition 1.2.2.3, we may identify the dimension r above with the Λ-rank of

$$\mathbf{Sym}_\varrho(L_1^\vee) := (L_1^\vee \underset{\Lambda}{\otimes} L_1^\vee) / \begin{pmatrix} x \otimes y - y \otimes x \\ (bx) \otimes z - x \otimes (b^\star z) \end{pmatrix}_{x,y,z \in L_1^\vee, b \in \mathcal{O}},$$

or equivalently the Λ-rank of

$$\mathbf{Sym}_\varrho(L_1) := (L_1 \underset{\Lambda}{\otimes} L_1) / \begin{pmatrix} x \otimes y - y \otimes x \\ (bx) \otimes z - x \otimes (b^\star z) \end{pmatrix}_{x,y,z \in L_1, b \in \mathcal{O}},$$

because both of the Λ-modules have no nonzero torsion. We may compute this Λ-rank by tensoring everything with a large field containing Λ and then computing the dimension of the corresponding vector space. Afterwards we may replace the large field with a smaller field over which every object is defined. In particular, its dimension can be calculated by replacing L_1 with $V_0 \cong L_0 \underset{\mathcal{O}_{F'_0}}{\otimes} \mathbb{C}$. Once we have calculated the dimension r of $\dim_k \mathsf{Def}_{(A_0,\lambda_0,i_0)}(k[\varepsilon]/(\varepsilon^2))$, we can conclude the proof as before by applying Propositions 2.2.3.9 and 2.2.4.9, and Theorem 2.2.1.4. □

Theorem 2.2.4.13. *The functor $\mathsf{Def}_{\xi_0} = \mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$ is formally smooth. Moreover, Def_{ξ_0} is (noncanonically) prorepresented by the formally smooth algebra $\Lambda[[x_1, \ldots, x_r]]$ over Λ, where r is the integer in Corollary 2.2.4.12.*

Proof. Combine Corollary 2.2.2.10 and Proposition 2.2.4.9. □

2.3 Algebraic Theory

2.3.1 Grothendieck's Formal Existence Theory

Let us summarize several useful results in [59, III-1, §5]. Let R be a noetherian ring I-adically complete for some ideal I of R. Then $S_{\mathrm{for}} := \mathrm{Spf}(R, I)$ is the formal completion of $S := \mathrm{Spec}(R)$ along the closed subscheme $S_0 := \mathrm{Spec}(R/I)$ defined by I. Let $f : X \to S$ be a separated morphism of schemes of finite type. Let $X_{\mathrm{for}} := X \underset{S}{\times} S_{\mathrm{for}}$, and let $f_{\mathrm{for}} : X_{\mathrm{for}} \to S_{\mathrm{for}}$ be the morphism canonically induced by f. For each coherent sheaf $\mathscr{F}$ over X, we denote by $\mathscr{F}_{\mathrm{for}}$ its pullback to X_{for}, which is a coherent sheaf over X_{for}.

Proposition 2.3.1.1 (see [59, III-1, 5.1.2])**.** *With assumptions and notation as above, if $\mathscr{F}$ is a coherent sheaf over X with support proper over S, then the canonical homomorphisms $H^i(X, \mathscr{F}) \to H^i(X_{\mathrm{for}}, \mathscr{F}_{\mathrm{for}})$ are isomorphisms.*

Theorem 2.3.1.2 (see [59, III-1, 5.1.4])**.** *With assumptions and notation as above, the functor $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{for}}$ is an equivalence between the category of coherent sheaves over X with support proper over S, and the category of coherent sheaves over X_{for} with support proper over S_{for}.*

Theorem 2.3.1.3 (see [59, III-1, 5.4.1])**.** *With assumptions and notation as above, suppose moreover that $Y \to S := \mathrm{Spec}(R)$ is a separated morphism of finite type, with formal completion $Y_{\mathrm{for}} := Y \underset{S}{\times} S_{\mathrm{for}}$. Then the map $\mathrm{Hom}_S(X, Y) \to \mathrm{Hom}_{S_{\mathrm{for}}}(X_{\mathrm{for}}, Y_{\mathrm{for}}) : h \mapsto h_{\mathrm{for}} := h \underset{S}{\times} S_{\mathrm{for}}$ is a bijection.*

Thus, the functor $Z \mapsto Z_{\mathrm{for}}$ from the category of schemes proper over S to the category of formal schemes proper over S_{for} is *fully faithful*. We say that a formal scheme $\mathfrak{Z}$ proper over S_{for} is *algebraizable* if it is in the essential image of this functor; that is, there exists (up to unique isomorphism) a scheme Z proper over S such that $\mathfrak{Z} \cong Z_{\mathrm{for}}$ over S_{for}.

Theorem 2.3.1.4 (see [59, III-1, 5.4.5])**.** *With assumptions and notation as above, let $\mathfrak{Z}$ be a formal scheme proper over S_{for}. Suppose there exists an invertible sheaf $\mathscr{L}$ over $\mathfrak{Z}$ such that $\mathscr{L}_0 := \mathscr{L} \underset{R}{\otimes} (R/I)$ is **ample** over $Z_0 := \mathfrak{Z} \underset{S_{\mathrm{for}}}{\times} S_0$. Then $\mathfrak{Z}$ is algebraizable. Moreover, if Z is a proper scheme over S such that $Z_{\mathrm{for}} \cong \mathfrak{Z}$, then there exists an ample invertible sheaf $\mathscr{M}$ over Z such that $\mathscr{M}_{\mathrm{for}} \cong \mathscr{L}$. (In particular, Z is **projective** over S.)*

2.3.2 Effectiveness of Local Moduli

With the definitions of Def_{A_0}, $\mathsf{Def}_{(A_0,\lambda_0)}$, $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$, and $\mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$ as in Section 2.2.1, we have shown the prorepresentability of these deformation functors in Section 2.2.3. A natural question is whether they are effectively prorepresentable.

For each of these functors, the prorepresentability means there is a noetherian complete local ring R in $\hat{\mathsf{C}}$ and a compatible system of abelian schemes A_i over $R/\mathfrak{m}_R^{i+1}$ (with some additional structures if necessary), indexed by integers $i \geq 0$, which induce an isomorphism h_R to Def_{A_0} via the natural isomorphism $\hat{\mathsf{Def}}_{A_0} \xrightarrow{\sim} \mathrm{Hom}(h_R, \mathsf{Def}_{A_0})$ (see Section B.1 for more details). In particular, this means there is a formal abelian scheme $\hat{A}_R \to \mathrm{Spf}(R_{A_0})$. Whether this moduli problem is effectively prorepresentable means whether the formal abelian scheme $\hat{A}_R \to \mathrm{Spf}(R_{A_0})$ is *algebraizable* in the sense of Section 2.3.1. According to Theorem 2.3.1.4, the answer is affirmative if there exists a compatible system of ample invertible sheaves $\mathcal{L}_i$ over A_i.

Proposition 2.3.2.1. *All the three functors* $\mathsf{Def}_{(A_0,\lambda_0)}$, $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$, *and* $\mathsf{Def}_{\xi} = \mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$ *are* ***effectively*** *prorepresentable.*

Proof. Let us first study the case $\mathsf{Def}_{(A_0,\lambda_0)}$. Suppose that this functor is prorepresented by a complete local ring R in $\hat{\mathsf{C}}$ and a compatible system of polarized abelian schemes (A_i, λ_i) over $R/\mathfrak{m}^{i+1}$. Over each of the A_i, take $\mathcal{L}_i := (\mathrm{Id}_{A_i}, \lambda_i)^*\mathcal{P}_{A_i}$, which is ample by definition of polarizations (see Definition 1.3.2.16 and Proposition 1.3.2.15). Therefore Theorem 2.3.1.4 implies $\{(A_i, \mathcal{L}_i)\}_i$ is algebraizable by some algebraic object $(A, \mathcal{L})$ over R. This implies that $\{A_i^\vee\}_i$ is also algebraizable. By Theorem 2.3.1.3, the morphisms $\{\lambda_i : A_i \to A_i^\vee\}_i$ are algebraizable by a unique $\lambda : A \to A^\vee$, which is necessarily a polarization (by Definition 1.3.2.16 and Proposition 1.3.2.15 again). This proves that $\mathsf{Def}_{(A_0,\lambda_0)}$ is effectively prorepresentable. Since the other two moduli problems $\mathsf{Def}_{(A_0,\lambda_0,i_0)}$ and $\mathsf{Def}_{\xi_0} = \mathsf{Def}_{(A_0,\lambda_0,i_0,\alpha_{\mathcal{H},0})}$ only involve more algebraizations of morphisms, the same argument as above implies they are effectively prorepresentable as well. $\square$

2.3.3 Automorphisms of Objects

Lemma 2.3.3.1. *Let A be an abelian scheme over a base scheme S. For each $n \geq 3$, and each polarization $\lambda : A \to A^\vee$ of A, the restriction homomorphism*
$$\mathrm{Aut}_S(A, \lambda) := \{f \in \mathrm{Aut}_S(A) : f^\vee \circ \lambda \circ f = \lambda\} \to \mathrm{Aut}_S(A[n])$$
is injective, and its image acts via a subgroup of the roots of unity.

Proof. If the homomorphism is not injective, then it must be so after pullback to some Artinian local subscheme of S. Therefore, by Lemma 2.2.2.1, it suffices to treat the case that S is a single point, in which case we can apply the theory of abelian varieties explained in [94, §21, Thm. 5 and its proof]. $\square$

Corollary 2.3.3.2. *Let S be a scheme over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ and let $(A, \lambda, i, \alpha_{\mathcal{H}})$ be an object of $\mathsf{M}_{\mathcal{H}}(S)$. Then $(A, \lambda, i, \alpha_{\mathcal{H}})$ has no nontrivial automorphism if $\mathcal{H}$ is* ***neat*** *(see Definition 1.4.1.8).*

Proof. Let $\bar{s}$ be any geometric point of S. It suffices to show that the restriction of every automorphism of $(A, \lambda, i, \alpha_{\mathcal{H}})$ to $\bar{s}$ is trivial. Let f be any automorphism of $(A_{\bar{s}}, \lambda_{\bar{s}}, i_{\bar{s}})$, the pullback of (A, λ, i) to $\bar{s}$. By Lemma 2.3.3.1, the restriction $\mathrm{Aut}_{\bar{s}}(A_{\bar{s}}, \lambda_{\bar{s}}, i_{\bar{s}}) \to \mathrm{Aut}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box}(\mathrm{T}^\Box A_{\bar{s}})$ is an injection, and its image acts via a subgroup of the roots of unity. If the image of f also preserves the $\mathcal{H}$-orbit of any $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$-equivariant symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box \xrightarrow{\sim} \mathrm{T}^\Box A_{\bar{s}}$ lifting $\alpha_{\mathcal{H}}$ over $\bar{s}$, then it lies in the intersection of $\mathcal{H}$ and a subgroup of the roots of unity, which is $\{1\}$ by neatness of $\mathcal{H}$. $\square$

2.3.4 Proof of Representability

Let us prove Theorem 1.4.1.11 using Artin's criterion in Appendix B. According to Theorems B.3.7 and B.3.9, to show that $\mathsf{M}_{\mathcal{H}}$ is an algebraic stack locally of finite type over the base scheme $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, which is either a field or an excellent Dedekind domain as in the assumption of Artin's criterion, it suffices to verify the following conditions:

1. $\mathsf{M}_{\mathcal{H}}$ is a stack for the étale topology, locally of finite presentation.
2. Suppose ξ and η are two objects in $\mathsf{M}_{\mathcal{H}}(U)$, where U is some scheme of finite type over S_0. Then $\underline{\mathrm{Isom}}_U(\xi, \eta)$ is an algebraic space locally of finite type.
3. For k and ξ_0 as above, which define a functor $\mathsf{Def}_{\xi_0} : \hat{\mathsf{C}} \to (\mathrm{Sets})$ as in Section 2.2.1, the functor Def_{ξ_0} is effectively prorepresentable.
4. If ξ is a (1-)morphism from a scheme U of finite type over S_0 to $\mathsf{M}_{\mathcal{H}}$, and if ξ is formally étale at a point u (of U) of finite type over S_0, then ξ is formally étale in a neighborhood of u (in U).

By Theorems 2.2.4.13 and B.3.11, condition 4 can be suppressed if $\mathsf{M}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ has either one or infinitely many closed points. By Lemma 1.4.4.2, it is harmless for our purpose to replace $\mathsf{M}_{\mathcal{H}}$ with the moduli problem defined by the same PEL-type $\mathcal{O}$-lattice with a larger set $\Box$. Therefore, we shall assume that the set $\Box$ is either empty or infinite.

Let us begin with condition 1. Consider any tuple $(A, \lambda, i, \alpha_{\mathcal{H}})$ parameterized by $\mathsf{M}_{\mathcal{H}}$. Let $\mathcal{L} := (\mathrm{Id}_A \times \lambda)^*\mathcal{P}_A$. Then the pair $(A, \mathcal{L})$ satisfies étale descent (or rather fpqc descent) by [56, VIII, 7.8]. The additional structures λ, i, and $\alpha_{\mathcal{H}}$ also satisfy étale descent, because they are defined by a collection of morphisms, or étale-locally-defined orbits of morphisms. Hence $\mathsf{M}_{\mathcal{H}}$ is a stack for the étale topology, which is locally of finite presentation (by Lemma 1.4.1.10).

To verify condition 2, note that the functor $\underline{\mathrm{Isom}}_U(\xi, \eta)$ is representable by an algebraic space by the general theory of Hilbert schemes and by [6, Cor. 6.2]. Moreover, it is quasi-finite by Lemma 2.3.3.1. Therefore, it is proper (and hence finite) over U by the valuative criterion over discrete valuation rings, using the theory of Néron models.

Remark 2.3.4.1. As soon as $\mathsf{M}_{\mathcal{H}}$ is an algebraic stack, what we have shown will imply that the diagonal morphism $\Delta : \mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{M}_{\mathcal{H}}$ is *finite*, and hence that $\mathsf{M}_{\mathcal{H}}$ is *separated* over S_0. Moreover, if the objects parameterized by $\mathsf{M}_{\mathcal{H}}$ have no nontrivial automorphisms, which by Corollary 2.3.3.2 is the case when $\mathcal{H}$ is *neat*, then Δ is a closed immersion.

Finally, condition 3 is already proved as Proposition 2.3.2.1, and condition 4 can be suppressed by our technical assumption that the set $\Box$ is either empty or infinite. Hence we see that $\mathsf{M}_{\mathcal{H}}$ is an algebraic stack separated and locally of finite presentation over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$.

By Corollary 2.3.3.2, the objects of $\mathsf{M}_{\mathcal{H}}$ do not admit automorphisms when $\mathcal{H}$ is neat. Hence $\mathsf{M}_{\mathcal{H}}$ is also representable by an algebraic space when $\mathcal{H}$ is neat (or whenever the existence of additional structures forces automorphisms of objects to be trivial).

Let us show that $\mathsf{M}_{\mathcal{H}}$ is of finite type. By Definition A.7.2.7, an algebraic stack is of finite type if it is quasi-compact and locally of finite type. By Lemma A.7.2.6, it suffices to show that there is a surjection from a quasi-compact scheme to $\mathsf{M}_{\mathcal{H}}$. Let us describe briefly how such a surjection can be constructed.

Let S be any locally noetherian scheme, and let $(A, \lambda, i, \alpha_{\mathcal{H}})$ be any object in $\mathsf{M}_{\mathcal{H}}(S)$. Since λ is a polarization, the invertible sheaf $\mathcal{L} := (\mathrm{Id}_A, \lambda)^* \mathcal{P}_A$ is relatively ample over S (see Definition 1.3.2.16). Moreover, $\pi_*(\mathcal{L}^{\otimes 3})$ is locally free of finite rank over S by [96, Prop. 6.13]. Let $m := \mathrm{rank}_{\mathscr{O}_S}(\pi_*(\mathcal{L}^{\otimes 3})) - 1$ (as a locally constant function over S). Whenever a local basis of sections of $\pi_*(\mathcal{L}^{\otimes 3})$ is chosen, the basis vectors define an isomorphism $r : \mathbb{P}_S(\pi_*(\mathcal{L}^{\otimes 3})) \xrightarrow{\sim} \mathbb{P}^m_S$ and define an embedding of A into $\mathbb{P}^m_S$, as $\mathcal{L}^{\otimes 3}$ is very ample over each fiber of A. Different choices give different embeddings r, which differ by an automorphism of $\mathbb{P}^m_S$ induced by the action of $\mathrm{PGL}(m+1)_S$. The images of the embeddings form a family of closed subschemes of $\mathbb{P}^m_S$, parameterized by the so-called Hilbert schemes. By Proposition 1.3.3.7, the additional structures such as endomorphisms and level structures are parameterized by schemes quasi-compact over the above-mentioned Hilbert schemes. By limit arguments (using Theorem 1.3.1.3), this shows that if we consider the moduli problem $\tilde{\mathsf{M}}_{\mathcal{H}}$ parameterizing tuples of the form $(A, \lambda, i, \alpha_{\mathcal{H}}, r)$, then $\tilde{\mathsf{M}}_{\mathcal{H}}$ is representable by a quasi-compact scheme over S_0. On the other hand, we have a natural surjection $\tilde{\mathsf{M}}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}}$ defined by forgetting the structure r. This shows that $\mathsf{M}_{\mathcal{H}}$ is also quasi-compact, as desired.

Remark 2.3.4.2. For more details on the use of Hilbert schemes, see [96, Ch. 7], where geometric invariant theory is used to carry out the construction of moduli of polarized abelian schemes. We could have proceeded in the same way, but this is logically unnecessary (see Remark 1.4.1.13).

Let us return to the proof of Theorem 1.4.1.11. Since $\mathsf{M}_{\mathcal{H}}$ is locally of finite presentation, its subset of points of finite type is dense. Since it is formally smooth at all of its points of finite type by Theorem 2.2.4.13, it is smooth everywhere. Thus the moduli problem $\mathsf{M}_{\mathcal{H}}$ is an algebraic stack separated, smooth, and of finite type over S_0. If the objects parameterized by $\mathsf{M}_{\mathcal{H}}$ have no nontrivial automorphisms, then $\mathsf{M}_{\mathcal{H}}$ is representable by an algebraic space (separated, smooth, and of finite type over S_0) by Proposition A.7.4 and Remark 2.3.4.1. This concludes the proof of Theorem 1.4.1.11.

2.3.5 Properties of Kodaira–Spencer Morphisms

Definition 2.3.5.1. *Let $(A, \lambda, i, \alpha_{\mathcal{H}})$ be a tuple over S parameterized by $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. Then we define the $\mathscr{O}_S$-module $\underline{\mathrm{KS}}_{(A,\lambda,i)/S} = \underline{\mathrm{KS}}_{(A,\lambda,i,\alpha_{\mathcal{H}})/S}$ as the ($\mathscr{O}_S$-module) quotient*

$$(\underline{\mathrm{Lie}}^\vee_{A/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/S}) / \begin{pmatrix} \lambda^*(y) \otimes z - \lambda^*(z) \otimes y \\ i(b)^*(x) \otimes y - x \otimes (i(b)^\vee)^*(y) \end{pmatrix}_{\substack{x \in \underline{\mathrm{Lie}}^\vee_{A/S}, \\ y,z \in \underline{\mathrm{Lie}}^\vee_{A^\vee/S}, \\ b \in \mathcal{O}}}.$$

(Here the quotient expression with $x \in \underline{\mathrm{Lie}}^\vee_{A/S}$ etc. means the sheafification of the presheaf with sections given by quotients of the same form, with x a section of $\underline{\mathrm{Lie}}^\vee_{A/S}$ etc.)

Proposition 2.3.5.2. *Let $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ be the base scheme over which $\mathsf{M}_{\mathcal{H}}$ is defined. Let $f : S \to \mathsf{M}_{\mathcal{H}}$ be any morphism over S_0, and let $(A, \lambda, i, \alpha_{\mathcal{H}})$ be the tuple over S associated with f by the universal property of $\mathsf{M}_{\mathcal{H}}$. Suppose that $\Omega^1_{S/\mathsf{S}_0}$ is locally free over $\mathscr{O}_S$. Then the Kodaira–Spencer morphism*

$$\mathrm{KS} = \mathrm{KS}_{A/S/\mathsf{S}_0} : \underline{\mathrm{Lie}}^\vee_{A/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/S} \to \Omega^1_{S/\mathsf{S}_0}$$

(see Definition 2.1.7.9) satisfies

$$\mathrm{KS}(\lambda^*(y) \otimes z) = \mathrm{KS}(\lambda^*(z) \otimes y) \tag{2.3.5.3}$$

and

$$\mathrm{KS}((i(b)^*(x)) \otimes y) = \mathrm{KS}(x \otimes ((i(b)^\vee)^*(y))) \tag{2.3.5.4}$$

for all $x \in \underline{\mathrm{Lie}}^\vee_{A/S}$, $y, z \in \underline{\mathrm{Lie}}^\vee_{A^\vee/S}$, and $b \in \mathcal{O}$, and hence induces a morphism

$$\mathrm{KS} : \underline{\mathrm{KS}}_{(A,\lambda,i)/S} \to \Omega^1_{S/\mathsf{S}_0}, \tag{2.3.5.5}$$

*where $\underline{\mathrm{KS}}_{(A,\lambda,i)/S}$ is defined as in Definition 2.3.5.1. Moreover, the morphism f is **étale** if and only if it is flat and KS is an **isomorphism**.*

Proof. Let $\tilde{S}$ be the first infinitesimal neighborhood of the image of S under the diagonal morphism $S \to S \underset{\mathsf{S}_0}{\times} S$ as in Section 2.1.7. Then the two relations (2.3.5.3) and (2.3.5.4) are satisfied because of Proposition 2.1.3.2, as each of the morphisms λ and $i(b)$ (for every $b \in \mathcal{O}$) can be lifted to the two pullbacks $\tilde{A}_1 := \mathrm{pr}_1^*(A)$ and $\tilde{A}_2 := \mathrm{pr}_2^*(A)$ of A under the two projections $\mathrm{pr}_1, \mathrm{pr}_2 : \tilde{S} \to S$.

Suppose the morphism f is étale. To show that (2.3.5.5) is an isomorphism over S, it suffices to show it (universally) over $\mathsf{M}_{\mathcal{H}}$, or rather over the formal completions of any étale presentation of $\mathsf{M}_{\mathcal{H}}$ at its points of finite type over the base S_0. Let us replace S with the spectrum $\mathrm{Spec}(R)$ of any such complete local ring R with maximal ideal $\mathfrak{m}$ and residue field k. Then we may assume that the tuple $(A, \lambda, i, \alpha_{\mathcal{H}})$ over S prorepresents the local deformation of the object $(A_0, \lambda_0, i_0, \alpha_{\mathcal{H},0})$ associated with some $\xi_0 : s = \mathrm{Spec}(k) = \mathrm{Spec}(R/\mathfrak{m}) \to \mathsf{M}_{\mathcal{H}}$ studied in Section 2.2.1. Since S is the formal completion of a scheme smooth and locally of finite type over S_0 (cf. Section 2.3.4), the completion $\widehat{\Omega}^1_{S/\mathsf{S}_0}$ of $\Omega^1_{S/\mathsf{S}_0}$ with respect to the topology defined by the maximal ideal $\mathfrak{m}$ of R is locally free of finite rank over S (cf. [59, 0_{IV}, 20.4.9]). By considering the image of the Kodaira–Spencer class $\mathsf{KS}_{A/S/\mathsf{S}_0}$ (cf. Proposition 2.1.7.3) under the canonical morphism

$$H^1(A, \underline{\mathrm{Der}}_{A/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{S}_0}) \to H^1(A, \underline{\mathrm{Der}}_{A/S} \underset{\mathscr{O}_S}{\otimes} \widehat{\Omega}^1_{S/\mathsf{S}_0})$$

induced by the canonical morphism $\Omega^1_{S/\mathsf{S}_0} \to \widehat{\Omega}^1_{S/\mathsf{S}_0}$, we obtain, as in Section 2.1.7, a canonical morphism

$$\mathrm{KS} : \underline{\mathrm{KS}}_{(A,\lambda,i)/S} \to \widehat{\Omega}^1_{S/\mathsf{S}_0}, \tag{2.3.5.6}$$

which agrees with the pullback of the analogous morphism over $\mathsf{M}_{\mathcal{H}}$. Thus, to show that (2.3.5.5) is an isomorphism, it suffices to show that (2.3.5.6) is an isomorphism. Since R is noetherian, since both sides of (2.3.5.6) is finitely generated, and since $\widehat{\Omega}^1_{S/\mathsf{S}_0}$ is flat over R, by Nakayama's lemma and by an argument analogous to that in the proof of Lemma 2.1.1.1, it suffices to show that the reduction of (2.3.5.6) modulo $\mathfrak{m}$ is an isomorphism, which follows from Corollaries 2.2.2.10 and 2.2.4.10. (Note that we do not need Corollaries 2.2.2.10 and 2.2.4.10 to know that $\underline{\mathrm{KS}}$ is a locally free sheaf. The local freeness of $\underline{\mathrm{KS}}$ follows from Proposition 1.2.2.3 and the assumption that (A, λ, i) satisfies the Lie algebra condition.)

Conversely, suppose $\mathrm{KS} : \underline{\mathrm{KS}}_{(A,\lambda,i)/S} \to \Omega^1_{S/\mathsf{S}_0}$ is an isomorphism. By the previous paragraph, we have an isomorphism $\mathrm{KS} : \underline{\mathrm{KS}}_{(A,\lambda,i)/\mathsf{M}_{\mathcal{H}}} \xrightarrow{\sim} \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$, where by abuse of notation we have also used (A, λ, i) to denote the tautological objects over $\mathsf{M}_{\mathcal{H}}$. Since the construction of $\underline{\mathrm{KS}}_{(A,\lambda,i)/S}$ commutes with base change, and since the association of Kodaira–Spencer morphisms is functorial, the first morphism in the exact sequence $f^*\Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0} \to \Omega^1_{S/\mathsf{S}_0} \to \Omega^1_{S/\mathsf{M}_{\mathcal{H}}} \to 0$ is an isomorphism. This shows that f is unramified, and hence étale because it is flat by assumption. $\square$

Chapter 3

Structures of Semi-Abelian Schemes

In this chapter, we review notions that are of fundamental importance in the study of degeneration of abelian varieties. The main objective is to understand the statement and the proof of the theory of degeneration data, to be presented in Chapter 4. Our main references for these will be [40], [57], and in particular, [93].

Technical results worth noting are Theorem 3.1.3.3, Propositions 3.1.5.1, 3.3.1.5, 3.3.1.7, and 3.3.3.6, Theorems 3.4.2.4 and 3.4.3.2, Lemma 3.4.3.1, and Proposition 3.4.4.1.

3.1 Groups of Multiplicative Type, Tori, and Their Torsors

3.1.1 Groups of Multiplicative Type

Definition 3.1.1.1 ([40, IX, 1.1])**.** *A **group (scheme) of multiplicative type** over a scheme S is a commutative group scheme over S that is **fpqc locally** of the form $\underline{\mathrm{Hom}}(X, \mathbf{G}_{\mathrm{m}})$ for some commutative group X.*

A fundamental property of groups of multiplicative type is that they are *rigid* in the sense that they cannot be deformed. We describe this phenomenon as follows:

Theorem 3.1.1.2 (see [40, IX, 3.6 and 3.6 bis])**.** *Let S be a quasi-compact scheme, and let S_0 be a closed subscheme of S defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$. Let H_0 be a group of multiplicative type over S_0, let G be a commutative group scheme smooth over S, and let $G_0 := G \underset{S}{\times} S_0$. Then*

1. *there exists (up to unique isomorphism) a group H of multiplicative type over S such that $H \underset{S}{\times} S_0 \cong H_0$;*
2. *each homomorphism $u_0 : H_0 \to G_0$ can be uniquely lifted to a homomorphism $u : H \to G$. If u_0 is a closed embedding, then so is u.*

The definition of groups of multiplicative type can be weakened when we talk about group schemes of finite type over the base scheme S:

Theorem 3.1.1.3 (see [40, X, 4.5])**.** *Every group of multiplicative type that is **of finite type** over a base scheme S is **étale locally** isomorphic to a group scheme of the form $\underline{\mathrm{Hom}}(X, \mathbf{G}_{\mathrm{m}})$ for some commutative group X.*

Definition 3.1.1.4. *For a group H of multiplicative type of finite type, we denote by $\underline{\mathbf{X}}(H) := \underline{\mathrm{Hom}}_S(H, \mathbf{G}_{\mathrm{m},S})$ the **character group** of H. It is an étale sheaf of finitely generated commutative groups over S. We say the group H is **split** if $\underline{\mathbf{X}}(H)$ is a constant sheaf. (In this case, we shall denote the constant value group by $\mathbf{X}(H)$.) We say the group H is **isotrivial** if there is a **finite étale** covering $S' \to S$ such that $H \underset{S}{\times} S'$ is split.*

Definition 3.1.1.5. *A **torus** T over a scheme S is a group of multiplicative type of finite type such that $\underline{\mathbf{X}}(T)$ is an étale sheaf of finitely generated **free** commutative groups. A torus is **split** (resp. **isotrivial**) if it is split (resp. isotrivial) as a group of multiplicative type of finite type. The **rank** of a torus T is the rank of the geometric stalks of $\underline{\mathbf{X}}(T)$ (as a locally constant function) over S, which can be identified with the dimensions of the fibers of T over S.*

Lemma 3.1.1.6. *The category of groups of multiplicative type of finite type (resp. tori) over a scheme S is antiequivalent to the category of étale sheaves of finitely generated commutative groups (resp. finitely generated free commutative groups) over S, the equivalence being given by sending a group H to the étale sheaf $\underline{\mathbf{X}}(H) := \underline{\mathrm{Hom}}_S(H, \mathbf{G}_{\mathrm{m},S})$, with a quasi-inverse given by sending an étale sheaf $\underline{X}$ to $\underline{\mathrm{Hom}}_S(\underline{X}, \mathbf{G}_{\mathrm{m},S})$.*

3.1.2 Torsors and Invertible Sheaves

Let us begin by reviewing the notion of torsors. Our main reference is [22, §6.4].

Definition 3.1.2.1. *Given a scheme $f : Z \to S$ over some base scheme S, a scheme $\mathcal{M}$ over Z, and a group scheme H over S acting on $\mathcal{M}$ by a morphism*

$$H_Z \underset{Z}{\times} \mathcal{M} \to \mathcal{M} : (g, x) \mapsto gx,$$

*where $H_Z := H \underset{S}{\times} Z$, assume that H_Z is (faithfully) flat and locally of finite presentation over Z. Then $\mathcal{M}$ is called an **H-torsor over** Z (with respect to the fppf topology) if*

1. *the structural morphism $\mathcal{M} \to Z$ is faithfully flat and locally of finite presentation, and*
2. *the morphism $H_Z \underset{Z}{\times} \mathcal{M} \to \mathcal{M} \underset{Z}{\times} \mathcal{M}$ defined by $(g, x) \mapsto (gx, x)$ is an isomorphism over Z.*

Viewing $H_Z \underset{Z}{\times} \mathcal{M}$ and $\mathcal{M} \underset{Z}{\times} \mathcal{M}$ as schemes over $\mathcal{M}$ with respect to the second projections, we see that the isomorphism in 2 of Definition 3.1.2.1 is an isomorphism over $\mathcal{M}$. In other words, $\mathcal{M}$ and H_Z become isomorphic after making the base change under $\mathcal{M} \to Z$. The same is true for the base change under every $Y \to Z$ that factors through $\mathcal{M}$. As a result, if $H_Z \to Z$ satisfies any of the properties listed in [59, IV-2, 2.7.1 and IV-4, 17.7.4], such as being smooth or of finite type, then $\mathcal{M} \to Z$ also does. (The references are applicable because we may assume that Z is affine and replace $\mathcal{M}$ with a quasi-compact subscheme when verifying the properties.)

If $\mathcal{M}(Z) \neq \emptyset$, the choice of any Z-valued point of $\mathcal{M}$ gives an isomorphism over Z from H_Z to $\mathcal{M}$. We say that the torsor $\mathcal{M}$ is *trivial* in this case. In general, if $\mathcal{M}(Z')$ is nonempty for some scheme $Z' \to Z$, then the pullback of $\mathcal{M}$ under $Z' \to Z$ is trivial.

Proposition 3.1.2.2 (see [22, §2.2, Prop. 14, and §6.4]). *With assumptions on H_Z and $\mathcal{M}$ as in Definition 3.1.2.1, suppose moreover that H_Z is **smooth**. Then there exists an étale covering $Z' \to Z$ such that $\mathcal{M}(Z') \neq \emptyset$. Therefore $\mathcal{M}$ is trivial after étale localization.*

In the special case of $H = \mathbf{G}_{\mathrm{m},Z}$, we have the following generalization of *Hilbert's Theorem 90* in the absolute setting:

Theorem 3.1.2.3. *Let Z be a scheme. We have an isomorphism $H^1_{\text{ét}}(Z, \mathbf{G}_{\mathrm{m}}) \overset{\sim}{\to} \mathrm{Pic}(Z)$, where $\mathrm{Pic}(Z)$ is the **(absolute) Picard group**; namely, the group of isomorphism classes of invertible sheaves over Z.*

See, for example, [14, IX, 3.3], [44, 2.10], or [91, III, 4.9] for the proof.

Since we are interested in the relative setting over S, we shall introduce the comparison between different topologies for the relative Picard functor.

Definition 3.1.2.4. *Let $f : Z \to S$ be a morphism of finite presentation. We say that $\mathscr{O}_S \overset{\sim}{\to} f_*\mathscr{O}_Z$ **holds universally** if for every morphism $S' \to S$, the canonical morphism $\mathscr{O}_{S'} \to (f \underset{S}{\times} S')_* \mathscr{O}_{Z \underset{S}{\times} S'}$ is an isomorphism.*

Theorem 3.1.2.5. *Suppose that $f : Z \to S$ is a morphism of finite presentation and that $\mathscr{O}_S \overset{\sim}{\to} f_*\mathscr{O}_Z$ holds universally. Then we have a series of natural injections*
$$\underline{\mathrm{Pic}}(Z/S) \hookrightarrow R^1 f_{*\mathrm{zar}} \mathbf{G}_{\mathrm{m}} \hookrightarrow R^1 f_{*\text{ét}} \mathbf{G}_{\mathrm{m}} \hookrightarrow R^1 f_{*\mathrm{fppf}} \mathbf{G}_{\mathrm{m}}.$$
If f has a section globally (resp. locally in the Zariski topology), then all three (resp. the latter two) injections are isomorphisms. If f has a section locally in the étale topology, then the last injection is an isomorphism.

See [71, Thm. 9.2.5] for the proof. (The assumption that schemes are locally noetherian in the beginning of [71] does not interfere there.)

Remark 3.1.2.6. The assumption that $\mathscr{O}_S \overset{\sim}{\to} f_*\mathscr{O}_Z$ holds universally is true, for example, if Z is an abelian scheme over S.

Assumption 3.1.2.7. *Let us assume from now on that the scheme $f : Z \to S$ is of finite presentation, that $\mathscr{O}_S \overset{\sim}{\to} f_*\mathscr{O}_Z$ holds universally, and that f admits a section $e_Z : S \to Z$.*

Under this assumption, $\underline{\mathrm{Pic}}(Z/S)$ is canonically isomorphic to $\underline{\mathrm{Pic}}_{e_Z}(Z/S)$ by rigidifications.

Now suppose H is a split group of multiplicative type *of finite type* over S. Then $H \to S$ is of finite presentation by its smoothness, and the techniques of reduction to the noetherian case by Theorem 1.3.1.3 apply. We shall assume that all our H-torsors $\mathcal{M}$ are rigidified, that is, each of them is equipped with an isomorphism $H \cong e_Z^*\mathcal{M}$ (over S).

By Proposition 3.1.2.2, we can define alternatively an H-torsor $\mathcal{M}$ by giving an étale covering $\{U_i\}_i$ of Z over S, together with trivializations of H-torsors $\mathcal{M}|_{U_i} \cong (H_Z \underset{Z}{\times} U_i)$ and gluing isomorphisms $a_{ij} \in H(U_i \underset{Z}{\times} U_j)$ giving isomorphisms $H_Z \underset{Z}{\times} (U_i \underset{Z}{\times} U_j) \overset{a_{ij}}{\underset{\sim}{\to}} H_Z \underset{Z}{\times} (U_i \underset{Z}{\times} U_j)$ over $U_i \underset{Z}{\times} U_j$. We write such data as $\{\{U_i\}_i, \{a_{ij}\}_{ij}\}$ and call it the gluing data for $\mathcal{M}$ in the étale topology.

Since H is a split group of multiplicative type of finite type over S, which is naturally embedded as a subgroup of a split torus over S, the gluing data $\{\{U_i\}_i, \{a_{ij}\}_{ij}\}$ define a global section of $R^1 f_{*\text{ét}}(H)$, which can be identified with a global section of $R^1 f_{*\mathrm{zar}}(H)$ by Theorem 3.1.2.5, that is, we may assume that there is a Zariski open covering $\{U'_i\}_i$ of Z over S trivializing the H-torsor $\mathcal{M}$ with gluing isomorphisms $\{a'_{ij}\}_{ij}$ as above. Let us summarize the above as follows:

Corollary 3.1.2.8. *If H is a split group of multiplicative type of finite type over S, then every H-torsor over Z can be defined by some gluing data $\{\{U_i\}_i, \{a_{ij}\}_{ij}\}$ in the Zariski (or étale, or fppf) topology over S.*

Suppose that we have an H-torsor $\mathcal{M}$. Regarding the torsor as a scheme relatively affine over Z, with structural morphism $\pi : \mathcal{M} \to Z$, we can consider the push-forward $\pi_*\mathscr{O}_{\mathcal{M}}$ of the structural sheaf $\mathscr{O}_{\mathcal{M}}$ over $\mathscr{O}_Z$. Then $\pi_*\mathcal{M}$ is an $\mathscr{O}_Z$-algebra (over Z), and $\mathcal{M} \cong \underline{\mathrm{Spec}}_{\mathscr{O}_Z}(\pi_*\mathscr{O}_{\mathcal{M}})$.

Convention 3.1.2.9. *By abuse of notation, and for simplicity, we shall often write $\mathscr{O}_{\mathcal{M}}$ for $\pi_*\mathscr{O}_{\mathcal{M}}$ in similar cases (when π is relatively affine), and say that we consider $\mathscr{O}_{\mathcal{M}}$ as an $\mathscr{O}_Z$-algebra (over Z).*

Since H is of multiplicative type, the H_Z-action on $\mathcal{M}$ defines a decomposition of the quasi-coherent $\mathscr{O}_Z$-module $\mathscr{O}_{\mathcal{M}}$ into *weight subsheaves* (or *eigensheaves*)
$$\mathscr{O}_{\mathcal{M}} = \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} \mathscr{O}_{\mathcal{M},\chi},$$
each $\mathscr{O}_{\mathcal{M},\chi}$ being a quasi-coherent $\mathscr{O}_Z$-submodule of $\mathscr{O}_{\mathcal{M}}$ (see [39, I, Prop. 4.7.3]). Since $\mathcal{M}$ is an H-torsor, it is isomorphic to H after some étale surjective base change Therefore, the quasi-coherent sheaf $\mathscr{O}_{\mathcal{M},\chi}$ is invertible for each $\chi \in \underline{\mathbf{X}}(H)$, and the canonical morphism $\mathscr{O}_{\mathcal{M},\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M},\chi'} \to \mathscr{O}_{\mathcal{M},\chi+\chi'}$ is an isomorphism for each $\chi, \chi' \in \underline{\mathbf{X}}(H)$, because these two statements can be verified by étale descent (see also [40, VIII, Prop. 4.1]).

Suppose we have a Zariski open covering $\{U_i\}_i$ of Z over S such that we have $\mathcal{M}|_{U_i} \cong H_Z \underset{Z}{\times} U_i$ over U_i and gluing isomorphisms $H_Z \underset{Z}{\times} (U_i \underset{Z}{\times} U_j) \overset{a_{ij}}{\underset{\sim}{\to}} H_Z \underset{Z}{\times} (U_i \underset{Z}{\times} U_j)$ over $U_i \underset{Z}{\times} U_j$. Then we have isomorphisms $\mathscr{O}_{\mathcal{M}}|_{U_i} \cong \mathscr{O}_{H_Z}|_{U_i}$ over U_i, with isomorphisms a_{ij} corresponding to multiplications $\mathscr{O}_{H_Z,\chi}(U_i \underset{Z}{\times} U_j) \overset{\chi(a_{ij})}{\underset{\sim}{\leftarrow}} \mathscr{O}_{H_Z,\chi}(U_i \underset{Z}{\times} U_j)$ in the reverse direction. Thus, the invertible sheaf $\mathscr{O}_{\mathcal{M},\chi}$ (defining a global section of $R^1 f_{*\mathrm{zar}}(Z, \mathbf{G}_{\mathrm{m}})$) can be defined by the gluing data $\{\{U_i\}_i, \{(-\chi)(a_{ij})\}_{ij}\}$.

Definition 3.1.2.10. *Let $\mathcal{M}$ be an H-torsor $\mathcal{M}$ over Z. The **push-out** $\mathcal{M}_\chi$ of $\mathcal{M}$ by a character $\chi : H \to \mathbf{G}_{\mathrm{m},Z}$ is the quotient of $\mathbf{G}_{\mathrm{m},Z} \underset{Z}{\times} \mathcal{M}$ by the relations $(x, tl) \sim (x\chi(t), l)$, where x, l, and t are functorial points of $\mathbf{G}_{\mathrm{m},Z}$, $\mathcal{M}$, and H_Z, respectively. (Then H acts on $\mathcal{M}_\chi$ by $\chi : H \to \mathbf{G}_{\mathrm{m},Z}$.)*

By definition, the formation of push-outs is functorial and compatible with arbitrary base change. Therefore, if the H-torsor $\mathcal{M}$ is defined by some Zariski gluing data $\{\{U_i\}_i, \{a_{ij}\}_{ij}\}$, then the push-out $\mathcal{M}_\chi$ is defined by the induced gluing data $\{\{U_i\}_i, \{\chi(a_{ij})\}_{ij}\}$.

Note that $\mathcal{M}_\chi$ is naturally rigidified if $\mathcal{M}$ is. By comparing the gluing data, we obtain the following two propositions:

Proposition 3.1.2.11. *Under Assumption 3.1.2.7, let $\mathcal{M}$ be a rigidified H-torsor over some scheme Z over S, where H is a split group of multiplicative type of finite type over S. Then the push-out operation defines a homomorphism*

$$\underline{\mathbf{X}}(H) \to \underline{\mathrm{Pic}}_{e_Z}(Z/S) : \chi \mapsto \mathcal{M}_\chi.$$

In particular, for each $\chi, \chi' \in \underline{\mathbf{X}}(H)$, there is a canonical isomorphism

$$\mathcal{M}_\chi \otimes \mathcal{M}_{\chi'} \cong \mathcal{M}_{\chi+\chi'} \tag{3.1.2.12}$$

respecting the rigidifications.

Proposition 3.1.2.13. *Under Assumption 3.1.2.7, let $\mathcal{M}$ be a rigidified H-torsor over some scheme Z over S, where H is a split group of multiplicative type of finite type over S. Let $\mathscr{O}_{\mathcal{M},\chi}$ be the weight-χ subsheaf of $\mathscr{O}_{\mathcal{M}}$ under the H_Z-action, where $\chi \in \underline{\mathbf{X}}(H)$. Let $\mathcal{M}_{-\chi}$ be the push-out of $\mathcal{M}$ by $(-\chi) : H \to \mathbf{G}_{\mathrm{m},Z}$. Then we have a (necessarily unique) isomorphism $\mathcal{M}_{-\chi} \cong \underline{\mathrm{Isom}}_{\mathscr{O}_Z}(\mathscr{O}_Z, \mathscr{O}_{\mathcal{M},\chi})$ respecting the rigidifications.*

Corollary 3.1.2.14. *Under Assumption 3.1.2.7, let $\mathcal{L}$ be a $\mathbf{G}_{\mathrm{m}}$-torsor over some scheme Z over S. Then we have a (necessarily unique) isomorphism $\mathcal{L} = \mathcal{L}_1 \cong \underline{\mathrm{Isom}}_{\mathscr{O}_Z}(\mathscr{O}_Z, \mathscr{O}_{\mathcal{L},-1})$ respecting the rigidifications. Here the character $1 : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ (resp. $-1 : \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$) is the identity (resp. inverse) isomorphism on $\mathbf{G}_{\mathrm{m}}$.*

3.1.3 Construction Using Sheaves of Algebras

Proposition 3.1.3.1. *Under Assumption 3.1.2.7, let H be a split group of multiplicative type of finite type over S, with character group $\underline{\mathbf{X}}(H)$, and suppose that we are given a homomorphism*

$$\underline{\mathbf{X}}(H) \to \underline{\mathrm{Pic}}_{e_Z}(Z/S) : \chi \mapsto \mathscr{M}_\chi.$$

In other words, suppose that we are given a family of rigidified invertible sheaves $\mathscr{M}_\chi$ over Z, indexed by $\chi \in \underline{\mathbf{X}}(H)$, together with the unique isomorphisms $\Delta^*_{\chi,\chi'} : \mathscr{M}_\chi \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_{\chi'} \cong \mathscr{M}_{\chi+\chi'}$ inducing the canonical isomorphisms $\mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_Z \cong \mathscr{O}_Z$ respecting the rigidifications. By abuse of notation, set $\mathscr{O}_{\mathcal{M}} := \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} \mathscr{M}_\chi$, which is equipped with the structure of an $\mathscr{O}_Z$-algebra $\Delta^* : \mathscr{O}_{\mathcal{M}} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \to \mathscr{O}_{\mathcal{M}}$ defined by the isomorphisms $\Delta^*_{\chi,\chi'}$. Then $\mathcal{M} := \underline{\mathrm{Spec}}_{\mathscr{O}_Z}(\mathscr{O}_{\mathcal{M}})$ has the natural structure of a rigidified H-torsor such that $\mathscr{O}_{\mathcal{M},\chi} = \mathscr{M}_\chi$, where $\mathscr{O}_{\mathcal{M},\chi}$ is the weight-χ subsheaf under the H_Z-action.

Proof. Let us first define an H_Z-action on $\mathcal{M}$, namely, a morphism $m : H_Z \underset{Z}{\times} \mathcal{M} \to \mathcal{M}$ satisfying the usual requirement for an action. (In what follows, all morphisms will be over Z or $\mathscr{O}_Z$, unless otherwise specified.)

Let us write $\mathscr{O}_{H_Z} = \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} \mathscr{O}_{H_Z,\chi}$ as in the last section. Then $\mathscr{O}_{H_Z,\chi} \cong \mathscr{O}_Z$, because H_Z is a trivial H-torsor, and we can write

$$\mathscr{M}_\chi \overset{\mathrm{can.}}{\cong} \mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi \cong \mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi.$$

All these isomorphisms are uniquely determined if we require the invertible modules to be rigidified. In particular, we get a morphism

$$m^* : \mathscr{O}_{\mathcal{M}} = \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} \mathscr{M}_\chi \to \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \subset \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}},$$

making the diagrams

$$\begin{tikzcd}
\mathscr{O}_{\mathcal{M}} = \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} \mathscr{M}_\chi \arrow[r, "m^*"] \arrow[dd, "m^*"'] & \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \arrow[d, "\mathrm{Id} \otimes m^*"] \\
& \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi)) \arrow[d, "\wr"', "\mathrm{can.}"] \\
\underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \arrow[r, "m^*_{H_Z} \otimes \mathrm{Id}"'] & \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} ((\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{H_Z,\chi}) \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathscr{O}_{\mathcal{M}} = \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} \mathscr{M}_\chi \arrow[r, "m^*"] \arrow[dd, "\mathrm{Id}"'] & \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \arrow[d, hook] \\
& \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \arrow[d, "e^*_{H_Z} \otimes \mathrm{Id}"] \\
\mathscr{O}_{\mathcal{M}} = \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} \mathscr{M}_\chi \arrow[r, "\sim", "\mathrm{can.}"'] & \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} \mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi
\end{tikzcd}$$

commutative.

Translating the above diagrams back to diagrams of schemes, we get a morphism $m : H_Z \underset{Z}{\times} \mathcal{M} \to \mathcal{M}$, which is an action because the diagrams

$$\begin{tikzcd}
(H_Z \underset{Z}{\times} H_Z) \underset{Z}{\times} \mathcal{M} \arrow[rr, "m_{H_Z} \times \mathrm{Id}"] \arrow[d, "\mathrm{can.}"', "\wr"] & & H_Z \underset{Z}{\times} \mathcal{M} \arrow[d, "m"] \\
H_Z \underset{Z}{\times} (H_Z \underset{Z}{\times} \mathcal{M}) \arrow[r, "\mathrm{Id} \times m"'] & H_Z \underset{Z}{\times} \mathcal{M} \arrow[r, "m"'] & \mathcal{M}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\mathrm{Id}"] \arrow[d, "(e_{H_Z}, \mathrm{Id})"'] & \mathcal{M} \arrow[d, "\mathrm{Id}"] \\
(H_Z \underset{Z}{\times} \mathcal{M}) \arrow[r, "m"'] & \mathcal{M}
\end{tikzcd}$$

are commutative.

Now we check that $(m, \mathrm{pr}_2) : H_Z \underset{Z}{\times} \mathcal{M} \to \mathcal{M} \times \mathcal{M}$ is an isomorphism. This is true because the diagram

$$\begin{tikzcd}
\mathscr{O}_{\mathcal{M}} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \arrow[d, "\wr", "m^* \otimes \mathrm{Id}"'] \arrow[dr, "m^* \otimes \mathrm{Id}"] & \\
\Big(\bigoplus\limits_{\chi \in \underline{\mathbf{X}}(H)} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi)\Big) \underset{\mathscr{O}_Z}{\otimes} \Big(\bigoplus\limits_{\chi \in \underline{\mathbf{X}}(H)} \mathscr{M}_\chi\Big) \arrow[r, hook] \arrow[d, "\wr", "\mathrm{can.}"'] & (\mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}}) \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \arrow[d, "\wr", "\mathrm{can.}"'] \\
\Big(\bigoplus\limits_{\chi \in \underline{\mathbf{X}}(H)} \mathscr{O}_{H_Z,\chi}\Big) \underset{\mathscr{O}_Z}{\otimes} \Big(\bigoplus\limits_{\chi' \in \underline{\mathbf{X}}(H)} \mathscr{M}_\chi \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_{\chi'}\Big) \arrow[r, hook] \arrow[d, "\wr", "\mathrm{Id} \otimes \Delta^*"'] & \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} (\mathscr{O}_{\mathcal{M}} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}}) \arrow[d, "\mathrm{Id} \otimes \Delta^*"] \\
\Big(\bigoplus\limits_{\chi \in \underline{\mathbf{X}}(H)} \mathscr{O}_{H_Z,\chi}\Big) \underset{\mathscr{O}_Z}{\otimes} \Big(\bigoplus\limits_{\chi'' \in \underline{\mathbf{X}}(H)} \mathscr{M}_{\chi''}\Big) \arrow[r, equal] & \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}}
\end{tikzcd}$$

is commutative.

The fact that $\mathscr{O}_{\mathcal{M}}$ is rigidified follows because we have compatible isomorphisms $e_Z^* \mathscr{M}_\chi \cong \mathscr{O}_S$ respecting the canonical isomorphism $\mathscr{O}_S \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S \cong \mathscr{O}_S$ induced by the rigidifications. These isomorphisms patch together and give an isomorphism

$$e_Z^* \mathscr{O}_{\mathcal{M}} = \bigoplus_{\chi \in \underline{\mathbf{X}}(H)} e_Z^* \mathscr{M}_\chi \cong \bigoplus_{\chi \in \underline{\mathbf{X}}(H)} \mathscr{O}_S \cong \mathscr{O}_H,$$

which gives the rigidification we want.

Finally, let us verify the statement that $\mathscr{O}_{\mathcal{M},\chi} = \mathscr{M}_\chi$. Given any Z-valued point $h : Z \to H_Z = H \underset{S}{\times} Z$, the multiplication by h on H_Z is given by

$$H_Z \xrightarrow[\sim]{\mathrm{can.}} Z \underset{Z}{\times} H_Z \xrightarrow{h \times \mathrm{Id}} H_Z \underset{Z}{\times} H_Z \xrightarrow{m_{H_Z}} H_Z \ ,$$

while the multiplication by h on $\mathcal{M}$ is given by

$$\mathcal{M} \xrightarrow[\sim]{\mathrm{can.}} Z \underset{Z}{\times} \mathcal{M} \xrightarrow{h \times \mathrm{Id}} H_Z \underset{Z}{\times} \mathcal{M} \xrightarrow{m} \mathcal{M} \ .$$

In terms of $\mathscr{O}_Z$-algebras, the commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{H_Z} \arrow[r, "m_{H_Z}^*"] \arrow[d, equal] & \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{H_Z} \arrow[r, "h^* \otimes \mathrm{Id}"] & \mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{H_Z} \cong \mathscr{O}_{H_Z} \\
\bigoplus\limits_{\chi \in \underline{\mathbf{X}}(H)} \mathscr{O}_{H_Z,\chi} \arrow[r, "m_{H_Z}^*"] & \bigoplus\limits_{\chi \in \underline{\mathbf{X}}(H)} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{H_Z,\chi}) \arrow[r, "h^* \otimes \mathrm{Id}"] \arrow[u, hook] & \bigoplus\limits_{\chi \in \underline{\mathbf{X}}(H)} (\mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{H_Z,\chi}) \arrow[u, "\mathrm{can.}", "\wr"']
\end{tikzcd}$$

explains the multiplication on H_Z, which implies that

$$h^* : \bigoplus_{\chi \in \underline{\mathbf{X}}(H)} \mathscr{O}_{H_Z,\chi} \to \mathscr{O}_Z$$

is given by

$$\sum_{\chi \in \underline{\mathbf{X}}(H)} c_\chi X^\chi \mapsto \sum_{\chi \in \underline{\mathbf{X}}(H)} c_\chi \chi(h),$$

if we denote symbolically by $c_\chi X^\chi$ an element in $\mathscr{O}_{H_Z,\chi}$ sent to c_χ under $\mathscr{O}_{H_Z,\chi} \cong \mathscr{O}_Z$, and if we view $\chi(h)$ as a section of $\mathbf{G}_{\mathrm{m}}(Z) = \mathscr{O}_Z^\times \subset \mathscr{O}_Z$. The commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{\mathcal{M}} \arrow[r, "m^*"] \arrow[d, equal] & \mathscr{O}_{H_Z} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \arrow[r, "h^* \otimes \mathrm{Id}"] & \mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M}} \cong \mathscr{O}_{\mathcal{M}} \\
\bigoplus\limits_{\chi \in \underline{\mathbf{X}}(H)} \mathscr{M}_\chi \arrow[r, "m_{H_Z}^*"] & \bigoplus\limits_{\chi \in \underline{\mathbf{X}}(H)} (\mathscr{O}_{H_Z,\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \arrow[r, "h^* \otimes \mathrm{Id}"] \arrow[u, hook] & \bigoplus\limits_{\chi \in \underline{\mathbf{X}}(H)} (\mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{M}_\chi) \arrow[u, "\mathrm{can.}", "\wr"']
\end{tikzcd}$$

explains the H_Z-action on $\mathcal{M}$, which by comparison implies that $\mathscr{M}_\chi$ is the weight-χ subsheaf of $\mathscr{O}_{\mathcal{M}}$ under the H_Z-action, as desired. □

Corollary 3.1.3.2. *Under Assumption 3.1.2.7, let $\mathscr{L}$ be a rigidified invertible sheaf over Z. Identify $\underline{\mathbf{X}}(\mathbf{G}_{\mathrm{m}}) = \mathbb{Z}$ naturally by sending $\mathrm{Id}_{\mathbf{G}_{\mathrm{m}}}$ to 1, and let $\Delta_{i,j}^* : \mathscr{L}^{\otimes i} \underset{\mathscr{O}_Z}{\otimes} \mathscr{L}^{\otimes j} \cong \mathscr{L}^{\otimes i+j}$ be the canonical isomorphism, which respects the canonical isomorphism $\mathscr{O}_Z \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_Z \cong \mathscr{O}_Z$ by the rigidification. Then $\mathcal{L}$ has the natural structure of a rigidified $\mathbf{G}_{\mathrm{m}}$-torsor such that $\mathscr{O}_{\mathcal{L},i} = \mathscr{L}^{\otimes -i}$. In particular, $\mathscr{O}_{\mathcal{L},-1} = \mathscr{L}$.*

Theorem 3.1.3.3. *Let H be a group of multiplicative type of finite type over S. Let $Z \to S$ be a scheme that satisfies Assumption 3.1.2.7. Then the category of rigidified H-torsors up to H-equivariant isomorphisms over Z is antiequivalent to the category of homomorphisms*

$$\underline{\mathbf{X}}(H) \to \underline{\mathrm{Pic}}_{e_Z}(Z/S)$$

between étale sheaves of groups over S.

Proof. By Propositions 3.1.2.11 and 3.1.3.1 (and their proofs), we have the antiequivalence for split H, naturally functorial in H. Thus, by étale descent, we have the antiequivalence for nonsplit H as well. □

3.1.4 Group Structures on Torsors

Let us develop a theory of *relative Hopf algebras* that describes group schemes G over S that are relatively affine over some (possibly nonaffine) group schemes A over S.

Suppose that we are given group schemes $G \to S$ and $A \to S$, together with a *relatively affine* group scheme homomorphism $G \to A$ over S. By taking push-forward and by abuse of language, we may view the structural sheaf $\mathscr{O}_G$ of G as an $\mathscr{O}_A$-algebra. Let us denote by pr_1, pr_2, pr_3, pr_{12}, pr_{23}, etc. the projections from products of copies of A. Then the group structure of G (covering the group structure of A) can be described using the following morphisms:

1. *Comultiplication* $m^* : m_A^* \mathscr{O}_G \to \mathscr{O}_{G \underset{S}{\times} G} = \mathrm{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_G$ as a homomorphism of $\mathscr{O}_{A \underset{S}{\times} A}$-algebras.

2. *Counit* $e^* : e_A^* \mathscr{O}_G \to \mathscr{O}_S$ as a homomorphism of $\mathscr{O}_S$-algebras.

3. *Coinverse* $[-1]^* : [-1]_A^* \mathscr{O}_G \to \mathscr{O}_G$ as a homomorphism of $\mathscr{O}_A$-algebras.

4. *Codiagonal* $\Delta^* : \Delta_A^*(\mathrm{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_G) = \mathscr{O}_G \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_G \to \mathscr{O}_G$ as a homomorphism of $\mathscr{O}_A$-algebras.

5. *Structural morphism* $(\mathrm{str.})^* : (\mathrm{str.}_A)^*\mathscr{O}_S = \mathscr{O}_A \to \mathscr{O}_G$ as a homomorphism of $\mathscr{O}_A$-algebras.

The associativity of the comultiplication can be checked by the commutativity of the diagram

$$\begin{tikzcd}
(m_A \times \mathrm{Id})^* m_A^* \mathscr{O}_G \arrow[rr, "\text{can.}", "\sim"'] \arrow[d, "(m_A \times \mathrm{Id})^*(m^*)"'] & & (\mathrm{Id} \times m_A)^* m_A^* \mathscr{O}_G \arrow[d, "(\mathrm{Id} \times m_A)^*(m^*)"] \\
(m_A \times \mathrm{Id})^*(\mathrm{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_G) \arrow[d, "\text{can.}"', "\wr"] & & (\mathrm{Id} \times m_A)^*(\mathrm{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_G) \arrow[d, "\wr"', "\text{can.}"] \\
\mathrm{pr}_{12}^*(m_A^* \mathscr{O}_G) \otimes \mathrm{pr}_3^* \mathscr{O}_G \arrow[d, "\mathrm{pr}_{12}^*(m^*)"'] & & \mathrm{pr}_1^* \mathscr{O}_G \otimes \mathrm{pr}_{23}^*(m_A^* \mathscr{O}_G) \arrow[d, "\mathrm{pr}_{23}^*(m^*)"] \\
(\mathrm{pr}_1^* \mathscr{O}_G \otimes \mathrm{pr}_2^* \mathscr{O}_G) \otimes \mathrm{pr}_3^* \mathscr{O}_G \arrow[rr, "\sim", "\text{can.}"'] & & \mathrm{pr}_1^* \mathscr{O}_G \otimes (\mathrm{pr}_2^* \mathscr{O}_G \otimes \mathrm{pr}_3^* \mathscr{O}_G)
\end{tikzcd} \tag{3.1.4.1}$$

of $\mathscr{O}_{A \underset{S}{\times} A \underset{S}{\times} A}$-algebras (with unspecified tensor products defined over $\mathscr{O}_{A \underset{S}{\times} A \underset{S}{\times} A}$).

The validity of the counit can be checked by the commutativity of the diagram

$$\begin{tikzcd}
(\mathrm{Id} \times e_A)^* m_A^* \mathscr{O}_G \arrow[rr, "\text{can.}", "\sim"'] \arrow[d, "(\mathrm{Id} \times e_A)^*(m^*)"'] & & \mathscr{O}_G \\
(\mathrm{Id} \times e_A)^*(\mathrm{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_G) \arrow[d, "\text{can.}"', "\wr"] & & \\
\mathscr{O}_G \underset{\mathscr{O}_S}{\otimes} e_A^* \mathscr{O}_G \arrow[rr, "\mathrm{Id} \otimes e^*"'] & & \mathscr{O}_G \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S \arrow[uu, "\wr"', "\text{can.}"]
\end{tikzcd} \tag{3.1.4.2}$$

of $\mathscr{O}_S$-algebras.

The validity of the coinverse can be checked by the commutativity of the diagram

$$\begin{tikzcd}
\Delta_A^*(\mathrm{Id} \times [-1]_A)^* m_A^* \mathscr{O}_G \arrow[rr, "\text{can.}", "\sim"'] \arrow[d, "\Delta_A^*(\mathrm{Id} \times [-1]_A)^*(m^*)"'] & & \mathscr{O}_G \\
\Delta_A^*(\mathrm{Id} \times [-1]_A^*)^*(\mathrm{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_G) \arrow[d, "\text{can.}"', "\wr"] & & \\
\Delta_A^*(\mathrm{pr}_1^* \mathscr{O}_G \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^*([-1]_A^* \mathscr{O}_G)) \arrow[rr, "\Delta_A^*(\mathrm{Id} \otimes [-1]^*)"'] & & \mathscr{O}_G \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_G \arrow[uu, "\Delta^*"']
\end{tikzcd} \tag{3.1.4.3}$$

of $\mathscr{O}_A$-algebras.

If the $\mathscr{O}_A$-algebra $\mathscr{O}_G$ satisfies all these conditions, then we get a group structure on G covering the one on A.

As an application, let us assume that $A \to S$ is a group scheme that satisfies Assumption 3.1.2.7, and that G is an H-torsor $\mathcal{M}$ over A, where H is a split group of multiplicative type of finite type over S.

Proposition 3.1.4.4. *Under the above hypothesis, suppose that we are given a homomorphism $\underline{\mathbf{X}}(H) \to \underline{\mathrm{Pic}}_e(A/S) : \chi \mapsto \mathscr{M}_\chi$ such that we have a unique isomorphism*

$$m_\chi^* : m_A^* \mathscr{M}_\chi \xrightarrow{\sim} \mathrm{pr}_1^* \mathscr{M}_\chi \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{M}_\chi \tag{3.1.4.5}$$

respecting the rigidifications for each $\chi \in \underline{\mathbf{X}}(H)$. Then the H-torsor $\mathcal{M} = \underline{\mathrm{Spec}}_{\mathscr{O}_A}\Big(\underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} \mathscr{M}_\chi\Big)$ defined in Proposition 3.1.3.1 has a group structure covering the one of A.

Proof. Recall that in the proof of Proposition 3.1.3.1, the morphism $\Delta^* : \mathscr{O}_\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\mathcal{M} \to \mathscr{O}_\mathcal{M}$ describing the $\mathscr{O}_A$-algebra structure of $\mathscr{O}_\mathcal{M}$ is given by the unique isomorphisms $\Delta_{\chi,\chi'}^* : \mathscr{M}_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{M}_{\chi'} \to \mathscr{M}_{\chi+\chi'}$ respecting the rigidifications. Moreover, there are the unique isomorphisms (3.1.4.5), which induce under pullback by $(\mathrm{Id}, [-1]_A)^*$ the isomorphisms $[-1]_\chi^* : [-1]_A^* \mathscr{M}_\chi \xrightarrow{\sim} \mathscr{M}_\chi^{\otimes -1} \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{M}_{-\chi}$ respecting the rigidifications $e_A^* \mathscr{M}_\chi \xrightarrow{\sim} \mathscr{O}_S$. Let $e_H^* : \mathscr{O}_H = \underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} \mathscr{O}_{H,\chi} \to \mathscr{O}_S$ be the counit for the group H, which induces a natural isomorphism $\mathscr{O}_{H,\chi} \xrightarrow{\sim} \mathscr{O}_S$ by restriction. Let us compose the rigidifications with the inverse isomorphisms $\mathscr{O}_S \cong \mathscr{O}_{H,\chi}$ and write $e_\chi^* : e_A^* \mathscr{M}_\chi \xrightarrow{\sim} \mathscr{O}_{H,\chi}$.

Let us define the comultiplication $m^* : m_A^* \mathscr{O}_\mathcal{M} \to \mathrm{pr}_1^* \mathscr{O}_\mathcal{M} \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_\mathcal{M}$ by mapping for each $\chi \in \underline{\mathbf{X}}(H)$ the subsheaf $m_A^* \mathscr{M}_\chi$ to the subsheaf $\mathrm{pr}_1^* \mathscr{M}_\chi \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{M}_\chi$ using the isomorphism m_χ^*. Let us define the counit $e^* : e_A^* \mathscr{O}_\mathcal{M} \to \mathscr{O}_S$ by mapping for each $\chi \in \underline{\mathbf{X}}(H)$ the subsheaf $e_A^* \mathscr{M}_\chi$ to $\mathscr{O}_{H,\chi} \cong \mathscr{O}_S$ using the isomorphism e_χ^*. Let us define the coinverse $[-1]^* : [-1]_A^* \mathscr{O}_\mathcal{M} \to \mathscr{O}_\mathcal{M}$ by mapping for each $\chi \in \underline{\mathbf{X}}(H)$ the subsheaf $[-1]_A^* \mathscr{M}_\chi$ to the subsheaf $\mathscr{M}_{-\chi}$ using the isomorphism $[-1]_\chi^*$.

These explicit choices of m^*, e^*, and $[-1]^*$ make the diagrams (3.1.4.1), (3.1.4.2), and (3.1.4.3) commutative, and hence define a group structure on $\mathcal{M}$ covering the one on A. □

Corollary 3.1.4.6. *Suppose that $A \to S$ is a group scheme satisfying Assumption 3.1.2.7, and that we have a rigidified $\mathbf{G}_\mathrm{m}$-torsor $\mathcal{L}$ together with an isomorphism $m_A^* \mathcal{L} \xrightarrow{\sim} \mathrm{pr}_1^* \mathcal{L} \underset{\mathscr{O}_A}{\otimes} \mathrm{pr}_2^* \mathcal{L}$ respecting the rigidifications. Then $\mathcal{L}$ has a group structure covering the one of A.*

Proof. This follows from Proposition 3.1.4.4 and Corollary 3.1.3.2. □

Corollary 3.1.4.7. *Suppose that A is an abelian scheme over a base scheme S, that H is a split group of multiplicative type of finite type over S, and that we have a homomorphism*

$$\underline{\mathbf{X}}(H) \to \underline{\mathrm{Pic}}_e^0(A/S) : \chi \mapsto \mathscr{M}_\chi.$$

Then the H-torsor $\mathcal{M} = \underline{\mathrm{Spec}}_{\mathscr{O}_A}\Big(\underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} \mathscr{M}_\chi\Big)$ defined as in Proposition 3.1.3.1 has a group structure covering the one of A.

Proof. This is true because $\mathscr{L} \in \underline{\mathrm{Pic}}_e^0(A/S)$ if and only if there exists an isomorphism $m_A^* \mathscr{L} \xrightarrow{\sim} \mathrm{pr}_1^* \mathscr{L} \underset{\mathscr{O}_A}{\otimes} \mathrm{pr}_2^* \mathscr{L}$. □

3.1.5 Group Extensions

Let H be a split group of multiplicative type of finite type over S as before. Let $\mathcal{M}$ be an H-torsor over an abelian scheme A over a base scheme S. (Then the structural morphism $A \to S$ satisfies Assumption 3.1.2.7.) Suppose that $\mathcal{M}$ admits a group structure over S such that the structural morphism $\mathcal{M} \to A$ is a group scheme homomorphism over S. Let $e : S \to \mathcal{M}$ be the identity section of $\mathcal{M}$. Then the orbit of e under H gives an embedding of H into $\mathcal{M}$, whose image is isomorphic to the kernel of $\mathcal{M} \to A$. In other words, we have an exact sequence

$$0 \to H \to \mathcal{M} \to A \to 0$$

of group schemes over S, giving an extension of the abelian scheme A by the group scheme H over S.

Proposition 3.1.5.1. *The category of commutative group scheme extensions E of A by H over S is antiequivalent to the category of homomorphisms $\underline{\mathbf{X}}(H) \to \underline{\mathrm{Pic}}^0_e(A/S) = A^\vee$ between étale sheaves of groups over S.*

Proof. Let $\mathscr{M}_\chi$ be the rigidified invertible sheaf corresponding to the rigidified $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{M}_{-\chi}$ defined by push-out by $(-\chi) \in \underline{\mathbf{X}}(H)$. We have already seen that this defines a homomorphism $\underline{\mathbf{X}}(H) \to \underline{\mathrm{Pic}}_e(A/S) : \chi \mapsto \mathscr{M}_\chi$, and conversely that each such homomorphism defines an H-torsor by $\mathcal{M} := \underline{\mathrm{Spec}}_{\mathscr{O}_A}\Big(\underset{\chi \in \underline{\mathbf{X}}(H)}{\oplus} \mathscr{M}_\chi\Big)$. The question is whether $\mathscr{M}_\chi$ is in $\underline{\mathrm{Pic}}^0_e(A/S)$.

We claim that $\mathscr{M}_\chi$ is in $\underline{\mathrm{Pic}}^0_e(A/S)$ if $\mathcal{M}$ has a group structure covering the one of A. In this case, the $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{M}_{-\chi}$ would have a group structure covering the one of A, which gives an isomorphism $\mathrm{pr}_1^* \mathcal{M}_{-\chi} \underset{\mathscr{O}_A}{\otimes} \mathrm{pr}_2^* \mathcal{M}_{-\chi} \cong m_A^* \mathcal{M}_{-\chi}$ over $A \underset{S}{\times} A$. This isomorphism corresponds to an isomorphism $\mathrm{pr}_1^* \mathscr{M}_\chi \underset{\mathscr{O}_A}{\otimes} \mathrm{pr}_2^* \mathscr{M}_\chi \cong m_A^* \mathscr{M}_\chi$ in $\underline{\mathrm{Pic}}_e(A \underset{S}{\times} A/S)$. In other words,

$$m_A^* \mathscr{M}_\chi \underset{\mathscr{O}_A}{\otimes} \mathrm{pr}_1^* \mathscr{M}_\chi^{\otimes -1} \underset{\mathscr{O}_A}{\otimes} \mathrm{pr}_2^* \mathscr{M}_\chi^{\otimes -1} \tag{3.1.5.2}$$

is trivial in $\underline{\mathrm{Pic}}_e(A \underset{S}{\times} A/S)$. By the theory of abelian varieties [94, §8] and the rigidity lemma [96, Prop. 6.1], this is true if and only if $\mathscr{M}_\chi$ is in $\underline{\mathrm{Pic}}^0_e(A/S)$.

The converse is just Corollary 3.1.4.7. □

3.2 Biextensions and Cubical Structures

3.2.1 Biextensions

Definition 3.2.1.1. *Let G, H, C be three commutative group schemes over a base scheme S. A* ***biextension*** *of $G \underset{S}{\times} H$ by C is defined by the data of a C-torsor $\mathcal{M}$ over $G \times H$, and two sections of the induced C-torsors $(m_G \times \mathrm{Id})^* \mathcal{M} \otimes p_{13}^* \mathcal{M}^{-1} \otimes p_{23}^* \mathcal{M}^{-1}$ over $G \times G \times H$ and $(\mathrm{Id} \times m_H)^* \mathcal{M} \otimes p_{12}^* \mathcal{M}^{-1} \otimes p_{13}^* \mathcal{M}^{-1}$ over $G \times H \times H$, corresponding to isomorphisms*

$$c_1 : p_{13}^* \mathcal{M} \otimes p_{23}^* \mathcal{M} \xrightarrow{\sim} (m_G \times \mathrm{Id})^* \mathcal{M}$$

over $G \times G \times H$ and

$$c_2 : p_{12}^* \mathcal{M} \otimes p_{13}^* \mathcal{M} \xrightarrow{\sim} (\mathrm{Id} \times m_H)^* \mathcal{M}$$

over $G \times H \times H$, respectively, such that the following diagrams of C-torsors are commutative:

1. (***Associativity*** *of c_1*)

$$\begin{tikzcd}
(p_{14}^*\mathcal{M} \otimes p_{24}^*\mathcal{M}) \otimes p_{34}^*\mathcal{M} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "p_{124}^*(c_1) \otimes \mathrm{Id}"', "\wr"] & p_{14}^*\mathcal{M} \otimes (p_{24}^*\mathcal{M} \otimes p_{34}^*\mathcal{M}) \arrow[d, "\wr"', "\mathrm{Id} \otimes p_{234}^*(c_1)"] \\
p_{124}^*(m_G \times \mathrm{Id})^*\mathcal{M} \otimes p_{34}^*\mathcal{M} \arrow[d, "(m_G \times \mathrm{Id} \times \mathrm{Id})^*(c_1)"', "\wr"] & p_{14}^*\mathcal{M} \otimes p_{234}^*(m_G \times \mathrm{Id})^*\mathcal{M} \arrow[d, "\wr"', "(\mathrm{Id} \times m_G \times \mathrm{Id})^*(c_1)"] \\
(m_G \times \mathrm{Id} \times \mathrm{Id})^*(m_G \times \mathrm{Id})^*\mathcal{M} \arrow[r, "\sim", "\text{can.}"'] & (\mathrm{Id} \times m_G \times \mathrm{Id})^*(m_G \times \mathrm{Id})^*\mathcal{M}
\end{tikzcd} \tag{3.2.1.2}$$

2. (***Associativity*** *of c_2*)

$$\begin{tikzcd}
(p_{12}^*\mathcal{M} \otimes p_{13}^*\mathcal{M}) \otimes p_{14}^*\mathcal{M} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "p_{123}^*(c_2) \otimes \mathrm{Id}"', "\wr"] & p_{12}^*\mathcal{M} \otimes (p_{13}^*\mathcal{M} \otimes p_{14}^*\mathcal{M}) \arrow[d, "\wr"', "\mathrm{Id} \otimes p_{134}^*(c_2)"] \\
p_{123}^*(\mathrm{Id} \times m_H)^*\mathcal{M} \otimes p_{14}^*\mathcal{M} \arrow[d, "(\mathrm{Id} \times m_H \times \mathrm{Id})^*(c_2)"', "\wr"] & p_{12}^*\mathcal{M} \otimes p_{134}^*(\mathrm{Id} \times m_H)^*\mathcal{M} \arrow[d, "\wr"', "(\mathrm{Id} \times \mathrm{Id} \times m_H)^*(c_2)"] \\
(\mathrm{Id} \times m_H \times \mathrm{Id})^*(\mathrm{Id} \times m_H)^*\mathcal{M} \arrow[r, "\sim", "\text{can.}"'] & (\mathrm{Id} \times \mathrm{Id} \times m_H)^*(\mathrm{Id} \times m_H)^*\mathcal{M}
\end{tikzcd} \tag{3.2.1.3}$$

3. (***Commutativity*** *of c_1*)

$$\begin{tikzcd}
p_{23}^*\mathcal{M} \otimes p_{13}^*\mathcal{M} \arrow[r, "(s_G \times \mathrm{Id})^* c_1"] \arrow[d, "\text{can.}"', "\wr"] & (s_G \times \mathrm{Id})^*(m_G \times \mathrm{Id})^*\mathcal{M} \arrow[d, "\wr"', "\text{can.}"] \\
p_{13}^*\mathcal{M} \otimes p_{23}^*\mathcal{M} \arrow[r, "c_1"'] & (m_G \times \mathrm{Id})^*\mathcal{M}
\end{tikzcd} \tag{3.2.1.4}$$

Here s_G is the symmetry automorphism of $G \underset{S}{\times} G$ switching the two factors.

4. (***Commutativity*** *of c_2*)

$$\begin{tikzcd}
p_{13}^*\mathcal{M} \otimes p_{12}^*\mathcal{M} \arrow[r, "(\mathrm{Id} \times s_H)^* c_2"] \arrow[d, "\text{can.}"', "\wr"] & (\mathrm{Id} \times s_H)^*(\mathrm{Id} \times m_H)^*\mathcal{M} \arrow[d, "\wr"', "\text{can.}"] \\
p_{12}^*\mathcal{M} \otimes p_{13}^*\mathcal{M} \arrow[r, "c_2"'] & (\mathrm{Id} \times m_H)^*\mathcal{M}
\end{tikzcd} \tag{3.2.1.5}$$

Here s_H is the symmetry automorphism of $H \underset{S}{\times} H$ switching the two factors.

5. (***Compatibility between the two composition laws***)

$$\begin{tikzcd}
(p_{13}^*\mathcal{M}\otimes p_{23}^*\mathcal{M})\otimes(p_{14}^*\mathcal{M}\otimes p_{24}^*\mathcal{M}) \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "p_{123}^*(c_1)\otimes p_{124}^*(c_1)"', "\wr"] & (p_{13}^*\mathcal{M}\otimes p_{14}^*\mathcal{M})\otimes(p_{23}^*\mathcal{M}\otimes p_{24}^*\mathcal{M}) \arrow[d, "\wr"', "p_{134}^*(c_2)\otimes p_{234}^*(c_2)"] \\
p_{123}^*(m_G\times \mathrm{Id})^*\mathcal{M}\otimes p_{124}^*(m_G\times \mathrm{Id})^*\mathcal{M} \arrow[d, "\text{can.}"', "\wr"] & p_{134}^*(\mathrm{Id}\times m_H)^*\mathcal{M}\otimes p_{234}^*(\mathrm{Id}\times m_H)^*\mathcal{M} \arrow[d, "\wr"', "\text{can.}"] \\
(m_G\times \mathrm{Id}\times \mathrm{Id})^*(p_{12}^*\mathcal{M}\otimes p_{13}^*\mathcal{M}) \arrow[d, "(m_G\times \mathrm{Id}\times \mathrm{Id})^*(c_2)"', "\wr"] & (\mathrm{Id}\times \mathrm{Id}\times m_H)^*(p_{13}^*\mathcal{M}\otimes p_{23}^*\mathcal{M}) \arrow[d, "\wr"', "(\mathrm{Id}\times \mathrm{Id}\times m_H)^*(c_1)"] \\
(m_G\times \mathrm{Id}\times \mathrm{Id})^*(\mathrm{Id}\times m_H)^*\mathcal{M} \arrow[r, "\sim", "\text{can.}"'] & (\mathrm{Id}\times \mathrm{Id}\times m_H)^*(\mathrm{Id}\times m_H)^*\mathcal{M}
\end{tikzcd} \tag{3.2.1.6}$$

The two sections c_1 and c_2 in Definition 3.2.1.1 are called the two ***partial multiplication laws*** *of $\mathcal{M}$.*

The most important examples of biextensions are given by Poincaré $\mathbf{G}_{\mathrm{m}}$-torsors over abelian schemes:

Proposition 3.2.1.7. *For each abelian scheme $A \to S$, the Poincaré $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{P}_A$ over $A \underset{S}{\times} A^\vee$ (defined using Theorem 3.1.2.5) has the canonical structure of a biextension of $A \underset{S}{\times} A^\vee$ by $\mathbf{G}_{\mathrm{m},S}$.*

This is essentially the *theorem of the square* for (rigidified) invertible sheaves over the abelian scheme A over S.

3.2.2 Cubical Structures

Let G and C be group schemes over a base scheme S. Let $\mathcal{L}$ be any rigidified C-torsor over a group scheme G. Then we may define functorially

$$\mathcal{D}_2(\mathcal{L}) := m^*\mathcal{L}\otimes \mathrm{pr}_1^*\mathcal{L}^{\otimes -1}\otimes \mathrm{pr}_2^*\mathcal{L}^{\otimes -1},$$

which is a C-torsor over $G \underset{S}{\times} G$, and similarly

$$\mathcal{D}_3(\mathcal{L}) := m_{123}^*\mathcal{L}\otimes m_{12}^*\mathcal{L}^{\otimes -1}\otimes m_{23}^*\mathcal{L}^{\otimes -1}\otimes m_{13}^*\mathcal{L}^{\otimes -1}\otimes \mathrm{pr}_1^*\mathcal{L}\otimes \mathrm{pr}_2^*\mathcal{L}\otimes \mathrm{pr}_3^*\mathcal{L},$$

(where m_{123}, m_{12}, etc. are the multiplication morphisms summing up the corresponding components) which is a C-torsor over $G \underset{S}{\times} G \underset{S}{\times} G$. Here functoriality means the same operations define morphisms $\mathcal{D}_2(f)$ and $\mathcal{D}_3(f)$ for morphisms between C-torsors (and hence also for sections of $\mathcal{L}$ because they are by definition isomorphisms between $\mathcal{L}$ and the trivial C-torsor).

Lemma 3.2.2.1. *There is a canonical* ***symmetry isomorphism***

$$s_A^*\mathcal{D}_2(\mathcal{L}) \xrightarrow{\sim} \mathcal{D}_2(\mathcal{L})$$

of the invertible sheaf $\mathcal{D}_2(\mathcal{L})$ covering the symmetry automorphism $s_A : A \underset{S}{\times} A \xrightarrow{\sim} A \underset{S}{\times} A$ switching the two factors of $A \underset{S}{\times} A$.

Remark 3.2.2.2. There are also canonical symmetry isomorphisms for $\mathcal{D}_3(\mathcal{L})$ covering the permutations of the factors of $A \underset{S}{\times} A \underset{S}{\times} A$, which we will not use explicitly. For more information, and also more formal properties of $\mathcal{D}_n$ in general, see [93, I].

Note that $\mathcal{D}_3(\mathcal{L})$ can be constructed from $\mathcal{D}_2(\mathcal{L})$ in two ways. Thus we have two canonical isomorphisms:

$$\xi_1 : (m \underset{S}{\times} \mathrm{pr}_3)^*\mathcal{D}_2(\mathcal{L})\otimes \mathrm{pr}_{13}^*\mathcal{D}_2(\mathcal{L})^{\otimes -1}\otimes \mathrm{pr}_{23}^*\mathcal{D}_2(\mathcal{L})^{\otimes -1} \xrightarrow{\sim} \mathcal{D}_3(\mathcal{L}) \tag{3.2.2.3}$$

and

$$\xi_2 : (\mathrm{pr}_1 \underset{S}{\times} m)^*\mathcal{D}_2(\mathcal{L})\otimes \mathrm{pr}_{12}^*\mathcal{D}_2(\mathcal{L})^{\otimes -1}\otimes \mathrm{pr}_{13}^*\mathcal{D}_2(\mathcal{L})^{\otimes -1} \xrightarrow{\sim} \mathcal{D}_3(\mathcal{L}). \tag{3.2.2.4}$$

Lemma 3.2.2.5. *Commutative group scheme structures on a C-torsor $\mathcal{M}$ over a group scheme G correspond bijectively to sections of $\mathcal{D}_2(\mathcal{M})$ over $G \underset{S}{\times} G$.*

Proof. This is a special case of [57, VII, 1.1.6 and 1.2]. □

Each section τ of $\mathcal{D}_3(\mathcal{L})$ defines sections $\xi_1^{-1}(\tau)$ and $\xi_2^{-1}(\tau)$ of

$$(m \underset{S}{\times} \mathrm{pr}_3)^*\mathcal{D}_2(\mathcal{L})\otimes \mathrm{pr}_{13}^*\mathcal{D}_2(\mathcal{L})^{\otimes -1}\otimes \mathrm{pr}_{23}^*\mathcal{D}_2(\mathcal{L})^{\otimes -1}$$

and

$$(\mathrm{pr}_1 \underset{S}{\times} m)^*\mathcal{D}_2(\mathcal{L})\otimes \mathrm{pr}_{12}^*\mathcal{D}_2(\mathcal{L})^{\otimes -1}\otimes \mathrm{pr}_{13}^*\mathcal{D}_2(\mathcal{L})^{\otimes -1},$$

respectively.

Definition 3.2.2.6. *A* ***cubical structure*** *on a C-torsor $\mathcal{L}$ over a group scheme G over S is a section τ of the C-torsor $\mathcal{D}_3(\mathcal{L})$ over $G \underset{S}{\times} G \underset{S}{\times} G$ such that the two sections $\xi_1^{-1}(\tau)$ and $\xi_2^{-1}(\tau)$ define two partial multiplication laws making $\mathcal{D}_2(\mathcal{L})$ a biextension of $G \underset{S}{\times} G$ by C (see Definition 3.2.1.1).*

Definition 3.2.2.7. *A* ***cubical*** ***C-torsor*** *is a C-torsor equipped with a cubical structure τ. A* ***morphism*** *of cubical C-torsors $(\mathcal{L},\tau) \to (\mathcal{L}',\tau')$ is a morphism $f : \mathcal{L} \to \mathcal{L}'$ of C-torsors such that $\mathcal{D}_3(f) : \mathcal{D}_3(\mathcal{L}) \to \mathcal{D}_3(\mathcal{L}')$ satisfies $\mathcal{D}_3(f)(\tau) = \tau'$. We shall denote by $\mathrm{CUB}_S(G,C)$ the category of cubical C-torsors over G.*

We leave it to the reader to make explicit the notions of pullbacks of a cubical C-torsor by a homomorphism $G' \to G$, of the image by a change of group structure $C \to C'$, of trivial cubical C-torsors, of tensor products (or sums) of C-torsors, of inverse cubical C-torsors, etc. With the operation of tensor products, $\mathrm{CUB}_S(G,C)$ is a *strictly commutative Picard category* in the sense of [14, XVIII, 1.4.2].

Definition 3.2.2.8. *A* ***trivialization*** *of an object $(\mathcal{L},\tau)$ in $\mathrm{CUB}_S(G,C)$ is an isomorphism from the trivial object to $(\mathcal{L},\tau)$, namely, a* ***section*** *σ of $\mathcal{L}$ over G such that $\mathcal{D}_3(\sigma) = \tau$.*

Remark 3.2.2.9. The set of trivializations of an object $(\mathcal{L},\tau)$ in $\mathrm{CUB}_S(G,C)$ is, if nonempty, a torsor under the group $\mathrm{Hom}^{(2)}(G,C)$, the pointed morphisms of degree two from G to C, namely, morphisms $f : G \to C$ such that $\mathcal{D}_3(f) = 1$. In particular, there might be more than one way to trivialize a cubical C-torsor if $\mathrm{Hom}^{(2)}(G,C)$ is nontrivial.

Remark 3.2.2.10. When $C = \mathbf{G}_{\mathrm{m},S}$, the group $\mathrm{Hom}^{(2)}(G,C)$ can be identified with $\mathrm{Hom}(G,C)$ under some mild assumptions on G and S, by Lemma 3.2.2.11 below.

Let us recall the following form of *Rosenlicht's lemma*:

Lemma 3.2.2.11 ([57, VIII, 4.1] and [30])**.** *Let k be a field. Let Z and W be two schemes over k, which are of finite type, separated, geometrically connected, geometrically reduced, and equipped with k-rational points e_Z and e_W, respectively. Then every morphism $Z \underset{k}{\times} W \to \mathbf{G}_{\mathrm{m},k}$ that is trivial over the two subschemes $Z \underset{k}{\times} e_W$ and $e_Z \underset{k}{\times} W$ is trivial.*

3.2.3 Fundamental Example

The most important example of cubical structures is given by the so-called *theorem of the cube* on abelian schemes, which can be generalized in the following form:

Proposition 3.2.3.1 (see [93, I, 2.6]). *Let G be a smooth commutative group scheme with geometrically connected fibers over a base scheme S. Suppose that (at least) one of the following conditions is satisfied:*

1. *G is an abelian scheme over S.*
2. *(cf. [24, 2.4]) S is normal, and fibers of G over* ***maximal points*** *of S (see [60, 0, 2.1.2]) are multiple extensions of abelian varieties, tori (not necessarily split), and (pullbacks of) the group $\mathbf{G}_{\mathrm{a}}$.*

Let us denote by $\underline{\mathrm{Pic}}_{e_G}(G/S)$ the category of $\mathbf{G}_{\mathrm{m}}$-torsors over G rigidified along identity section of G. Then the forgetful functor

$$\mathrm{CUB}_S(G, \mathbf{G}_{\mathrm{m},S}) \to \underline{\mathrm{Pic}}_{e_G}(G/S)$$

is an equivalence of categories.

Remark 3.2.3.2 ([24, p. 17]). By Chevalley's theorem (see [26] and [108], or see [29, Thm. 1.1] for a modern proof), and by taking successive commutators of the unipotent radical of the maximal linear subgroup, condition 2 of Proposition 3.2.3.1 is automatic when the fields defined by the maximal points of S are *perfect*.

3.2.4 The Group $\mathcal{G}(\mathcal{L})$ for Abelian Schemes

Let S be a scheme, A an abelian scheme over S, and $\mathcal{L}$ a $\mathbf{G}_{\mathrm{m}}$-torsor over A, rigidified at the identity section (hence cubical, by Proposition 3.2.3.1). Let $A^\vee$ be the dual abelian scheme of A over S. By Construction 1.3.2.7, we know that $\mathcal{L}$ defines canonically a symmetric homomorphism $\lambda_{\mathcal{L}} : A \to A^\vee$. Set

$$K(\mathcal{L}) := \ker \lambda_{\mathcal{L}}. \tag{3.2.4.1}$$

This is a closed subgroup scheme of A. The restriction of $\mathrm{Id}_A \times \lambda_{\mathcal{L}} : A \underset{S}{\times} A \to A \underset{S}{\times} A^\vee$ to $A \underset{S}{\times} K(\mathcal{L})$ induces an isomorphism of biextensions

$$\mathcal{D}_2(\mathcal{L})|_{A \underset{S}{\times} K(\mathcal{L})} \xrightarrow{\sim} [(\mathrm{Id}_A \times \lambda_{\mathcal{L}})^* \mathcal{P}_A]|_{A \underset{S}{\times} K(\mathcal{L})} \xrightarrow{\sim} (\mathrm{Id}_A \times \lambda_{\mathcal{L}}|_{K(\mathcal{L})})^*(\mathcal{P}_A|_{A \underset{S}{\times} e_{A^\vee}}).$$

As $\mathcal{P}_A|_{A \underset{S}{\times} e_{A^\vee}}$ is the trivial biextension, we obtain a *canonical trivialization* of $\mathcal{D}_2(\mathcal{L})|_{A \underset{S}{\times} K(\mathcal{L})}$, from which we deduce a structure of a commutative group scheme on $\mathcal{L}|_{K(\mathcal{L})}$, as a central extension

$$0 \to \mathbf{G}_{\mathrm{m},S} \to \mathcal{L}|_{K(\mathcal{L})} \to K(\mathcal{L}) \to 0,$$

as well as a *left action* (by switching the factors)

$$* : \mathcal{L}|_{K(\mathcal{L})} \underset{S}{\times} \mathcal{L} \to \mathcal{L}$$

of $\mathcal{L}|_{K(\mathcal{L})}$ on the $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}$.

On the other hand, we have the familiar group scheme $\mathcal{G}(\mathcal{L})$ over S defined as follows: For each scheme S' over S, the group $\mathcal{G}(\mathcal{L})(S')$ consists of pairs $(a, \tilde{a})$, where $a \in K(\mathcal{L})(S')$, and where $\tilde{a}$ is an automorphism of the $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}_{S'}$ covering the translation action $T_a : A_{S'} \xrightarrow{\sim} A_{S'}$ (see, for example, [94, §23] or [92]).

Proposition 3.2.4.2 ([93, I, 4.4]). *With the above setting, the assignment of $(a, \tilde{a}_u)$ to each point $u \in \mathcal{L}|_{K(\mathcal{L})}$ over $a \in K(\mathcal{L})$, where $\tilde{a}_u(v) = u * v$ for all $v \in \mathcal{L}$ (the symbol $*$ denotes the left action of $\mathcal{L}|_{K(\mathcal{L})}$ on $\mathcal{L}$ defined above), defines an isomorphism $\mathcal{L}|_{K(\mathcal{L})} \xrightarrow{\sim} \mathcal{G}(\mathcal{L})$ of central extensions (of $K(\mathcal{L})$ by $\mathbf{G}_{\mathrm{m},S}$) over S. The inverse isomorphism is given by the association $(a, \tilde{a}) \mapsto \tilde{a}(\varepsilon_{\mathcal{L}})$, where $\varepsilon_{\mathcal{L}} \in \mathcal{L}|_{e_A}(S)$ is the rigidification at the identity section $e_A \in K(L)$, namely, the identity section of the extension $\mathcal{L}|_{K(\mathcal{L})} \cong \mathbf{G}_{\mathrm{m},K(\mathcal{L})}$.*

3.2.5 Descending Structures

If G, H, and C are three commutative group schemes over a base scheme S, we denote by $\mathrm{EXT}_S(G, C)$ (resp. $\mathrm{BIEXT}_S(G, H; C)$) the category of commutative extensions of G by C (resp. the biextensions of $G \underset{S}{\times} H$ by C), in accordance with [57].

Let us begin by including results in [57, VIII] concerning the descent of biextensions:

Proposition 3.2.5.1 (see [57, VIII, 3.4]). *Let P be a smooth group scheme of finite presentation over S, with connected geometric fibers. Let T be a torus over S. Then every biextension of $P \underset{S}{\times} T$ by $\mathbf{G}_{\mathrm{m},S}$ is trivial.*

Corollary 3.2.5.2 (see [57, VIII, 3.5]). *With assumptions as in Proposition 3.2.5.1, let $0 \to T \to Q \to Q' \to 0$ be an exact sequence of commutative group schemes over S. Then the pullback functor $\mathrm{BIEXT}_S(P, Q'; \mathbf{G}_{\mathrm{m},S}) \to \mathrm{BIEXT}_S(P, Q; \mathbf{G}_{\mathrm{m},S})$ is an equivalence of categories.*

Now let us turn to cubical $\mathbf{G}_{\mathrm{m}}$-torsors. Let T be a *torus* over S, and let

$$0 \to T \xrightarrow{i} G \xrightarrow{\pi} H \to 0$$

be an exact sequence of *smooth* commutative group schemes *with connected geometric fibers* over S. Then we have the following results:

Proposition 3.2.5.3 ([93, I, 7.2.1]). *For each torus T over S, the category $\mathrm{CUB}_S(T, \mathbf{G}_{\mathrm{m},S})$ is equivalent to the category $\mathrm{EXT}_S(T, \mathbf{G}_{\mathrm{m},S})$ of commutative group extensions of T by $\mathbf{G}_{\mathrm{m},S}$.*

Proposition 3.2.5.4 ([93, I, 7.2.2]). *The category $\mathrm{CUB}_S(H, \mathbf{G}_{\mathrm{m},S})$ is equivalent to the category of pairs $(\mathcal{L}, s)$, where $\mathcal{L} \in \mathrm{Ob}\,\mathrm{CUB}_S(G, \mathbf{G}_{\mathrm{m},S})$ and where s is a trivialization of $i^*\mathcal{L}$ in $\mathrm{CUB}_S(T, \mathbf{G}_{\mathrm{m},S})$ (namely, by Proposition 3.2.5.3, in $\mathrm{EXT}_S(T, \mathbf{G}_{\mathrm{m},S})$).*

Corollary 3.2.5.5 ([93, I, 7.2.3]). *In the above setting, let $\mathcal{L}$ be a cubical $\mathbf{G}_{\mathrm{m}}$-torsor over G. Then*

1. *$\mathcal{L} \otimes [-1]_G^* \mathcal{L}$ comes canonically from a cubical $\mathbf{G}_{\mathrm{m}}$-torsor over H;*
2. *there exists an étale surjective morphism $S' \to S$ such that $\mathcal{L}_{S'} = \mathcal{L} \underset{S}{\times} S'$ comes from a cubical $\mathbf{G}_{\mathrm{m}}$-torsor over $H_{S'} = H \underset{S}{\times} S'$;*
3. *if all tori over S are isotrivial (see Definition 3.1.1.5), then we may suppose in 2 that the surjection $S' \to S$ is finite étale.*

Remark 3.2.5.6 ([93, I, 7.2.3]). The assumption in 3 is satisfied, in particular, when S is locally noetherian and *normal* by [40, X, 5.16], or when S is the spectrum of a *complete* noetherian local ring by [40, X, 3.3].

Corollary 3.2.5.7 (cf. [93, I, 7.2.4]). *Suppose S is locally noetherian and normal, and suppose $S' \to S$ is a finite étale covering that splits T. (Such an $S' \to S$ exists by Remark 3.2.5.6 above.) Then $\mathrm{EXT}_{S'}(\mathbf{G}_{\mathrm{m},S'}, \mathbf{G}_{\mathrm{m},S'}) = 0$, and hence every cubical $\mathbf{G}_{\mathrm{m}}$-torsor over $G_{S'}$ comes from $H_{S'}$ (by Proposition 3.2.5.4).*

Proof. It suffices to show that $\mathrm{EXT}_{S'}(\mathbf{G}_{\mathrm{m},S'}, \mathbf{G}_{\mathrm{m},S'}) = 0$ under the assumptions. Note that the group $\mathrm{EXT}_{S'}(\mathbf{G}_{\mathrm{m},S'}, \mathbf{G}_{\mathrm{m},S'})$ is canonically isomorphic to $H^1_{\mathrm{fppf}}(S', \underline{\mathbb{Z}}_{S'})$ because group schemes are sheaves in the fppf topology. By [55, Thm. 11.7], whose assumptions are trivially satisfied by $\underline{\mathbb{Z}}_{S'}$, we have a canonical isomorphism $H^1_{\text{ét}}(S', \underline{\mathbb{Z}}_{S'}) \xrightarrow{\sim} H^1_{\mathrm{fppf}}(S', \underline{\mathbb{Z}}_{S'})$. By [14, IX, 3.6(ii)], we have $H^1_{\text{ét}}(S', \underline{\mathbb{Z}}_{S'}) = 0$ when S' is geometrically unibranch, or equivalently by [59, IV-4, 18.8.15] when all strict localizations of S' are irreducible. This is certainly satisfied when S' is locally noetherian and normal. Hence $\mathrm{EXT}_{S'}(\mathbf{G}_{\mathrm{m},S'}, \mathbf{G}_{\mathrm{m},S'}) = 0$, as desired. □

3.3 Semi-Abelian Schemes

3.3.1 Generalities

Definition 3.3.1.1. *A **semi-abelian scheme** is a separated smooth commutative group scheme $G \to S$ such that each fiber G_s (for $s \in S$) is an extension $0 \to T_s \to G_s \to A_s \to 0$ of an abelian variety A_s by a torus T_s.*

Remark 3.3.1.2. In the above definition, A_s and T_s are uniquely determined by G_s as follows: T_s is the largest smooth connected affine subgroup scheme of G_s, and A_s is the quotient of G_s by T_s. The torus T_s is called the *torus part* of G_s, and the abelian scheme A_s is called the *abelian part* of G_s.

Remark 3.3.1.3. For each integer $n \geq 1$, we have an extension of group schemes $0 \to T_s[n] \to G_s[n] \to A_s[n] \to 0$, which implies that $\mathrm{rk}(G_s[n]) = \mathrm{rk}(T_s[n])\,\mathrm{rk}(A_s[n]) = n^{\dim(T_s)} n^{2\dim(A_s)}$. This shows that we can calculate $\dim(T_s) = \mathrm{rk}_{\mathbb{Z}}(\underline{\mathbf{X}}(T_s))$ by calculating the rank of the subscheme $G_s[n]$ of G_s over s (for $n > 1$).

Lemma 3.3.1.4. *Let $G \to S$ be a semi-abelian scheme. With notation as above, the function $s \mapsto \dim(T_s)$ is upper semicontinuous on S.*

Proof. By pulling back to completions of localizations, we may assume that S is the spectrum of a complete discrete valuation ring. Let $n \geq 1$ be any integer. By [57, IX, 2.2.1], the scheme $G[n]$ is quasi-finite flat over S. By [57, IX, 2.2.3], the function $s \mapsto G_s[n] \cong G[n]_s$ is lower semicontinuous on S. Hence the claim follows from Remark 3.3.1.3. □

Let us quote some other useful results from [42, Ch. I, §2]:

Proposition 3.3.1.5. *Let S be a noetherian normal scheme, and G and H two semi-abelian schemes over S. Suppose that, over a dense open subscheme U of S, there is a homomorphism $\phi_U : H_U \to G_U$. Then ϕ_U extends uniquely to a homomorphism $\phi : H \to G$ over S.*

This is originally proved in a more general setting in [105, IX, 1.4], and proved more directly for semi-abelian schemes in [42, Ch. I, Prop. 2.7].

Corollary 3.3.1.6 ([42, Ch. I, Rem. 2.8]). *A semi-abelian scheme $G \to S$ whose geometric fibers are all abelian varieties is proper (and hence an abelian scheme) over S.*

Proof. We may assume that S is the spectrum of a discrete valuation ring by the valuative criterion. Then G is isomorphic to the Néron model of its generic fiber by Proposition 3.3.1.5. □

Proposition 3.3.1.7. *Let S be a noetherian normal scheme, U a dense open subscheme of S, and G a semi-abelian scheme over S. If a torus H_U over U is a closed subgroup of G_U, then the closure of H_U in G is a torus H over S (contained in G).*

This is originally proved in a more general setting in [105, IX, 2.4], and proved more directly for semi-abelian schemes in [42, Ch. I, Prop. 2.9].

Remark 3.3.1.8. If we drop the assumption that S is normal, then Propositions 3.3.1.5 and 3.3.1.7 both become *false*. Let us include an instructive counterexample in [42, p. 12]: Let S be the nodal curve over an algebraically closed field k defined by identifying the points 0 and ∞ of $\mathbb{P}^1_k$. Then S has an infinite étale cover by an infinite chain $\tilde{S}$ of $\mathbb{P}^1_k$ indexed by the integers. Take the rank-two *nonconstant* étale sheaf of free commutative group $\underline{X}$ over S corresponding to $\tilde{S} \to S$. Let $G := \underline{\mathrm{Hom}}_S(\underline{X}, \mathbf{G}_{\mathrm{m},S})$ and $H := \mathbf{G}^2_{\mathrm{m},S}$. Take U to be an affine open subscheme of S complementing the unique singular point of S. Then the restriction $\underline{X}_U$ of $\underline{X}$ to U is constant. Hence $H_U \xrightarrow{\sim} G_U$ gives a counterexample to Proposition 3.3.1.5, and taking the closure of the graph of $H_U \xrightarrow{\sim} G_U$ gives a counterexample of Proposition 3.3.1.7.

Let $G \to S$ be a semi-abelian scheme, where S is the spectrum of a discrete valuation ring V with generic point η and special point s. Then the generic torus T_η extends to a subtorus of G, whose special fiber is contained in T_s and defines canonically a surjection $\underline{\mathbf{X}}(T_s) \to \underline{\mathbf{X}}(T_\eta)$. As a result, there exists an étale constructible sheaf $\underline{\mathbf{X}}(G)$ over $S = \mathrm{Spec}(V)$ such that, for each geometric point $\bar{\eta}$ above η specializing to a geometric point $\bar{s}$ above s, the homomorphism $\underline{\mathbf{X}}(T_{\bar{s}}) \to \underline{\mathbf{X}}(T_{\bar{\eta}})$ induced by the surjection above is the specialization homomorphism (see [13, VIII, 7.7]) associated with $\underline{\mathbf{X}}(G)$.

In general,

Theorem 3.3.1.9 ([42, Ch. I, Thm. 2.10]). *Let $G \to S$ be a semi-abelian scheme, where S is an arbitrary scheme (which does not have to be normal). There exists a unique étale constructible sheaf $\underline{\mathbf{X}}(G)$ over S such that for every $s \in S$ the pullback of $\underline{\mathbf{X}}(G)$ to s is equal to $\underline{\mathbf{X}}(T_s)$, and such that for every morphism $\mathrm{Spec}(V) \to S$ with V a discrete valuation ring, the pullback of $\underline{\mathbf{X}}(G)$ to $\mathrm{Spec}(V)$ is the constructible sheaf above. Furthermore, the formation of $\underline{\mathbf{X}}(G)$ is functorial and commutes with arbitrary base change. For every torus T over S with character group $\underline{\mathbf{X}}(T)$, the group $\mathrm{Hom}_S(T, G)$ is isomorphic to $\mathrm{Hom}_S(\underline{\mathbf{X}}(G), \underline{\mathbf{X}}(T))$.*

Corollary 3.3.1.10 ([42, Ch. I, Cor. 2.11]). *Assume that $\dim(T_s)$ is locally constant on S. Then G is globally an extension of an abelian scheme by a torus. That is, there exists an exact sequence $0 \to T \to G \to A \to 0$ in which T is a torus and A is an abelian scheme. In particular, G is a torus if all its fibers are.*

For each semi-abelian scheme $G \to S$ we obtain a stratification of S by locally closed subsets $S_r = \{s \in S : \dim(T_s) = r\}$. The closure of S_r is contained in the union of the S_i for $i \geq r$, and over each S_r the semi-abelian scheme G is globally the extension of an abelian scheme by a torus.

3.3.2 Extending Structures

Let G be a semi-abelian scheme over a base scheme S. For each scheme S' over S, we denote by

$$\mathrm{Res}_{S,S'} : \mathrm{CUB}_S(G, \mathbf{G}_{\mathrm{m},S}) \to \mathrm{CUB}_{S'}(G_{S'}, \mathbf{G}_{\mathrm{m},S'}) \tag{3.3.2.1}$$

the restriction (or pullback) functor. We denote by U either an *open dense subscheme* of S, or the *generic point* of S when S is irreducible. It is clear that the functor $\mathrm{Res}_{S,U}$ is already *faithful* in this case. Then natural questions are about when $\mathrm{Res}_{S,U}$ is fully faithful, and when it is an equivalence of categories.

Proposition 3.3.2.2 (see [93, II, 3.2.1, 3.2.2, 3.2.3])**.** *Let G be a semi-abelian scheme over a **normal** base scheme S. Then*

1. *the functor $\mathrm{Res}_{S,U}$ is **fully faithful**;*

2. *if S is **regular** at points of $S - U$, then $\mathrm{Res}_{S,U}$ is an equivalence.*

Theorem 3.3.2.3 (cf. [93, II, 3.3])**.** *Let G be a semi-abelian scheme over a **normal** base scheme S. Let $\mathcal{L}_U$ be a cubical $\mathbf{G}_{\mathrm{m}}$-torsor over G_U, satisfying either of the following properties:*

1. *The underlying $\mathbf{G}_{\mathrm{m}}$-torsor of $\mathcal{L}_U$ is **of finite order** in $\mathrm{Pic}(G_U)$.*

2. *The underlying $\mathbf{G}_{\mathrm{m}}$-torsor of $\mathcal{L}_U$ is **symmetric**. Namely, there exists an isomorphism $\mathcal{L}_U \overset{\sim}{\to} [-1]^*_{G_U}\mathcal{L}_U$ of $\mathbf{G}_{\mathrm{m}}$-torsors over G_U.*

Then $\mathcal{L}_U$ is in the essential image of $\mathrm{Res}_{S,U}$.

Now let us state the important *semistable reduction theorem*:

Theorem 3.3.2.4 ([42, Ch. I, Thm. 2.6])**.** *Let V be a discrete valuation ring, let $K := \mathrm{Frac}(V)$, and let G_K be a semi-abelian variety over K. Then there exists a finite extension $V \to V'$ of discrete valuation rings such that $G_{K'} := G_K \underset{K}{\otimes} K'$, where $K' := \mathrm{Frac}(V')$, extends to a semi-abelian scheme over $\mathrm{Spec}(V')$.*

Let us include the proof for the convenience of readers.

Proof of Theorem 3.3.2.4. When G_K is an abelian variety, the theorem is well known and well documented, in which case it means that the Néron model of $G_{K'}$ contains a semi-abelian open subgroup scheme over some finite extension K' of K (see, for example, [57, IX] or [15]).

When G_K is an extension of an abelian scheme A_K by some torus T_K, we may take some finite extension V' of V (and hence K' of K) such that $T_{K'}$ becomes a split torus, and such that $A_{K'}$ extends to a semi-abelian scheme A over $\mathrm{Spec}(V')$. Suppose G' is any smooth group scheme over $\mathrm{Spec}(V')$ whose geometric fibers are connected. By Lemma 3.2.2.5, each group scheme extension $G''_{K'}$ of $G'_{K'}$ by $\mathbf{G}_{\mathrm{m},K'}$ defines a rigidified $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}_{K'}$ over $G'_{K'}$, together with a section of $\mathcal{D}_2(\mathcal{L}_{K'})$. In particular, $\mathcal{L}_{K'}$ admits a cubical structure. Hence, by Proposition 3.3.2.2, $\mathcal{L}_{K'}$ extends uniquely to a cubical $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}$ over G', with a unique section of $\mathcal{D}_2(\mathcal{L})$ extending the one of $\mathcal{D}_2(\mathcal{L}_{K'})$ by Theorem 3.3.2.6. By Lemma 3.2.2.5 again, this shows that $G''_{K'}$ extends to a group scheme extension G'' of G' by $\mathbf{G}_{\mathrm{m},V'}$ over $\mathrm{Spec}(V')$. Since $T_{K'}$ is a split torus, the above argument of extending G'' over G' proves the theorem by induction on the number of copies of $\mathbf{G}_{\mathrm{m},K'}$ in $T_{K'}$. $\square$

Remark 3.3.2.5 ([42, p. 9])*.* The proof of Theorem 3.3.2.4 shows that, after a finite extension of K, there exists an extension G of G_K such that the torus part of G_K extends to a closed subtorus of G.

To finish, let us include the following result concerning biextensions:

Theorem 3.3.2.6 (cf. [93, II, 3.6])**.** *Let S be a normal integral scheme with generic point η, and let G and H be two semi-abelian schemes over S. Then the restriction functor $\mathrm{BIEXT}_S(G, H; \mathbf{G}_{\mathrm{m},S}) \to \mathrm{BIEXT}_\eta(G_\eta, H_\eta; \mathbf{G}_{\mathrm{m},\eta})$ is an equivalence of categories.*

3.3.3 Raynaud Extensions

Let R be a noetherian integral domain complete with respect to an ideal I, with $\mathrm{rad}(I) = I$ for convenience. Let $S = \mathrm{Spec}(R)$ and let η denote the generic point of S. Let $G \to S$ be a semi-abelian scheme. For each integer $i \geq 0$, let $R_i := R/I^{i+1}$, $S_i = \mathrm{Spec}(R/I^{i+1})$, and $G_i := G \underset{S}{\times} S_i$. Let S_{for} (resp. G_{for}) be the formal scheme formed by the compatible (inductive) system $(S_i)_{i\geq 0}$ (resp. $(G_i)_{i\geq 0}$), which can be identified with the formal completion of S (resp. G) along its subscheme S_0 (resp. G_0). Alternatively, we may set $S_{\mathrm{for}} := \mathrm{Spf}(R, I)$ and $G_{\mathrm{for}} := G \underset{S}{\times} S_{\mathrm{for}}$.

*Let us assume that the group scheme $G_0 \to S_0$ is an **extension** of an abelian variety A_0 by an **isotrivial** torus T_0 (see Definition 3.1.1.5) over S_0.*

By Theorem 3.1.1.2, T_0 can be lifted uniquely to a multiplicative subgroup scheme T_i of G_i for every i. The quotient of G_i by T_i is an abelian scheme because it is smooth and trivially proper. Therefore we obtain (as in [57, IX, 7]) an exact sequence of formal group schemes

$$0 \to T_{\mathrm{for}} \to G_{\mathrm{for}} \to A_{\mathrm{for}} \to 0 \tag{3.3.3.1}$$

over S_{for}, which is a compatible system (for all $i \geq 0$) of exact sequences of group schemes

$$0 \to T_i \to G_i \to A_i \to 0 \tag{3.3.3.2}$$

over S_i, where A_i is an abelian scheme over S_i and where T_i is a torus over S_i. Then we have the compatible system $(c_i : \underline{\mathbf{X}}(T_i) \to A_i^\vee)_{i\geq 0}$ of morphisms corresponding to the sequence (3.3.3.1) (or rather the sequences (3.3.3.2)) under Proposition 3.1.5.1.

The notions of $\mathbf{G}_{\mathrm{m}}$-biextensions and cubical $\mathbf{G}_{\mathrm{m}}$-torsors have their natural analogues over formal schemes. We have the following formal version of Corollary 3.2.5.5 (see [93, IV, 2.1] for the first two statements):

Corollary 3.3.3.3. *Let $\mathcal{L}_{\mathrm{for}}$ be a cubical $\mathbf{G}_{\mathrm{m}}$-torsor over G_{for} (namely, a compatible system $(\mathcal{L}_i)_{i\geq 0}$ of cubical $\mathbf{G}_{\mathrm{m}}$-torsors $\mathcal{L}_i$ over G_i, for $i \geq 0$). Then*

1. *the cubical $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}_{\mathrm{for}} \otimes [-1]^*\mathcal{L}_{\mathrm{for}}$ is canonically isomorphic to the pullback of a canonical cubical $\mathbf{G}_{\mathrm{m}}$-torsor over A_{for};*

2. *there exists an étale covering $S'_{\mathrm{for}} \twoheadrightarrow S_{\mathrm{for}}$ such that $(\mathcal{L}_{\mathrm{for}})_{S'_{\mathrm{for}}}$ is isomorphic to the pullback of a cubical $\mathbf{G}_{\mathrm{m}}$-torsor over $(A_{\mathrm{for}})_{S'_{\mathrm{for}}}$; this surjection algebraizes uniquely to a morphism $S' \to S$, which is not necessarily étale (due to potential issues of finiteness);*

3. *if all tori over S_0 are isotrivial (see Definition 3.1.1.5 and Remark 3.2.5.6), then we may suppose in 2 that $S'_{\mathrm{for}} \twoheadrightarrow S_{\mathrm{for}}$ algebraizes to a finite étale covering $S' \to S$.*

Proof. Statement 1 follows by applying 1 of Corollary 3.2.5.5 to the base schemes S_i.

As for 2, by Propositions 3.2.5.4 and 3.2.5.3, it suffices to trivialize the restriction of $\mathcal{L}_{\mathrm{for}}$ to T_{for} as an *extension*. By [40, X, 3.2], it suffices to trivialize $\mathcal{L}_0$ over T_0, which can be achieved after making some étale base change $R_0 \hookrightarrow R'_0$. Then the unique formally étale I-adically complete R-algebra R' (which we shall explain in Remark 3.3.3.4 below) defines the surjection $S' = \mathrm{Spec}(R') \to S = \mathrm{Spec}(R)$ whose formal completion along S_0 gives the étale surjective morphism $S'_{\mathrm{for}} \to S_{\mathrm{for}}$.

Under the assumption on S_0 in 3, we may assume that R'_0 is finite over R_0 in the above argument, and accordingly that the unique R' above is finite over R (by Theorem 2.3.1.4, or rather by [59, III-1, 5.4.5]); that is, we may assume that the covering $S' \to S$ is finite étale. □

Remark 3.3.3.4. The unique lifting R' in the proof of Corollary 3.3.3.3 can be realized concretely as follows: For each integer $i \geq 0$, let $R_i := R/I^{i+1}$. Start with the given étale R_0-algebra R'_0. For each integer $i \geq 0$, suppose $R'_{i'}$ is defined for every $0 \leq i' \leq i$. Note that $R_{i+1}/(I^{i+1} \cdot R_{i+1}) \cong R_i$ and $(I^{i+1})^2 = 0$ in R_{i+1}. Hence, by Lemma 2.1.1.6 (or rather by [59, IV-4, 18.1.2]), there is a unique étale R_{i+1}-algebra R'_{i+1} such that $R'_{i+1}/I^i \cdot R'_{i+1} \cong R'_i$. Repeating this process, we can define R'_i for each integer $i \geq 0$, forming a projective system compatible with their structural morphisms as R-algebras. Then we define the R-algebra $R' := \varprojlim_i R'_i$, which is formally étale over R by construction.

Remark 3.3.3.5. If R_0 is a separably closed field k, then the second assertion in Corollary 3.3.3.3 shows that the pullback functor $\mathrm{CUB}_{S_{\mathrm{for}}}(A_{\mathrm{for}}, \mathbf{G}_{\mathrm{m},S_{\mathrm{for}}}) \to \mathrm{CUB}_{S_{\mathrm{for}}}(G_{\mathrm{for}}, \mathbf{G}_{\mathrm{m},S_{\mathrm{for}}})$ is essentially surjective.

Proposition 3.3.3.6. *Let G be a semi-abelian scheme over S such that G_0 is an extension of an abelian scheme A_0 by an isotrivial torus T_0 (see Definition 3.1.1.5). Suppose that there exists an* ***ample*** *cubical invertible sheaf over G. Then the extension (3.3.3.1) is* ***algebraizable*** *by Theorem 2.3.1.4. Namely, there exist uniquely (over S) an abelian scheme A, a torus T, and an extension*

$$0 \to T \to G^\natural \to A \to 0 \tag{3.3.3.7}$$

whose associated extension of formal schemes is (3.3.3.1).

Definition 3.3.3.8. *The extension (3.3.3.7) is called the* ***Raynaud extension*** *associated with G (see [106] and [57, IX]).*

Remark 3.3.3.9. If S is (noetherian and) *normal*, then G is quasi-projective over S by [105, XI, 1.13], a theorem of Grothendieck; that is, there exists an ample invertible sheaf $\mathcal{L}$ over G. Moreover, by Proposition 3.2.3.1, the invertible sheaf $\mathcal{L}$ admits a cubical structure as soon as it is rigidified. Therefore, the hypothesis of the existence of an ample cubical sheaf in Proposition 3.3.3.6 is automatic when S is normal.

The following proof is adapted from [93, IV, 2.2] (see also [57, X, 7]).

Proof of Proposition 3.3.3.6. Let $\mathcal{L}$ be any *ample* cubical invertible sheaf over G. Then, by 1 of Corollary 3.3.3.3, $\mathcal{L}_{\mathrm{for}} \otimes [-1]^*\mathcal{L}_{\mathrm{for}}$ is canonically isomorphic to the pullback of a canonical cubical $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{M}_{\mathrm{for}}$ over A_{for}, which is *ample* by [105, XI, 1.11]. By Theorem 2.3.1.4 (or rather [59, III-1, 5.4.5]), the pair $(A_{\mathrm{for}}, \mathcal{M}_{\mathrm{for}})$ is algebraizable; that is, there exist an abelian scheme A over S and an ample invertible sheaf $\mathcal{M}$ over A such that $(A_{\mathrm{for}}, \mathcal{M}_{\mathrm{for}}) \cong (A, \mathcal{M}) \underset{S}{\times} S_{\mathrm{for}}$.

On the other hand, by [40, X, 3.2], there is a torus T over S such that $T_{\mathrm{for}} \cong T \underset{S}{\times} S_{\mathrm{for}}$, so that $\underline{\mathbf{X}}(T_i) \cong \underline{\mathbf{X}}(T) \underset{S}{\times} S_i$ for all $i \geq 0$. Hence we can make sense of $\underline{\mathbf{X}}(T)_{\mathrm{for}} := \underline{\mathbf{X}}(T) \underset{S}{\times} S_{\mathrm{for}}$ and interpret the compatible system $(c_i : \underline{\mathbf{X}}(T_i) \to A_i^\vee)_{i \geq 0}$ as a homomorphism $c_{\mathrm{for}} : \underline{\mathbf{X}}(T)_{\mathrm{for}} \to A^\vee_{\mathrm{for}}$ of formal group schemes. Moreover, we may treat the underlying scheme of $\underline{\mathbf{X}}(T)$ as a disjoint union of schemes that are *finite étale* over S. By Theorem 2.3.1.3 (or rather [59, III-1, 5.4.1]), the homomorphism $c_{\mathrm{for}} : \underline{\mathbf{X}}(T)_{\mathrm{for}} \to A^\vee_{\mathrm{for}}$ is algebraizable by a unique homomorphism $c : \underline{\mathbf{X}}(T) \to A^\vee$. This gives an extension as in (3.3.3.7) whose formal completion is (3.3.3.1), as desired. □

Remark 3.3.3.10. In [42, Ch. II, §1, p. 33], they considered also those G such that G_0 is an extension of an abelian scheme A_0 by a torus T_0, without the isotriviality assumption on T_0, and without an explanation of the algebraizability of T_{for}. Since the isotriviality assumption on T_0 is also made by Grothendieck in [57, IX, 7.2.1, 7.2.2], we shall be content with our understanding of this more restricted context (which nevertheless suffices for the construction of compactifications).

Proposition 3.3.3.11. *With the same setting as in Proposition 3.3.3.6, the natural functor*

$$\mathrm{CUB}_S(G^\natural, \mathbf{G}_{\mathrm{m},S}) \to \mathrm{CUB}_{S_{\mathrm{for}}}(G_{\mathrm{for}}, \mathbf{G}_{\mathrm{m},S_{\mathrm{for}}}) \tag{3.3.3.12}$$

induced by the isomorphism $G^\natural_{\mathrm{for}} \cong G_{\mathrm{for}}$ is fully faithful, and the essential image of (3.3.3.12) contains all $\mathcal{L}_{\mathrm{for}}$ coming from a cubical $\mathbf{G}_{\mathrm{m}}$-torsor over A_{for}. In particular, if all tori over S_0 are isotrivial, then (3.3.3.12) is an equivalence of categories.

The same proof as in [93, IV, 2.2] applies to our case. We do not include the proof here because it does not involve information that we will need later.

3.4 The Group $K(\mathcal{L})$ and Applications

3.4.1 Quasi-Finite Subgroups of Semi-Abelian Schemes over Henselian Bases

Let R be a *Henselian* local ring with residue field k. Let $S := \mathrm{Spec}(R)$, with closed point $S_0 := \mathrm{Spec}(k)$. We shall denote pullbacks of objects from S to S_0 by the subscript 0. Let G be a semi-abelian scheme over S (see Definition 3.3.1.1). If X is a scheme that is *quasi-finite* and separated over S, then we denote by X^{f} (the *finite part* of X) its largest finite subscheme over S. Thus we have a decomposition

$$X = X^{\mathrm{f}} \amalg X' \tag{3.4.1.1}$$

over S (as in [57, IX, 2.2.3]), where X^{f} is finite over S and where the closed fiber X'_0 of X' over S_0 is empty.

Now let us take X to be a closed subgroup scheme H of G, where H is quasi-finite over S. Then H^{f} is an open and closed subgroup scheme of H. If H_1 is a closed subgroup of H, then we verify immediately that

$$H_1^{\mathrm{f}} = H_1 \cap H^{\mathrm{f}}. \tag{3.4.1.2}$$

Suppose H is *flat* over S. Then H^{f} is also flat over S, because it is open in H. Let T_0 be the torus part of G_0 (see Remark 3.3.1.2). Since the group $H \cap T_0$ is of multiplicative type, it extends (by [57, IX, 6.1]) to a unique finite subgroup of H, flat and of multiplicative type, which we denote by

$$H^\mu \subset H \qquad \text{(the } \textit{torus part} \text{ of } H\text{)}. \tag{3.4.1.3}$$

By definition, $H^\mu \subset H^{\mathrm{f}}$. If $H_1 \subset H$ is closed and flat over S, then we have

$$H_1^\mu = H_1 \cap H^\mu. \tag{3.4.1.4}$$

Note that the formations of H^μ and H^{f} commute with base changes under *local* morphisms of Henselian local schemes. Finally, we set

$$H^{\mathrm{ab}} := H^{\mathrm{f}}/H^\mu. \tag{3.4.1.5}$$

This is a finite and flat group scheme over S, whose closed fiber H_0^{ab} can be identified with a subgroup of the *abelian part* $A_0 \cong G_0/T_0$ of G_0 (see Remark 3.3.1.2). We call this the *abelian part* of H.

Let U be a noetherian scheme over S, and let H_U be a *closed* subgroup scheme of $G_U = G \underset{S}{\times} U$ that is *quasi-finite and flat* over U. (In what follows, U will often be an open subscheme of S.) Then there exists an integer $m \geq 1$ such that $H_U \subset G_U[m]$. Since G is a semi-abelian scheme, the kernel $G[m]$ of multiplication by m (which is a closed subgroup of G) is *flat* and *quasi-finite* over S (see [57, IX, 2.2.1]). Thus we may set

$$H_{U/S}^{\mathrm{f}} := H_U \cap (G[m]^{\mathrm{f}})_U \tag{3.4.1.6}$$

and

$$H_{U/S}^\mu := H_U \cap (G[m]^\mu)_U. \tag{3.4.1.7}$$

(We denote here $G[m]^{\mathrm{f}} = (G[m])^{\mathrm{f}}$ etc.)

By applying (3.4.1.2) and (3.4.1.4) to the inclusions $G[m_1] \subset G[m_2]$ when $m_1|m_2$, we see that the definitions (3.4.1.6) and (3.4.1.7) do not depend on the choice of m. The two subgroups of H_U thus defined are *finite* over U, because they are closed in $(G[m]^{\mathrm{f}})_U$. Moreover, $H_{U/S}^{\mathrm{f}}$ is *open* in H_U, which implies that $H_{U/S}^{\mathrm{f}}$ is *finite and flat* over U. The same is true for $H_{U/S}^\mu$:

Lemma 3.4.1.8 ([93, IV, 1.3.3])**.** *The group $H_{U/S}^\mu$ defined in (3.4.1.7) is flat over U.*

By Lemma 3.4.1.8, we can define the *abelian part* of H_U by

$$H_{U/S}^{\mathrm{ab}} := H_{U/S}^{\mathrm{f}}/H_{U/S}^\mu.$$

This is a finite and flat group scheme over U.

When $U = S$, the groups $H_{U/S}^{\mathrm{f}}$, $H_{U/S}^\mu$, and $H_{U/S}^{\mathrm{ab}}$ coincide with the groups H_U^{f}, H_U^μ, and H_U^{ab} of (3.4.1.1), (3.4.1.3), and (3.4.1.5), respectively. Moreover, for a fixed scheme S, their formations commute with arbitrary base changes in schemes U over S.

3.4.2 Statement of the Theorem on the Group $K(\mathcal{L})$

Now we retain the hypotheses and notation of Section 3.4.1, and assume moreover that R is *complete*. (That is, we also incorporate the assumptions in Section 3.3.3, so that Raynaud extensions of G can be defined.) Proceeding explicitly as in [57, IX, 7.3] for the quasi-finite subscheme $G[m]$, we obtain canonical isomorphisms

$$(G[m])^{\mathrm{f}} \cong G^\natural[m], \quad (G[m])^\mu \cong T[m], \quad (G[m])^{\mathrm{ab}} \cong A[m] \tag{3.4.2.1}$$

of group schemes over S, characterized by the condition that they induce the (compatible) canonical isomorphisms $(G[m])^{\mathrm{f}}_{\mathrm{for}} \cong (G_{\mathrm{for}})[m] \cong (G^\natural_{\mathrm{for}})[m] \cong (G^\natural[m])_{\mathrm{for}}$, $(G[m])^\mu_{\mathrm{for}} \cong (T[m])_{\mathrm{for}}$, and $(G[m])^{\mathrm{ab}}_{\mathrm{for}} \cong (A[m])_{\mathrm{for}}$ over the formal completions.

Now let U be a scheme over S, and let H_U be a *closed* subscheme of G_U, as in Section 3.4.1, which is *flat and quasi-finite* over U. Then we have the finite and flat groups $H_{U/S}^\mu$, $H_{U/S}^{\mathrm{f}}$, and $H_{U/S}^{\mathrm{ab}}$ over U, contained in $(G[m]^\mu)_U$, $(G[m]^{\mathrm{f}})_U$, and $(G[m]^{\mathrm{ab}})_U$, respectively, as soon as $H_U \subset G[m]_U$. From the isomorphisms in (3.4.2.1), we have finite and flat group schemes over U denoted, following Grothendieck, as $H_{U/S}^\flat$, $H_{U/S}^\natural$, and $H_{U/S}^\sharp$, which are defined by the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H_{U/S}^\mu \arrow[r] \arrow[d, "\wr"', "(3.4.2.1)"] & H_{U/S}^{\mathrm{f}} \arrow[r] \arrow[d, "\wr"', "(3.4.2.1)"] & H_{U/S}^{\mathrm{ab}} \arrow[r] \arrow[d, "\wr"', "(3.4.2.1)"] & 0 \\
0 \arrow[r] & H_{U/S}^\flat \arrow[r] \arrow[d, hook] & H_{U/S}^\natural \arrow[r] \arrow[d, hook] & H_{U/S}^\sharp \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & (T[m])_U \arrow[r] & (G^\natural[m])_U \arrow[r] & (A[m])_U \arrow[r] & 0
\end{tikzcd} \tag{3.4.2.2}$$

with exact rows. Note that the subgroups $H_{U/S}^\flat$ etc., of T, $G^\natural$, and A are independent of the choice of m.

Remark 3.4.2.3. Although $H_{U/S}^{\mathrm{f}}$ and $H_{U/S}^\natural$ are canonically isomorphic, it is convenient (for our purpose) to retain the difference in notation.

We can now state the main theorem of [93, IV]:

Theorem 3.4.2.4 ([93, IV, 2.4])**.** *Let S be a* ***normal*** *locally noetherian integral scheme, with generic point η. Let $f : G \to S$ be a* ***semi-abelian*** *scheme (see Definition 3.3.1.1) such that its generic fiber G_η is an abelian variety, and let $\mathcal{L}_\eta$ be a cubical $\mathbf{G}_{\mathrm{m}}$-torsor over G_η. Suppose that the following conditions hold:*

- *$\mathcal{L}_\eta$ is* ***nondegenerate****. In other words, the group $K(\mathcal{L}_\eta)$ is* ***finite*** *(see (3.2.4.1)). (The group $K(\mathcal{L}_\eta)$ will be denoted K_η.)*
- *$\mathcal{L}_\eta$ admits a cubical extension $\mathcal{L}_S$ over S.*

The first hypothesis is satisfied, for example, if $\mathcal{L}_\eta$ is ***ample****. The second is satisfied, for example, if S is* ***regular*** *(by Proposition 3.3.2.2), or if $\mathcal{L}_\eta$ is* ***symmetric*** *(by Theorem 3.3.2.3). Then we have the following results:*

1. *$K_\eta := K(\mathcal{L}_\eta)$ extends to a unique closed subgroup K_S of G that is* ***flat*** *and quasi-finite over S. In this case, K_S is necessarily (uniquely isomorphic to) the schematic closure of K_η in G.*
2. *There exists a unique alternating pairing*
$$\mathrm{e}_S^{\mathcal{L}_\eta} : K_S \underset{S}{\times} K_S \to \mathbf{G}_{\mathrm{m},S}$$
extending the $\mathcal{L}_\eta$-Weil pairing $\mathrm{e}^{\mathcal{L}_\eta}$ (see [94, §23]). Moreover, if K_S is ***finite*** *over S, then $\mathrm{e}_S^{\mathcal{L}_\eta}$ is a perfect duality.*
3. *The restriction of the biextension $\mathcal{D}_2(\mathcal{L}_S)$ to $K_S \underset{S}{\times} G$ is equipped with a trivialization extending the one of $\mathcal{D}_2(\mathcal{L}_\eta)$ defined in Section 3.2.4. Hence, the $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{G}(\mathcal{L}_S) \cong \mathcal{L}_S|_{K_S}$ (see Proposition 3.2.4.2) is equipped with a natural structure of a central extension $0 \to \mathbf{G}_{\mathrm{m},S} \to \mathcal{G}(\mathcal{L}_S) \to K_S \to 0$, and with an action of this extension on $\mathcal{L}_S$. The pairing $\mathrm{e}_S^{\mathcal{L}_\eta}$ in 2 coincides with the commutator pairing of the extension $\mathcal{G}(\mathcal{L}_S)$.*

4. (***Theorem of orthogonality.***) *Suppose S is **Henselian local**, so that K_S is equipped with a filtration $K_S^{\mu} \subset K_S^{\mathrm{f}} \subset K_S$ (as in Section 3.4.1). Then K_S^{f} is the annihilator of K_S^{μ} under the pairing $\mathrm{e}_S^{\mathcal{L}_\eta}$, and the induced pairing on the quotient K_S^{ab} is a perfect duality. Moreover, if K_S is **finite** over S, then K_S^{μ} is trivial.*

5. *Suppose S is **complete local**. Let $0 \to T \to G^{\natural} \to A \to 0$ be the Raynaud extension associated with G (see Proposition 3.3.3.6). By Proposition 3.3.3.11, there exists a unique cubical $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}^{\natural}$ over $G^{\natural}$ whose formal completion is isomorphic to $(\mathcal{L}_S)_{\mathrm{for}}$. Suppose that $\mathcal{L}^{\natural}$ is isomorphic to the pullback of a (cubical) $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{M}$ over A. (By Corollary 3.2.5.5 and Remark 3.2.5.6, this can be achieved after making some **finite étale** surjective base change in S.) Then the subgroup $K_S^{\sharp}$ of A coincides with $K(\mathcal{M})$, and the pairing induced by $\mathrm{e}_S^{\mathcal{L}_\eta}$ on $K_S^{\sharp} \cong K_S^{\mathrm{ab}}$ in 4 coincides with $\mathrm{e}^{\mathcal{M}}$.*

3.4.3 Dual Semi-Abelian Schemes

Let S be a *normal* locally noetherian integral scheme, with generic point η. Let $G \to S$ be a semi-abelian scheme whose generic fiber G_η is an abelian scheme. A natural question is whether the dual abelian variety $(G_\eta)^\vee$ of G_η extends to a semi-abelian scheme $G^\vee$ over S.

Lemma 3.4.3.1 ([93, IV, 7.1.2])**.** *Let G be a semi-abelian scheme over a locally noetherian scheme S. Let K be a closed subgroup scheme of G, flat and quasi-finite over S. Suppose one of the following hypotheses is satisfied:*

1. *Locally for the étale topology over S, the scheme G is quasi-projective.*

2. *K is étale over S.*

Then the quotient G/K is representable by a semi-abelian scheme over S.

Proof. Let us first show that G/K is an *algebraic space* over S. This is clear in 2. In 1 the question is local over S for the étale topology. Hence we may suppose that G is quasi-projective over S and that we have an exact sequence $0 \to K^{\mathrm{f}} \to K \to E \to 0$ of group schemes, where K^{f} is finite and flat and E is étale over S. By [39, V, 4.1], and by the hypothesis of quasi-projectivity, the quotient $G' := G/K^{\mathrm{f}}$ is a *scheme*. Hence $G/K \cong G'/E$ is an algebraic space.

To see that G/K is a scheme, it suffices to remark that (locally over S) there exists $N \geq 1$ such that $K \subset G[N]$. Then there exists a *quasi-finite* homomorphism $G/K \to G$, and we can conclude by applying [73, II, 6.16] (which says that an algebraic space is a scheme if it is quasi-finite and separated over a scheme). □

Theorem 3.4.3.2 (cf. [93, IV, 7.1])**.** *Let S be a **normal** locally noetherian integral scheme with generic point η. Let G be a semi-abelian scheme over S whose generic fiber G_η is an abelian variety. Then we have the following:*

1. *There exists a unique semi-abelian scheme over S, denoted by $G^\vee$, extending the dual $(G_\eta)^\vee$ of G_η. Therefore, by Theorem 3.3.2.6, the Poincaré biextension $\mathcal{P}_\eta$ over $G_\eta \underset{\eta}{\times} (G_\eta)^\vee$ extends to a unique biextension $\mathcal{P}$ over $G \underset{S}{\times} G^\vee$ by $\mathbf{G}_{\mathrm{m},S}$.*

2. *Let $\mathcal{L}$ be a cubical $\mathbf{G}_{\mathrm{m}}$-torsor over G such that $\mathcal{L}_\eta$ is nondegenerate over G_η. Let $K_S(\mathcal{L}_\eta)$ be the schematic closure of $K(\mathcal{L}_\eta)$ in G over S. Then $\lambda_{\mathcal{L}_\eta} : G_\eta \to (G_\eta)^\vee$ extends to a unique homomorphism $\lambda_{\mathcal{L}_\eta,S} : G \to G^\vee$ such that $\ker(\lambda_{\mathcal{L}_\eta,S}) = K_S(\mathcal{L}_\eta)$. Moreover, we have a canonical isomorphism $\mathcal{D}_2(\mathcal{L}) \overset{\sim}{\to} (\mathrm{Id}_G \times \lambda_{\mathcal{L}_\eta,S})^*\mathcal{P}$ of biextensions over $G \underset{S}{\times} G$, extending the usual isomorphism over $G_\eta \underset{\eta}{\times} G_\eta$.*

Proof. First let us prove 1. The uniqueness of $G^\vee$ is a consequence of Proposition 3.3.1.5. As for the existence of $G^\vee$, we may suppose that S is local. By [105, IX, 1.13], there exists an *ample* invertible sheaf $\mathcal{L}$ over G. We may require that $\mathcal{L}$ is rigidified, so that a cubical structure of $\mathcal{L}$ exists by Proposition 3.2.3.1. By Theorem 3.4.2.4, $K(\mathcal{L}_\eta)$ extends to a flat and quasi-finite subscheme $K_S(\mathcal{L}_\eta)$ of G over S. By Lemma 3.4.3.1, the quotient sheaf $G/K_S(\mathcal{L}_\eta)$ is representable by a semi-abelian scheme $G^\vee$, extending the quotient $G_\eta/K(\mathcal{L}_\eta)$ which can be identified with $(G_\eta)^\vee$ according to the usual theory of abelian varieties (in, for example, [94]).

Next let us prove 2. Since $K_S(\mathcal{L}_\eta)$ is flat and quasi-finite over S by Theorem 3.4.2.4, the quotient $\lambda_{\mathcal{L}_\eta,S} : G \to G/K_S(\mathcal{L}_\eta)$ is between semi-abelian schemes by Lemma 3.4.3.1. Therefore, by Proposition 3.3.1.5, $G/K_S(\mathcal{L}_\eta)$ is uniquely isomorphic to $G^\vee$, and $\lambda_{\mathcal{L}_\eta,S}$ is the unique homomorphism extending $\lambda_{\mathcal{L}_\eta}$. Now the isomorphism $\mathcal{D}_2(\mathcal{L}) \overset{\sim}{\to} (\mathrm{Id}_G \times \lambda_{\mathcal{L}_\eta,S})^*\mathcal{P}$ of biextensions over $G \underset{S}{\times} G$ follows from Theorem 3.3.2.6. □

3.4.4 Dual Raynaud Extensions

Suppose R is a noetherian normal domain complete with respect to an ideal I, with $\mathrm{rad}(I) = I$ for convenience. Let $S := \mathrm{Spec}(R)$, $K := \mathrm{Frac}(K)$, $\eta := \mathrm{Spec}(K)$ the generic point of S, $R_i := R/I^{i+1}$, and $S_i := \mathrm{Spec}(R_i)$, for each integer $i \geq 0$. Let G be a semi-abelian scheme over S whose generic fiber G_η is an abelian scheme, such that G_0 is an *extension* of an abelian scheme A_0 by an *isotrivial* torus T_0 (see Definition 3.1.1.5). By Theorem 3.4.3.2 and its proof, there is a semi-abelian scheme $G^\vee$, whose generic fiber is the dual abelian variety of G_η, such that the torus part of $G_0^\vee := G^\vee \underset{S}{\times} S_0$ is isotrivial because it is so for G. Hence we may consider the Raynaud extensions (see Definition 3.3.3.8) associated with G and $G^\vee$, denoted $0 \to T \to G^{\natural} \to A \to 0$ and $0 \to T^\vee \to G^{\vee,\natural} \to A^\vee \to 0$, respectively, by abuse of notation.

Proposition 3.4.4.1. *The abelian part $A^\vee$ of $G^{\vee,\natural}$ is the dual abelian scheme of the abelian part A of G. In fact, the extension $\mathcal{P}$ over $G \underset{S}{\times} G^\vee$ of the Poincaré invertible sheaf $\mathcal{P}_{G_\eta}$ over $G_\eta \underset{\eta}{\times} (G_\eta)^\vee$ constructed above (in Theorem 3.4.3.2) can be descended to the Poincaré invertible sheaf $\mathcal{P}_A$ over $A \underset{S}{\times} A^\vee$ after first passing to the formal completion.*

Proof. By [57, VIII, 3.5] (or Corollary 3.2.5.2), the formal completion $\mathcal{P}_{\mathrm{for}} = \{\mathcal{P} \underset{S}{\times} S_i\}_{i\geq 0}$ of $\mathcal{P}$ is a biextension over $G_{\mathrm{for}} \times (G^\vee)_{\mathrm{for}}$ that can be uniquely descended to a biextension $\mathcal{P}_{A,\mathrm{for}}$ over $A_{\mathrm{for}} \times (A^\vee)_{\mathrm{for}}$. Since A_{for} and $(A^\vee)_{\mathrm{for}}$ are both proper and algebraizable, so is $\mathcal{P}_{A,\mathrm{for}}$, by Theorem 2.3.1.2. Denote by $\mathcal{P}_A$ the algebraization of $\mathcal{P}_{A,\mathrm{for}}$. It remains to check that $\mathcal{P}_A$ actually defines a duality between A and $A^\vee$. It suffices to check this over η. Hence it suffices to check that the pairing $A[\ell^\infty] \times A^\vee[\ell^\infty] \to \mathbf{G}_{\mathrm{m},S}[\ell^\infty]$ induced by $\mathcal{P}_A$ is a perfect pairing over η, for every prime ℓ. The pairing $G^\natural[\ell^\infty] \times G^{\vee,\natural}[\ell^\infty] \to \mathbf{G}_{\mathrm{m},S}[\ell^\infty]$ induced by $\mathcal{P}_{G,\mathrm{for}}$ factors through the previous pairing, because $\mathcal{P}_{G,\mathrm{for}}$ is the pullback of $\mathcal{P}_{A,\mathrm{for}}$. On the other hand, $G^\natural[\ell^\infty] \subset G[\ell^\infty]$ and $G^{\vee,\natural}[\ell^\infty] \subset G^\vee[\ell^\infty]$ canonically, and the pairing $G[\ell^\infty] \times G^\vee[\ell^\infty] \to \mathbf{G}_{\mathrm{m},S}[\ell^\infty]$ induced by $\mathcal{P}$ extends the previous pairing. Since G and $G^\vee$ are abelian varieties over η, $G[\ell^\infty] \times G^\vee[\ell^\infty] \to \mathbf{G}_{\mathrm{m},S}[\ell^\infty]$ is a perfect pairing over η. If we can show that the annihilator of $G_\eta^\natural[\ell^\infty]$ in $G_\eta^\vee[\ell^\infty]$ is $T_\eta^\vee[\ell^\infty]$, the claim will follow. But this follows from similar statements for pairings defined by $\mathcal{D}_2(\mathcal{L})$. Then we can conclude the proof because we may take $\mathcal{L}$ to be of the form $\mathcal{L}' \underset{\mathscr{O}_G}{\otimes} [-1]^*\mathcal{L}'$ for some ample cubical invertible sheaf $\mathcal{L}'$ so that $\mathcal{L}_{\mathrm{for}}$ descends to an ample $\mathcal{M}_{\mathrm{for}}$ over A_{for}. (This is the argument in [42, Ch. II, §2]. More details about various facts used in the above argument can be found in [57, IX].) □

By Proposition 3.1.5.1, the two Raynaud extensions $0 \to T \to G^\natural \to A \to 0$ and $0 \to T^\vee \to G^{\vee,\natural} \to A^\vee \to 0$ are encoded by two homomorphisms $c : \underline{\mathbf{X}}(T) \to A^\vee$ and $c^\vee : \underline{\mathbf{X}}(T^\vee) \to A$. Let us denote $G^\natural \to A$ (resp. $G^{\vee,\natural} \to A^\vee$) by π (resp. $\pi^\vee$), and let $\underline{X} := \underline{\mathbf{X}}(T)$ and $\underline{Y} := \underline{\mathbf{X}}(T^\vee)$.

By Proposition 3.3.1.5, each polarization $\lambda_\eta : G_\eta \to G_\eta^\vee$ extends uniquely to a homomorphism $\lambda_G : G \to G^\vee$. The functoriality of Raynaud extensions then gives us a homomorphism $\lambda^\natural : G^\natural \to G^{\vee,\natural}$, which induces a homomorphism $\lambda^\natural|_T : T \to T^\vee$ between the torus parts, and a polarization $\lambda_A : A \to A^\vee$ between the abelian parts. By Lemma 3.1.1.6, the homomorphism $\lambda^\natural|_T$ determines (and is determined by) a homomorphism $\phi : \underline{Y} \to \underline{X}$, and the two morphisms ϕ and λ_A satisfy the compatible relation $\lambda_A c^\vee = c\phi$. Note that the two étale sheaves of finitely generated free commutative groups $\underline{X}$ and $\underline{Y}$ have the same rank, and ϕ is injective with finite cokernel (because $\lambda^\natural|_T$ is surjective with finite kernel).

Conversely, the two homomorphisms ϕ and λ_A satisfying $\lambda_A c^\vee = c\phi$ determines $\lambda^\natural$ uniquely, because every homomorphism from an abelian scheme to a torus is trivial, and more concretely because we can describe $\lambda^\natural$ explicitly as follows: Consider $\mathscr{M}_{c(\chi)} := (\mathrm{Id}, c(\chi))^*\mathcal{P}_A \in \underline{\mathrm{Pic}}_e^0(A/S)$ (corresponding to $c(\chi) \in A^\vee$) and $\mathscr{M}_{c^\vee(\chi)} := (c^\vee(\chi), \mathrm{Id})^*\mathcal{P}_A \in \underline{\mathrm{Pic}}_e^0(A^\vee/S)$ (corresponding to $c^\vee(\chi) \in (A^\vee)^\vee \cong A$). Then $\lambda^\natural : G^\natural \to G^{\vee,\natural}$ can be described explicitly as

$$\begin{aligned} G^\natural &\cong \underline{\mathrm{Spec}}_{\mathscr{O}_A}\Big(\underset{\chi\in\underline{X}}{\oplus} \mathscr{M}_{c(\chi)}\Big) \\ &\to \lambda_A^*(G^{\vee,\natural}) := G^{\vee,\natural} \underset{A^\vee,\lambda_A}{\times} A \cong \underline{\mathrm{Spec}}_{\mathscr{O}_A}\Big(\underset{\chi\in\underline{Y}}{\oplus} \lambda_A^*\mathscr{M}_{c^\vee(\chi)}\Big) \\ &\cong \underline{\mathrm{Spec}}_{\mathscr{O}_A}\Big(\underset{\chi\in\underline{Y}}{\oplus} \mathscr{M}_{\lambda_A c^\vee(\chi)}\Big) = \underline{\mathrm{Spec}}_{\mathscr{O}_A}\Big(\underset{\chi\in\underline{Y}}{\oplus} \mathscr{M}_{c(\phi(\chi))}\Big) \end{aligned}$$

(see Propositions 3.1.2.11 and 3.1.5.1). Let us record this observation as the following lemma:

Lemma 3.4.4.2. *The datum of a homomorphism $\lambda^\natural : G^\natural \to G^{\vee,\natural}$ is equivalent to the datum of a pair of homomorphisms $\phi : \underline{Y} \to \underline{X}$ and $\lambda_A : A \to A^\vee$ such that $\lambda_A c^\vee = c\phi$.*

(The argument explained above does not require that λ_A is a polarization.)

Lemma 3.4.4.3. *We have the relations $\deg(\lambda^\natural) = [\underline{X} : \phi(\underline{Y})]\deg(\lambda_A)$ and $\deg(\lambda_{G_\eta}) = [\underline{X} : \phi(\underline{Y})]^2 \deg(\lambda_A)$.*

Proof. These can be verified after replacing S with complete discrete valuation rings centered at S_0, which then follow from Theorem 3.4.2.4. □

Chapter 4

Theory of Degeneration for Polarized Abelian Schemes

In this chapter we reproduce the theory of degeneration data for abelian varieties, following Mumford's original paper [95] and the first three chapters of Faltings and Chai's monograph [42]. Although there is essentially nothing new in this chapter, some modifications have been introduced to make the statements compatible with our understanding of the proofs. Moreover, since [95] and [42] have supplied full details only in the completely degenerate case, we will balance the literature by avoiding the special case and treating all cases equally.

The main objective in this chapter will be to state and prove Theorems 4.2.1.14, 4.4.16, and 4.6.3.43. Technical results worth noting are Propositions 4.3.2.10, 4.3.3.6, 4.3.4.5, and 4.5.1.15, Theorems 4.5.3.6 and 4.5.3.10, Proposition 4.5.3.11, Theorem 4.5.4.17, Propositions 4.5.5.1, 4.5.6.1, 4.5.6.3, 4.5.6.5, 4.6.1.5 and 4.6.2.11, and Theorem 4.6.3.16. Some of the differences among our work and the corresponding parts in [95] and [42], notably in the statements of Definitions 4.2.1.1 and 4.5.1.2, and Theorem 4.2.1.14, are explained in Remarks 4.2.1.2, 4.2.1.3, 4.2.1.17 and 4.5.1.4.

4.1 The Setting for This Chapter

Let R be a noetherian normal domain complete with respect to an ideal I, with $\mathrm{rad}(I) = I$ for convenience. Let $S := \mathrm{Spec}(R)$, $K := \mathrm{Frac}(R)$, $\eta := \mathrm{Spec}(K)$ the generic point of S, $R_i := R/I^{i+1}$, $S_i := \mathrm{Spec}(R_i)$, for each integer $i \geq 0$, and $S_{\mathrm{for}} = \mathrm{Spf}(R, I)$.

This setting will be assumed throughout the chapter, unless otherwise specified.

4.2 Ample Degeneration Data

The idea of introducing periods to degenerating abelian varieties is originally due to Tate in the case of elliptic curves. In the context of rigid analytic geometry, the idea is generalized by Raynaud [106, §1]. In the context of schemes over noetherian complete adic bases, the idea is generalized by Mumford [95] for degeneration into tori, and then generalized by Faltings and Chai [42, Ch. II] for degeneration into semi-abelian varieties.

In this section, we will introduce the notion of *ample degeneration data*, and state the main theorem of [42, Ch. II] associating such data with degenerating abelian varieties with ample cubical invertible sheaves. The proof will be given in Section 4.3 using the Fourier expansions of theta functions.

4.2.1 Main Definitions and Main Theorem of Degeneration

Let us begin with the category $\mathrm{DEG}_{\mathrm{ample}}(R, I)$:

Definition 4.2.1.1. *With assumptions as in Section 4.1, the category* $\mathrm{DEG}_{\mathrm{ample}}(R, I)$ *has objects of the form* $(G, \mathcal{L})$*, where*

1. *G is a semi-abelian scheme over S such that G_η is an abelian variety, and such that $G_0 = G \underset{S}{\times} S_0$ is an **extension** $0 \to T_0 \to G_0 \to A_0 \to 0$ of an abelian scheme A_0 by an **isotrivial** torus T_0 (see Definition 3.1.1.5) over S_0;*
2. *$\mathcal{L}$ is an ample cubical invertible sheaf over G rigidified along the identity section such that $\mathcal{L}_{\mathrm{for}}$ is in the essential image of (3.3.3.12) (the cubical structure on $\mathcal{L}$ exists uniquely by Proposition 3.2.3.1).*

The morphisms of $\mathrm{DEG}_{\mathrm{ample}}(R, I)$ *are naive isomorphisms (over S) respecting all structures. (We shall often omit this statement in similar definitions.)*

If R is local and I is the maximal ideal of R, we shall drop I from the notation $\mathrm{DEG}_{\mathrm{ample}}(R, I)$*. (We shall adopt the same convention for similar definitions.)*

Remark 4.2.1.2. Since we assume the isotriviality of T_0 in Definition 4.2.1.1, we do *not* need the following technical assumption in [42, Ch. II, §3, p. 36]: For each étale R_0-algebra R'_0, its unique lifting R' to a formally étale I-adically complete R-algebra (as in Remark 3.3.3.4) is *normal* (see Remark 3.3.3.10).

Remark 4.2.1.3. In [42, Ch. II, §3, p. 36], they only assume that $\mathcal{L}_\eta$ is ample over G_η. However, they claim in the same section that étale locally $\mathcal{L}_{\mathrm{for}}$ descends to an *ample* invertible sheaf $\mathcal{M}_{\mathrm{for}}$, and they stated explicitly in [42, Ch. II, §5, p. 49] that $\mathcal{L}$ is ample over G. Therefore, we believe they mean to assume that $\mathcal{L}$ is ample over G. (This is harmless when $\mathcal{L}_\eta$ is symmetric. See Lemma 4.2.1.6 below and [93, VI, 3.1 and 3.1.1].)

Lemma 4.2.1.4. *Let G be a semi-abelian scheme over S such that G_η is an abelian variety (but does not necessarily satisfy other conditions in 1 of Definition 4.2.1.1), and let $\mathcal{L}$ be a cubical invertible sheaf over G such that $\mathcal{L}_\eta$ is nondegenerate. As in Theorem 3.4.3.2, let $\mathcal{P}$ be the unique $\mathbf{G}_{\mathrm{m}}$-biextension of $G \underset{S}{\times} G$ extending the Poincaré $\mathbf{G}_{\mathrm{m}}$-biextension $\mathcal{P}_\eta$ of $G_\eta \underset{\eta}{\times} G_\eta$, and let $\lambda = \lambda_{\mathcal{L}_\eta, S} : G \to G^\vee$ be unique homomorphism extending $\lambda_\eta = \lambda_{\mathcal{L}_\eta}$ (see Proposition 3.3.1.5). Then $(\mathrm{Id}_G, \lambda)^* \mathcal{P} \cong \mathcal{L} \underset{\mathscr{O}_G}{\otimes} [-1]^* \mathcal{L}$.*

Proof. Since $(\mathrm{Id}_G \times \lambda)^*\mathcal{P} \cong \mathcal{D}_2(\mathcal{L})$ by Theorem 3.4.3.2, we have to show that $(\mathrm{Id}_G, \mathrm{Id}_G)^*\mathcal{D}_2(\mathcal{L}) \cong \mathcal{L} \underset{\mathscr{O}_G}{\otimes} [-1]^*\mathcal{L}$. Since $\mathcal{L}$ is cubical, this follows by pulling back $\mathcal{D}_3(\mathcal{L})$ (which is trivialized by the cubical structure of $\mathcal{L}$) under $(\mathrm{Id}_G, \mathrm{Id}_G, [-1]) : G \to G \underset{S}{\times} G \underset{S}{\times} G$ (cf. [94, §6, Cor. 3]). □

Convention 4.2.1.5. *With the assumptions in Lemma 4.2.1.4, we say that $\lambda = \lambda_{\mathcal{L}_\eta,S}$ is the homomorphism induced by $\mathcal{L}$, and write $\lambda = \lambda_{\mathcal{L}}$.*

Lemma 4.2.1.6. *With the assumptions in Lemma 4.2.1.4, if $\mathcal{L}_\eta$ is* ***symmetric****, then $\mathcal{L}$ is ample if and only if $\mathcal{L}_\eta$ is ample, and if and only if $\lambda_\eta = \lambda_{\mathcal{L}_\eta}$ is a polarization (see Definition 1.3.2.16).*

Proof. The implication from $\mathcal{L}_\eta$ to $\mathcal{L}$ follows from [105, XI, 1.16] and the uniqueness of cubical extensions (see Proposition 3.3.2.2 and Theorem 3.3.2.3; see also [93, VI, 3.1]). The remaining implications then follow from the definitions. □

Next let us describe the category $\mathrm{DD}_{\mathrm{ample}}(R, I)$ of *ample degeneration data*:

First, we need a semi-abelian scheme $G^\natural$ (see Definition 3.3.1.1) that is globally an extension of an abelian scheme A by a torus T. What we have in mind is that $G^\natural$ should be the Raynaud extension associated with G (see Section 3.3.3), so that $G^\natural_{\mathrm{for}} = G_{\mathrm{for}}$ and so that, in particular, $T_0 \cong T \underset{S}{\times} S_0$ and $A_0 \cong A \underset{S}{\times} S_0$ if T_0 and A_0 are as in Definition 4.2.1.1. By Proposition 3.1.5.1, we know that the structure of $G^\natural$ as a commutative group scheme extension of A by T is determined by a homomorphism

$$c : \underline{X} = \underline{\mathbf{X}}(T) \to \underline{\mathrm{Pic}}^0_e(A/S) = A^\vee.$$

Second, we need a notion of a period homomorphism $\iota : \underline{Y}_\eta \to G^\natural_\eta$. For reasons that will be seen later, we need $\underline{Y}$ to be an étale sheaf of free commutative groups of rank $r = \dim_S(T) = \mathrm{rank}_S(\underline{X})$. If we compose this ι with the canonical homomorphism $G^\natural_\eta \to A_\eta$, then we obtain a homomorphism $\underline{Y}_\eta \to A_\eta$. What we have in mind is that this should come from the homomorphism

$$c^\vee : \underline{Y} \to A$$

describing the Raynaud extension $G^{\vee,\natural}$ of the dual $G^\vee$ of G, as described by Theorem 3.4.3.2 and Proposition 3.4.4.1.

Lemma 4.2.1.7. *With the setting as above, a group homomorphism $\iota : \underline{Y}_\eta \to G^\natural_\eta$ lifting $c^\vee : \underline{Y} \to A$ as above determines and is determined by a trivialization*

$$\tau^{-1} : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X},\eta} \xrightarrow{\sim} (c^\vee \times c)^*\mathcal{P}_{A,\eta}$$

of the biextension $(c^\vee \times c)^*\mathcal{P}_{A,\eta}$ over the étale group scheme $(\underline{Y} \underset{S}{\times} \underline{X})_\eta$.

The proof of Lemma 4.2.1.7 will be given in Section 4.2.2. For convenience, we shall denote the (tensor) inverse of τ^{-1} by

$$\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X},\eta} \xrightarrow{\sim} (c^\vee \times c)^*\mathcal{P}^{\otimes -1}_{A,\eta}.$$

So far we have described data that are related to G. Next, we need data that are related to the ample cubical invertible sheaf $\mathcal{L}$.

First, we need an ample cubical invertible sheaf $\mathcal{L}^\natural$ over $G^\natural$. What we have in mind is that $\mathcal{L}^\natural_{\mathrm{for}}$ and $\mathcal{L}_{\mathrm{for}}$ should be canonically isomorphic over $G^\natural_{\mathrm{for}} \cong G_{\mathrm{for}}$, and there is a unique such $\mathcal{L}^\natural$ by Proposition 3.3.3.11 and by our assumption that $\mathcal{L}_{\mathrm{for}}$ is in the essential image of (3.3.3.12). By Corollary 3.2.5.2, the $\mathbf{G}_{\mathrm{m}}$-biextension $\mathcal{D}_2(\mathcal{L}^\natural)$ over $G^\natural \underset{S}{\times} G^\natural$ descends uniquely to a $\mathbf{G}_{\mathrm{m}}$-biextension over $A \underset{S}{\times} A$, which is isomorphic to $(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A$ for a unique homomorphism $\lambda_A : A \to A^\vee$ (as in Construction 1.3.2.7). These statements are compatible with the corresponding statements over S_{for}. Moreover, we need a homomorphism $\phi : \underline{Y} \hookrightarrow \underline{X}$ which is dual to an isogeny from T to the torus part $T^\vee$ of $G^{\vee,\natural}$. We shall require that $\lambda_A c^\vee = c\phi$. Then, by Lemma 3.4.4.2, we have a homomorphism $\lambda^\natural : G^\natural \to G^{\vee,\natural}$ inducing λ_A and ϕ.

Lemma 4.2.1.8. *The homomorphism λ_A above is a polarization. Equivalently, $(\mathrm{Id}_A, \lambda_A)^*\mathcal{P}_A$ is an ample invertible sheaf (see Definition 1.3.2.16).*

The proof of Lemma 4.2.1.8 will be given after the next paragraph.

Let $i : T \to G^\natural$ and $\pi : G^\natural \to A$ denote the canonical morphisms. By Corollary 3.2.5.7, which is applicable because S is noetherian and normal, after making a finite étale surjective base change in S if necessary, we may assume that the étale sheaf $\underline{X}$ is constant and that the cubical invertible sheaf $i^*\mathcal{L}^\natural$ is trivial. In this case, each cubical trivialization $s : i^*\mathcal{L}^\natural \xrightarrow{\sim} \mathscr{O}_T$ determines a cubical invertible sheaf $\mathcal{M}$ over A and a cubical isomorphism $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$, both depending uniquely on the choice of s. By [105, XI, 1.11], $\mathcal{M}$ is *ample* because $\mathcal{L}^\natural$ is.

Proof of Lemma 4.2.1.8. After making a finite étale surjective base change in S if necessary, we may assume that $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$ for some ample $\mathcal{M}$. Then Corollary 3.2.5.2 shows that $(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A \cong \mathcal{D}_2(\mathcal{M})$, because they both descend from $\mathcal{D}_2(\mathcal{L}^\natural)$. Hence $(\mathrm{Id}_A, \lambda_A)^*\mathcal{P}_A \cong \mathcal{M} \underset{\mathscr{O}_A}{\otimes} [-1]^*\mathcal{M}$ is ample, as desired. □

Second, we need a $\underline{Y}_\eta$-action on $\mathcal{L}^\natural_\eta$ covering the $\underline{Y}_\eta$-action on $G^\natural_\eta$ defined by ι, which commutes with the T-action up to a character. Let us make this precise by stating the following lemma:

Lemma 4.2.1.9. *With the setting as above, such an action determines and is determined by a cubical trivialization*

$$\psi : \mathbf{1}_{\underline{Y},\eta} \xrightarrow{\sim} \iota^*(\mathcal{L}^\natural_\eta)^{\otimes -1}$$

compatible with $(\mathrm{Id}_{\underline{Y}} \times \phi)^*\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{Y},\eta} \xrightarrow{\sim} (c^\vee \times c\phi)^*\mathcal{P}^{\otimes -1}_{A,\eta}$ in the sense that $\mathcal{D}_2(\psi) = (\mathrm{Id}_{\underline{Y}} \times \phi)^*\tau$.

The proof will be given in Section 4.2.3.

This compatibility makes sense because the biextension $\mathcal{D}_2(\mathcal{L}^\natural)$ uniquely descends to the biextension $(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A$ of $A \underset{S}{\times} A$ over S. Moreover, it forces τ to be *symmetric* with respect to ϕ in the sense that $(\mathrm{Id}_{\underline{Y}} \times \phi)^*\tau$ is a trivialization of the biextension $(c^\vee \times c\phi)^*\mathcal{P}^{\otimes -1}_{A,\eta} \cong (c^\vee \times \lambda_A c^\vee)^*\mathcal{P}^{\otimes -1}_{A,\eta} \cong (c^\vee \times c^\vee)^*(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}^{\otimes -1}_{A,\eta}$ that is invariant under the symmetric isomorphism of $(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A$. Here $(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A$ has a symmetric isomorphism because λ_A is symmetric by the definition of a polarization (see Definitions 1.3.2.12 and 1.3.2.16), or equivalently because it is étale locally of the form $\mathcal{D}_2(\mathcal{M})$ for some invertible sheaf $\mathcal{M}$ over A (see Lemma 3.2.2.1).

The trivializations τ and ψ will be useful only if we have some suitable positivity conditions. After making a finite étale surjective base change in S if necessary, let us assume that both $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively, and that there is a cubical invertible sheaf $\mathcal{M}$ over A such that $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$. Then,

Definition 4.2.1.10. *The* ***positivity condition*** *for τ can be stated as follows: For each element y in Y, the section $\tau(y, \phi(y))$ extends to a section of the invertible sheaf $(c^\vee(y) \times c\phi(y))^*\mathcal{P}^{\otimes -1}_A$ over S, and for each nonzero element y in Y, the section $\tau(y, \phi(y))$ is congruent to zero modulo I in the sense that $\tau(y, \phi(y))$ induces a* *morphism $(c^\vee(y) \times c\phi(y))^*\mathcal{P}_A \to \mathscr{O}_S$ whose image factors through $\underline{I}$, where $\underline{I}$ is the invertible subsheaf of $\mathscr{O}_S$ corresponding to the ideal $I \subset R$.*

Definition 4.2.1.11. *The **positivity condition** for ψ can be stated as follows: For each element $y \in Y$, the section $\psi(y)$ extends to a section of the invertible sheaf $c^\vee(y)^*\mathcal{M}^{\otimes -1} = \mathcal{M}(c^\vee(y))^{\otimes -1}$ over S. Moreover, given any integer $n \geq 1$, for all but finitely many y in Y, the section $\psi(y)$ is congruent to zero modulo I^n in the sense that $\psi(y)$ induces a morphism $\mathcal{M}(c^\vee(y)) \to \mathscr{O}_S$ whose image factors through $\underline{I}^n$, where $\underline{I}$ is defined as above.*

Lemma 4.2.1.12. *The positivity condition for τ (in Definition 4.2.1.10) and the positivity condition for ψ (in Definition 4.2.1.11) are equivalent to each other.*

The proof of Lemma 4.2.1.12 will be given in Section 4.2.4.
Now let us state the definition of the category of degeneration data:

Definition 4.2.1.13. *With assumptions as in Section 4.1, the category $\mathrm{DD}_{\mathrm{ample}}(R, I)$ has objects of the form $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$, with entries described as follows:*

1. *An abelian scheme A and a torus T over S, and an extension*
$$0 \to T \to G^\natural \to A \to 0$$
over S, which determines and is determined by a homomorphism $c : \underline{X} = \underline{\mathbf{X}}(T) \to A^\vee$ via (-1 times the) push-out.

2. *An étale sheaf of free commutative groups $\underline{Y}$ of rank $r = \dim_S(T)$.*

3. *A homomorphism $c^\vee : \underline{Y} \to A$, which determines an extension*
$$0 \to T^\vee \to G^{\vee,\natural} \to A^\vee \to 0$$
over S, where $T^\vee$ is a torus with character group $\underline{Y}$.

4. *An injective homomorphism $\phi : \underline{Y} \to \underline{X}$ with finite cokernel.*

5. *An ample cubical invertible sheaf $\mathcal{L}^\natural$ over $G^\natural$ inducing a polarization $\lambda_A : A \to A^\vee$ of A over S such that $\lambda_A c^\vee = c\phi$, or equivalently a homomorphism from $G^\natural$ to $G^{\vee,\natural}$ inducing a polarization $\lambda_A : A \to A^\vee$ of A over S.*

6. *A trivialization*
$$\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X}, \eta} \xrightarrow{\sim} (c^\vee \times c)^*\mathcal{P}_{A,\eta}^{\otimes -1}$$
of the biextension $(c^\vee \times c)^*\mathcal{P}_{A,\eta}^{\otimes -1}$ over the étale group scheme $(\underline{Y} \underset{S}{\times} \underline{X})_\eta$ symmetric with respect to ϕ (i.e., $(\mathrm{Id}_{\underline{Y}} \times \phi)^*\tau$ is symmetric; see the second paragraph after Lemma 4.2.1.9), which determines a trivialization $\tau^{-1} : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X}, \eta} \xrightarrow{\sim} (c^\vee \times c)^*\mathcal{P}_{A,\eta}$ and hence a homomorphism $\iota : \underline{Y}_\eta \to G^\natural_\eta$ lifting $c^\vee$ by Lemma 4.2.1.7.

7. *A cubical trivialization*
$$\psi : \mathbf{1}_{\underline{Y},\eta} \xrightarrow{\sim} \iota^*(\mathcal{L}^\natural_\eta)^{\otimes -1}$$
compatible with $(\mathrm{Id}_{\underline{Y}} \times \phi)^*\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{Y}, \eta} \xrightarrow{\sim} (c^\vee \times c\phi)^*\mathcal{P}_{A,\eta}^{\otimes -1}$ in the sense that $\mathcal{D}_2(\psi) = (\mathrm{Id}_{\underline{Y}} \times \phi)^*\tau$, which by Lemma 4.2.1.9 determines and is determined by a $\underline{Y}_\eta$-action on $\mathcal{L}^\natural_\eta$ covering the $\underline{Y}_\eta$-action on $G^\natural_\eta$ defined by ι, which commutes with the T-action up to a character.

The two trivializations τ and ψ are required to satisfy their respective positivity conditions (see Definitions 4.2.1.10 and 4.2.1.11) which are equivalent to each other by Lemma 4.2.1.12. (As in Definition 4.2.1.1, the morphisms of $\mathrm{DD}_{\mathrm{ample}}(R, I)$ are naive isomorphisms over S respecting all structures. We shall often omit this statement in similar definitions. Similarly, if R is local and I is the maximal ideal of R, we shall drop I from the notation $\mathrm{DEG}_{\mathrm{ample}}(R, I)$. We shall adopt the same convention for similar definitions.)

Let us state our version of [42, Ch. II, Thm. 6.2] as follows:

Theorem 4.2.1.14. *With assumptions as in Section 4.1, there is a functor*
$$\mathrm{F}_{\mathrm{ample}}(R, I) : \mathrm{DEG}_{\mathrm{ample}}(R, I) \to \mathrm{DD}_{\mathrm{ample}}(R, I) :$$
$$(G, \mathcal{L}) \mapsto (A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$$
*called the association of **ample degeneration data**. Moreover, the association of A, $\underline{X}$, $\underline{Y}$, c, and $c^\vee$ does not depend on the choice of $\mathcal{L}$, but only on its existence (see Remark 4.2.1.15 below). The association of ϕ, τ, and the homomorphism $\lambda_A : A \to A^\vee$ induced by $\mathcal{L}^\natural$ (as explained above) depends on the homomorphism $\lambda : G \to G^\vee$ induced by $\mathcal{L}$, but not on the precise choice of $\mathcal{L}$ (see Remark 4.2.1.16 below).*

Apart from several basic justifications in Sections 4.2.2, 4.2.3, and 4.2.4, the proof of Theorem 4.2.1.14 will be given in Section 4.3.

Remark 4.2.1.15. The association of A, $\underline{X}$, and c in Theorem 4.2.1.14 depends only on G and not on $\mathcal{L}$, as they exist by the association of the Raynaud extension $G^\natural$ with G. The association of $\underline{Y}$ and $c^\vee$ in Theorem 4.2.1.14 depends also only on G, because it depends only on the dual $G^\vee$ of G defined in Theorem 3.4.3.2, and then on the association of $G^{\vee,\natural}$ with $G^\vee$ as described by Proposition 3.4.4.1.

Remark 4.2.1.16. The association of ι and equivalently τ in Theorem 4.2.1.14 depends a priori on the choice of $\mathcal{L}$, but we will show in Section 4.3.4 that it depends only on the homomorphism $\lambda : G \to G^\vee$ induced by $\mathcal{L}$. In fact, two polarizations $\lambda_1, \lambda_2 : G \to G^\vee$ induce the same τ if $N_1\lambda_1 = N_2\lambda_2$ for some positive integers N_1 and N_2 (see also Remark 4.2.1.17 below).

Remark 4.2.1.17. The claim in [42] that the independence of ι on the choice of $\mathcal{L}$ should follow after proving the equivalences of categories in Theorem 4.4.16 using Mumford's constructions requires some further explanation. In the proof of [42, Ch. III, Thm. 7.1], they set $\mathcal{L}_1 = f^*(\mathcal{L}_2)$ after assuming that f is an isomorphism, but it is still not clear that the object $(G_1^\natural, \iota_1)$ in $\mathrm{DD}(R, I)$ is part of the degeneration data in $\mathrm{DD}_{\mathrm{ample}}(R, I)$ associated with $(G_1, \mathcal{L}_1)$. (This is the independence of ι_1 on the choice of $\mathcal{L}_1$ that we wanted to know.) Therefore a circulation of reasoning is invoked if we conclude as in [42, Ch. III, Cor. 7.2]. We consulted Chai about this issue, and he kindly suggested a different approach that could repair the argument. We will explain an argument based on his suggestion in Section 4.5.5.

4.2.2 Equivalence between ι and τ

Proof of Lemma 4.2.1.7. For simplicity, after making a finite étale surjective base change in S if necessary, let us assume that the étale sheaves $\underline{X} = \underline{\mathbf{X}}(T)$ and $\underline{Y} = \underline{\mathbf{X}}(T^\vee)$ are constant with values X and Y, respectively. Let us interpret τ as a collection $\{\tau(y, \chi)\}_{y \in Y, \chi \in X}$ over η, where each $\tau(y, \chi)$ is a section of $\mathcal{P}_A(c^\vee(y), c(\chi))_\eta^{\otimes -1}$, satisfying the bimultiplicative conditions coming from the axioms of biextensions. Namely, for each $y_1, y_2 \in Y$ and $\chi \in X$, the section $\tau(y_1, \chi) \otimes \tau(y_2, \chi)$ is mapped to the section $\tau(y_1+y_2,\chi)$ under the isomorphism

$$\mathcal{P}_A(c^\vee(y_1),c(\chi))_\eta^{\otimes -1} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{P}_A(c^\vee(y_2),c(\chi))_\eta^{\otimes -1} \xrightarrow{\sim} \mathcal{P}_A(c^\vee(y_1+y_2),c(\chi))_\eta^{\otimes -1}$$

given by the first partial multiplication law of the biextension structure of $\mathcal{P}_{A,\eta}$, and, for each $y\in Y$ and $\chi_1,\chi_2\in X$, the section $\tau(y,\chi_1)\otimes\tau(y,\chi_2)$ is mapped to the section $\tau(y,\chi_1+\chi_2)$ under the isomorphism

$$\mathcal{P}_A(c^\vee(y),c(\chi_1))_\eta^{\otimes -1} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{P}_A(c^\vee(y),c(\chi_2))_\eta^{\otimes -1} \xrightarrow{\sim} \mathcal{P}_A(c^\vee(y),c(\chi_1+\chi_2))_\eta^{\otimes -1}$$

given by the second partial multiplication law of the biextension structure of $\mathcal{P}_{A,\eta}$. By abuse of notation, we shall denote these bimultiplicative conditions symbolically by

$$\tau(y_1,\chi)\tau(y_2,\chi)=\tau(y_1+y_2,\chi)$$

and

$$\tau(y,\chi_1)\tau(y,\chi_2)=\tau(y,\chi_1+\chi_2).$$

Then it follows from the compatibility between the two partial multiplication laws of the biextension structure of $\mathcal{P}_{A,\eta}$ that it makes sense to write

$$\tau(y_1,\chi_1)\tau(y_2,\chi_1)\tau(y_1,\chi_2)\tau(y_2,\chi_2)=\tau(y_1+y_2,\chi_1+\chi_2),$$

verifying the compatibility between the two partial multiplication laws.

For each $y\in Y$, we have a collection $\{\tau(y,\chi)\}_{\chi\in X}$ of sections $\tau(y,\chi)$ of $\mathcal{P}_A(c^\vee(y),c(\chi))_\eta^{\otimes -1}$. Let us denote the invertible sheaf $(\mathrm{Id},c(\chi))^*\mathcal{P}_A$ over A by $\mathscr{O}_\chi$, and denote the invertible sheaf $\mathcal{P}_A(c^\vee(y),c(\chi))=(c^\vee(y)\times c(\chi))^*\mathcal{P}_A$ over S by $\mathscr{O}_\chi(c^\vee(y))=(c^\vee(y))^*\mathscr{O}_\chi$. Considering the canonical isomorphism $\mathscr{O}_\chi(c^\vee(y))_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes -1} \cong \mathscr{O}_{S,\eta}$ for each $\chi\in X$, we can interpret $\tau(y,\chi)\in\mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes -1}$ as "multiplication by $\tau(y,\chi)$":

$$\tau(y,\chi):\mathscr{O}_\chi(c^\vee(y))_\eta\to\mathscr{O}_{S,\eta}.$$

Putting together the morphisms corresponding to all $\chi\in X$, we obtain

$$c^\vee(y)^*\mathscr{O}_{G,\eta}=c^\vee(y)^*(\underset{\chi\in X}{\oplus}\mathscr{O}_\chi)_\eta \xrightarrow{\sum_\chi \tau(y,\chi)} \mathscr{O}_{S,\eta}.$$

Here the notation makes sense if we interpret $\mathscr{O}_{G^\natural}$ as an $\mathscr{O}_A$-module (as in Section 3.1.4) using the fact that $G^\natural$ is relatively affine over A, and if we interpret $\mathscr{O}_{S,\eta}$ as an $\mathscr{O}_S$-module. For this to define a homomorphism of $\mathscr{O}_S$-algebras, we need to map $\tau(y,\chi_1)\otimes\tau(y,\chi_2)$ to $\tau(y,\chi_1+\chi_2)$ for each $y\in Y$ and $\chi_1,\chi_2\in X$ under the isomorphism

$$(c^\vee(y)^*\mathscr{O}_{\chi_1})_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} (c^\vee(y)^*\mathscr{O}_{\chi_2})_\eta \xrightarrow{\sim} (c^\vee(y)^*\mathscr{O}_{\chi_1+\chi_2})_\eta,$$

which is exactly the isomorphism

$$\mathcal{P}_A(c^\vee(y),c(\chi_1))_\eta^{\otimes -1} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{P}_A(c^\vee(y),c(\chi_2))_\eta^{\otimes -1} \xrightarrow{\sim} \mathcal{P}_A(c^\vee(y),c(\chi_1+\chi_2))_\eta^{\otimes -1}$$

given by the biextension structure of $\mathcal{P}_{A,\eta}$. Namely, we need the multiplicative condition

$$\tau(y,\chi_1)\tau(y,\chi_2)=\tau(y,\chi_1+\chi_2). \qquad (4.2.2.1)$$

This is indeed the same as giving a point $\iota(y)$ of $G_\eta^\natural$ lifting the point $c^\vee(y)$ of A.

It still remains to show that τ defines a homomorphism $\iota:Y\to G_\eta$ lifting $c^\vee:Y\to A$, namely, $\iota(y_1)+\iota(y_2)=\iota(y_1+y_2)$ for all $y_1,y_2\in Y$ under the multiplication of $G^\natural$. Recall that the multiplication of $G^\natural$ is defined by

$$m^*:m_A^*\mathscr{O}_{G^\natural}\cong m_A^*(\underset{\chi\in X}{\oplus}\mathscr{O}_\chi)\cong\underset{\chi\in X}{\oplus}m_A^*\mathscr{O}_\chi$$

$$\overset{\text{can.}}{\xrightarrow{\sim}} \underset{\chi\in X}{\oplus}(\mathrm{pr}_1^*\mathscr{O}_\chi \underset{\mathscr{O}_{A\times_S A}}{\otimes} \mathrm{pr}_2^*\mathscr{O}_\chi) \overset{\text{can.}}{\hookrightarrow} \mathrm{pr}_1^*\mathscr{O}_{G^\natural} \underset{\mathscr{O}_{A\times_S A}}{\otimes} \mathrm{pr}_2^*\mathscr{O}_{G^\natural} \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_{G^\natural\times_S G^\natural},$$

where each of the isomorphisms $m_A^*\mathscr{O}_\chi\xrightarrow{\sim}\mathrm{pr}_1^*\mathscr{O}_\chi \underset{\mathscr{O}_{A\times_S A}}{\otimes} \mathrm{pr}_2^*\mathscr{O}_\chi$ exists uniquely because $\mathscr{O}_\chi\in\underline{\mathrm{Pic}}_e^0(A/S)$ and because we require it to respect the rigidifications. Applying $(c^\vee(y_1)\times c^\vee(y_2))^*$ to this morphism, we obtain $(c^\vee(y_1)\times c^\vee(y_2))^*m^*$:

$$(c^\vee(y_1)\times c^\vee(y_2))^*m_A^*\mathscr{O}_{G^\natural}\xrightarrow{\sim}\underset{\chi\in X}{\oplus}(c^\vee(y_1)\times c^\vee(y_2))^*(\mathrm{pr}_1^*\mathscr{O}_\chi \underset{\mathscr{O}_{A\times_S A}}{\otimes} \mathrm{pr}_2^*\mathscr{O}_\chi),$$

which is essentially

$$c^\vee(y_1+y_2)^*\mathscr{O}_{G^\natural}\xrightarrow{\sim}\underset{\chi\in X}{\oplus}(c^\vee(y_1)^*\mathscr{O}_\chi \underset{\mathscr{O}_S}{\otimes} c^\vee(y_2)^*\mathscr{O}_\chi).$$

Therefore the compatibility $\iota(y_1)+\iota(y_2)=\iota(y_1+y_2)$ follows because the diagram

$$\begin{tikzcd}
\underset{\chi\in X}{\oplus} c^\vee(y_1+y_2)^*\mathscr{O}_{\chi,\eta} \arrow[r, "\iota(y_1+y_2)^*"] \arrow[d, "(c^\vee(y_1)\times c^\vee(y_2))^*m^*"', "\wr"] & \mathscr{O}_{S,\eta} \arrow[dd, "\wr"', "\text{can.}"] \\
\underset{\chi\in X}{\oplus}(c^\vee(y_1)^*\mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_S}{\otimes} c^\vee(y_2)^*\mathscr{O}_{\chi,\eta}) \arrow[d, hook] & \\
(\underset{\chi\in X}{\oplus} c^\vee(y_1)^*\mathscr{O}_{\chi,\eta}) \underset{\mathscr{O}_S}{\otimes} (\underset{\chi\in X}{\oplus} c^\vee(y_2)^*\mathscr{O}_{\chi,\eta}) \arrow[r, "\iota(y_1)^*\otimes\iota(y_2)^*"'] & \mathscr{O}_{S,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{S,\eta}
\end{tikzcd}$$

is commutative. Since the morphism $\iota(y_1+y_2)^*$ is defined by the "multiplication by $\tau(y_1+y_2,\chi)$", while the morphism $\iota(y_1)^*\otimes\iota(y_2)^*$ is defined by the "multiplication by $\tau(y_1,\chi)\otimes\tau(y_2,\chi)$", the commutativity follows if $\tau(y_1,\chi)\otimes\tau(y_2,\chi)$ is mapped to $\tau(y_1+y_2,\chi)$ under the isomorphism

$$\mathcal{P}_A(c^\vee(y_1),c(\chi))_\eta^{\otimes -1} \underset{\mathscr{O}_S}{\otimes} \mathcal{P}_A(c^\vee(y_2),c(\chi))_\eta^{\otimes -1} \xrightarrow{\sim} \mathcal{P}_A(c^\vee(y_1+y_2),c(\chi))_\eta^{\otimes -1}$$

given by the biextension structure of $\mathcal{P}_{A,\eta}$. Namely, we need the multiplicative condition

$$\tau(y_1+y_2,\chi)=\tau(y_1,\chi)\tau(y_2,\chi), \qquad (4.2.2.2)$$

together with the compatibility between (4.2.2.2) and (4.2.2.1) making τ a trivialization of the biextension $(c^\vee\times c)^*\mathcal{P}_{A,\eta}^{\otimes -1}$.

Now that we have settled the case where $\underline{X}$ and $\underline{Y}$ are constant, the general case follows by étale descent.

To summarize, a homomorphism $\iota:\underline{Y}_\eta\to G_\eta^\natural$ determines and is determined by a trivialization $\tau:\mathbf{1}_{\underline{Y}\times_S\underline{X},\eta}\xrightarrow{\sim}(c^\vee\times c)^*\mathcal{P}_{A,\eta}^{\otimes -1}$ of biextensions, which is exactly the statement of Lemma 4.2.1.7. □

4.2.3 Equivalence between ψ and Actions on $\mathcal{L}_\eta^\natural$

The proof of Lemma 4.2.1.9 is similar to the proof of Lemma 4.2.1.7 in nature, but needs some more preparation.

For simplicity, assume again that the étale sheaves $\underline{X}=\underline{\mathbf{X}}(T)$ and $\underline{Y}=\underline{\mathbf{X}}(T^\vee)$ are constant with values X and Y, respectively, by making a finite étale surjective base change in S. Assume further that we have made a choice of a cubical trivialization $s:i^*\mathcal{L}\cong\mathscr{O}_T$, so that by Proposition 3.2.5.4 we have a cubical isomorphism $\mathcal{L}^\natural\cong\pi^*\mathcal{M}$ for some invertible sheaf $\mathcal{M}$ over A. These assumptions are justified by étale descent, as in Section 4.2.2.

Since $\mathcal{L}^\natural$ is an $\mathscr{O}_{G^\natural}$-module, and since $G^\natural$ is relatively affine over A, we would like to think of $\mathcal{L}^\natural$ as an $\mathscr{O}_A$-module with the structure of an $\mathscr{O}_{G^\natural}$-module, where $\mathscr{O}_{G^\natural}$ is interpreted as an $\mathscr{O}_A$-algebra (as in Section 3.1.4). This is the same as considering $\pi_*\mathcal{L}^\natural$ as a $\pi_*\mathscr{O}_{G^\natural}$-module, with the abuse of language of suppressing all the π_*'s in the notation. From now on, we shall adopt this abuse of language whenever possible.

By defining $\mathscr{O}_\chi$ as in Section 4.2.2, we can write $\mathscr{O}_{G^\natural} \cong \underset{\chi\in X}{\oplus} \mathscr{O}_\chi$ and $\mathcal{L}^\natural \cong \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{G^\natural} \cong \underset{\chi\in X}{\oplus} (\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi)$. Let us denote $\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi$ by $\mathcal{M}_\chi$. Then the $\mathscr{O}_{G^\natural}$-module structure of $\mathcal{L}^\natural$, given by a morphism

$$\mathcal{L}^\natural \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{G^\natural} \cong (\underset{\chi\in X}{\oplus} \mathcal{M}_\chi) \underset{\mathscr{O}_A}{\otimes} (\underset{\chi\in X}{\oplus} \mathscr{O}_\chi) \to \mathcal{L}^\natural \cong \underset{\chi\in X}{\oplus} \mathcal{M}_\chi$$

of $\mathscr{O}_A$-modules, can be obtained by the canonical isomorphisms

$$\mathcal{M}_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi'} \cong (\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi) \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi'} \overset{\sim}{\to} \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi+\chi'} \cong \mathcal{M}_{\chi+\chi'}$$

determined by the unique isomorphisms $\mathscr{O}_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi'} \overset{\sim}{\to} \mathscr{O}_{\chi+\chi'}$ respecting the rigidifications (and giving the $\mathscr{O}_{G^\natural}$ the structure of an $\mathscr{O}_A$-algebra) for all $\chi, \chi' \in X$. To summarize, the $\mathscr{O}_{G^\natural}$-module structure of $\mathcal{L}^\natural$ is obtained by making $\mathscr{O}_\chi$ act on $\mathcal{L}^\natural$ by *translation of weights by* χ.

If we take $\mathcal{M}' := \mathcal{M}_{\chi_0}$ for some $\chi_0 \in X$ and consider $\mathcal{L}^{\natural\prime} := \pi^*\mathcal{M}'$, then we get $\mathcal{L}^{\natural\prime} \cong \underset{\chi\in X}{\oplus} ((\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi_0}) \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi) \cong \underset{\chi\in X}{\oplus} \mathcal{M}_{\chi+\chi_0}$. The $\mathscr{O}_{G^\natural}$-module structure of $\mathcal{L}^{\natural\prime}$ is again given by *translation of weights by* χ, namely, the canonical isomorphisms $\mathcal{M}_{\chi+\chi_0} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi'} \overset{\sim}{\to} \mathcal{M}_{\chi+\chi'+\chi_0}$. The only difference is that every weight is shifted by χ_0.

As an $\mathscr{O}_A$-module, $\mathcal{L}^\natural$ can be canonically identified with $\mathcal{L}^{\natural\prime}$, by sending the subsheaf $\mathcal{M}_\chi$ of $\mathcal{L}^\natural$ identically to $\mathcal{M}_\chi$ of $\mathcal{L}^{\natural\prime}$ for all χ. Let us denote this isomorphism by Id_{χ_0}, because it is the *identity* on the *same* underlying $\mathscr{O}_{G^\natural}$-module. We claim that this isomorphism is also a cubical isomorphism. This is because Id_{χ_0} is given by putting together the canonical isomorphisms $\mathcal{M}_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi_0} \overset{\sim}{\to} \mathcal{M}_{\chi+\chi_0}$, and so $\mathcal{D}_3(\mathrm{Id}_{\chi_0})$ is given by the canonical isomorphisms $\mathcal{D}_3(\mathcal{M}_\chi) \underset{\mathscr{O}_{A \underset{S}{\times} A \underset{S}{\times} A}}{\otimes} \mathcal{D}_3(\mathscr{O}_{\chi_0}) \overset{\sim}{\to} \mathcal{D}_3(\mathcal{M}_{\chi+\chi_0})$. Now $\mathcal{D}_3(\mathscr{O}_{\chi_0})$, $\mathcal{D}_3(\mathcal{M}_\chi)$, and $\mathcal{D}_3(\mathcal{M}_{\chi+\chi_0})$ are all isomorphic to the trivial invertible sheaf by the usual theorem of the cube (see Proposition 3.2.3.1), with unique choices of isomorphisms respecting the rigidifications. Therefore, the isomorphism $\mathcal{D}_3(\mathrm{Id}_{\chi_0})$ must agree with the canonical isomorphism between trivial invertible sheaves. On the other hand, the cubical structures on $\mathcal{L}$ and on $\mathcal{L}'$ differ by the pullback of the canonical cubical structure of $\mathscr{O}_{\chi_0}$, which is again built up by the same canonical isomorphisms in $\mathcal{D}_3(\mathrm{Id}_{\chi_0})$. This justifies the claim.

By Rosenlicht's lemma (see Lemma 3.2.2.11) and by Remark 3.2.2.10, the set of cubical trivializations $s : i^*\mathcal{L}^\natural \cong \mathscr{O}_T$ is a torsor under $\mathrm{Hom}^{(1)}(T, \mathbf{G}_{\mathrm{m},S})$, namely, the character group X of T. By the above arguments, we see that this set as an X-module is equivalent to the set of choices of $\mathscr{O}_{G^\natural}$-module structures on $\mathcal{L}^\natural$, on which a character $\chi \in X$ acts by *translation of weights by* χ.

Let us record the above observation:

Lemma 4.2.3.1. *With assumptions as in Definition 4.2.1.13, each two cubical invertible sheaves $\mathcal{M}$ and $\mathcal{M}'$ over A such that $\mathcal{L}^\natural \cong \pi^*\mathcal{M} \cong \pi^*\mathcal{M}'$ (as cubical invertible sheaves) are related to each other by $\mathcal{M}' \cong \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi$ for some $\chi \in \underline{X} = \underline{\mathbf{X}}(T)$.*

Proof of Lemma 4.2.1.9. To define a Y-action on $\mathcal{L}^\natural$ covering the Y-action on $G^\natural_\eta$ defined by (multiplication by) $\iota : Y \to G^\natural_\eta$, let us first understand the multiplication by $\iota(y)$ covering $c^\vee(y)$, which makes the diagram

$$\begin{tikzcd}
G^\natural_\eta \arrow[r, "\mathrm{can}._{G^\natural}"] \arrow[d] & \eta \underset{\eta}{\times} G^\natural_\eta \arrow[r, "\iota(y)\times \mathrm{Id}"] \arrow[d] & G^\natural_\eta \underset{\eta}{\times} G^\natural_\eta \arrow[r, "m_{G^\natural}"] \arrow[d] & G^\natural_\eta \arrow[d] \\
A_\eta \arrow[r, "\mathrm{can}._A"'] & \eta \underset{\eta}{\times} A_\eta \arrow[r, "c^\vee(y)\times \mathrm{Id}"'] & A_\eta \underset{\eta}{\times} A_\eta \arrow[r, "m_A"'] & A_\eta
\end{tikzcd}$$

commutative. By pulling back all sheaves to the lower-left A_η, we get a diagram

$$\begin{tikzcd}
T^*_{c^\vee(y)}\mathscr{O}_{G^\natural,\eta} \arrow[r, equal] \arrow[dddd, "T^*_{\iota(y)}"', "\wr"] & (\mathrm{can}._A)^*(c^\vee(y)\times \mathrm{Id})^* m_A^* \mathscr{O}_{G^\natural,\eta} \arrow[d, "(\mathrm{can}._A)^*(c^\vee(y)\times\mathrm{Id})^*(m^*)"] \\
& (\mathrm{can}._A)^*(c^\vee(y)\times \mathrm{Id})^*(\mathrm{pr}_1^*\mathscr{O}_{G^\natural,\eta} \underset{\mathscr{O}_{A_\eta \underset{\eta}{\times} A_\eta}}{\otimes} \mathrm{pr}_2^*\mathscr{O}_{G^\natural,\eta}) \arrow[d, equal] \\
& (\mathrm{can}._A)^*(\mathrm{pr}_1^* c^\vee(y)^*\mathscr{O}_{G^\natural,\eta} \underset{\mathscr{O}_{\eta \underset{\eta}{\times} A_\eta}}{\otimes} \mathrm{pr}_2^*\mathscr{O}_{G^\natural,\eta}) \arrow[d, "(\mathrm{can}._A)^*(\iota(y)^*\otimes\mathrm{Id})"] \\
& (\mathrm{can}._A)^*(\mathrm{pr}_1^*\mathscr{O}_{S,\eta} \underset{\mathscr{O}_{\eta \underset{\eta}{\times} A_\eta}}{\otimes} \mathrm{pr}_2^*\mathscr{O}_{G^\natural,\eta}) \arrow[dl, "(\mathrm{can}._G)^*"] \\
\mathscr{O}_{G^\natural,\eta} &
\end{tikzcd}$$

of $\mathscr{O}_{A,\eta}$-algebras. More precisely, this is defined by a diagram

$$\begin{tikzcd}
T^*_{c^\vee(y)}\mathscr{O}_{G^\natural,\eta} \arrow[r, equal] \arrow[dddd, "T^*_{\iota(y)}"', "\wr"] & (\mathrm{can}._A)^*(c^\vee(y)\times \mathrm{Id})^* m_A^* (\underset{\chi\in X}{\oplus}\mathscr{O}_{X,\eta}) \arrow[d, "(\mathrm{can}._A)^*(c^\vee(y)\times\mathrm{Id})^*(m^*)", "\wr"'] \\
& \underset{\chi\in X}{\oplus}(\mathrm{can}._A)^*(c^\vee(y)\times \mathrm{Id})^*(\mathrm{pr}_1^*\mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{A_\eta \underset{\eta}{\times} A_\eta}}{\otimes} \mathrm{pr}_2^*\mathscr{O}_{\chi,\eta}) \arrow[d, equal] \\
& \underset{\chi\in X}{\oplus}(\mathrm{can}._A)^*(\mathrm{pr}_1^* c^\vee(y)^*\mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{\eta \underset{\eta}{\times} A_\eta}}{\otimes} \mathrm{pr}_2^*\mathscr{O}_{\chi,\eta}) \arrow[d, "(\mathrm{can}._A)^*(\iota(y)^*\otimes\mathrm{Id})", "\wr"'] \\
& \underset{\chi\in X}{\oplus}(\mathrm{can}._A)^*(\mathrm{pr}_1^*\mathscr{O}_{S,\eta} \underset{\mathscr{O}_{\eta \underset{\eta}{\times} A_\eta}}{\otimes} \mathrm{pr}_2^*\mathscr{O}_{\chi,\eta}) \arrow[dl, "\sim", "(\mathrm{can}._G)^*"'] \\
\underset{\chi\in X}{\oplus}\mathscr{O}_{\chi,\eta} = \mathscr{O}_{G^\natural,\eta} &
\end{tikzcd}$$

of isomorphisms, which relies essentially on the isomorphisms

$$\begin{aligned} T^*_{c^\vee(y)}\mathscr{O}_{\chi,\eta} &\cong \mathscr{O}_\chi(c^\vee(y))_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(0)_\eta^{\otimes -1} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \\ &\quad (\text{unique isomorphism in } \underline{\mathrm{Pic}}^0_e(A_\eta/\eta)) \\ &\cong \mathscr{O}_\chi(c^\vee(y))_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \\ &\quad (\text{rigidification of } \mathscr{O}_\chi(0)_\eta) \\ &\cong \mathscr{O}_{\chi,\eta} \\ &\quad (\text{multiplication by } \tau(y,\chi) \in \mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes -1}) \end{aligned} \tag{4.2.3.2}$$

for each $\chi \in X$.

If we want to define a Y-action on $\mathcal{L}^\natural_\eta$ (which commutes with the T-action up to a character), we need to write down an isomorphism between $T^*_{c^\vee(y)}\mathcal{L}^\natural_\eta$ and $\mathcal{L}^\natural_\eta$. Again, the essential point is to understand the restriction of this isomorphism to the *weight-χ subsheaf* $T^*_{c^\vee(y)}\mathcal{M}_{\chi,\eta}$. Since $\mathcal{M}$ induces λ_A, we know that the invertible sheaf $T^*_{c^\vee(y)}\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathcal{M}^{\otimes -1}$ is equivalent to the invertible sheaf in $\underline{\mathrm{Pic}}^0(A/S)$ corresponding to the point $\lambda_A c^\vee(y) = c\phi(y)$ in $A^\vee$, namely, the invertible sheaf $\mathscr{O}_{\phi(y)}$. By matching the rigidifications (along the identity section of A), we obtain a uniquely determined isomorphism

$$T^*_{c^\vee(y)}\mathcal{M} \cong \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\phi(y)} \underset{\mathscr{O}_S}{\otimes} \mathcal{M}(c^\vee(y)) \cong \mathcal{M}_{\phi(y)} \underset{\mathscr{O}_S}{\otimes} \mathcal{M}(c^\vee(y)).$$

Then we have

$$T^*_{c^\vee(y)}\mathcal{M}_\chi \cong T^*_{c^\vee(y)}\mathcal{M} \underset{\mathscr{O}_A}{\otimes} T^*_{c^\vee(y)}\mathscr{O}_\chi \cong \mathcal{M}_{\chi+\phi(y)} \underset{\mathscr{O}_S}{\otimes} \mathcal{M}(c^\vee(y)) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_\chi(c^\vee(y)).$$

Now, for each $y \in Y$, we can interpret the section $\psi(y)$ of $\iota^*(y)^*(\mathcal{L}^\natural_\eta)^{\otimes -1} \cong \mathcal{M}(c^\vee(y))_\eta^{\otimes -1}$ as "multiplication by $\psi(y)$":

$$\psi(y) : \mathcal{M}(c^\vee(y))_\eta \to \mathscr{O}_{S,\eta}.$$

Similarly, for each $\chi \in X$, we have "multiplication by $\tau(y,\chi)$":

$$\tau(y,\chi) : \mathscr{O}_\chi(c^\vee(y))_\eta \to \mathscr{O}_{S,\eta}.$$

Putting these together, we have (symbolically) "multiplication by $\psi(y)\tau(y,\chi)$":

$$\psi(y)\tau(y,\chi) : \mathcal{M}_\chi(c^\vee(y))_\eta \cong \mathcal{M}(c^\vee(y))_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(c^\vee(y))_\eta \to \mathscr{O}_{S,\eta},$$

which enables us to define

$$\psi(y)\tau(y,\chi) : T^*_{c^\vee(y)}\mathcal{M}_{\chi,\eta} \cong \mathcal{M}_{\chi+\phi(y),\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y))_\eta \to \mathcal{M}_{\chi+\phi(y),\eta}.$$

These isomorphisms sum together into an isomorphism

$$\tilde{T}_{\iota(y)} : T^*_{c^\vee(y)}\mathcal{L}^\natural_\eta \to \mathcal{L}^\natural_\eta,$$

with the shift of weights by $\phi(y)$ described above.

For this to define a Y-action on $\mathcal{L}^\natural_\eta$, we need to show that the diagram

$$\begin{tikzcd} T^*_{c^\vee(y_1+y_2)}\mathcal{L}^\natural_\eta \arrow[rr, "\tilde{T}_{\iota(y_1+y_2)}", "\sim"'] \arrow[dr, "\sim", "T^*_{c^\vee(y_1)}(\tilde{T}_{\iota(y_2)})"'] & & \mathcal{L}^\natural_\eta \\ & T^*_{c^\vee(y_1)}\mathcal{L}^\natural_\eta \arrow[ur, "\sim", "\tilde{T}_{\iota(y_1)}"'] & \end{tikzcd}$$

is commutative, or equivalently, that the diagram

$$\begin{tikzcd} T^*_{c^\vee(y_1+y_2)}\mathcal{M}_{\chi,\eta} \arrow[rr, "\psi(y_1+y_2)\tau(y_1+y_2,\chi)", "\sim"'] \arrow[dr, "\sim", "T^*_{c^\vee(y_1)}(\psi(y_2)\tau(y_2,\chi))"'] & & \mathcal{M}_{\chi+\phi(y_1)+\phi(y_2),\eta} \\ & T^*_{c^\vee(y_1)}\mathcal{M}_{\chi+\phi(y_2),\eta} \arrow[ur, "\sim", "\psi(y_1)\tau(y_1,\chi+\phi(y_2))"'] & \end{tikzcd} \tag{4.2.3.3}$$

is commutative.

The first object $T^*_{c^\vee(y_1+y_2)}\mathcal{M}_\chi$ is isomorphic to $\mathcal{M}_\chi(c^\vee(y_1)+c^\vee(y_2)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_{\chi+\phi(y_1)+\phi(y_2)}$. On the other hand, by pulling back the isomorphism $T_{c^\vee(y_2)}\mathcal{M}_\chi \overset{\sim}{\to} \mathcal{M}_{\chi+\phi(y_2)} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y_2))$ under $c^\vee(y_1)$, we obtain an isomorphism

$$\mathcal{M}_\chi(c^\vee(y_1)+c^\vee(y_2)) \overset{\sim}{\to} \mathcal{M}_{\chi+\phi(y_2)}(c^\vee(y_1)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y_2)).$$

If we pull back the isomorphism

$$\psi(y_2)\tau(y_2,\chi) : T^*_{c^\vee(y_2)}\mathcal{M}_\chi \cong \mathcal{M}_{\chi+\phi(y_2)} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y_2)) \to \mathcal{M}_{\chi+\phi(y_2)}$$

under $T^*_{c^\vee(y_1)}$, then we obtain

$$\begin{aligned} &T^*_{c^\vee(y_1)}(\psi(y_2)\tau(y_2,\chi)) : \\ &T^*_{c^\vee(y_1)+c^\vee(y_2)}\mathcal{M}_\chi \cong (T^*_{c^\vee(y_1)}\mathcal{M}_{\chi+\phi(y_2)}) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y_2)) \\ &\qquad \cong \mathcal{M}_{\chi+\phi(y_1)+\phi(y_2)} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_{\chi+\phi(y_2)}(c^\vee(y_1)) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y_2)) \\ &\to T^*_{c^\vee(y_1)}\mathcal{M}_{\chi+\phi(y_2)} \cong \mathcal{M}_{\chi+\phi(y_1)+\phi(y_2)} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_{\chi+\phi(y_2)}(c^\vee(y_1)), \end{aligned}$$

which is the same "multiplication by $\psi(y_2)\tau(y_2,\chi)$" applied to a different invertible sheaf. Therefore, for the diagram (4.2.3.3) to be commutative, the essential point is to have (symbolically)

$$\psi(y_1+y_2)\tau(y_1+y_2,\chi) = \psi(y_1)\tau(y_1,\chi)\psi(y_2)\tau(y_2,\chi+\phi(y_1)),$$

or equivalently, the compatibility

$$\psi(y_1+y_2)\psi(y_1)^{-1}\psi(y_2)^{-1} = \tau(y_1,\phi(y_2)) \tag{4.2.3.4}$$

verifying $\mathcal{D}_2(\psi)(y_1,y_2) = (\mathrm{Id}_{\underline{Y}} \times \phi)^*\tau(y_1,y_2)$ in Lemma 4.2.1.9. □

Remark 4.2.3.5. By symmetry, we obtain

$$\tau(y_1,\phi(y_2)) = \mathcal{D}_2(\psi)(y_1,y_2) = \mathcal{D}_2(\psi)(y_2,y_1) = \tau(y_2,\phi(y_1)),$$

that is, $(\mathrm{Id}_{\underline{Y}} \times \phi)^*\tau$ is a symmetric trivialization because $\mathcal{D}_2(\psi)$ is.

4.2.4 Equivalence between the Positivity Condition for ψ and the Positivity Condition for τ

Definition 4.2.4.1. *Let R be a noetherian integral domain. Then we denote by* $\mathrm{Inv}(R)$ *the group of invertible R-submodules of $K := \mathrm{Frac}(R)$.*

Let υ be a valuation of K. For an invertible R-submodule J of K, we define $\upsilon(J)$ to be the minimal value of υ on nonzero elements of J.

Since R is noetherian and *normal*, we know that R is the intersection of the valuation rings of its discrete valuations defined by height-one primes, namely, $R = \underset{\substack{\mathfrak{p} \text{ prime} \\ \mathrm{ht}\,\mathfrak{p}=1}}{\cap} R_\mathfrak{p}$ (see, for example, [88, Thm. 11.5]). Let us denote by Υ_1 the set of valuations of K defined by height-one primes of R. Then,

Lemma 4.2.4.2. *An invertible R-submodule J of K satisfies $J \subset R$ if and only if $\upsilon(J) \geq 0$ for all $\upsilon \in \Upsilon_1$.*

Lemma 4.2.4.3 ([116, pp. 95–96, proof of Thm. 35, §14, Ch. VI])**.** *Let R be a noetherian integral domain with fractional field K. Then, for each prime ideal $\mathfrak{p}$ of R, there is a discrete valuation $\upsilon : K^\times \to \mathbb{Z}$ of K such that $R \subset R_\upsilon$ and $\mathfrak{p} = R \cap \mathfrak{m}_\upsilon$, where R_υ is the valuation ring of υ, and where $\mathfrak{m}_\upsilon$ is the maximal ideal of R_υ.*

Let us denote by Υ_I the set of discrete valuations υ of K such that $R \subset R_\upsilon$ and $I_\upsilon := I \underset{R}{\otimes} R_\upsilon \subsetneq R_\upsilon$. In other words, Υ_I is the set of discrete valuations of K that are nonnegative on R and has center on $S_0 = \mathrm{Spec}(R_0) = \mathrm{Spec}(R/I)$. This is determined essentially only by $\mathrm{rad}(I) = I$. Then,

Lemma 4.2.4.4. *An invertible R-submodule J of R satisfies $J \subset \mathrm{rad}(I) = I$ if and only if $\upsilon(J) > 0$ for all $\upsilon \in \Upsilon_I$.*

The section $\psi(y)$ of $\mathcal{M}(c^\vee(y))_\eta^{\otimes -1}$ is an isomorphism $\mathscr{O}_{S,\eta} \xrightarrow{\sim} \mathcal{M}(c^\vee(y))_\eta^{\otimes -1}$, which induces an isomorphism $\mathcal{M}(c^\vee(y))_\eta \xrightarrow{\sim} \mathscr{O}_{S,\eta}$. Since $\mathcal{M}(c^\vee(y))_\eta$ has an integral structure given by $\mathcal{M}(c^\vee(y))^{\otimes -1}$, the isomorphism $\psi(y) : \mathcal{M}(c^\vee(y))_\eta \xrightarrow{\sim} \mathscr{O}_{S,\eta}$ carries this integral structure to an invertible R-submodule of $K = \mathscr{O}_{S,\eta}$.

Definition 4.2.4.5. *We shall denote this submodule of K by I_y.*

In this case, we can interpret $\psi(y)$ as an isomorphism $\psi(y) : \mathcal{M}(c^\vee(y)) \xrightarrow{\sim} \underline{I}_y$. If $I_y \subset I^n \subset R$ for some nonnegative integer n, we obtain a morphism $\mathcal{M}(c^\vee(y)) \to \mathscr{O}_S$ whose image factors through $\underline{I}^n$, where $\underline{I}$ is the invertible subsheaf of $\mathscr{O}_S$ corresponding to the ideal $I \subset R$. We will write symbolically $\upsilon(\psi(y)) = \upsilon(I_y)$, as if $\psi(y)$ were an invertible R-submodule of K.

Similarly, the section $\tau(y, \chi)$ of $\mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes -1}$ is an isomorphism $\mathscr{O}_{S,\eta} \xrightarrow{\sim} \mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes -1}$, which induces an isomorphism $\mathscr{O}_\chi(c^\vee(y))_\eta \xrightarrow{\sim} \mathscr{O}_{S,\eta}$. Since $\mathscr{O}_\chi(c^\vee(y))_\eta$ has an integral structure given by $\mathscr{O}_\chi(c^\vee(y))^{\otimes -1}$, the isomorphism $\tau(y, \chi) : \mathscr{O}_\chi(c^\vee(y))_\eta \xrightarrow{\sim} \mathscr{O}_{S,\eta}$ carries this integral structure to an invertible R-submodule of $K = \mathscr{O}_{S,\eta}$.

Definition 4.2.4.6. *We shall denote this submodule of K by $I_{y,\chi}$.*

In this case, we can interpret $\tau(y, \chi)$ as an isomorphism $\tau(y, \chi) : \mathscr{O}_\chi(c^\vee(y)) \xrightarrow{\sim} \underline{I}_{y,\chi}$. If $I_{y,\chi} \subset I^n \subset R$ for some nonnegative integer n, we obtain a morphism $\mathscr{O}_\chi(c^\vee(y)) \to \mathscr{O}_S$ whose image factors through $\underline{I}^n$, where $\underline{I}$ is as above. The valuation $\upsilon(I_{y,\chi})$ is defined to be the minimal valuation of υ on nonzero elements in $I_{y,\chi}$. We will write symbolically $\upsilon(\tau(y, \chi)) = \upsilon(I_{y,\chi})$, as if $\tau(y, \chi)$ were an invertible R-submodule of K.

Proof of Lemma 4.2.1.12. By the symbolic relation (4.2.3.4), $\psi(y)$ can be interpreted as a quadratic function in y, with associated bilinear pairing $\tau(y_1, \phi(y_2))$. For each valuation υ of K and each $y \in Y$, define a function $f_{\upsilon,y} : \mathbb{Z} \to \mathbb{Z}$ by $f_{\upsilon,y}(k) := \upsilon(\psi(ky))$. Then we have the quadratic relation

$$f_{\upsilon,y}(k+1) - 2f_{\upsilon,y}(k) + f_{\upsilon,y}(k-1) = \upsilon(\tau(y, \phi(y))). \tag{4.2.4.7}$$

The implication from the positivity condition for ψ (in Definition 4.2.1.11) to the positivity condition for τ (in Definition 4.2.1.10) can be justified as follows:

Suppose $\upsilon \in \Upsilon_1$. The first half of the positivity condition for ψ implies that $\upsilon(\psi(y)) \leq 0$ for all but finitely many y in Y. If $y \in Y$, $y \neq 0$, and $\upsilon(\tau(y, \phi(y))) < 0$, then the relation (4.2.4.7) shows that it is impossible that $f_{\upsilon,y}(k) \geq 0$ for all but finitely many k. As a result, $\upsilon(\tau(y, \phi(y))) \geq 0$ for all $\upsilon \in \Upsilon_1$, which implies that $I_{y,\phi(y)} \subset R$ by Lemma 4.2.4.2 and by noetherian normality of R.

Suppose $\upsilon \in \Upsilon_I$. The second half of the positivity condition for ψ implies that for each given value n_0 and each $y \in Y$ such that $y \neq 0$, there can only be finitely many integers k such that $f_{\upsilon,y}(k) \leq n_0$. If $\upsilon(\tau(y, \phi(y))) \leq 0$, then the relation (4.2.4.7) shows that there are infinitely many integers k such that $f_{\upsilon,y}(k) \leq f(0)$, which is a contradiction. As a result, $\upsilon(\tau(y, \phi(y))) > 0$ for all $\upsilon \in \Upsilon_I$, which implies that $I_{y,\phi(y)} \subset I$ by Lemma 4.2.4.4 and by the known fact that $I_{y,\phi(y)} \subset R$. This verifies the positivity condition for τ.

Conversely, the positivity condition for τ shows that $\upsilon(\tau(y_1, \phi(y_2)))$ defines a positive semidefinite form for all $\upsilon \in \Upsilon_1$ and defines a positive definite form for all $\upsilon \in \Upsilon_I$. Therefore, the associated quadratic form $\upsilon(\tau(y, \phi(y)))$ is positive semidefinite for all $\upsilon \in \Upsilon_1$ and is positive definite for all $\upsilon \in \Upsilon_I$. This implies the positivity condition for ψ, using the fact that Y is finitely generated and the assumption that R is noetherian. □

4.3 Fourier Expansions of Theta Functions

In this section we investigate the Fourier expansions of theta functions, namely, the sections of $\Gamma(G, \mathcal{L})$, and use the result to prove Theorem 4.2.1.14.

4.3.1 Definition of ψ and τ

With the setting as in Section 4.1, suppose that the Raynaud extension of G is

$$0 \to T \xrightarrow{i} G^\natural \xrightarrow{\pi} A \to 0$$

over S, and suppose that the Raynaud extension of $G^\vee$ is

$$0 \to T^\vee \to G^{\vee,\natural} \to A^\vee \to 0,$$

so that $\underline{X} = \underline{\mathbf{X}}(T)$ and $\underline{Y} = \underline{\mathbf{X}}(T^\vee)$. Let us suppose that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively, and suppose that a cubical trivialization $s : i^*\mathcal{L}^\natural \cong \mathscr{O}_T$ is chosen, so that we have a cubical isomorphism $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$ for some ample invertible sheaf $\mathcal{M}$ over A. Then we have an isomorphism $\pi_*\mathcal{L}^\natural \cong \underset{\chi \in X}{\oplus} (\mathscr{O}_\chi \underset{\mathscr{O}_A}{\otimes} \mathcal{M}) = \underset{\chi \in X}{\oplus} \mathcal{M}_\chi$, where $\pi_*\mathscr{O}_{G^\natural} = \underset{\chi \in X}{\oplus} \mathscr{O}_\chi$ is the decomposition into weight subsheaves under the T-action, introduced in Section 4.2.3. (Recall that in Section 4.2.2 we defined $\mathscr{O}_\chi := (\mathrm{Id}, c(\chi))^*\mathcal{P}_A$, the pullback of the Poincaré invertible sheaf $\mathcal{P}_A$ over $A \underset{S}{\times} A^\vee$ under the morphism $(\mathrm{Id}, c(\chi)) : A \to A \underset{S}{\times} A^\vee$, and that in Section 4.2.3 we defined $\mathcal{M}_\chi := \mathscr{O}_\chi \underset{\mathscr{O}_A}{\otimes} \mathcal{M}$.)

Let us consider the formal completions

$$0 \to T_{\mathrm{for}} \to G_{\mathrm{for}} \xrightarrow{\pi_{\mathrm{for}}} A_{\mathrm{for}} \to 0,$$

over S_{for} as a compatible system (for all $i \geq 0$) of exact sequences of group schemes $0 \to T_i \to G_i \xrightarrow{\pi_i} A_i \to 0$ over S_i, where A_i is an abelian scheme over S_i, and where T_i is a torus over S_i (see Section 3.3.3).

Since $\Gamma(G_i, \mathscr{O}_{G_i}) \cong \Gamma(A_i, \pi_*\mathscr{O}_{G_i}) \cong \underset{\chi \in X}{\oplus} \Gamma(A_i, \mathscr{O}_\chi)$ (compatibly for each i), we can write

$$\Gamma(G_{\mathrm{for}}, \mathscr{O}_{G,\mathrm{for}}) \cong \varprojlim_i \Gamma(G_i, \mathscr{O}_{G_i}) \cong \underset{\chi \in X}{\hat{\oplus}} \Gamma(A_{\mathrm{for}}, \mathscr{O}_{\chi,\mathrm{for}}) \cong \underset{\chi \in X}{\hat{\oplus}} \Gamma(A, \mathscr{O}_\chi),$$

where $\hat{\bigoplus}_{\chi\in X}$ stands for *I-adic completion*, and where $\Gamma(A_{\mathrm{for}},\mathscr{O}_{\chi,\mathrm{for}})\cong\Gamma(A,\mathscr{O}_\chi)$ follows from Proposition 2.3.1.1 for each $\chi\in X$. Then we can form the *Fourier expansion of regular functions* by assigning to each $f\in\Gamma(G_{\mathrm{for}},\mathscr{O}_{G,\mathrm{for}})$ the infinite sum

$$f=\sum_{\chi\in X}\sigma_\chi(f),$$

where $\sigma_\chi(f)$ lies in $\Gamma(A,\mathscr{O}_\chi)$ for each $\chi\in X$, and where the sum is *I-adically convergent* in the sense that if we consider f as a limit of $(f \bmod I^{i+1})\in\Gamma(G_i,\mathscr{O}_{G_i})$, then the corresponding sum $\sum\limits_{\chi\in X}\sigma_\chi(f) \bmod I^{i+1}$ has only finitely many nonzero terms for each i.

Similarly, since we have

$$\Gamma(G_i,\mathcal{L}_i)\cong\Gamma(G_i,\pi_i^*\mathcal{M}_i)\cong\Gamma(A_i,\pi_{i,*}\pi_i^*\mathcal{M}_i)\cong\bigoplus_{\chi\in X}\Gamma(A_i,\mathcal{M}_{\chi,i})$$

(compatibly for each i), we can write

$$\Gamma(G_{\mathrm{for}},\mathcal{L}_{\mathrm{for}})\cong\varprojlim_i\Gamma(G_i,\mathcal{L}_i)\cong\hat{\bigoplus_{\chi\in X}}\Gamma(A_{\mathrm{for}},\mathcal{M}_{\chi,\mathrm{for}})\cong\hat{\bigoplus_{\chi\in X}}\Gamma(A,\mathcal{M}_\chi).$$

Then we can form the *Fourier expansion of theta functions* by assigning to each $s\in\Gamma(G_{\mathrm{for}},\mathcal{L}_{\mathrm{for}})$ the infinite sum

$$s=\sum_{\chi\in X}\sigma_\chi(s),$$

where $\sigma_\chi(s)$ lies in $\Gamma(A,\mathcal{M}_\chi)$ for each $\chi\in X$, and where the sum is I-adically convergent in the sense that it has only finitely many nonzero terms modulo each power I^{i+1} of I as in the case of $\Gamma(G_{\mathrm{for}},\mathscr{O}_{G,\mathrm{for}})$. Symbolically, we shall write $\sigma_\chi(s)\equiv 0 \pmod{I^{i+1}}$ *for all but finitely many $\chi\in X$, for each fixed i.*

If we consider the canonical embedding $\Gamma(G,\mathcal{L})\hookrightarrow\Gamma(G_{\mathrm{for}},\mathcal{L}_{\mathrm{for}})$, then we obtain a morphism

$$\sigma_\chi:\Gamma(G,\mathcal{L})\to\Gamma(A,\mathcal{M}_\chi),$$

which extends naturally to

$$\sigma_\chi:\Gamma(G,\mathcal{L})\underset{R}{\otimes}K\to\Gamma(A,\mathcal{M}_\chi)\underset{R}{\otimes}K.$$

(This is now a morphism between K-vector spaces.) Since $\Gamma(G_\eta,\mathcal{L}_\eta)\cong\Gamma(G,\mathcal{L})\underset{R}{\otimes}K$ and $\Gamma(A_\eta,\mathcal{M}_{\chi,\eta})\cong\Gamma(A,\mathcal{M}_\chi)\underset{R}{\otimes}K$, the morphism σ_χ above can be written as

$$\sigma_\chi:\Gamma(G_\eta,\mathcal{L}_\eta)\to\Gamma(A_\eta,\mathcal{M}_{\chi,\eta}).$$

Note that (in the case of either $\Gamma(G,\mathcal{L})$ or $\Gamma(G_\eta,\mathcal{L}_\eta)$) the morphisms σ_χ do depend on the choice of $\mathcal{M}$. We shall write $\sigma_\chi=\sigma_\chi^{\mathcal{M}}$ to signify this choice when necessary.

Remark 4.3.1.1. By Lemma 4.2.3.1, each different choice of $s:i^*\mathcal{L}^\natural\cong\mathscr{O}_T$ gives a different choice of $\mathcal{M}'$ such that $\mathcal{L}^\natural\cong\pi^*\mathcal{M}$, which is necessarily of the form $\mathcal{M}'\cong\mathcal{M}_\chi=\mathcal{M}\underset{\mathscr{O}_A}{\otimes}\mathscr{O}_\chi$ for some $\chi\in\underline{X}$. This results in a shift of the indices for the σ_χ's above, and we will see later (in Lemmas 4.3.1.14 and 4.3.1.15) that this is harmless for defining the trivializations τ and ψ.

Since $\Gamma(G_\eta,\mathcal{L}_\eta)$ is finite-dimensional (by properness of G_η), we expect some redundancy in the full collection $\{\sigma_\chi\}_{\chi\in X}$ indexed by X (which is an infinite group when it is nontrivial). To make this precise, we shall compare

$$\begin{aligned}T^*_{c^\vee(y)}\circ\sigma_\chi:\Gamma(G_\eta,\mathcal{L}_\eta)&\to\Gamma(A_\eta,T^*_{c^\vee(y)}\mathcal{M}_{\chi,\eta})\\&\cong\Gamma(A_\eta,\mathcal{M}_{\chi+\phi(y),\eta})\underset{K}{\otimes}\mathcal{M}_\chi(c^\vee(y))_\eta\end{aligned}$$

(given by $T^*_{c^\vee(y)}\mathcal{M}_\chi\cong\mathcal{M}_{\chi+\phi(y)}\underset{R}{\otimes}\mathcal{M}_\chi(c^\vee(y))$) with

$$\sigma_{\chi+\phi(y)}:\Gamma(G_\eta,\mathcal{L}_\eta)\to\Gamma(A_\eta,\mathcal{M}_{\chi+\phi(y),\eta}).$$

We claim that (as morphisms over the generic fiber η)

$$\sigma_\chi\neq 0 \tag{4.3.1.2}$$

for *all* $\chi\in X$, and we claim that for each $y\in Y$ and $\chi\in X$ there exists a unique section $\psi(y,\chi)$ in $\mathcal{M}_\chi(c^\vee(y))_\eta^{\otimes-1}$ defining an isomorphism

$$\psi(y,\chi):\mathcal{M}_\chi(c^\vee(y))\xrightarrow{\sim}\mathscr{O}_{S,\eta},$$

or rather a section of $\mathcal{M}_\chi(c^\vee(y))^{\otimes-1}$, such that

$$\psi(y,\chi)\,T^*_{c^\vee(y)}\circ\sigma_\chi=\sigma_{\chi+\phi(y)}. \tag{4.3.1.3}$$

For each $y\in Y$, let us define a section $\psi(y)$ of $\mathcal{M}(c^\vee(y))_\eta^{\otimes-1}$ by setting

$$\psi(y):=\psi(y,0). \tag{4.3.1.4}$$

For each $y\in Y$ and $\chi\in X$, under the canonical isomorphism $\mathcal{M}_\chi(c^\vee(y))_\eta^{\otimes-1}\cong\mathcal{M}(c^\vee(y))_\eta^{\otimes-1}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes-1}$, let us also define a section $\tau(y,\chi)$ of $\mathscr{O}_\chi(c^\vee(y))_\eta^{\otimes-1}\cong\mathcal{M}(c^\vee(y))_\eta\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\chi(c^\vee(y))_\eta^{\otimes-1}$ by setting

$$\tau(y,\chi):=\psi(y)^{-1}\psi(y,\chi), \tag{4.3.1.5}$$

so that we have the (symbolic) relation

$$\psi(y,\chi)=\psi(y)\tau(y,\chi). \tag{4.3.1.6}$$

Then we can rewrite the above relation (4.3.1.3) as

$$\psi(y)\tau(y,\chi)\,T^*_{c^\vee(y)}\circ\sigma_\chi=\sigma_{\chi+\phi(y)}. \tag{4.3.1.7}$$

Lemma 4.3.1.8. *1. We have (symbolically) $\psi(0)=1$ in the sense that $\psi(0):\mathscr{O}_{S,\eta}\xrightarrow{\sim}\mathcal{M}(c^\vee(0))_\eta^{\otimes-1}=\mathcal{M}(0)_\eta^{\otimes-1}$ coincides with the rigidification of $\mathcal{M}_\eta^{\otimes-1}$.*

2. For all $\chi\in X$, we have (symbolically) $\tau(0,\chi)=1$, in the sense that $\tau(0,\chi):\mathscr{O}_{S,\eta}\xrightarrow{\sim}\mathscr{O}_\chi(c^\vee(0))_\eta^{\otimes-1}=\mathscr{O}_\chi(0)_\eta^{\otimes-1}$ coincides with the rigidification of $\mathscr{O}_{\chi,\eta}^{\otimes-1}$.

3. For all $y\in Y$, we have (symbolically) $\tau(y,0)=1$, in the sense that the inverse morphism of $\tau(y,0):\mathscr{O}_{S,\eta}\xrightarrow{\sim}\mathscr{O}_0(c^\vee(y))_\eta^{\otimes-1}\cong c^\vee(y)^*\mathscr{O}_{A,\eta}$ coincides with the structural isomorphism given by the section $c^\vee(y):\eta\to A_\eta$. Here $\mathscr{O}_0^{\otimes-1}\cong\mathscr{O}_A$ is the unique isomorphism given by the rigidification of $\mathcal{P}_A$, and the inverse of $\tau(y,0)$ is interpreted as an isomorphism $c^\vee(y)^*\mathscr{O}_{A,\eta}\xrightarrow{\sim}\mathscr{O}_{S,\eta}$.

(Under (4.3.1.6), 3 implies 1, and 1 implies 2.)

Proposition 4.3.1.9. *If we assume the relations (4.3.1.2) and (4.3.1.7) above, then the following relations are natural consequences of the definitions:*

1. For all $y_1,y_2\in Y$, we have (symbolically)

$$\psi(y_1+y_2)=\psi(y_1)\psi(y_2)\tau(y_1,\phi(y_2)) \tag{4.3.1.10}$$

under the $\mathbf{G}_{\mathrm{m}}$-torsor isomorphism

$$(c^\vee\times c^\vee)^*\mathcal{D}_2(\mathcal{M})_\eta^{\otimes-1}\cong(c^\vee\times c\phi)^*\mathcal{P}_{A,\eta}^{\otimes-1}.$$

By symmetry, we also have

$$\psi(y_1+y_2)=\psi(y_2)\psi(y_1)\tau(y_2,\phi(y_1)). \tag{4.3.1.11}$$

2. For all $y_1,y_2\in Y$, we have (symbolically)

$$\tau(y_1,\phi(y_2))=\tau(y_2,\phi(y_1))$$

under the symmetry isomorphism of

$$(c^\vee\times c^\vee)^*\mathcal{D}_2(\mathcal{M})_\eta^{\otimes-1}\cong(c^\vee\times c\phi)^*\mathcal{P}_{A,\eta}^{\otimes-1}.$$

(This is a formal consequence of (4.3.1.10) and (4.3.1.11).)

3. *If we have (symbolically)*
$$\tau(y, \chi_1 + \chi_2) = \tau(y, \chi_1)\tau(y, \chi_2) \tag{4.3.1.12}$$
(under the biextension structure of $\mathcal{P}_{A,\eta}^{\otimes -1}$ as in Section 4.2.2) for all $\chi_1, \chi_2 \in X$ and $y \in Y$, then we have (symbolically)
$$\tau(y_1 + y_2, \chi) = \tau(y_1, \chi)\tau(y_2, \chi) \tag{4.3.1.13}$$
(under the biextension structure of $\mathcal{P}_{A,\eta}^{\otimes -1}$ as in Section 4.2.2) for all $\chi \in X$ and $y_1, y_2 \in Y$. (Note that (4.3.1.12) has to be proved independently later.)

4. *For all but finitely many $y \in Y$, the section $\psi(y)$ extends to a section of $\mathcal{M}(c^\vee(y))^{\otimes -1}$ and is congruent to zero modulo I.*

   This is a special case of the stronger statement: For each integer $n > 0$, for all but finitely many $y \in Y$, the section $\psi(y)$ extends to a section of (the invertible sheaf) $\mathcal{M}(c^\vee(y))^{\otimes -1}$ and is congruent to zero modulo I^n. (This is the positivity condition for ψ; see Definition 4.2.1.11.)

5. *For all nonzero $y \in Y$, the section $\tau(y, \phi(y))$ extends to a section of (the invertible sheaf) $(c^\vee(y) \times c\phi(y))^*\mathcal{P}_A^{\otimes -1}$ and is congruent to zero modulo I. (This is the positivity condition for τ; see Definition 4.2.1.10.)*

Proof. Claims 1 and 2 can be verified as follows: Consider the relations
$$\begin{aligned} &\psi(y_1 + y_2)\, T^*_{c^\vee(y_1)+c^\vee(y_2)} \circ \sigma_0 \\ &= \psi(y_1 + y_2)\tau(y_1 + y_2, 0)\, T^*_{c^\vee(y_1)+c^\vee(y_2)} \circ \sigma_0 = \sigma_{\phi(y_1+y_2)} \end{aligned}$$
and
$$\begin{aligned} &\psi(y_1)\psi(y_2)\tau(y_1, \phi(y_2))\, T^*_{c^\vee(y_1)+c^\vee(y_2)} \circ \sigma_0 \\ &= \psi(y_1)\tau(y_1, \phi(y_2))\psi(y_2)\tau(y_2, 0)\, T^*_{c^\vee(y_1)} T^*_{c^\vee(y_2)} \circ \sigma_0 \\ &= \psi(y_1)\tau(y_1, \phi(y_2))\, T^*_{c^\vee(y_2)}\sigma_{\phi(y_2)} = \sigma_{\phi(y_1)+\phi(y_2)}. \end{aligned}$$
Then the uniqueness of $\psi(y, \chi) = \psi(y)\tau(y, \chi)$ implies (4.3.1.10).

Claim 3 can be verified as follows: Consider the relations
$$\psi(y_1 + y_2)\tau(y_1 + y_2, \chi)\, T^*_{c^\vee(y_1)+c^\vee(y_2)} \circ \sigma_\chi = \sigma_{\chi+\phi(y_1+y_2)}$$
and
$$\begin{aligned} &\psi(y_1)\psi(y_2)\tau(y_1, \chi + \phi(y_2))\tau(y_2, \chi)\, T^*_{c^\vee(y_1)+c^\vee(y_2)} \circ \sigma_\chi \\ &= \psi(y_1)\tau(y_1, \chi + \phi(y_2))\psi(y_2)\tau(y_2, \chi)\, T^*_{c^\vee(y_1)} T^*_{c^\vee(y_2)} \circ \sigma_\chi \\ &= \psi(y_1)\tau(y_1, \chi + \phi(y_2))\, T^*_{c^\vee(y_2)}\sigma_{\chi+\phi(y_2)} = \sigma_{\chi+\phi(y_1)+\phi(y_2)}. \end{aligned}$$
Then the uniqueness of $\psi(y, \chi) = \psi(y)\tau(y, \chi)$ and (4.3.1.12) imply
$$\begin{aligned} &\psi(y_1 + y_2)\tau(y_1 + y_2, \chi) = \psi(y_1)\psi(y_2)\tau(y_1, \chi + \phi(y))\tau(y_2, \chi) \\ &= \psi(y_1)\psi(y_2)\tau(y_1, \chi)\tau(y_1, \phi(y_2))\tau(y_2, \chi). \end{aligned}$$
By cancellation using (4.3.1.10), we obtain (4.3.1.13).

Claims 4 and 5 can be verified as follows: It suffices to establish the positivity condition for τ, as the equivalence between the positivity conditions for τ and for ψ has already been established in Section 4.2.4.

By (4.3.1.2), there exists $s_0 \in \Gamma(G_\eta, \mathcal{L}_\eta)$ such that $\sigma_0(s_0) \neq 0$. Since $\Gamma(G_\eta, \mathcal{L}_\eta) \cong \Gamma(G, \mathcal{L}) \underset{R}{\otimes} K$, we may and we shall assume that $s_0 \in \Gamma(G, \mathcal{L})$ instead. Then we have $T^*_{c^\vee(y)} \circ \sigma_0(s_0) \neq 0$ for all $y \in Y$. On the other hand, we have $\sigma_{\phi(y)}(s_0) = \psi(y)\, T^*_{c^\vee(y)} \circ \sigma_0(s_0) \in \Gamma(A, \mathcal{M}_{\phi(y)})$ for all $y \in Y$. As a result, we have $\sigma_0(s_0) \in \Gamma(A, \mathcal{M}_0) \underset{R}{\otimes} I_y^{\otimes -1}$ for all $y \in Y$.

Let us fix a $y \neq 0$ in Y. Suppose there is a discrete valuation $\upsilon \in \Upsilon_1$ such that $\upsilon(\tau(y, \phi(y))) < 0$. Then, by (4.2.4.7), we have $\lim\limits_{k\to\infty} \upsilon(\psi(ky)) \to -\infty$, and hence $\sigma_0(s_0) \in \Gamma(A, \mathcal{M}_0) \underset{R}{\otimes} \mathfrak{m}_\upsilon^N$ for all $N > 0$. Since $\Gamma(A, \mathcal{M}_0)$ is a finitely generated R-module, this is possible only when $\sigma_0(s_0) = 0$, which is a contradiction. Thus we see that $\upsilon(\tau(y, \phi(y))) \geq 0$ for every $\upsilon \in \Upsilon_1$, which implies that $I_{y,\phi(y)} \subset R$ by noetherian normality of R. In other words, the section $\tau(y, \phi(y))$ of $\mathscr{O}_{\phi(y)}(c^\vee(y))_\eta^{\otimes -1}$ extends to a section of $\mathscr{O}_{\phi(y)}(c^\vee(y))^{\otimes -1} \cong (c^\vee(y) \times c\phi(y))^*\mathcal{P}_A^{\otimes -1}$. This shows the first half of the positivity condition for τ.

For the second half, suppose that there is a $y \neq 0$ in Y such that $\tau(y, \phi(y))$ is not congruent to zero modulo I. Then $\upsilon(\tau(y, \phi(y))) = 0$ for some $\upsilon \in \Upsilon_I$, and hence $\upsilon(\psi(ky)) = k\upsilon(\psi(y))$ for all $k \in \mathbb{Z}$. Let $i > 0$ be an integer such that symbolically $\sigma_0(s_0) \not\equiv 0 \pmod{\mathfrak{m}_\upsilon^{i+1}}$. That is, the image of $\sigma_0(s_0)$ under the pullback $\Gamma(A, \mathcal{M}) \to \Gamma(A_{\upsilon,i}, \mathcal{M}_{\upsilon,i})$ is nonzero, where $A_{\upsilon,i}$ and $\mathcal{M}_{\upsilon,i}$ are the pullbacks of A and $\mathcal{M}$ to $S_{\upsilon,i} = \mathrm{Spec}(R_\upsilon/\mathfrak{m}_\upsilon^{i+1})$, respectively. After composition with the pullback of the isomorphism $T^*_{c^\vee(ky)}$ to $S_{\upsilon,i}$, we have symbolically $T^*_{c^\vee(ky)} \circ \sigma_0(s_0) \not\equiv 0 \pmod{\mathfrak{m}_\upsilon^{i+1}}$. Since $\sigma_{\phi(ky)}(s_0) = \psi(ky)\, T^*_{c^\vee(ky)} \circ \sigma_0(s_0)$, we have $\upsilon(\sigma_{\phi(ky)}(s_0)) = \upsilon(\sigma_0(s_0)) + k\upsilon(\psi(y))$ for all $k \in \mathbb{Z}$. In particular, there exist infinitely many k such that $\sigma_{\phi(ky)}(s_0) \not\equiv 0 \pmod{\mathfrak{m}_\upsilon^{i+1}}$. This implies that there exist infinitely many k such that $\sigma_{\phi(ky)}(s_0) \not\equiv 0 \pmod{I^{i+1}}$, which contradicts the fact that $s_0 \equiv \sum\limits_{\chi\in X} \sigma_\chi(s_0) \pmod{I^{i+1}}$ is a finite sum for all i. This shows the full positivity condition for τ. □

Lemma 4.3.1.14. *The definition of τ is independent of the $\mathcal{M}$ we choose.*

Proof. If we replace $\mathcal{M}$ above with $\mathcal{M}' := \mathcal{M}_{\chi_0} = \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi_0}$ for some $\chi_0 \in \underline{X}$, then the indices of σ_χ are all shifted by χ_0. As a result, if we denote the new τ by τ', then $\psi(y)\tau(y, \chi) = \psi'(y)\tau'(y, \chi - \chi_0)$ for every χ. Hence $\tau(y, \chi - \chi') = \tau'(y, \chi - \chi')$ for every two $\chi, \chi' \in X$. This shows that $\tau = \tau'$. □

Lemma 4.3.1.15. *The definition of ψ as a cubical trivialization of $\iota^*(\mathcal{L}_\eta^\natural)^{\otimes -1}$ (rather than $(c^\vee)^*\mathcal{M}_\eta^{\otimes -1}$) is independent of the $\mathcal{M}$ we choose. (This is called the **invariant formulation** in [42, Ch. II, §5].)*

Proof. Continuing the proof of Lemma 4.3.1.14, the cubical trivialization $\mathbf{1}_{\underline{Y},\eta} \xrightarrow{\sim} \iota^*(\mathcal{L}_\eta^\natural)^{\otimes -1}$ remains the same because it is not affected by shifting the indices of the σ_χ's by χ_0. □

Corollary 4.3.1.16. *Assuming the relations (4.3.1.2) and (4.3.1.7) above, the association $(G, \mathcal{L}) \mapsto (A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ (described thus far) defines a functor $\mathrm{F}_{\mathrm{ample}}(R, I) : \mathrm{DEG}_{\mathrm{ample}}(R, I) \to \mathrm{DD}_{\mathrm{ample}}(R, I)$.*

Convention 4.3.1.17. *The tuple $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ is called the **degeneration datum** associated with $(G, \mathcal{L})$.*

We will see variants of this usage later when we study other kinds of additional structures.

Now that we have seen the rather formal consequences of the definitions, the proof of Theorem 4.2.1.14 will be completed by verifying the relations (4.3.1.2) and (4.3.1.7), and Proposition 4.3.4.5 below, using the theory of theta representations.

To conclude, let us record the following observation:

Lemma 4.3.1.18. *For every section $g : S \to G$, if we replace $\mathcal{L}$ with $\mathcal{L}' := T_g^* \mathcal{L}$, then the τ and ψ defined by $\mathcal{L}$ and by $\mathcal{L}'$ remain the same.*

Proof. Let $g_{\mathrm{for}} : S_{\mathrm{for}} \to G_{\mathrm{for}} \cong G^{\natural}_{\mathrm{for}}$ be the formal section defined by the I-completion of $g : S \to G$, which induces a formal section $\pi_{\mathrm{for}}(g_{\mathrm{for}}) : S_{\mathrm{for}} \to A_{\mathrm{for}}$. This formal section of A_{for} algebraizes to a section $S \to A$, which we denote by $\pi(g)$ by abuse of notation. Then we have $(\mathcal{L}')^{\natural} \cong \pi^* \mathcal{M}'$ for $\mathcal{M}' := T^*_{\pi(g)} \mathcal{M}$, and we can translate the morphism $\sigma_{\chi}^{\mathcal{M}} : \Gamma(G, \mathcal{L}) \to \Gamma(A, \mathcal{M}_{\chi})$ to $\sigma_{\chi}^{\mathcal{M}'} : \Gamma(G, \mathcal{L}') \to \Gamma(A, \mathcal{M}'_{\chi})$, where $\mathcal{M}'_{\chi} := \mathcal{M}' \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi}$. Since the definitions of τ and ψ are given by comparing σ_{χ} with $\sigma_{\chi + \phi(y)}$, they remain the same if we replace $\mathcal{L}$ and $\mathcal{M}$ with $\mathcal{L}'$ and $\mathcal{M}'$ (by translations), respectively. $\square$

4.3.2 Relations between Theta Representations

In this section we use the uniqueness of irreducible theta representations (which are algebro-geometric analogues of Heisenberg group representations) to deduce (4.3.1.7) from (4.3.1.2). Since (4.3.1.7) is an equality, in order to prove it, we may localize and make the convenient assumption that R is complete *local*.

By Theorem 3.4.2.4, the group $K(\mathcal{L}_{\eta})$ extends to a quasi-finite flat subgroup scheme $K(\mathcal{L})$ in G over S. The group scheme $K(\mathcal{L})$ has the *finite part* $K(\mathcal{L})^{\mathrm{f}}$, which is the largest finite subscheme of $K(\mathcal{L})$, and the *torus part* $K(\mathcal{L})^{\mu}$, which is isomorphic to $K(\mathcal{L})^{\flat} = K(\mathcal{L})^{\natural} \cap T$, a subgroup of T. The pairing $\mathrm{e}^{\mathcal{L}_{\eta}} : K(\mathcal{L}_{\eta}) \underset{\eta}{\times} K(\mathcal{L}_{\eta}) \to \mathbf{G}_{\mathrm{m},\eta}$ extends to a pairing $\mathrm{e}^{\mathcal{L}} := \mathrm{e}_S^{\mathcal{L}_{\eta}} : K(\mathcal{L}) \underset{S}{\times} K(\mathcal{L}) \to \mathbf{G}_{\mathrm{m},S}$, which can be identified with the commutator pairing of the central extension structure

$$0 \to \mathbf{G}_{\mathrm{m},S} \to \mathcal{G}(\mathcal{L}) \to K(\mathcal{L}) \to 0. \tag{4.3.2.1}$$

(Here $\mathcal{G}(\mathcal{L}) \cong \mathcal{L}|_{K(\mathcal{L})}$ by Proposition 3.2.4.2.) Under this commutator pairing, $K(\mathcal{L})^{\mu}$ is totally isotropic, and $K(\mathcal{L})^{\mathrm{f}}$ is the annihilator of $K(\mathcal{L})^{\mu}$. Let $\mathcal{G}(\mathcal{L})^{\mu} := \mathcal{G}(\mathcal{L})|_{K(\mathcal{L})^{\mu}} \cong \mathcal{L}|_{K(\mathcal{L})^{\mu}}$ and $\mathcal{G}(\mathcal{L})^{\mathrm{f}} := \mathcal{G}(\mathcal{L})|_{K(\mathcal{L})^{\mathrm{f}}} \cong \mathcal{L}|_{K(\mathcal{L})^{\mathrm{f}}}$. Then we have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}(\mathcal{L})^{\mu} \arrow[r] \arrow[d, "\mathrm{can.}"'] & \mathcal{G}(\mathcal{L})^{\mathrm{f}} \arrow[r] \arrow[d, "\mathrm{can.}"'] & K(\mathcal{M}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K(\mathcal{L})^{\mu} \arrow[r] & K(\mathcal{L})^{\mathrm{f}} \arrow[r] & K(\mathcal{M}) \arrow[r] & 0
\end{tikzcd}$$

Now $K(\mathcal{L})^{\mu}$ being totally isotropic in $K(\mathcal{L})$ under $\mathrm{e}^{\mathcal{L}}$ implies that the extension (4.3.2.1) splits over $K(\mathcal{L})^{\mu}$, namely, there exists a splitting

$$K(\mathcal{L})^{\mu} \hookrightarrow \mathcal{G}(\mathcal{L})^{\mu} = \mathcal{G}(\mathcal{L})|_{K(\mathcal{L})^{\mu}}. \tag{4.3.2.2}$$

Among all possible splittings as above, there is a natural choice coming from the cubical isomorphism $s : i^* \mathcal{L}^{\natural} \cong \mathscr{O}_T$, which can be explained as follows: Recall (from (3.4.2.2)) that we have isomorphisms $K(\mathcal{L})^{\natural} \cong K(\mathcal{L})^{\mathrm{f}}$ and $K(\mathcal{L})^{\flat} \cong K(\mathcal{L})^{\mu}$ between finite flat group schemes over S, where $K(\mathcal{L})^{\natural} \subset G^{\natural}$, $K(\mathcal{L})^{\flat} \subset T \subset G^{\natural}$, $K(\mathcal{L})^{\mathrm{f}} \subset G$, and $K(\mathcal{L})^{\mu} \subset G$. Using the canonical isomorphisms $\mathcal{L}^{\natural}_{\mathrm{for}} \cong \mathcal{L}_{\mathrm{for}}$ and $K(\mathcal{L})^{\natural}_{\mathrm{for}} \cong K(\mathcal{L})^{\mathrm{f}}_{\mathrm{for}}$ over S_{for}, we obtain a canonical isomorphism $\mathcal{L}^{\natural}|_{K(\mathcal{L})^{\natural}_{\mathrm{for}}} \cong \mathcal{L}|_{K(\mathcal{L})^{\mathrm{f}}_{\mathrm{for}}}$, which by Theorem 2.3.1.2 (using finiteness of $K(\mathcal{L})^{\natural} \cong K(\mathcal{L})^{\mathrm{f}}$) algebraizes uniquely to a canonical isomorphism $\mathcal{L}^{\natural}|_{K(\mathcal{L})^{\natural}} \cong \mathcal{L}|_{K(\mathcal{L})^{\mathrm{f}}}$, inducing a canonical isomorphism $\mathcal{L}^{\natural}|_{K(\mathcal{L})^{\flat}} \cong \mathcal{L}|_{K(\mathcal{L})^{\mu}}$ by restriction. The cubical isomorphism s above can be interpreted as an isomorphism $s : \mathcal{L}^{\natural}|_T \cong \mathbf{G}_{\mathrm{m},T}$ of cubical $\mathbf{G}_{\mathrm{m}}$-torsors over T. Since $K(\mathcal{L})^{\mu} \subset T$, we obtain by restriction an isomorphism $s|_{K(\mathcal{L})^{\flat}} : \mathcal{L}^{\natural}|_{K(\mathcal{L})^{\flat}} \cong \mathbf{G}_{\mathrm{m},K(\mathcal{L})^{\flat}}$. This gives us a natural choice of a splitting of $\mathcal{L}^{\natural}|_{K(\mathcal{L})^{\flat}}$ over $K(\mathcal{L})^{\flat}$, and hence a natural choice of (4.3.2.2) via the canonical isomorphism $\mathcal{L}^{\natural}|_{K(\mathcal{L})^{\flat}} \cong \mathcal{L}|_{K(\mathcal{L})^{\mu}}$ above. Let us fix this choice of the splitting from now on. We say that this is the choice *compatible with the cubical trivialization $s : i^* \mathcal{L}^{\natural} \cong \mathscr{O}_T$*.

The set of choices of the splitting (4.3.2.2) form a torsor under the group of group homomorphisms $\mathrm{Hom}_S(K(\mathcal{L})^{\mu}, \mathbf{G}_{\mathrm{m},S})$, namely, the character group of $K(\mathcal{L})^{\mu}$. Since $K(\mathcal{L})^{\mu} \cong K(\mathcal{L})^{\flat} = K(\mathcal{L})^{\natural} \cap T$ is the kernel of $T \twoheadrightarrow T^{\vee}$ induced by $\lambda^{\natural} : G^{\natural} \to G^{\vee,\natural}$, and since $T \twoheadrightarrow T^{\vee}$ is by definition dual to $\phi : Y \to X$, we can identify the character group of $K(\mathcal{L})^{\mu}$ with $X/\phi(Y)$ such that the canonical embedding $K(\mathcal{L})^{\mu} \hookrightarrow T$ is dual to the canonical surjection $X \to X/\phi(Y)$.

Since $\Gamma(G, \mathcal{L})$ is a representation of $\mathcal{G}(\mathcal{L})$, we have an action of $K(\mathcal{L})^{\mu}$ on $\Gamma(G, \mathcal{L})$ via the above-chosen splitting (4.3.2.2). Since $K(\mathcal{L})^{\mu}$ is of multiplicative type, the representation $\Gamma(G, \mathcal{L})$ can be decomposed according to the character group $X/\phi(Y)$ of $K(\mathcal{L})^{\mu}$. Hence we can write

$$\Gamma(G, \mathcal{L}) \cong \bigoplus_{\bar{\chi} \in X/\phi(Y)} \Gamma(G, \mathcal{L})_{\bar{\chi}},$$

where $\Gamma(G, \mathcal{L})_{\bar{\chi}}$ is the *weight-$\bar{\chi}$ subspace* of $\Gamma(G, \mathcal{L})$ under the action of $K(\mathcal{L})^{\mu}$. Note that this depends on the choice of the splitting (4.3.2.2).

Lemma 4.3.2.3. *The weight subspaces under the actions of $K(\mathcal{L})^{\mu}$ and T are compatible in the sense that, for each $\chi \in X$ and $\bar{\chi} = \chi + \phi(Y)$, we have*

$$\Gamma(G, \mathcal{L})_{\bar{\chi}} \hookrightarrow \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})_{\bar{\chi}} \cong \hat{\bigoplus_{\chi' \in \chi + \phi(Y)}} \Gamma(A, \mathcal{M}_{\chi'})$$

under the canonical morphisms

$$\Gamma(G, \mathcal{L}) \hookrightarrow \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) \cong \hat{\bigoplus_{\chi' \in X}} \Gamma(A, \mathcal{M}_{\chi'}).$$

Proof. Since weight subspaces under both actions are determined by the cubical trivialization $s : i^* \mathcal{L}_{\mathrm{for}} \cong \mathscr{O}_{T_{\mathrm{for}}}$, which algebraizes to a cubical trivialization $i^* \mathcal{L}^{\natural} \cong \mathscr{O}_T$ equivalent to an algebraic T-action on the sections of $\mathcal{L}^{\natural}$, the lemma follows because the restriction of T to $K(\mathcal{L})^{\flat}$ induces the canonical homomorphism $X \to X/\phi(Y) : \chi \mapsto \bar{\chi} = \chi + \phi(Y)$. $\square$

The fact that $K(\mathcal{L})^{\mu}$ is totally isotropic under the pairing $\mathrm{e}_S^{\mathcal{L}}$ implies that $\mathcal{G}(\mathcal{L})^{\mu}$ is commutative. Moreover, the choice (4.3.2.2) of a splitting of $\mathcal{G}(\mathcal{L})^{\mu} \to K(\mathcal{L})^{\mu}$ gives us an action of $K(\mathcal{L})^{\mu}$ on $\Gamma(G, \mathcal{L})$, as we saw above. Let

$$h_{\bar{\chi}} : \mathcal{G}(\mathcal{L})^{\mu} \to \mathbf{G}_{\mathrm{m},S}$$

be the group scheme homomorphism that is the identity homomorphism on $\mathbf{G}_{\mathrm{m},S}$ and is $\bar{\chi}$ on $K(\mathcal{L})^{\mu}$ via the above-chosen splitting, so that $h_{\bar{\chi}}$ reflects the character of the (commutative) action of $\mathcal{G}(\mathcal{L})^{\mu}$ on $\Gamma(G, \mathcal{L})$. Then the push-out

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{G}(\mathcal{L})^{\mu} \arrow[r] \arrow[d, "h_{\bar{\chi}}"'] & \mathcal{G}(\mathcal{L})^{\mathrm{f}} \arrow[r] \arrow[d, dotted] & K(\mathcal{M}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathbf{G}_{\mathrm{m},S} \arrow[r] & \mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}} \arrow[r] & K(\mathcal{M}) \arrow[r] & 0
\end{tikzcd}$$

by $h_{\bar{\chi}} : \mathcal{G}(\mathcal{L})^{\mu} \to \mathbf{G}_{\mathrm{m},S}$ defines a group scheme $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$.

Lemma 4.3.2.4. *The push-out $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$ is naturally isomorphic to*
$$0 \to \mathbf{G}_{\mathrm{m},S} \to \mathcal{G}(\mathcal{M}_\chi) \to K(\mathcal{M}_\chi) \to 0$$
as extensions of $K(\mathcal{M}) = K(\mathcal{M}_\chi)$ by $\mathbf{G}_{\mathrm{m},S}$.

Proof. Given the chosen cubical trivialization $s : i^*\mathcal{L}^\natural \cong \mathscr{O}_T$ or $s : \mathcal{L}^\natural|_T \cong \mathbf{G}_{\mathrm{m},T}$, we obtain a splitting of $\mathcal{L}^\natural|_T$ over T given by the identity section of $\mathbf{G}_{\mathrm{m},T}$, and all the other possible cubical trivializations can be identified with the character group of T. Since the cubical isomorphism $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$ restricts to s over T, the $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{M}$ over A can be identified with the weight-0 subsheaf of $\pi_*\mathcal{L}$ under the T-action (defined by the splitting from T to $\mathbf{G}_{\mathrm{m},T} \cong \mathcal{L}^\natural|_T$), or the invariant subsheaf of $\mathcal{L}$ under the T-action described above. If the splitting is modified by adding a character $-\chi$ of T to the splitting of $\mathbf{G}_{\mathrm{m},T}$ over T, which we denote by $s_\chi : i^*\mathcal{L}^\natural \cong \mathscr{O}_T$, then T acts by $-\chi$ on $\mathcal{M}$ as a subsheaf of $\mathcal{L}^\natural$, which implies that the invariant subsheaf should be given by $\mathcal{M}_\chi = \mathscr{O}_\chi \underset{\mathscr{O}_A}{\otimes} \mathcal{M}$, together with an isomorphism $\mathcal{L}^\natural \cong \pi^*\mathcal{M}_\chi$ that restricts to s_χ over T.

If we restrict s_χ to $s_\chi|_{K(\mathcal{L})^\flat} : \mathcal{L}^\natural|_{K(\mathcal{L})^\flat} \cong \mathbf{G}_{\mathrm{m},K(\mathcal{L})^\flat}$, then we obtain a splitting of $\mathcal{G}(\mathcal{L})^\mu$ over $K(\mathcal{L})^\mu$, compatible with the cubical trivialization s_χ (instead of s). Note that this splitting maps $K(\mathcal{L})^\mu_\eta$ isomorphically onto the kernel of $h_{\bar{\chi}}$ over the generic fiber. The group $K(\mathcal{L})^\flat$ is the kernel of the homomorphism $\bar{\lambda} : G^\natural \to \lambda_A^*(G^{\vee,\natural}) = G^{\vee,\natural} \underset{A^\vee,\lambda_A}{\times} A$ induced by $\lambda^\natural : G^\natural \to G^{\vee,\natural}$. These homomorphisms fit into the following commutative diagram.
$$\begin{tikzcd}
0 \arrow[r] & T \arrow[r] \arrow[d, "\text{dual to } \phi"'] & G^\natural \arrow[r] \arrow[d, dotted, "\bar{\lambda}"] \arrow[dd, dotted, bend right, "\lambda^\natural"'] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T^\vee \arrow[r] \arrow[d, equal] & \lambda_A^*(G^{\vee,\natural}) \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\lambda_A"] & 0 \\
0 \arrow[r] & T^\vee \arrow[r] & G^{\vee,\natural} \arrow[r] & A^\vee \arrow[r] & 0
\end{tikzcd}$$
Each cubical trivialization s_χ as above corresponds to a cubical $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}^\natural_{\bar{\chi}}$ over $\lambda_A^*(G^{\vee,\natural})$, together with a cubical isomorphism $\mathcal{L}^\natural \cong \bar{\lambda}^*\mathcal{L}^\natural_{\bar{\chi}}$ that restricts to $s_\chi|_{K(\mathcal{L})^\flat}$ over $K(\mathcal{L})^\flat$. (This is just finite flat descent, which is simpler than, but certainly consistent with, the descent used in proving Proposition 3.2.5.4 in [57, VIII, 3.4] and [93, I, 7.2].) We can interpret $\mathcal{L}^\natural_{\bar{\chi}}$ as the subsheaf of $\bar{\lambda}_*\mathcal{L}^\natural$ on which $K(\mathcal{L})^\flat$ acts by $\bar{\chi}$.

Let $\bar{\pi} : \lambda_A^*(G^{\vee,\natural}) \to A$ be the structural morphism. The compatibility of the splitting of $\mathcal{G}(\mathcal{L})^\mu$ over $K(\mathcal{L})^\mu$ with s_χ gives us cubical isomorphisms $\mathcal{L}^\natural \cong \bar{\lambda}^*\mathcal{L}^\natural_{\bar{\chi}}$, $\mathcal{L}^\natural_{\bar{\chi}} \cong \bar{\pi}^*\mathcal{M}_\chi$, and $\mathcal{L}^\natural \cong \pi^*\mathcal{M}_\chi$ such that the cubical isomorphism $\mathcal{L}^\natural \cong \bar{\lambda}^*\bar{\pi}^*\mathcal{M}_\chi$ induced by the first two cubical isomorphisms agrees with the third one. By restriction, we obtain a commutative diagram
$$\begin{tikzcd}
\mathcal{L}^\natural|_{K(\mathcal{L})^\natural} \arrow[rr] \arrow[dr] & & \mathcal{L}^\natural_{\bar{\chi}}|_{K(\mathcal{L})^\natural/K(\mathcal{L})^\flat} \arrow[dl] \\
& \mathcal{G}(\mathcal{M}_\chi) = \mathcal{M}_\chi|_{K(\mathcal{M}_\chi)} &
\end{tikzcd}$$
of group schemes, where $K(\mathcal{L})^\natural/K(\mathcal{L})^\flat$ is identified with the image of $K(\mathcal{L})^\natural$ in $\lambda_A^*(G^{\vee,\natural})$ (under $\bar{\lambda}$). The rigidifications give compatible homomorphisms from $\mathbf{G}_{\mathrm{m},S}$ to these group schemes, inducing a commutative diagram
$$\begin{tikzcd}
0 \arrow[r] & \mathbf{G}_{\mathrm{m},S} \arrow[r] \arrow[d, equal] & \mathcal{L}^\natural_{\bar{\chi}}|_{K(\mathcal{L})^\natural/K(\mathcal{L})^\flat} \arrow[r] \arrow[d] & K(\mathcal{L})^\natural/K(\mathcal{L})^\flat \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \mathbf{G}_{\mathrm{m},S} \arrow[r] & \mathcal{G}(\mathcal{M}_\chi) \arrow[r] & K(\mathcal{M}_\chi) \arrow[r] & 0
\end{tikzcd}$$
in which the middle vertical arrow is forced to be an isomorphism. The group scheme $\mathcal{L}^\natural_{\bar{\chi}}|_{K(\mathcal{L})^\natural/K(\mathcal{L})^\flat}$ can be identified with the quotient of $\mathcal{L}^\natural|_{K(\mathcal{L})^\natural} \cong \mathcal{L}|_{K(\mathcal{L})^{\mathrm{f}}} = \mathcal{G}(\mathcal{L})^{\mathrm{f}}$ by $\ker(h_{\bar{\chi}})$, which is the push-out $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$. Therefore $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$ is naturally isomorphic to $\mathcal{G}(\mathcal{M}_\chi)$, as desired. ☐

Since $\mathcal{G}(\mathcal{L})^{\mathrm{f}}$ is the annihilator of $\mathcal{G}(\mathcal{L})^\mu$ under the commutator pairing of $\mathcal{G}(\mathcal{L})$, or equivalently since $K(\mathcal{L})^{\mathrm{f}}$ is the annihilator of $K(\mathcal{L})^\mu$ under the pairing $\mathrm{e}^{\mathcal{L}}$ induced by the commutator pairing of $\mathcal{G}(\mathcal{L})$, we may interpret $\Gamma(G,\mathcal{L})_{\bar{\chi}}$ as a $\mathcal{G}(\mathcal{L})^{\mathrm{f}}$-invariant submodule of $\Gamma(G,\mathcal{L})$. Therefore the push-out group scheme $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$ acts naturally on $\Gamma(G,\mathcal{L})_{\bar{\chi}}$, because $\ker(h_{\bar{\chi}})$ acts trivially, and because push-out by $h_{\bar{\chi}}$ just means forming the quotient by $\ker(h_{\bar{\chi}})$.

Lemma 4.3.2.5. *Under the identification $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}} \cong \mathcal{G}(\mathcal{M}_\chi)$ given by Lemma 4.3.2.4, the morphism $\sigma_\chi : \Gamma(G,\mathcal{L})_{\bar{\chi}} \to \Gamma(A,\mathcal{M}_\chi)$ is $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}} \cong \mathcal{G}(\mathcal{M}_\chi)$-equivariant.*

Proof. By definition, the morphism σ_χ above factors through
$$\begin{aligned}
&\Gamma(G,\mathcal{L})_{\bar{\chi}} \hookrightarrow \Gamma(G_{\mathrm{for}},\mathcal{L}_{\mathrm{for}})_{\bar{\chi}} \cong \Gamma(\lambda_A^*(G^{\vee,\natural})_{\mathrm{for}},\mathcal{L}^\natural_{\bar{\chi},\mathrm{for}}) \\
&\cong \hat{\bigoplus_{y\in Y}} \Gamma(A_{\mathrm{for}},\mathcal{M}_{\chi+\phi(y),\mathrm{for}}) \to \Gamma(A_{\mathrm{for}},\mathcal{M}_{\chi,\mathrm{for}}) \cong \Gamma(A,\mathcal{M}_\chi),
\end{aligned} \tag{4.3.2.6}$$
where the first inclusion is $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$-equivariant.

Since $\mathcal{D}_2(\mathcal{L}^\natural_{\bar{\chi}})$ (together with its canonical trivialization descended from $\mathcal{D}_2(\mathcal{L}^\natural)$) descends down to $\mathcal{D}_2(\mathcal{M}_\chi)$, the diagram
$$\begin{tikzcd}[column sep=small]
(\mathcal{L}^\natural_{\bar{\chi}}|_{K(\mathcal{L})^{\mathrm{f}}/K(\mathcal{L})^\mu}) \underset{S}{\times} \mathcal{L}^\natural_{\bar{\chi}} \arrow[rrr] \arrow[dr] \arrow[dd] & & & \mathcal{L}^\natural_{\bar{\chi}} \arrow[dr] \arrow[dd] & \\
& (\mathcal{M}_\chi|_{K(\mathcal{M}_\chi)}) \underset{S}{\times} \mathcal{M}_\chi \arrow[rrr] \arrow[dd] & & & \mathcal{M}_\chi \arrow[dd] \\
(K(\mathcal{L})^{\mathrm{f}}/K(\mathcal{L})^\mu) \underset{S}{\times} \lambda_A^*(G^{\vee,\natural}) \arrow[rrr] \arrow[dr] & & & \lambda_A^*(G^{\vee,\natural}) \arrow[dr] & \\
& K(\mathcal{M}_\chi) \underset{S}{\times} A \arrow[rrr] & & & A
\end{tikzcd} \tag{4.3.2.7}$$
is commutative. By identifying $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}} \cong \mathcal{L}^\natural_{\bar{\chi}}|_{K(\mathcal{L})^{\mathrm{f}}/K(\mathcal{L})^\mu}$ with $\mathcal{G}(\mathcal{M}_\chi) \cong \mathcal{M}_\chi|_{K(\mathcal{M}_\chi)}$ as in the proof of Lemma 4.3.2.4, we may interpret the two rectangles in the diagram (4.3.2.7) as describing the respective actions of $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}}$ and $\mathcal{G}(\mathcal{M}_\chi)$ on $\Gamma(G,\mathcal{L})_{\bar{\chi}}$ and

$\Gamma(A, \mathcal{M}_\chi)$. Then the $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}} \cong \mathcal{G}(\mathcal{M}_\chi)$-equivariance of (4.3.2.6) follows from the commutativity of the diagram (4.3.2.7), as desired. $\square$

Let us pullback everything to the generic point η. Then we obtain an equivariant morphism

$$\sigma_\chi : \Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}} \to \Gamma(A_\eta, \mathcal{M}_{\chi,\eta}) \tag{4.3.2.8}$$

between two representations of the *same* group.

Lemma 4.3.2.9. *Both representations in (4.3.2.8) are nonzero and irreducible. As a result, the morphism σ_χ in (4.3.2.8) is an intertwining operator between two irreducible representations, which is either zero or unique up to a nonzero scalar multiple in K.*

Proof. For every $\chi \in X$, since $\mathcal{L}^\natural_{\bar{\chi}}$ is the subsheaf of $\bar{\lambda}_* \mathcal{L}^\natural$ on which $K(\mathcal{L})^\flat$ acts by $\bar{\chi}$, we have $\Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}} \cong \Gamma((\lambda_A^*(G^{\vee,\natural}))_\eta, (\mathcal{L}^\natural_{\bar{\chi}})_\eta) \neq 0$. Then $\dim_K \Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}} \geq \dim_K \Gamma(A_\eta, \mathcal{L}_{\chi,\eta})$ because $\Gamma(A_\eta, \mathcal{M}_{\chi,\eta})$ is an irreducible representation of $\mathcal{G}(\mathcal{M}_{\chi,\eta})$ by [93, VI, §2]. By the Riemann–Roch theorem [94, §16], $\dim_K \Gamma(A_\eta, \mathcal{M}_{\chi,\eta}) = \deg(\lambda_A)^{1/2}$. By Lemma 3.4.4.3, $\dim_K \Gamma(G_\eta, \mathcal{L}_\eta) = \deg(\lambda_G)^{1/2} = [X : \phi(Y)] \deg(\lambda_A)^{1/2}$. Since the two sides of the inequality $\dim_K \Gamma(G_\eta, \mathcal{L}_\eta) = \sum\limits_{\bar{\chi} \in X/\phi(Y)} \dim_K \Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}} \geq \sum\limits_{\bar{\chi} \in X/\phi(Y)} \dim_K \Gamma(A_\eta, \mathcal{M}_{\chi,\eta})$ are equal (where for $\Gamma(A_\eta, \mathcal{M}_{\chi,\eta})$ we can take any representative χ of $\bar{\chi}$), we must have $\dim_K \Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}} = \dim_K \Gamma(A_\eta, \mathcal{L}_{\chi,\eta})$ for each $\chi \in X$, as desired. $\square$

Proposition 4.3.2.10. *If (4.3.1.2) (i.e., $\sigma_\chi \neq 0$) is true for every $\chi \in X$, then (4.3.1.3) is true.*

Proof. For each $\chi \in X$, by (4.3.2.6) then $\sigma_\chi = 0$ on $\Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}'}$ if $\bar{\chi}' \neq \bar{\chi}$ in $X/\phi(Y)$. Since $\sigma_\chi \neq 0$ on the whole of $\Gamma(G_\eta, \mathcal{L}_\eta)$, the induced morphism (4.3.2.8) is nonzero. By Lemma 4.3.2.9, we see that σ_χ is a nonzero intertwining operator between two irreducible representations, which is unique up to a nonzero multiple in K.

By restricting the two morphisms $T^*_{c^\vee(y)} \circ \sigma_\chi : \Gamma(G_\eta, \mathcal{L}_\eta) \to \Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta}) \underset{K}{\otimes} \mathcal{M}(c^\vee(y))_\eta$ and $\sigma_{\chi+\phi(y)} : \Gamma(G_\eta, \mathcal{L}_\eta) \to \Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta})$ to the *weight-$\bar{\chi}$ subspace* $\Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}} = \Gamma(G_\eta, \mathcal{L}_\eta)_{\overline{\chi+\phi(y)}}$, we obtain

$$T^*_{c^\vee(y)} \circ \sigma_\chi : \Gamma(G_\eta, \mathcal{L}_\eta)_{\bar{\chi}} \to \Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta}) \underset{K}{\otimes} \mathcal{M}(c^\vee(y))_\eta$$

and

$$\sigma_{\chi+\phi(y)} : \Gamma(G_\eta, \mathcal{L}_\eta)_{\overline{\chi+\phi(y)}} \to \Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta}),$$

respectively, both of which are nonzero equivariant homomorphisms between irreducible representations of $\mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\bar{\chi}} = \mathcal{G}(\mathcal{L})^{\mathrm{f}}_{\overline{\chi+\phi(y)}} \cong \mathcal{G}(\mathcal{M}_{\chi,\eta})$ by Lemma 4.3.2.9. Therefore they must be proportional (up to a nonzero multiple in K) if we identity the two spaces $\Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta}) \underset{K}{\otimes} \mathcal{M}(c^\vee(y))_\eta$ and $\Gamma(A_\eta, \mathcal{M}_{\chi+\phi(y),\eta})$ as K-vector spaces. This is equivalent to saying that there exists a section $\psi(y,\chi)$ of $\mathcal{M}(c^\vee(y))_\eta^{\otimes -1}$ satisfying $\psi(y,\chi)\ T^*_{c^\vee(y)} \circ \sigma_\chi = \sigma_{\chi+\phi(y)}$, which is just the desired relation (4.3.1.3). $\square$

The proof for the assumption that (4.3.1.2) (i.e., $\sigma_\chi \neq 0$) is true for all $\chi \in X$ will be given in the next section using the so-called *addition formula for theta functions*.

4.3.3 Addition Formulas

In this section, we introduce the addition formula for theta functions, and prove both (4.3.1.2) and (4.3.1.12).

Since (4.3.1.2) and (4.3.1.12) are about inequalities and equalities, after making base changes under continuous injections, we may and we will assume that R is a complete discrete valuation ring, with maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$. In this case, the normalizations of R in finite algebraic extensions of $K = \mathrm{Frac}(R)$ are again discrete valuation rings. Hence, we may and we will also make further base changes $R \to R'$ to finite flat extensions of complete discrete valuation rings whenever necessary.

Remark 4.3.3.1. The assumption that $S_0 = \mathrm{Spec}(k)$ is the spectrum of a field k is convenient for the following purpose. Later we will have to replace $\mathcal{L}$ with some other cubical invertible sheaf, and for $(G, \mathcal{L})$ to qualify as an object of $\mathrm{DEG}_{\mathrm{ample}}(R, I)$ (see Definition 4.2.1.1), we have to verify that $\mathcal{L}_{\mathrm{for}}$ lies in the essential image of (3.3.3.12). Let us claim that this is automatic (under the assumption that $S_0 = \mathrm{Spec}(k)$). By Corollary 3.2.5.7, there is a finite étale extension of S_0 over which $\mathcal{L}_0 \cong \pi_0^* \mathcal{M}_0$ for some $\mathcal{M}_0$ over A_0. As explained in the proof of Corollary 3.3.3.3, we have accordingly a finite formally étale extension of S_{for} over which $\mathcal{L}_{\mathrm{for}} \cong \pi_{\mathrm{for}}^* \mathcal{M}_{\mathrm{for}}$ for some $\mathcal{M}_{\mathrm{for}}$ over A_{for}. By Theorems 2.3.1.3 and 2.3.1.2, there is a finite étale extension of S over which $\mathcal{M}_{\mathrm{for}}$ algebraizes to some $\mathcal{M}$ over A. Hence $\mathcal{L}^\natural := \pi^* \mathcal{M}$ satisfies $\mathcal{L}^\natural_{\mathrm{for}} \cong \mathcal{L}_{\mathrm{for}}$ and descends to S, which shows that $\mathcal{L}_{\mathrm{for}}$ lies in the essential image of (3.3.3.12), as desired.

Definition 4.3.3.2. *Under the above assumptions, we say that two cubical invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ are **algebraically equivalent** if $\mathcal{N} := \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2^{\otimes -1}$ as a $\mathbf{G}_{\mathrm{m}}$-torsor has the structure of a commutative group scheme. In other words, $\mathcal{L}_1$ and $\mathcal{L}_2$ are algebraically equivalent over the generic fiber G_η in the usual sense of algebraic equivalence for invertible sheaves over an abelian variety. (By Proposition 3.3.2.2, it suffices to verify the statements over the generic fibers.)*

Let us consider the isogeny $\Phi : G \underset{S}{\times} G \to G \underset{S}{\times} G$ defined by $(x, y) \mapsto (x+y, x-y)$ for all functorial points x and y of G.

Lemma 4.3.3.3. *If $\mathcal{L}_1$ and $\mathcal{L}_2$ are algebraically equivalent cubical invertible sheaves, then we have a canonical isomorphism*

$$\Phi^*(\mathrm{pr}_1^* \mathcal{L}_1 \underset{\mathscr{O}_{G \underset{S}{\times} G}}{\otimes} \mathrm{pr}_2^* \mathcal{L}_2) \xrightarrow{\sim} \mathrm{pr}_1^*(\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2) \underset{\mathscr{O}_{G \underset{S}{\times} G}}{\otimes} \mathrm{pr}_2^*(\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} [-1]^* \mathcal{L}_2), \tag{4.3.3.4}$$

where pr_1 and pr_2 are the two projections.

Proof. By Proposition 3.3.2.2, it suffices to verify the statements over the generic fibers. If $\mathcal{L}_1 \cong \mathcal{L}_2$, then (4.3.3.4) follows from the pullback of the theorem of the cube under $G_\eta \underset{\eta}{\times} G_\eta \to G_\eta \underset{\eta}{\times} G_\eta \underset{\eta}{\times} G_\eta : (x, y) \mapsto (x, y, -y)$. Hence we may assume that $\mathcal{L}_1$ is trivial and that $\mathcal{L}_2$ is algebraically equivalent to the trivial invertible sheaf. Then (4.3.3.4) follows from the theorem of the square. $\square$

Lemma 4.3.3.5 (see [42, Ch. II, Lem. 4.3])**.** *Let $f : H' \to H$ be an isogeny of semi-abelian schemes over R such that both H_η and H'_η are abelian schemes, and such that both the torus parts of $H_0 = H \underset{R}{\otimes} k$ and $H'_0 = H' \underset{R}{\otimes} k$ are split tori. Let $H^\natural$ (resp. $(H')^\natural$) be the Raynaud extension of H (resp. H'), with torus part T (resp.*

T') and abelian part A (resp. A'). The isogeny f induces an isogeny

$$\begin{tikzcd} 0 \arrow[r] & T' \arrow[r, "i'"] \arrow[d, "f_T"'] & (H')^\natural \arrow[r, "\pi'"] \arrow[d, "f^\natural"'] & A' \arrow[r] \arrow[d, "f_A"'] & 0 \\ 0 \arrow[r] & T \arrow[r, "i"'] & H^\natural \arrow[r, "\pi"'] & A \arrow[r] & 0 \end{tikzcd}$$

between Raynaud extensions, where the homomorphism f_T between tori is dual to the homomorphism $f_T^* : \mathbf{X}(T) \to \mathbf{X}(T')$ between character groups.

Let $\mathcal{F}$ be a cubical invertible sheaf over H, and let $\mathcal{F}' := f^*\mathcal{F}$. Choose a trivialization $s : i^*\mathcal{F}^\natural \cong \mathscr{O}_T$, which determines a cubical invertible sheaf $\mathcal{N}$ over A such that $\mathcal{F}^\natural \cong \pi^*\mathcal{N}$. Then the pullback $s' := f_T^*(s)$ is a trivialization $s' : (i')^*(\mathcal{F}')^\natural \cong \mathscr{O}_{T'}$, which determines a cubical invertible sheaf $\mathcal{N}'$ over A' such that $(\mathcal{F}')^\natural \cong (\pi')^*(\mathcal{N}')$. Let the invertible sheaf $\mathcal{N}_\chi := \mathcal{N} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi$ (resp. $\mathcal{N}'_{\chi'} := \mathcal{N}' \underset{\mathscr{O}_{A'}}{\otimes} \mathscr{O}_{\chi'}$) be determined as usual for each $\chi \in \underline{\mathbf{X}}(T)$ (resp. $\chi' \in \underline{\mathbf{X}}(T')$) as the weight-$\chi$ subsheaf (resp. weight-χ' subsheaf) of $\pi_*\mathcal{F}^\natural$ (resp. $(\pi')_*(\mathcal{F}')^\natural$). Then we have the natural compatibility $\mathcal{N}'_{f_T^*(\chi)} \cong f_A^*\mathcal{N}_\chi$. In particular, $\mathcal{N}' \cong f_A^*\mathcal{N}$.

For each section s of $\Gamma(H, \mathcal{F})$, we have a decomposition $s = \sum\limits_{\chi \in \mathbf{X}(T)} \sigma_\chi^{\mathcal{N}}(s)$, where each $\sigma_\chi^{\mathcal{N}}(s)$ is an element in $\Gamma(A, \mathcal{N}_\chi)$. Similarly, for each section s' of $\Gamma(H', f^*\mathcal{F})$, we have a decomposition $s' = \sum\limits_{\chi' \in \mathbf{X}(T')} \sigma_{\chi'}^{\mathcal{N}'}(s')$, where each $\sigma_{\chi'}^{\mathcal{N}'}(s')$ is an element in $\Gamma(A', \mathcal{N}'_{\chi'})$. Then, with the compatible choices above, we have

$$f_A^*(\sigma_\chi^{\mathcal{N}}(s)) = \sigma_{f_T^*(\chi)}^{\mathcal{N}'}(f^*s),$$

where $f_A^* : \Gamma(A, \mathcal{N}_\chi) \to \Gamma(A', \mathcal{N}'_{f_T{}^*(\chi)})$ is the canonical morphism.

The proof of this lemma follows immediately from the definitions.

Applying this lemma to the isogeny $\Phi : G \underset{S}{\times} G \to G \underset{S}{\times} G$, we obtain the following proposition:

Proposition 4.3.3.6 (addition formula; see [42, Ch. II, p. 40])**.** *Let $\pi : G^\natural \to A$ be the structural morphism, and let $\mathcal{L}_1$ and $\mathcal{L}_2$ be algebraically equivalent cubical invertible sheaves over G such that $\mathcal{L}_1^\natural \cong \pi^*\mathcal{M}_1$ and $\mathcal{L}_2^\natural \cong \pi^*\mathcal{M}_2$ for some cubical invertible sheaves $\mathcal{M}_1$ and $\mathcal{M}_2$ over A. Let $\mathcal{F} := \mathrm{pr}_1^*\mathcal{L}_1 \underset{\mathscr{O}_{G \underset{S}{\times} G}}{\otimes} \mathrm{pr}_2^*\mathcal{L}_2$ and $\mathcal{F}' := \Phi^*\mathcal{F}$. Let $\mathcal{N} := \mathrm{pr}_1^*\mathcal{M}_1 \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^*\mathcal{M}_2$ and let $\mathcal{N}' := \Phi_A^*\mathcal{N}$, where Φ_A is the isogeny induced by Φ on the abelian part, given similarly by $(x, y) \mapsto (x + y, x - y)$ for all functorial points x and y of A. Then we have $\mathcal{F} \cong (\pi \times \pi)^*\mathcal{N}$ and $\mathcal{F}' \cong (\pi \times \pi)^*\mathcal{N}'$, and*

$$\Phi_A^*(\sigma_{(\chi,\mu)}^{\mathcal{N}}(\mathrm{pr}_1^* s_1 \otimes \mathrm{pr}_2^* s_2)) = \sigma_{(\chi+\mu,\chi-\mu)}^{\mathcal{N}'}(\Phi^*(\mathrm{pr}_1^* s_1 \otimes \mathrm{pr}_2^* s_2)) \tag{4.3.3.7}$$

for all $s_1, s_2 \in \Gamma(G, \mathcal{L})$ and $\chi, \mu \in X$.

Since $\sigma_{(\chi,\mu)}^{\mathcal{N}} = \sigma_\chi^{\mathcal{M}_1} \otimes \sigma_\mu^{\mathcal{M}_2}$ and $\sigma_{(\chi+\mu,\chi-\mu)}^{\mathcal{N}'} = \sigma_{\chi+\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \otimes \sigma_{\chi-\mu}^{\mathcal{M}_1 \otimes [-1]^*\mathcal{M}_2}$, we may rewrite the addition formula (4.3.3.7) as

$$\Phi_A^* \circ (\sigma_\chi^{\mathcal{M}_1} \otimes \sigma_\mu^{\mathcal{M}_2}) = (\sigma_{\chi+\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \otimes \sigma_{\chi-\mu}^{\mathcal{M}_1 \otimes [-1]^*\mathcal{M}_2}) \circ \Phi^*. \tag{4.3.3.8}$$

Here the domains and codomains of the morphisms can be described in the following commutative diagram.

$$\begin{tikzcd} \Gamma(G, \mathcal{L}_1) \underset{R}{\otimes} \Gamma(G, \mathcal{L}_2) \arrow[d, "\sigma_\chi^{\mathcal{M}_1} \otimes \sigma_\mu^{\mathcal{M}_2}"'] \arrow[dr, bend left, "\Phi^*"] & \\ \Gamma(A, \mathcal{M}_{1,\chi}) \underset{R}{\otimes} \Gamma(A, \mathcal{M}_{2,\mu}) \arrow[dr, bend right, "\Phi_A^*"'] & \Gamma(G, \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2) \underset{R}{\otimes} \Gamma(G, \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} [-1]^*\mathcal{L}_2) \arrow[d, "\sigma_{\chi+\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \otimes \sigma_{\chi-\mu}^{\mathcal{M}_1 \otimes [-1]^*\mathcal{M}_2}"] \\ & \Gamma(A, (\mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_2)_{\chi+\mu}) \underset{R}{\otimes} \Gamma(A, (\mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} [-1]^*\mathcal{M}_2)_{\chi-\mu}) \end{tikzcd}$$

Now assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ are ample and algebraically equivalent to $\mathcal{L}$, so that the arguments in Section 4.3.2 (such as restriction to weight subspaces under the $K(\mathcal{L})^\mu$-action) apply. For $\chi, \mu \in X$, the above diagram induces the following commutative diagram.

$$\begin{tikzcd} \Gamma(G_\eta, \mathcal{L}_{1,\eta})_{\bar{\chi}} \underset{K}{\otimes} \Gamma(G_\eta, \mathcal{L}_{2,\eta})_{\bar{\mu}} \arrow[d, "\sigma_\chi^{\mathcal{M}_1} \otimes \sigma_\mu^{\mathcal{M}_2}"'] \arrow[dr, bend left, "\mathrm{pr}_{\overline{\chi+\mu}, \overline{\chi-\mu}} \circ \Phi^*"] & \\ \Gamma(A_\eta, \mathcal{M}_{1,\chi,\eta}) \underset{K}{\otimes} \Gamma(A_\eta, \mathcal{M}_{2,\mu,\eta}) \arrow[dr, bend right, "\Phi_A^*"'] & \Gamma(G_\eta, \mathcal{L}_{1,\eta} \underset{\mathscr{O}_{G_\eta}}{\otimes} \mathcal{L}_{2,\eta})_{\overline{\chi+\mu}} \underset{K}{\otimes} \Gamma(G_\eta, \mathcal{L}_{1,\eta} \underset{\mathscr{O}_{G_\eta}}{\otimes} [-1]^*\mathcal{L}_{2,\eta})_{\overline{\chi-\mu}} \arrow[d, "\sigma_{\chi+\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \otimes \sigma_{\chi-\mu}^{\mathcal{M}_1 \otimes [-1]^*\mathcal{M}_2}"] \\ & \Gamma(A_\eta, (\mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_2)_{\chi+\mu,\eta}) \underset{K}{\otimes} \Gamma(A_\eta, (\mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} [-1]^*\mathcal{M}_2)_{\chi-\mu,\eta}) \end{tikzcd}$$

Here the $\bar{\chi}$ and $\bar{\mu}$ on the left-hand side are classes in $X/\phi(Y)$, while the $\overline{\chi+\mu}$ and $\overline{\chi-\mu}$ on the right-hand side are classes in $X/2\phi(Y)$. We have to use 2ϕ instead of ϕ because the polarizations defined by $\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2$ and $\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} [-1]^*\mathcal{L}_2$ are both twice that defined by $\mathcal{L}$. The morphism $\mathrm{pr} = \mathrm{pr}_{\overline{\chi+\mu}, \overline{\chi-\mu}}$ above is the projection to the weight-$(\overline{\chi+\mu}, \overline{\chi-\mu})$ subspace.

For our purpose, the following observation will be useful:

Lemma 4.3.3.9. *The left-hand side of (4.3.3.8), when restricted to $\Gamma(G_\eta, \mathcal{L}_{1,\eta})_{\bar{\chi}} \underset{K}{\otimes} \Gamma(G_\eta, \mathcal{L}_{2,\eta})_{\bar{\mu}}$, is nonzero if and only if $\sigma_\chi^{\mathcal{M}_1} \neq 0$ and $\sigma_\mu^{\mathcal{M}_2} \neq 0$. For the same reason, the right-hand side of (4.3.3.8), when restricted to $\Gamma(G_\eta, \mathcal{L}_{1,\eta} \underset{\mathscr{O}_{G_\eta}}{\otimes} \mathcal{L}_{2,\eta})_{\overline{\chi+\mu}} \underset{K}{\otimes} \Gamma(G_\eta, \mathcal{L}_{1,\eta} \underset{\mathscr{O}_{G_\eta}}{\otimes} [-1]^*\mathcal{L}_{2,\eta})_{\overline{\chi-\mu}}$, is nonzero if and only if $\sigma_{\chi+\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \neq 0$, $\sigma_{\chi-\mu}^{\mathcal{M}_1 \otimes \mathcal{M}_2} \neq 0$, and $\mathrm{pr}_{\overline{\chi+\mu}, \overline{\chi-\mu}} \circ \Phi^* \neq 0$.*

Lemma 4.3.3.10. *Given any ample cubical invertible sheaf $\mathcal{L}$ over G, there exists a finite flat homomorphism $R \to R'$ of complete discrete valuation rings such that $\mathcal{L} \underset{R}{\otimes} R'$ is algebraically equivalent to a* ***symmetric*** *ample cubical invertible sheaf $\mathcal{L}'$ over R'. Moreover, we can assume that $(\mathcal{L}')^\natural \cong \pi^*\mathcal{M}'$ for some symmetric (ample* *cubical) invertible sheaf* $\mathcal{M}$ *over* $A \underset{R}{\otimes} R'$ *(see Remark 4.3.3.1 for the justification of* $(\mathcal{L}')^\natural$*).*

Proof. First let us show that, after replacing $K = \mathrm{Frac}(R)$ with a finite extension $K \to K'$ of fields, the generic fiber $\mathcal{L}_\eta$ of $\mathcal{L}$ is algebraically equivalent to some symmetric invertible sheaf $\mathcal{L}'_\eta$. Then we can take R' to be the normalization of R in K', and take $\mathcal{L}'$ to be the unique cubical invertible sheaf extending $\mathcal{L}'_\eta$ (by Proposition 3.2.3.1 and Theorem 3.3.2.3).

Since the polarizations induced by $\mathcal{L}_\eta$ and $[-1]^*\mathcal{L}_\eta$ are the same, the invertible sheaf $\mathcal{L}_\eta \underset{\mathscr{O}_{G_\eta}}{\otimes} [-1]^*\mathcal{L}_\eta^{\otimes -1}$ defines a point of the dual abelian variety $G_\eta^\vee$. Since abelian varieties are 2-divisible, by replacing K with a finite extension field, we may assume that there exists some invertible sheaf $\mathcal{N}$ defining a point of $G_\eta^\vee$ such that $\mathcal{N}^{\otimes 2} \cong \mathcal{L}_\eta \underset{\mathscr{O}_{G_\eta}}{\otimes} [-1]^*\mathcal{L}_\eta^{\otimes -1}$ (and $[-1]^*\mathcal{N} \cong \mathcal{N}^{\otimes -1}$). Then $\mathcal{L}'_\eta := \mathcal{L}_\eta \underset{\mathscr{O}_{G_\eta}}{\otimes} \mathcal{N}^{\otimes -1} \cong [-1]^*\mathcal{L}_\eta \underset{\mathscr{O}_{G_\eta}}{\otimes} \mathcal{N} \cong [-1]^*\mathcal{L}'_\eta$ is symmetric.

It remains to explain the last statement. We shall enlarge R' freely to make the following work: By 3 of Corollary 3.2.5.5, we may assume that $(\mathcal{L}')^\natural$ descends to some (possibly nonsymmetric) $\mathcal{M}'$. Since $\mathcal{M}'$ is ample, $T_a^*\mathcal{M}'$ is symmetric for some torsion point a of A. Let $\tilde{a}$ be a torsion point of $G^\natural$ mapping to a, which is canonically identified with a torsion point of G as in Section 3.4.1. Then it suffices to replace $\mathcal{L}'$ with $T_{\tilde{a}}^*\mathcal{L}'$. □

By Lemma 4.3.3.10, let us assume from now on that the ample cubical invertible sheaf $\mathcal{L}$ is *symmetric* and that $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$ for some *symmetric* $\mathcal{M}$.

For every $\bar{\chi} \in X/\phi(Y)$, since $\Gamma(G,\mathcal{L})_{\bar{\chi}} \neq 0$ by Lemma 4.3.2.9, there exists some χ_0 in $\bar{\chi}$ such that $\sigma_{\chi_0}^{\mathcal{M}} \neq 0$. In particular, $\sigma_{\phi(y_0)}^{\mathcal{M}} \neq 0$ for some $y_0 \in Y$. Then, by (4.3.3.8) (and Lemma 4.3.3.9), we obtain $\mathrm{pr}_{\bar{0},\bar{0}} \circ \Phi^* \neq 0$, $\sigma_{\phi(2y_0)}^{\mathcal{M}^{\otimes 2}} \neq 0$, and $\sigma_0^{\mathcal{M}^{\otimes 2}} \neq 0$. By (4.3.3.8) again, we obtain $\sigma_0^{\mathcal{M}} \neq 0$.

Lemma 4.3.3.11. *The subset* $Z := \{\chi \in \phi(Y) : \sigma_\chi^{\mathcal{M}} \neq 0\}$ *is a subgroup of* $\phi(Y)$*.*

Proof. We have seen that $0 \in Z$. If $\chi,\ \mu \in Z$, then (4.3.3.8) gives $\sigma_{2\chi}^{\mathcal{M}^{\otimes 2}} \neq 0$ and $\sigma_{2\mu}^{\mathcal{M}^{\otimes 2}} \neq 0$. Applying (4.3.3.8) again, we get $\sigma_{\chi+\mu}^{\mathcal{M}} \neq 0$ and $\sigma_{\chi-\mu}^{\mathcal{L}} \neq 0$. Hence $\chi + \mu \in Z$ and $\chi - \mu \in Z$, as desired. □

Lemma 4.3.3.12. *We have* $Z = \phi(Y)$ *in Lemma 4.3.3.11.*

Proof. Let W be any subgroup of finite index in $\phi(Y)$ containing Z. We would like to show that any such W is the whole group $\phi(Y)$.

Let H be the finite flat subgroup of T containing $K(\mathcal{L})^\mu$ such that W is the kernel of the restriction $X = \underline{\mathbf{X}}(T) \to \underline{\mathbf{X}}(H)$. As in Section 4.3.2, the trivialization of $i^*\mathcal{L}_{\mathrm{for}}$ induces a $K(\mathcal{L})^\mu$-action on $\mathcal{L}$, so that $\mathcal{L}$ descends to an ample cubical invertible sheaf $\underline{\mathcal{L}}$ over the quotient semi-abelian scheme $\underline{G} := G/K(\mathcal{L})^\mu$. By replacing G, H, and $\mathcal{L}$ with $\underline{G}$, $H/K(\mathcal{L})^\mu$, and $\underline{\mathcal{L}}$, respectively, we are reduced to the case that $\phi(Y) = X$ (and $K(\mathcal{L})^\mu = 1$). Since W contains Z, each section $s \in \Gamma(G,\mathcal{L})$ has a Fourier expansion $s = \sum\limits_{\chi\in W} \sigma_\chi^{\mathcal{M}}(s)$ involving only terms with $\chi \in W$. Translating $\mathcal{L}$ by elements of $G(R)$, we see that the same is true for every $\mathcal{L}'$ algebraically equivalent to $\mathcal{L}$ (with a suitable trivialization of $i^*\mathcal{L}'$).

Now we quote the following result of Mumford:

Proposition 4.3.3.13 (Mumford; recorded as [42, Ch. I, Prop. 5.3])**.** *Suppose* Z *is an abelian variety over an algebraically closed field* k*, and suppose* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are two ample line bundles over* Z*. Then* $\Gamma(Z, \mathcal{L}_1 \underset{\mathscr{O}_Z}{\otimes} \mathcal{L}_2)$ *is spanned by the images of* $\Gamma(Z,\mathcal{L}_1 \underset{\mathscr{O}_Z}{\otimes} \mathcal{N}) \underset{k}{\otimes} \Gamma(Z,\mathcal{L}_2 \underset{\mathscr{O}_Z}{\otimes} \mathcal{N}^{\otimes -1}) \to \Gamma(Z,\mathcal{L}_1 \underset{\mathscr{O}_Z}{\otimes} \mathcal{L}_2)$*, for* $\mathcal{N}$ *running over a Zariski dense subset of* $Z^\vee(k)$*.*

Proof. The images span a subspace stable under $\mathcal{G}(\mathcal{L}_1 \underset{\mathscr{O}_Z}{\otimes} \mathcal{L}_2)$. □

Let us return to the proof of Lemma 4.3.3.12. After making a suitable base change $R \hookrightarrow R'$ if necessary, we may assume that $\Gamma(G,\mathcal{L}_\eta^{\otimes n})$ is generated by elements of the form $T_{x_1}^*(s)\otimes\cdots\otimes T_{x_n}^*(s)$ for $n \geq 1$, where $x_1,\ldots,x_n \in G(R)$ satisfy $x_1+\cdots+x_n = 0$. Then Proposition 4.3.3.13 implies that every element of $\Gamma(G,\mathcal{L}^{\otimes n})$ has a Fourier expansion involving only terms with $\chi \in W$. As a result, elements of $\Gamma(G,\mathcal{L}^{\otimes n})$ factor through G/H. (This conclusion is independent of the base change we made.) Since $\mathcal{L}$ is ample over G, this is possible only if H is trivial and $W = \phi(Y)$, as desired. □

Now we are ready to prove (4.3.1.2) and (4.3.1.12).

Proof of (4.3.1.2). As in the context of (4.3.1.2), let $\mathcal{L}$ be any given ample cubical invertible sheaf over G such that $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$ for some cubical invertible sheaf $\mathcal{M}$ over A. Given any $\chi \in X$, since $\Gamma(G,\mathcal{L})_{\bar{\chi}} \neq 0$ by Lemma 4.3.2.9, there exists ξ such that $\xi \equiv \chi \pmod{\phi(Y)}$ and such that $\sigma_\xi^{\mathcal{M}_1} \neq 0$. On the other hand, by Lemmas 4.3.3.10 and 4.3.3.12 (and also Remark 4.3.3.1), after making a finite flat (surjective) base change if necessary, there exists an ample cubical invertible sheaf $\mathcal{L}'$ over G that is algebraically equivalent to $\mathcal{L}$, such that $(\mathcal{L}')^\natural \cong \pi^*\mathcal{M}'$ for some cubical invertible sheaf $\mathcal{M}'$ over A, with the additional property that $\sigma_\mu^{\mathcal{M}'} \neq 0$ for every $\mu \in \phi(Y)$. Then (4.3.3.8) implies that $\mathrm{pr}_{\overline{\xi+\mu},\overline{\xi-\mu}} \circ \Phi^* \neq 0$, $\sigma_{\xi+\mu}^{\mathcal{M}\otimes\mathcal{M}'} \neq 0$, and $\sigma_{\xi-\mu}^{\mathcal{M}\otimes[-1]^*\mathcal{M}'} \neq 0$ for every $\mu \in \phi(Y)$. Writing $\chi = \xi + \mu = \xi - (-\mu)$, we obtain $\mathrm{pr}_{\bar{\chi},\bar{\chi}} \circ \Phi^* \neq 0$, $\sigma_\chi^{\mathcal{M}\otimes\mathcal{M}'} \neq 0$, and $\sigma_\chi^{\mathcal{M}\otimes[-1]^*\mathcal{M}'} \neq 0$. By (4.3.3.8) again, we obtain $\sigma_\chi^{\mathcal{M}} \neq 0$, as desired. □

Proof of (4.3.1.12). Let us denote by $\tau^{\mathcal{L}}$ and $\psi^{\mathcal{L}}$ (resp. $\tau^{\mathcal{L}^{\otimes 2}}$ and $\psi^{\mathcal{L}^{\otimes 2}}$) the trivializations associated with $\mathcal{L}$ (resp. $\mathcal{L}^{\otimes 2}$). (As explained in the proofs of Lemmas 4.3.1.14 and 4.3.1.15, $\tau^{\mathcal{L}}$ and $\psi^{\mathcal{L}}$ do not depend on the choice of $\mathcal{M}$, because a different choice only results in a shift of the indices of the $\sigma_\chi^{\mathcal{M}}$'s. Similarly, $\tau^{\mathcal{L}^{\otimes 2}}$ and $\psi^{\mathcal{L}^{\otimes 2}}$ do not depend on the choice of $\mathcal{M}^{\otimes 2}$.)

For all $\chi,\mu \in X$ and $y \in Y$, we have by definition,

$$\sigma_{\chi+\phi(y)}^{\mathcal{M}} = \psi^{\mathcal{L}}(y)\tau^{\mathcal{L}}(y,\chi)\, T_{c^\vee(y)}^* \circ \sigma_\chi^{\mathcal{M}}$$

and

$$\sigma_{\chi+\mu+\phi(y)}^{\mathcal{M}} = \psi^{\mathcal{L}}(y)\tau^{\mathcal{L}}(y,\chi+\mu)\, T_{c^\vee(y)}^* \circ \sigma_{\chi+\mu}^{\mathcal{M}}.$$

We have similar formulas for $\tau^{\mathcal{L}^{\otimes 2}}$ and $\psi^{\mathcal{L}^{\otimes 2}}$. By (4.3.3.8), we have

$$\Phi_A^* \circ (\sigma_{\chi+\phi(y)}^{\mathcal{M}} \otimes \sigma_{\mu+\phi(y)}^{\mathcal{M}}) = (\sigma_{\chi+\mu+2\phi(y)}^{\mathcal{M}\otimes\mathcal{M}} \otimes \sigma_{\chi-\mu}^{\mathcal{M}\otimes[-1]^*\mathcal{M}}) \circ \Phi^*$$

and

$$\Phi_A^* \circ (\sigma_\chi^{\mathcal{M}} \otimes \sigma_\mu^{\mathcal{M}}) = (\sigma_{\chi+\mu}^{\mathcal{M}\otimes\mathcal{M}} \otimes \sigma_{\chi-\mu}^{\mathcal{M}\otimes[-1]^*\mathcal{M}}) \circ \Phi^*.$$

Since translation by $(\iota(y), \iota(y)) \in G_\eta \underset{\eta}{\times} G_\eta$ corresponds to translation by $(2\iota(y), 0)$ under Φ, we obtain

$$\psi^{\mathcal{L}}(y)^2 \tau^{\mathcal{L}}(y,\chi)\tau^{\mathcal{L}}(y,\mu) = \psi^{\mathcal{L}^{\otimes 2}}(2y)\tau^{\mathcal{L}^{\otimes 2}}(2y, \chi+\mu).$$

By replacing $(\chi+\mu, 0)$ with (χ,μ), we obtain similarly

$$\psi^{\mathcal{L}}(y)^2 \tau^{\mathcal{L}}(y,\chi+\mu) = \psi^{\mathcal{L}^{\otimes 2}}(2y)\tau^{\mathcal{L}^{\otimes 2}}(2y, \chi+\mu).$$

Comparing these two relations, we obtain the multiplicativity

$$\tau^{\mathcal{L}}(y,\chi+\mu) = \tau^{\mathcal{L}}(y,\chi)\tau^{\mathcal{L}}(y,\mu)$$

of τ in the second variable, as desired. ☐

4.3.4 Dependence of τ on the Choice of $\mathcal{L}$

In this section our goal is to show that τ does not depend fully on $\mathcal{L}$, but only on the homomorphism $\lambda : G \to G^\vee$ induced by $\mathcal{L}$. More precisely, suppose we have two pairs $(G, \mathcal{L}_1)$ and $(G, \mathcal{L}_2)$ in $\mathrm{DEG}_{\mathrm{ample}}(R, I)$, with both $\mathcal{L}_{1,\eta}$ and $\mathcal{L}_{2,\eta}$ ample over G_η, such that the induced polarizations satisfy $N_1\lambda_{1,\eta} = N_2\lambda_{2,\eta}$ for some integers $N_1, N_2 > 0$. Suppose the associated degeneration data in $\mathrm{DD}_{\mathrm{ample}}(R, I)$ (using the constructions so far) are, respectively, $(A, \underline{X}, \underline{Y}, \phi_1, c, c^\vee, \mathcal{L}_1^\natural, \tau_1, \psi_1)$ and $(A, \underline{X}, \underline{Y}, \phi_2, c, c^\vee, \mathcal{L}_2^\natural, \tau_2, \psi_2)$. Then our goal is to show that $\tau_1 = \tau_2$.

The starting point is the following:

Lemma 4.3.4.1 (cf. [42, Ch. II, Lem. 6.1])**.** *Let $(G, \mathcal{L}_1)$, $(G, \mathcal{L}_2)$, and $(G, \mathcal{L})$ be three objects in $\mathrm{DEG}_{\mathrm{ample}}(R, I)$, such that $\mathcal{L} \cong \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2$, with respective associated degeneration data $(A, \underline{X}, \underline{Y}, \phi_1, c, c^\vee, \mathcal{L}_1^\natural, \tau_1, \psi_1)$, $(A, \underline{X}, \underline{Y}, \phi_2, c, c^\vee, \mathcal{L}_2^\natural, \tau_2, \psi_2)$, and $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ as in Corollary 4.3.1.16. Suppose we already know that $\tau_1 = \tau_2$. Then $\phi = \phi_1 + \phi_2$, $\tau = \tau_1 = \tau_2$, and $\psi = \psi_1\psi_2$.*

Proof. Let us assume that $\mathcal{L}_1^\natural \cong \pi^*\mathcal{M}_1$ and $\mathcal{L}_2^\natural \cong \pi^*\mathcal{M}_2$ for some cubical invertible sheaves $\mathcal{M}_1$ and $\mathcal{M}_2$ over A, where $\pi : G^\natural \to A$ is the structural morphism. Then we also have $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$ for $\mathcal{M} := \mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_2$. Since the morphisms λ, λ_1, and λ_2 defined by $\mathcal{L}$, $\mathcal{L}_1$, and $\mathcal{L}_2$, respectively, satisfy $\lambda = \lambda_1 + \lambda_2$, we have $\lambda^\natural = \lambda_1^\natural + \lambda_2^\natural$ and hence $\phi = \phi_1 + \phi_2$.

Suppose we have nonzero sections $s_1 \in \Gamma(G, \mathcal{L}_1)$ and $s_2 \in \Gamma(G, \mathcal{L}_2)$. Let $s := s_1 \otimes s_2 \in \Gamma(G, \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2)$. Then $\sigma_\chi^{\mathcal{M}}(s) = \sum\limits_{\chi_1+\chi_2=\chi} \sigma_{\chi_1}^{\mathcal{M}_1}(s_1) \otimes \sigma_{\chi_2}^{\mathcal{M}_2}(s_2)$. If we translate by $c^\vee(y)$ and compare the morphisms over η using (4.3.1.7), then $\sigma_\chi^{\mathcal{M}}(s)$ receives a factor $\psi(y)\tau(y,\chi)$, while each of the summand $\sigma_{\chi_1}^{\mathcal{M}_1}(s_1) \otimes \sigma_{\chi_2}^{\mathcal{M}_2}(s_2)$ receives a factor $\psi_1(y)\tau_1(y,\chi_1)\psi_2(y)\tau_2(y,\chi_2)$, which is equal to $\psi_1(y)\psi_2(y)\tau_1(y,\chi)$ because $\tau_1 = \tau_2$. If $\sigma_\chi^{\mathcal{M}}(s) \neq 0$, then we obtain $\psi(y)\tau(y,\chi) = \psi_1(y)\psi_2(y)\tau_1(y,\chi)$. As a result, we see that $\tau_1(y, \cdot\,) = \tau(y, \cdot\,)$ over the subset of X consisting of differences $\chi - \chi'$ between χ and χ' such that $\sigma_\chi^{\mathcal{M}}(s) \neq 0$ and $\sigma_{\chi'}^{\mathcal{M}}(s) \neq 0$. We claim that this subset of X is the whole X. Then $\psi = \psi_1\psi_2$ will also follow.

In the above argument, s has to be in the image of $\Gamma(G, \mathcal{L}_1) \underset{K}{\otimes} \Gamma(G, \mathcal{L}_2) \to \Gamma(G, \mathcal{L})$. By Lemma 4.3.1.18, for $i = 1, 2$, the two trivializations τ_i and ψ_i remain unchanged if we replace $\mathcal{L}_i$ and $\mathcal{M}_i$ respectively with $T_{g_i}^*\mathcal{L}_i$ and $T_{\pi(g_i)}^*\mathcal{M}_i$, where $\pi(g_i)$ is defined by algebraizing $\pi_{\mathrm{for}}(g_{i,for})$ by abuse of notation as in the proof of Lemma 4.3.1.18. Since (4.3.1.2) is true for all $\chi \in X$, it suffices to show that $\Gamma(G_\eta, \mathcal{L}_\eta)$ is spanned by images of $\Gamma(G_\eta, T_{g_1}^*\mathcal{L}_{1,\eta}) \underset{K}{\otimes} \Gamma(G_\eta, T_{g_2}^*\mathcal{L}_{2,\eta}) \to \Gamma(G_\eta, \mathcal{L}_\eta)$, with (g_1, g_2) running through a set of S-points of $G \underset{S}{\times} G$ such that $T_{g_1}^*\mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} T_{g_2}^*\mathcal{L}_2 \cong \mathcal{L}_1 \underset{\mathscr{O}_G}{\otimes} \mathcal{L}_2 \cong \mathcal{L}$, or equivalently $\lambda_1(g_1) + \lambda_2(g_2) = 0$. Since our goal is to compare morphisms, we may and we shall make a base change in R and restart with the assumption that R is a complete discrete valuation ring with algebraically closed residue field. Then S-points of $G^\vee$ (which are of the form $\lambda_1(g_1) = -\lambda_2(g_2)$) are Zariski-dense on $G^\vee$, and the claim follows from Proposition 4.3.3.13. ☐

Corollary 4.3.4.2. *Let $(G, \mathcal{L})$ be an object in $\mathrm{DEG}_{\mathrm{ample}}(R, I)$ with associated degeneration datum $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ in $\mathrm{DD}_{\mathrm{ample}}(R, I)$. Then $(A, \underline{X}, \underline{Y}, n\phi, c, c^\vee, (\mathcal{L}^\natural)^{\otimes n}, \tau, n\psi)$, with the same τ, is the degeneration datum associated with $(G, \mathcal{L}^{\otimes n})$.*

Lemma 4.3.4.3. *Let $(G, \mathcal{L}_1)$ and $(G, \mathcal{L}_2)$ be two objects in $\mathrm{DEG}_{\mathrm{ample}}(R, I)$, with respective associated degeneration data $(A, \underline{X}, \underline{Y}, \phi_1, c, c^\vee, \mathcal{L}_1^\natural, \tau_1, \psi_1)$ and $(A, \underline{X}, \underline{Y}, \phi_2, c, c^\vee, \mathcal{L}_2^\natural, \tau_2, \psi_2)$, such that $\mathcal{L}_1 = f^*\mathcal{L}_2$ for some $f : G \to G$ whose restriction $f_\eta : G_\eta \to G_\eta$ to η is an isogeny. Let $f_Y : \underline{Y} \to \underline{Y}$ and $f^\natural : G^\natural \to G^\natural$ be the homomorphisms induced by $f : G \to G$, and let $\iota_1 : \underline{Y}_\eta \to G_\eta^\natural$ and $\iota_2 : \underline{Y}_\eta \to G_\eta^\natural$ be the homomorphisms corresponding to τ_1 and τ_2, respectively (see Lemma 4.2.1.7). Then ι_1 and ι_2 are related by $f^\natural \circ \iota_1 = \iota_2 \circ f_Y$.*

In particular, if $f = [-1] : G \to G$, then $\iota_1 = \iota_2$ as homomorphisms from $\underline{Y}_\eta$ to $G_\eta^\natural$, or equivalently $\tau_1 = \tau_2$ as trivializations of $(c^\vee \times c)^*\mathcal{P}_{A,\eta}^{\otimes -1}$.

Proof. By étale descent if necessary, we may assume that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively, and that $\mathcal{L}_2^\natural \cong \pi^*\mathcal{M}_2$ for some cubical invertible sheaf $\mathcal{M}_2$ over A, where $\pi : G^\natural \to A$ is the structural morphism. Let $\mathcal{M}_1 := f_A^*\mathcal{M}_2$, so that naturally $\mathcal{L}_1^\natural \cong \pi^*\mathcal{M}_1$. Let us denote by $f_T : T \twoheadrightarrow T$ and $f_A : A \twoheadrightarrow A$ the isogenies induced by $f^\natural$, and let $f_{T^\vee} : T^\vee \to T^\vee$ be the isogeny induced by $f^{\vee,\natural}$. Let $f_X : X \hookrightarrow X$ and $f_Y : Y \hookrightarrow Y$ be the homomorphisms corresponding to f_T and $f_{T^\vee}$, respectively. Consider the morphism

$$f^* : \Gamma(G, \mathcal{L}_2) \to \Gamma(G, f^*\mathcal{L}_2) \cong \Gamma(G, \mathcal{L}_1)$$

given by pulling back the sections by f, and consider the corresponding weight subspaces

$$f_A^* : \Gamma(A, \mathcal{M}_{2,\chi}) \to \Gamma(A, f_A^*(\mathcal{M}_{2,\chi})) \cong \Gamma(A, \mathcal{M}_{1,f_X(\chi)})$$

for all $\chi \in X$, where the last isomorphism follows from $f_{A^\vee}c = cf_X$. Note that we have $\phi_1 = f_X\phi_2 f_Y$, coming from the relation $\lambda_1 = f^\vee\lambda_2 f$, where λ_1 and λ_2 are defined by $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. The comparison between $\sigma_\chi^{\mathcal{M}_2}$ and $\sigma_{\chi+\phi_2(f_Y(y))}^{\mathcal{M}_2}$ over η gives the relation

$$\psi_2(f_Y(y))\tau_2(f_Y(y),\chi)\; T_{c^\vee(f_Y(y))}^* \circ \sigma_\chi^{\mathcal{M}_2} = \sigma_{\chi+\phi_2(f_Y(y))}^{\mathcal{M}_2},$$

while the comparison between $\sigma_{f_X(\chi)}^{\mathcal{M}_1}$ and $\sigma_{f_X(\chi)+\phi_1(y)}^{\mathcal{M}_1}$ gives the relation

$$\psi_1(y)\tau_1(y, f_X(\chi))\; T_{c^\vee(y)}^* \circ \sigma_{f_X(\chi)}^{\mathcal{M}_1} = \sigma_{f_X(\chi)+\phi_1(y)}^{\mathcal{M}_1}.$$

Since $f_X(\phi_2(f_Y(y))) = \phi_1(y)$, pulling back by f_A matches the two relations and gives the natural functorial relation

$$\tau_1(y, f_X(\chi)) = \tau_2(f_Y(y), \chi).$$

This is exactly what it means by $f^\natural \circ \iota_1 = \iota_2 \circ f_Y$. ☐

Corollary 4.3.4.4. *Let $(G, \mathcal{L})$ be an object in* $\mathrm{DEG}_{\mathrm{ample}}(R, I)$*. Then, in the associated degeneration datum $(A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$, the trivialization τ depends only on $\mathcal{L} \underset{\mathcal{O}_G}{\otimes} [-1]^*\mathcal{L}$, and hence only on the λ induced by $\mathcal{L}$.*

Proof. Combine Lemmas 4.3.4.1, 4.3.4.3, and 4.2.1.4. □

Proposition 4.3.4.5. *Let $(G, \mathcal{L}_1)$ and $(G, \mathcal{L}_2)$ be two objects in* $\mathrm{DEG}_{\mathrm{ample}}(R, I)$*, with associated degeneration data $(A, \underline{X}, \underline{Y}, \phi_1, c, c^\vee, \mathcal{L}_1^\natural, \tau_1, \psi_1)$ and $(A, \underline{X}, \underline{Y}, \phi_2, c, c^\vee, \mathcal{L}_2^\natural, \tau_2, \psi_2)$, respectively. Then $\tau_1 = \tau_2$ if there are positive integers N_1 and N_2 such that $N_1\lambda_1 = N_2\lambda_2$, where $\lambda_1, \lambda_2 : G \to G$ are the homomorphisms induced by $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively.*

Proof. Combine Corollaries 4.3.4.2 and 4.3.4.4. □

Remark 4.3.4.6. The proof of Theorem 4.2.1.14 is now complete.

Remark 4.3.4.7. As mentioned in Remark 4.2.1.17, we will show in Section 4.5.5 that the association of τ is independent of the choice of polarizations (or their positive multiplies), using more details in Mumford's construction.

4.4 Equivalences of Categories

Our goal in this section is to define, following [42, Ch. III], the categories $\mathrm{DEG}_{\mathrm{pol}}(R, I)$, $\mathrm{DEG}_{\mathrm{IS}}(R, I)$, $\mathrm{DEG}(R, I)$, $\mathrm{DD}_{\mathrm{pol}}(R, I)$, $\mathrm{DD}_{\mathrm{IS}}(R, I)$, and $\mathrm{DD}(R, I)$, and state the main Theorem 4.4.16 that sets up the equivalences of categories $\mathrm{DEG}_{\mathrm{ample}}(R, I) \cong \mathrm{DD}_{\mathrm{ample}}(R, I)$, $\mathrm{DEG}_{\mathrm{pol}}(R, I) \cong \mathrm{DD}_{\mathrm{pol}}(R, I)$, $\mathrm{DEG}_{\mathrm{IS}}(R, I) \cong \mathrm{DD}_{\mathrm{IS}}(R, I)$, and $\mathrm{DEG}(R, I) \cong \mathrm{DD}(R, I)$, using *Mumford's construction* (so-called), including in particular, quasi-inverses of the functor $\mathrm{F}_{\mathrm{ample}}(R, I) : \mathrm{DEG}_{\mathrm{ample}}(R, I) \to \mathrm{DD}_{\mathrm{ample}}(R, I)$ in Theorem 4.2.1.14 (defined by Fourier expansions of theta functions, in Section 4.3) and of the functor $\mathrm{F}_{\mathrm{pol}}(R, I) : \mathrm{DEG}_{\mathrm{pol}}(R, I) \to \mathrm{DD}_{\mathrm{pol}}(R, I)$ to be defined below (in Definition 4.4.8, following Corollary 4.3.4.4).

First, let us define the categories $\mathrm{DEG}(R, I)$, $\mathrm{DEG}_{\mathrm{pol}}(R, I)$, and $\mathrm{DEG}_{\mathrm{IS}}(R, I)$ as follows:

Definition 4.4.1. *With assumptions as in Section 4.1, the category* $\mathrm{DEG}(R, I)$ *has objects G with same conditions as in Definition 4.2.1.1.*

By [105, XI, 1.13] (see also Remark 3.3.3.9), since S is noetherian and normal, each G in $\mathrm{DEG}(R, I)$ carries some ample cubical invertible sheaf $\mathcal{L}$.

Definition 4.4.2. *With assumptions as in Section 4.1, the category* $\mathrm{DEG}_{\mathrm{pol}}(R, I)$ *has objects of the form (G, λ_η), where G is an object in* $\mathrm{DEG}(R, I)$*, and where $\lambda_\eta : G_\eta \to G_\eta^\vee$ is a polarization of G_η.*

Remark 4.4.3. Since S is noetherian and normal as assumed in Section 4.1, $\lambda_\eta : G_\eta \to G_\eta^\vee$ extends to a unique homomorphism $\lambda : G \to G^\vee$ by Proposition 3.3.1.5. Thus it is unambiguous to write objects of $\mathrm{DEG}_{\mathrm{pol}}(R, I)$ as (G, λ).

Lemma 4.4.4. *Given any object (G, λ) in* $\mathrm{DEG}_{\mathrm{pol}}(R, I)$*, the invertible sheaf $\mathcal{L} := (\mathrm{Id}_G, \lambda)^*\mathcal{P}$, where $\mathcal{P}$ is as in Theorem 3.4.3.2, is a (symmetric) ample cubical invertible sheaf satisfying 2 in Definition 4.2.1.1. In other words, $(G, \mathcal{L})$ defines an object in* $\mathrm{DEG}_{\mathrm{ample}}(R, I)$*.*

Proof. By Lemma 4.2.1.6, the ampleness of $\mathcal{L}$ follows from the ampleness of the symmetric $\mathcal{L}_\eta$ (see Definition 1.3.2.16). By Proposition 3.3.3.11, since $\mathcal{L}_{\mathrm{for}}$ is the pullback of $\mathcal{M}_{\mathrm{for}} := (\mathrm{Id}_{A_{\mathrm{for}}}, \lambda_{A_{\mathrm{for}}})^*\mathcal{P}_{A_{\mathrm{for}}}$, it is in the essential image of (3.3.3.12), as desired. □

Definition 4.4.5. *With assumptions as in Section 4.1, the category* $\mathrm{DEG}_{\mathrm{IS}}(R, I)$ *has objects of the form $(G, \mathcal{F})$, where G is an object in* $\mathrm{DEG}(R, I)$*, and where $\mathcal{F}$ is an invertible sheaf over G rigidified along the identity section (and hence endowed with a unique cubical structure by Proposition 3.2.3.1).*

Next, let us define the categories $\mathrm{DD}(R, I)$, $\mathrm{DD}_{\mathrm{pol}}(R, I)$, and $\mathrm{DD}_{\mathrm{IS}}(R, I)$ as follows:

Definition 4.4.6. *With assumptions as in Section 4.1, the category* $\mathrm{DD}_{\mathrm{pol}}(R, I)$ *has objects of the form $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$, with entries (together with the positivity condition for τ) described as in Definition 4.2.1.13.*

Remark 4.4.7. Each object $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ of $\mathrm{DD}_{\mathrm{pol}}(R, I)$ defines an object of $\mathrm{DD}_{\mathrm{ample}}(R, I)$ as follows: Let $\mathcal{M} := (\mathrm{Id}, \lambda_A)^*\mathcal{P}_A$. Let $\mathcal{L}^\natural := \pi^*\mathcal{M}$, where $\pi : G^\natural \to A$ is the structural morphism. Then $\mathcal{L}^\natural$ admits a canonical $\underline{Y}$-action over η given by

$$\begin{aligned}\psi := (\mathrm{Id}_Y, \phi)^*\tau : \mathbf{1}_{\underline{Y},\eta} \xrightarrow{\sim} (c^\vee, c\phi)^*\mathcal{P}_{A,\eta}^{\otimes -1} &\cong (c^\vee, \lambda_A c^\vee)^*\mathcal{P}_{A,\eta}^{\otimes -1} \\ &\cong (c^\vee)^*\mathcal{M}_\eta^{\otimes -1} \cong \iota^*(\mathcal{L}_\eta^\natural)^{\otimes -1},\end{aligned}$$

and $(A, \underline{X}, \underline{Y}, 2\phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ defines an object of $\mathrm{DD}_{\mathrm{ample}}(R, I)$. (However, this object induces $2\lambda_A$ instead of λ_A.)

Definition 4.4.8. *Following Corollary 4.3.4.4, the functor* $\mathrm{F}_{\mathrm{ample}}(R, I) : \mathrm{DEG}_{\mathrm{ample}}(R, I) \to \mathrm{DD}_{\mathrm{ample}}(R, I)$ *induces a functor*

$$\mathrm{F}_{\mathrm{pol}}(R, I) : \mathrm{DEG}_{\mathrm{pol}}(R, I) \to \mathrm{DD}_{\mathrm{pol}}(R, I) : (G, \lambda) \mapsto (A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau),$$

where A, λ_A, $\underline{X}$, $\underline{Y}$, ϕ, c, and $c^\vee$ are defined canonically by (G, λ), and where τ is defined by any $(G, \mathcal{L})$ in $\mathrm{DEG}_{\mathrm{ample}}(R, I)$ *such that $\mathcal{L}$ induces a multiple of λ.*

Remark 4.4.9. For the definition of τ, we can always take $\mathcal{L}$ to be $(\mathrm{Id}, \lambda)^*\mathcal{P}$, where $\mathcal{P}$ is as in Theorem 3.4.3.2 (see Lemma 4.4.4 and Remark 4.4.7).

Definition 4.4.10. *With assumptions as in Section 4.1, the category* $\mathrm{DD}(R, I)$ *has objects of the form $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ (or equivalently of the form $(G^\natural, \iota : \underline{Y}_\eta \to G_\eta^\natural)$), with entries described as in Definition 4.2.1.13, that can be extended to objects of* $\mathrm{DD}_{\mathrm{pol}}(R, I)$ *(or, equivalently, of* $\mathrm{DD}_{\mathrm{ample}}(R, I)$*; cf. Remark 4.4.7).*

Remark 4.4.11. The extendability of an object in $\mathrm{DD}(R, I)$ to an object in $\mathrm{DD}_{\mathrm{pol}}(R, I)$ is crucial, because the statements of both the positivity and symmetry conditions for τ require some choices of λ_A and ϕ.

Definition 4.4.12. *With assumptions as in Section 4.1, the category* $\mathrm{DD}_{\mathrm{IS}}(R, I)$ *has objects of the form $(A, \underline{X}, \underline{Y}, f_Y, c, c^\vee, \mathcal{F}^\natural, \tau, \zeta)$, where $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ defines an object in* $\mathrm{DD}(R, I)$ *(see Definition 4.4.10), with the remaining entries explained as follows:*

1. *$\mathcal{F}^\natural$ is a cubical invertible sheaf over $G^\natural$ (where $G^\natural$ is defined by $c : \underline{X} \to A^\vee$), defining a $\mathbf{G}_{\mathrm{m}}$-biextension $\mathcal{D}_2(\mathcal{F}^\natural)$ of $G^\natural \underset{S}{\times} G^\natural$, which (by Corollary 3.2.5.2) descends uniquely to a $\mathbf{G}_{\mathrm{m}}$-biextension of $A \underset{S}{\times} A$ that (as an invertible sheaf*

over $A \underset{S}{\times} A$) induces a homomorphism $\lambda_A : A \to A^\vee$ of abelian schemes over S (by the universal property of $A^\vee$, as in Construction 1.3.2.7).

2. *$f_Y : \underline{Y} \to \underline{X}$ is a homomorphism such that $f_A c^\vee = c f_Y$. By Lemma 3.4.4.2, f_A and f_Y induce a homomorphism $f^\natural : G^\natural \to G^{\vee,\natural}$.*

3. *A cubical trivialization $\zeta : \mathbf{1}_{\underline{Y},\eta} \xrightarrow{\sim} \iota^*\mathcal{F}_\eta^{\otimes -1}$ compatible with $(\mathrm{Id}_{\underline{Y}} \times f_Y)^*\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{Y},\eta} \xrightarrow{\sim} (c^\vee \times c f_Y)^* \mathcal{P}_{A,\eta}^{\otimes -1}$ in the sense that $\mathcal{D}_2(\zeta) = (\mathrm{Id}_{\underline{Y}} \times f_Y)^*\tau$. This compatibility makes sense because the biextension $\mathcal{D}_2(\mathcal{F}^\natural)$ over $G^\natural \underset{S}{\times} G^\natural$ uniquely descends to a biextension over $A \underset{S}{\times} A$, which by 1 above is just $(\mathrm{Id}_A \times f_A)^*\mathcal{P}_A$.*

Moreover, ζ has to satisfy the following ***finiteness condition*** *(cf. Definition 4.2.1.11): After passing to a finite étale covering over S that makes $\underline{Y}$ constant, there is an integer $n \geq 1$ such that $I_y \cdot I_{y,\phi(y)}^{\otimes n} \subset R$ for every y in Y, where I_y is defined by ζ as in the case of ψ in Definition 4.2.4.5, and where $I_{y,\phi(y)}$ is defined by τ as in Definition 4.2.4.6.*

Definition 4.4.13. *The forgetful functor $\mathrm{DD}_{\mathrm{IS}}(R,I) \to \mathrm{DD}(R,I)$ has a natural structure of a (strictly commutative) Picard category (see [14, XVIII, 1.4.2]) defined by the following tensor operations: The tensor product*
$$(A,\underline{X},\underline{Y},f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta) \otimes (A,\underline{X},\underline{Y},f'_Y,c,c^\vee,\mathcal{F}^{\natural\prime},\tau,\zeta')$$
of two objects in $\mathrm{DD}_{\mathrm{IS}}(R,I)$ (over the same object in $\mathrm{DD}(R,I)$) is defined to be
$$(A,\underline{X},\underline{Y},f_Y+f'_Y,c,c^\vee,\mathcal{F}^\natural \underset{\mathscr{O}_{G^\natural}}{\otimes} \mathcal{F}^{\natural\prime},\tau,\zeta+\zeta').$$
The tensor inverse
$$(A,\underline{X},\underline{Y},f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta)^{\otimes -1}$$
of an object in $\mathrm{DD}_{\mathrm{IS}}(R,I)$ is defined to be
$$(A,\underline{X},\underline{Y},-f_Y,c,c^\vee,(\mathcal{F}^\natural)^{\otimes -1},\tau,\zeta^{-1}).$$
Similarly, there are natural tensor operations
$$(G,\mathcal{F}) \otimes (G,\mathcal{F}') := (G,\mathcal{F} \underset{\mathscr{O}_G}{\otimes} \mathcal{F}')$$
and
$$(G,\mathcal{F})^{\otimes -1} := (G,\mathcal{F}^{\otimes -1})$$
making the forgetful functor $\mathrm{DEG}_{\mathrm{IS}}(R,I) \to \mathrm{DEG}(R,I)$ a (strictly commutative) Picard category.

Since $\mathrm{DD}_{\mathrm{ample}}(R,I)$ is naturally a full subcategory of $\mathrm{DD}_{\mathrm{IS}}(R,I)$, we can talk about tensor products of objects in $\mathrm{DD}_{\mathrm{ample}}(R,I)$.

Lemma 4.4.14. *For each object $(A,\underline{X},\underline{Y},f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta)$ in $\mathrm{DD}_{\mathrm{IS}}(R,I)$, there exist (noncanonically) objects $(A,\underline{X},\underline{Y},\phi_1,c,c^\vee,\mathcal{L}_1^\natural,\tau,\psi_1)$ and $(A,\underline{X},\underline{Y},\phi_2,c,c^\vee,\mathcal{L}_2^\natural,\tau,\psi_2)$ in $\mathrm{DD}_{\mathrm{ample}}(R,I)$ such that $f_Y = \phi_1 - \phi_2$, $\mathcal{F}^\natural \cong \mathcal{L}_1^\natural \underset{\mathscr{O}_{G^\natural}}{\otimes} (\mathcal{L}_2^\natural)^{\otimes -1}$, and $\zeta = \psi_1\psi_2^{-1}$. In other words, we have*
$$\begin{aligned}&(A,\underline{X},\underline{Y},f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta)\\ &\cong (A,\underline{X},\underline{Y},\phi_1,c,c^\vee,\mathcal{L}_1^\natural,\tau,\psi_1) \otimes (A,\underline{X},\underline{Y},\phi_2,c,c^\vee,\mathcal{L}_2^\natural,\tau,\psi_2)^{\otimes -1}.\end{aligned} \tag{4.4.15}$$

Proof. By the positivity condition in Definition 4.4.12, the tuple $(A,\underline{X},\underline{Y},c,c^\vee,\tau)$ defines an object in $\mathrm{DD}(R,I)$. By Definition 4.4.10, this means there is an object $(A,\underline{X},\underline{Y},\phi,c,c^\vee,\mathcal{L}^\natural,\tau,\psi)$ in $\mathrm{DD}_{\mathrm{ample}}(R,I)$ that extends $(A,\underline{X},\underline{Y},c,c^\vee,\tau)$.

Let us translate the statements preceding Definition 4.2.1.13 almost verbatim to this context: Let $i : T \to G^\natural$ and $\pi : G^\natural \to A$ denote the canonical morphisms. By the normality assumption on S and by Corollary 3.2.5.7, after making a finite étale surjective base change in S if necessary, we may assume that the étale sheaf $\underline{X}$ is constant and that the cubical invertible sheaves $i^*\mathcal{F}^\natural$ and $i^*\mathcal{L}^\natural$ are trivial.

In this case, each cubical trivialization $s_{\mathcal{F}^\natural} : i^*\mathcal{F}^\natural \xrightarrow{\sim} \mathscr{O}_T$ (resp. $s : i^*\mathcal{L}^\natural \xrightarrow{\sim} \mathscr{O}_T$) determines a cubical invertible sheaf $\mathcal{N}$ (resp. an ample cubical invertible sheaf $\mathcal{M}$) over A and a cubical isomorphism $\mathcal{F}^\natural \cong \pi^*\mathcal{N}$ (resp. $\mathcal{L}^\natural \cong \pi^*\mathcal{M}$), depending uniquely on the choice of $s_{\mathcal{F}^\natural}$ (resp. s). Take a sufficiently large integer $N > 0$ such that $\mathcal{M}_1 := \mathcal{N} \underset{\mathscr{O}_A}{\otimes} \mathcal{M}^{\otimes N}$ and $\mathcal{M}_2 := \mathcal{M}^{\otimes N}$ are both ample. Let $\mathcal{L}_1^\natural := \pi^*\mathcal{M}_1$ and $\mathcal{L}_2^\natural := \pi^*\mathcal{M}_2$, so that $\mathcal{F}^\natural \cong \mathcal{L}_1^\natural \underset{\mathscr{O}_A}{\otimes} (\mathcal{L}_2^\natural)^{\otimes -1}$ because $\mathcal{N} \cong \mathcal{M}_1 \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_2^{\otimes -1}$. Let $\phi_2 = N\phi$, and let $\psi_2 = \psi^N$. Then $\phi_2 : Y \hookrightarrow X$ is injective because ϕ is, and ψ_2 satisfies the positivity and compatibility conditions because ψ does. Set $\phi_1 = f_Y + \phi_2$ and $\psi_1 = \zeta\psi_2$. By increasing N if necessary, we may assume that $\phi_1 : Y \hookrightarrow X$ is injective, and that ψ_1 satisfies the positivity and compatibility conditions as ψ_2 does (by the finiteness condition for ζ; see Definition 4.4.12).

Thus, by étale descent, we obtain two objects $(A,\underline{X},\underline{Y},\phi_1,c,c^\vee,\mathcal{L}_1^\natural,\tau,\psi_1)$ and $(A,\underline{X},\underline{Y},\phi_2,c,c^\vee,\mathcal{L}_2^\natural,\tau,\psi_2)$ in $\mathrm{DD}_{\mathrm{ample}}(R,I)$ (over the original base scheme) realizing the relation (4.4.15), as desired. $\square$

Theorem 4.4.16. *With assumptions as in Section 4.1, there are equivalences of categories*
$$\begin{aligned}\mathrm{M}_{\mathrm{ample}}(R,I) &: \mathrm{DD}_{\mathrm{ample}}(R,I) \to \mathrm{DEG}_{\mathrm{ample}}(R,I) :\\ &\quad (A,\underline{X},\underline{Y},\phi,c,c^\vee,\mathcal{L}^\natural,\tau,\psi) \mapsto (G,\mathcal{L}),\\ \mathrm{M}_{\mathrm{pol}}(R,I) &: \mathrm{DD}_{\mathrm{pol}}(R,I) \to \mathrm{DEG}_{\mathrm{pol}}(R,I) :\\ &\quad (A,\lambda_A,\underline{X},\underline{Y},\phi,c,c^\vee,\tau) \mapsto (G,\lambda_\eta),\\ \mathrm{M}_{\mathrm{IS}}(R,I) &: \mathrm{DD}_{\mathrm{IS}}(R,I) \to \mathrm{DEG}_{\mathrm{IS}}(R,I) :\\ &\quad (A,\underline{X},\underline{Y},f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta) \mapsto (G,\mathcal{F}),\\ \mathrm{M}(R,I) &: \mathrm{DD}(R,I) \to \mathrm{DEG}(R,I) :\\ &\quad (G^\natural,\iota : \underline{Y}_\eta \to G_\eta^\natural) \text{ or } (A,\underline{X},\underline{Y},c,c^\vee,\tau) \mapsto G,\end{aligned}$$
which are given by generalizations of Mumford's construction in [95] and are compatible with each other under the natural forgetful functors. The quasi-inverses of $\mathrm{M}_{\mathrm{ample}}(R,I)$ and $\mathrm{M}_{\mathrm{pol}}(R,I)$ are given, respectively, by the functors
$$\mathrm{F}_{\mathrm{ample}}(R,I) : \mathrm{DEG}_{\mathrm{ample}}(R,I) \to \mathrm{DD}_{\mathrm{ample}}(R,I)$$
and
$$\mathrm{F}_{\mathrm{pol}}(R,I) : \mathrm{DEG}_{\mathrm{pol}}(R,I) \to \mathrm{DD}_{\mathrm{pol}}(R,I)$$
defined above in Theorem 4.2.1.14 and Definition 4.4.8. Moreover, the functors $\mathrm{M}_{\mathrm{IS}}(R,I)$ and $\mathrm{M}(R,I)$ respect the tensor structures of $\mathrm{DD}_{\mathrm{IS}}(R,I) \to \mathrm{DD}(R,I)$ and $\mathrm{DEG}_{\mathrm{IS}}(R,I) \to \mathrm{DEG}(R,I)$ in Definition 4.4.13.

The proof will be given in Section 4.5 following Faltings and Chai's generalization of Mumford's construction.

4.5 Mumford's Construction

The aim of this section is to explain *Mumford's construction*, which is a construction designed by Mumford in [95] that assigns objects in $\mathrm{Ob}(\mathrm{DEG}_{\mathrm{ample}}(R,I))$ to a certain subcollection of the *split objects* in $\mathrm{Ob}(\mathrm{DD}_{\mathrm{ample}}(R,I))$ (to be described in Section 4.5.1). In [42, Ch. III], Faltings and Chai explained that Mumford's construction (for split objects) is sufficient for defining a functor from (all of) $\mathrm{DD}_{\mathrm{IS}}(R,I)$ to $\mathrm{DEG}_{\mathrm{IS}}(R,I)$ that induces quasi-inverses for $\mathrm{F}_{\mathrm{ample}}(R,I)$ and $\mathrm{F}_{\mathrm{pol}}(R,I)$. We will explain their arguments in Section 4.5.4.

4.5.1 Relatively Complete Models

Definition 4.5.1.1. *With assumptions as in Section 4.1, the category $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$ has objects consisting of 9-tuples $(A,\mathcal{M},\underline{X},\underline{Y},\phi,c,c^\vee,\tau,\psi)$ as in Definition 4.2.1.13 (with the positivity conditions for ψ and τ), where $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively, and where $\mathcal{M}$ is an ample invertible sheaf over A. (In this case, for simplicity, we shall often denote $\underline{X}$ and $\underline{Y}$ by X and Y, respectively.) There is a natural functor $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)\to\mathrm{DD}_{\mathrm{ample}}(R,I)$ defined by sending the tuple $(A,\mathcal{M},\underline{X},\underline{Y},\phi,c,c^\vee,\tau,\psi)$ to $(A,\underline{X},\underline{Y},\phi,c,c^\vee,\mathcal{L}^\natural:=\pi^*\mathcal{M},\tau,\psi)$, where $\pi:G^\natural\to A$ is the structural morphism.*

Then $\psi:\mathbf{1}_{\underline{Y},\eta}\overset{\sim}{\to}\iota^*(\mathcal{L}^\natural_\eta)^{\otimes -1}$ can be viewed as a trivialization of the cubical invertible sheaf $(c^\vee)^*\mathcal{M}_\eta^{\otimes -1}$ over the constant group Y_η, which extends to a section of $(c^\vee)^*\mathcal{M}^{\otimes -1}$ over Y_S and is congruent to zero modulo I (cf. Definition 4.2.1.11). Note that $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$ is not embedded as a subcategory of $\mathrm{DD}_{\mathrm{ample}}(R,I)$, because in general there is no unique choice of $\mathcal{M}$ satisfying $\mathcal{L}^\natural=\pi^*\mathcal{M}$. Indeed, the possible choices of such $\mathcal{M}$ form a torsor under X (by Lemma 3.2.2.11, Remark 3.2.2.10, and Proposition 3.2.5.4).

Our first major goal is to construct an object $(G,\mathcal{L})$ in $\mathrm{DEG}_{\mathrm{ample}}(R,I)$ for each object $(A,\mathcal{M},X,Y,\phi,c,c^\vee,\tau,\psi)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$, such that $(G,\mathcal{L})$ gives (up to isomorphism) the tuple $(A,X,Y,\phi,c,c^\vee,\phi,\mathcal{L}^\natural:=\pi^*\mathcal{M},\tau,\psi)$ via Theorem 4.2.1.14. One of Mumford's insights is to axiomatize the desired conditions of partial compactifications of $(G^\natural,\mathcal{L}^\natural)$ on which Y acts:

Definition 4.5.1.2. *A **relatively complete model** of an object $(A,\mathcal{M},X,Y,\phi,c,c^\vee,\tau,\psi)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$ consists of the following data:*

1. *An integral scheme $P^\natural$ locally of finite type over A (and hence over S) containing $G^\natural$ as an open dense subscheme. Again, we denote the structural morphism by $\pi:P^\natural\to A$.*

2. *An invertible sheaf over $P^\natural$ extending the invertible sheaf $\pi^*\mathcal{M}$ over $G^\natural$, which we again denote by $\mathcal{L}^\natural$ by abuse of notation.*

3. *An action of $G^\natural$ on $P^\natural$ extending the translation action of $G^\natural$ on itself. We shall denote this action by $T_g:P^\natural\overset{\sim}{\to}P^\natural$ for each functorial point g of $G^\natural$.*

4. *An action of Y on $(P^\natural,\mathcal{L}^\natural)$ extending the action of Y on $(G^\natural_\eta,\pi^*\mathcal{M}_\eta)$. We shall denote this action by $S_y:P^\natural\overset{\sim}{\to}P^\natural$ and $\tilde{S}_y:S_y^*\mathcal{L}^\natural\overset{\sim}{\to}\mathcal{L}^\natural$ for each $y\in Y$. (Since $P^\natural$ contains the identity section of $G^\natural$, this action defines for each $y\in Y$ an extension of $\iota(y)$ to a section of $P^\natural$ over S.)*

5. *An action of $G^\natural$ on the invertible sheaf $\mathcal{N}:=\mathcal{L}^\natural\underset{\mathscr{O}_{P^\natural}}{\otimes}\pi^*\mathcal{M}^{\otimes -1}$ over $P^\natural$ extending the translation action of $G^\natural$ on $\mathscr{O}_{G^\natural}$. We shall denote this action by $\tilde{T}_g:T_g^*\mathcal{N}\overset{\sim}{\to}\mathcal{N}$ for each functorial point g of $G^\natural$.*

Moreover, the above data are required to satisfy the following conditions:

(i) *There exists an open $G^\natural$-invariant subscheme $U\subset P^\natural$ of finite type over S such that $P^\natural=\underset{y\in Y}{\cup}S_y(U)$.*

(ii) *$\mathcal{L}^\natural$ is ample over $P^\natural$ in the sense that the complement of the zero sets of global sections of $(\mathcal{L}^\natural)^{\otimes n}$ for all $n\geq 1$, define a basis of the (Zariski) topology of $P^\natural$. (In [59, II, 4.5] Grothendieck gave several mutually equivalent definitions of ampleness, but only over quasi-compact schemes. As Mumford remarked in [95, §2], the generalization here seems to be the most suitable for our purpose.)*

(iii) *(**Completeness condition**) Let $\Upsilon(G^\natural)$ be the set of all valuations υ of the rational function field $K(G^\natural)$ of $G^\natural$ such that $\upsilon(R)\geq 0$, namely, the underlying set of **Zariski's Riemann surface** of $K(G^\natural)/R$. For each υ in $\Upsilon(G^\natural)$ with valuation ring R_υ, let $S_\upsilon:=\mathrm{Spec}(R_\upsilon)$, and denote by x_υ the center of υ on A (which exists because A is proper over S), which can be interpreted as an S_υ-valued point $\tilde{x}_\upsilon:S_\upsilon\to A$. For each $y\in Y$ and $\chi\in X$, let $I_{y,\chi}$ be the invertible R-submodule of K as in Definition 4.2.4.6. Then the completeness condition is: For each $\upsilon\in\Upsilon(G^\natural)$ such that $\upsilon(I)>0$, υ has a center on $P^\natural$ (which then necessarily lies on $P^\natural_0:=P^\natural\underset{S}{\times}S_0$) if, for each $y\in Y$, there exists an integer $n_y>0$ such that $\upsilon(I^{\otimes n_y}_{y,\phi(y)}\cdot\tilde{x}^*_\upsilon(\mathscr{O}_{\phi(y)}))\geq 0$.*

Remark 4.5.1.3. In the completeness condition (iii), we need to allow valuations of rank greater than one (see [116, Ch. VI] for a classical treatment on this topic).

Remark 4.5.1.4. Our completeness condition is much weaker than Mumford's [95, Def. 2.1(ii)] and Faltings–Chai's [42, Ch. III, Def. 3.1(3)], as we only require one direction of the implications in the special case that $\upsilon(I)>0$. We weaken the condition because the original condition is only used in Mumford's proof of [95, Prop. 3.3] to show that the irreducible components of $P^\natural_0$ are proper, and because our weaker version suffices.

We will prove the existence of relatively complete models by writing down explicitly the $\underline{\mathrm{Proj}}$ of an explicit graded $\mathscr{O}_A$-algebra, under a stronger assumption to be specified later. We shall follow the constructions in [95] and [42, Ch. III, §3] very closely.

Construction 4.5.1.5. Let $\pi:G^\natural\to A$ denote the canonical morphism, and let us write $\mathscr{O}_{G^\natural}$ instead of $\pi_*\mathscr{O}_{G^\natural}$ by abuse of notation as before. Let us define two graded $\mathscr{O}_A$-algebras

$$\mathcal{S}_1:=\mathscr{O}_A[\mathscr{O}_\chi\theta]_{\chi\in X}=\underset{n\geq 0}{\oplus}(\underset{\chi\in X}{\oplus}\mathscr{O}_\chi)\theta^n\cong\underset{n\geq 0}{\oplus}\mathscr{O}_{G^\natural}\theta^n$$

and

$$\mathcal{S}_2:=\mathscr{O}_A[\mathcal{M}\underset{\mathscr{O}_A}{\otimes}\mathscr{O}_\chi\theta]_{\chi\in X}=\underset{n\geq 0}{\oplus}(\underset{\chi\in X}{\oplus}(\mathcal{M}^{\otimes n}\underset{\mathscr{O}_A}{\otimes}\mathscr{O}_\chi))\theta^n,$$

where θ is a free variable of degree 1 giving the gradings.

Note that $G^\natural$ acts on $\mathscr{O}_{G^\natural}$ by translation, or by the $\mathscr{O}_{A \underset{S}{\times} A}$-algebra morphism $m^* : m_A^* \mathscr{O}_{G^\natural} \to \mathrm{pr}_1^* \mathscr{O}_{G^\natural} \underset{\mathscr{O}_{G^\natural \underset{S}{\times} G^\natural}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_{G^\natural}$, whose restriction to each weight-χ subspace factors through the isomorphism $m_\chi^* : m_A^* \mathscr{O}_\chi \overset{\sim}{\to} \mathrm{pr}_1^* \mathscr{O}_\chi \underset{\mathscr{O}_{A \underset{S}{\times} A}}{\otimes} \mathrm{pr}_2^* \mathscr{O}_\chi$ given by the theorem of the square over $A \underset{S}{\times} A$. For each $g \in G^\natural$, these can be written as morphisms of $\mathscr{O}_A$-modules

$$\tilde{T}_g : T_{\pi(g)}^* \mathscr{O}_\chi \overset{\sim}{\to} \mathscr{O}_\chi,$$

which induces an action of $G^\natural$ on $\mathcal{S}_1$ given by

$$\tilde{T}_g : T_{\pi(g)}^* \mathscr{O}_\chi \overset{\sim}{\to} \mathscr{O}_\chi$$

covering the translation by $\pi(g)$ on A. In particular, if $t \in T$, then $\pi(t) = e_A$, and we have

$$\tilde{T}_t : \mathscr{O}_\chi \overset{\sim}{\to} \mathscr{O}_\chi$$

given exactly by multiplication by $\chi(t)$.

Recall that (in Section 4.3.1) the Y-action on $(\pi^* \mathcal{M})_\eta \cong \underset{\chi \in X}{\oplus} (\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta})$ is given by

$$\begin{aligned} \psi(y)\tau(y,\chi) : T_{c^\vee(y)}^* (\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) & \\ \cong \mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\chi(c^\vee(y)) & \overset{\sim}{\to} \mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta}. \end{aligned}$$

This action extends to an action on the whole of $\mathcal{S}_{2,\eta}$ by

$$\tilde{S}_y = \psi(y)^n \tau(y,\chi) : \ T_{c^\vee(y)}^* (\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \overset{\sim}{\to} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+n\phi(y)},$$

which covers the translation by $c^\vee(y)$ on A. Note that this agrees with $\tilde{T}_{\iota(y)}$ when $n = 0$.

Following [95] and [42], we define a *star* $\bigstar$ in X to be a finite subset of X generating X (as a commutative group) such that $0 \in \bigstar$ and $-\bigstar = \bigstar$.

Let $\bigstar$ be such a star; we define two subsheaves of graded $\mathscr{O}_A$-algebras of $\mathcal{S}_{1,\eta}$ and $\mathcal{S}_{2,\eta}$, respectively, by

$$\mathcal{R}_{1,\bigstar} := \mathscr{O}_A[(I_y \cdot I_{y,\alpha} \cdot \mathscr{O}_{\alpha+\phi(y)})\theta]_{y \in Y, \alpha \in \bigstar}$$

and

$$\begin{aligned} \mathcal{R}_{2,\bigstar} &:= \mathscr{O}_A[(I_y \cdot I_{y,\alpha} \cdot \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\alpha+\phi(y)})\theta]_{y \in Y, \alpha \in \bigstar} \\ &\cong \mathscr{O}_A[\tilde{S}_y(T_{c^\vee(y)}^*(\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\alpha))\theta]_{y \in Y, \alpha \in \bigstar}. \end{aligned}$$

Here we have used the isomorphisms $\psi(y) : \mathcal{M}(c^\vee(y)) \overset{\sim}{\to} \underline{I}_y$ and $\tau(y,\alpha) : \mathscr{O}_\chi(c^\vee(y)) \overset{\sim}{\to} \underline{I}_{y,\alpha}$ by the very definitions of $\underline{I}_y$ and $\underline{I}_{y,\alpha}$ (see Definitions 4.2.4.5 and 4.2.4.6).

By construction of $\underline{\mathrm{Proj}}$, we have a canonical isomorphism $\underline{\mathrm{Proj}}_{\mathscr{O}_A}(\mathcal{R}_{1,\bigstar}) \cong \underline{\mathrm{Proj}}_{\mathscr{O}_A}(\mathcal{R}_{2,\bigstar})$. We shall denote this scheme (up to the above-mentioned isomorphism) by $P^\natural = P^\natural_{(\phi,\psi),\bigstar}$, with structural morphism $\pi : P^\natural \to A$. Regarding $P^\natural_{(\phi,\psi),\bigstar}$ as $\underline{\mathrm{Proj}}_{\mathscr{O}_A}(\mathcal{R}_{1,\bigstar})$ endows it with an invertible sheaf $\mathcal{N}$ that is relatively ample over A (in the appropriate sense for morphisms only locally of finite type as in Definition 4.5.1.2), because sections of powers of $\mathcal{N}$ must give a basis of $\underline{\mathrm{Proj}}_{\mathscr{O}_A}(\mathcal{R}_{1,\bigstar})$ by the very construction of $\underline{\mathrm{Proj}}$. On the other hand, regarding it as $\underline{\mathrm{Proj}}_{\mathscr{O}_A}(\mathcal{R}_{2,\bigstar})$ endows it with an *ample* invertible sheaf $\mathcal{L}^\natural = \mathcal{N} \underset{\mathscr{O}_{P^\natural}}{\otimes} \pi^* \mathcal{M}$ (again in the sense of Definition 4.5.1.2) because $\mathcal{M}$ is ample.

The pair $(P^\natural_{(\phi,\psi),\bigstar}, \mathcal{L}^\natural)$ inherits a natural $G^\natural$-action from the action $\tilde{T}$ on $\mathcal{S}_{1,\eta}$, because $T_{\pi(g)}^* \mathcal{R}_{1,\bigstar}$, for each functorial point g of $G^\natural$, is generated by $T_{\pi(g)}^*(I_y \cdot I_{y,\alpha} \cdot \mathscr{O}_{\alpha+\phi(y)})\theta$, and we have

$$\tilde{T}_g : T_{\pi(g)}^*(I_y \cdot I_{y,\alpha} \cdot \mathscr{O}_{\alpha+\phi(y)}) \overset{\sim}{\to} I_y \cdot I_{y,\alpha} \cdot \mathscr{O}_{\alpha+\phi(y)}$$

because I_y and $I_{y,\alpha}$ are invertible R-submodules in K, which do not intervene with the translation by $\pi(g) : S \to A$. We shall denote this action by $T_g : P^\natural \overset{\sim}{\to} P^\natural$ for each functorial point g of $G^\natural$.

Similarly, the pair $(P^\natural_{(\phi,\psi),\bigstar}, \mathcal{L}^\natural)$ inherits a natural Y-action from the action $\tilde{S}$ on $\mathcal{S}_{2,\eta}$, because $T_{c^\vee(z)}^* \mathcal{R}_{2,\bigstar}$, for $z \in Y$, is generated by $T_{c^\vee(z)}^* \tilde{S}_y(T_{c^\vee(y)}^*(\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\alpha))\theta$, and because $\tilde{S}_z T_{c^\vee(z)}^* \tilde{S}_y T_{c^\vee(y)}^* = \tilde{S}_{z+y} T_{c^\vee(z+y)}^*$ by the definition of $\tilde{S}$. We shall denote this action by $S_z : P^\natural \overset{\sim}{\to} P^\natural$ and $\tilde{S}_z : S_z^* \mathcal{L}^\natural \overset{\sim}{\to} \mathcal{L}^\natural$. (This finishes Construction 4.5.1.5.)

To make $P^\natural_{(\phi,\psi),\bigstar}$ a relatively complete model, we need one additional condition:

Condition 4.5.1.6. *We have $I_y \cdot I_{y,\alpha} \subset R$ for all $y \in Y$ and all $\alpha \in \bigstar$.*

This condition is not necessarily satisfied by an arbitrary split object $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R, I)$.

Lemma 4.5.1.7 (cf. [95, Lem. 1.4] and [42, Ch. III, Prop. 3.2])**.** *Let R be a noetherian normal domain with fraction field K. Let $a, a' : Y \to \mathrm{Inv}(R)$ be multiplicatively quadratic functions, $b : Y \times X \to \mathrm{Inv}(R)$ a bimultiplicative function, and $\phi, \phi' : Y \to X$ homomorphisms between free commutative groups of the same finite rank such that ϕ is injective and such that*

$$a(y_1 + y_2) = a(y_1) \cdot a(y_2) \cdot b(y_1, \phi(y_2))$$

and

$$a'(y_1 + y_2) = a'(y_1) \cdot a'(y_2) \cdot b(y_1, \phi'(y_2))$$

for all $y_1, y_2 \in Y$. Suppose moreover that $a(y) \subset R$ and $a'(y) \subset R$ for all but finitely many $y \in Y$. Then, for each α in X, there exists an integer $n_0 > 0$ such that, for all $y \in Y$ and $n \geq n_0$,

$$a(y) \cdot b(y, 2n\phi(y)) \cdot b(y, \alpha) \subset R,$$

and

$$a'(y) \cdot b(y, 2n\phi(y)) \cdot b(y, \alpha) \subset R.$$

Proof. Since X and Y are finitely generated, all but finitely many discrete valuations $\upsilon \in \Upsilon_1$ satisfy $\upsilon(a(y)) = 0$, $\upsilon(a'(y)) = 0$, and $\upsilon(b(y,\chi)) = 0$ for all $y \in Y$ and $\chi \in X$. Hence, by Lemma 4.2.4.2, we are reduced to the case where R is a discrete valuation ring, by taking the maximum of the n_0's obtained from (the finitely many) such cases. Let us take any $\upsilon \in \Upsilon_1$. Then $\upsilon \circ a$ is a real-valued quadratic function, whose associated bilinear pairing $\upsilon \circ b$ is symmetric and positive semidefinite on $Y \times \phi(Y) \subset Y \times X$.

Note that $\upsilon(b(\,\cdot\,, \phi(\,\cdot\,)))$ defines a positive semidefinite symmetric bilinear pairing on $(Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R})$, which restricts to $\upsilon(b(\,\cdot\,,\,\cdot\,))$ on $Y \times X$ when we realize both Y and X as lattices inside $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$.

Suppose $y \in Y$ is not in $\mathrm{Rad}(\upsilon(b(\,\cdot\,, \phi(\,\cdot\,))))$, the radical (namely, the annihilator of the whole space) of the positive semidefinite symmetric bilinear pairing

$\upsilon(b(\cdot,\phi(\cdot)))$ on Y. Then there is some $z \in Y$ such that $\upsilon(b(y,\phi(z))) \neq 0$. If $\upsilon(b(y,\phi(y))) = 0$, then $\upsilon(b(ny+z,\phi(ny+z))) = 2n\upsilon(b(y,\phi(z))) + \upsilon(b(z,\phi(z))) < 0$ for some $n \in \mathbb{Z}$, which contradicts the positive semidefiniteness of $\upsilon(b(\cdot,\phi(\cdot)))$. As a result, $\upsilon(b(y,\phi(y))) > 0$ if and only if $y \notin \mathrm{Rad}(\upsilon(b(\cdot,\phi(\cdot))))$.

We claim that it suffices to show that there is an integer $n_1 > 0$ such that $\upsilon(b(y,2n\phi(y)) \cdot b(y,\alpha)) \geq 0$ for all $y \in Y$ and $n \geq n_1$.

This is because there are only finitely many $y \in Y$ such that $\upsilon(a(y)) < 0$ or $\upsilon(a'(y)) < 0$. If $\upsilon(a(y)) < 0$, then, as in Section 4.2.4, the fact that $a(ky) \subset R$ for all but finitely many $k \in \mathbb{Z}$ forces $\upsilon(b(y,\phi(y))) > 0$. Similarly, if $\upsilon(a'(y)) < 0$, then $\upsilon(b(y,\phi'(y))) > 0$ shows that $y \notin \mathrm{Rad}(\upsilon(b(\cdot,\phi(\cdot))))$, and hence $\upsilon(b(y,\phi(y))) > 0$. In either case, there is an integer $n_2 > 0$ such that $\upsilon(a(y)b(y,n_2\phi(y))) \geq 0$ and $\upsilon(a'(y))b(y,n_2\phi(y)) \geq 0$ for all $y \in Y$, and it suffices to take $n_0 = n_1 + n_2$. This proves the claim.

Now let us assume that $\upsilon(b(\cdot,\phi(\cdot)))$ is positive definite by replacing $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ with its quotient by $\mathrm{Rad}(\upsilon(b(\cdot,\phi(\cdot))))$. The images of Y and X under this quotient are again lattices because $\mathrm{Rad}(\upsilon(b(\cdot,\phi(\cdot))))$ is rationally defined. Then $\upsilon(b(\cdot,\phi(\cdot)))$ defines a norm $\|\cdot\|_\upsilon$ defined by $\|y\|_\upsilon := \upsilon(b(y,\phi(y)))^{1/2}$ on the real vector space $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, and we have the Cauchy–Schwarz inequality $|\upsilon(b(y,\alpha))| \leq \|y\|_\upsilon \|\alpha\|_\upsilon$. As a result, we have

$$\upsilon(b(y,2n\phi(y))b(y,\alpha)) \geq 2n\|y\|_\upsilon^2 - \|y\|_\upsilon\|\alpha\|_\upsilon \geq (2n\|y\|_\upsilon - \|\alpha\|_\upsilon)\|y\|_\upsilon.$$

Now it suffices to show that there exists an integer $n_1 > 0$ such that $\|y\|_\upsilon \geq \frac{1}{2n_1}\|\alpha\|_\upsilon$ for all $y \neq 0$ in Y. Since Y is a lattice in $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, which is discrete with respect to the topology defined by $\|\cdot\|_\upsilon$, there is an open ball $\{y \in Y \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \|y\|_\upsilon < r\}$ that contains no element of Y but 0. Hence we can take any integer $n_1 > \frac{1}{2r}\|\alpha\|_\upsilon$. □

Corollary 4.5.1.8. *Given an object $(A,\mathcal{M},X,Y,\phi,c,c^\vee,\tau,\psi)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$, and given an invertible sheaf $\mathcal{M}'$ over A that is either ample or trivial, suppose that we have ϕ' and ψ' such that*

$$(A,X,Y,\phi',c,c^\vee,(\mathcal{L}^\natural)' = \pi^*\mathcal{M}',\tau,\psi')$$

defines an object in $\mathrm{DD}_{\mathrm{IS}}(R,I)$. Then there is an integer $n_0 > 0$ such that, for every $n \geq n_0$, both the tuples

$$(A,\mathcal{M}^{\otimes n+1} \underset{\mathscr{O}_A}{\otimes} [-1]^*\mathcal{M}^{\otimes n},X,Y,(2n+1)\phi,c,c^\vee,\tau,\psi^{n+1}[-1]^*\psi^n)$$

and

$$(A,\mathcal{M}' \underset{\mathscr{O}_A}{\otimes} \mathcal{M}^{\otimes n} \underset{\mathscr{O}_A}{\otimes} [-1]^*\mathcal{M}^{\otimes n},X,Y,\phi'+2n\phi,c,c^\vee,\tau,\psi'\psi^n[-1]^*\psi^n)$$

satisfy Condition 4.5.1.6.

Proof. Let I_y and $I_{y,\alpha}$ be defined by $(A,\mathcal{M},X,Y,\phi,c,c^\vee,\tau,\psi)$ as usual (see Definitions 4.2.4.5 and 4.2.4.6). Let I'_y be either trivial, in the case that $\mathcal{M}'$ is trivial, or otherwise defined by the tuple $(A,\mathcal{M}',X,Y,\phi',c,c^\vee,\tau,\psi')$ as usual.

Let us consider the functions $a : Y \to \mathrm{Inv}(R) : y \mapsto I_y$, $a' : Y \to \mathrm{Inv}(R) : y' \mapsto I'_y$, and $b : Y \times X \to \mathrm{Inv}(R) : (y,\chi) \mapsto I_{y,\chi}$, which together with ϕ and ϕ' satisfy the requirement of Lemma 4.5.1.7. Then there is an integer $n_0 \geq 2$ such that $I_y \cdot I_{y,2n\phi(y)} \cdot I_{y,\alpha} \subset R$ and $I'_y \cdot I_{y,2n\phi(y)} \cdot I_{y,\alpha} \subset R$ for all $n \geq n_0$ and all of the finitely many $\alpha \in \bigstar$. Note that Condition 4.5.1.6 for the tuple

$$(A,\mathcal{M}^{\otimes n+1} \underset{\mathscr{O}_A}{\otimes} [-1]^*\mathcal{M}^{\otimes n},X,Y,(2n+1)\phi,c,c^\vee,\tau,\psi^{n+1}[-1]^*\psi^n)$$

is given by

$$I_{(n+1)y} \cdot I_{-ny} \cdot I_{y,\alpha} = I_y \cdot I_{y,n(n+1)\phi(y)} \cdot I_{y,\alpha} \subset R,$$

and Condition 4.5.1.6 for the tuple

$$(A,\mathcal{M}' \underset{\mathscr{O}_A}{\otimes} \mathcal{M}^{\otimes n} \underset{\mathscr{O}_A}{\otimes} [-1]^*\mathcal{M}^{\otimes n},X,Y,\phi'+2n\phi,c,c^\vee,\tau,\psi'\psi^n[-1]^*\psi^n)$$

is given by

$$I'_y \cdot I_{ny} \cdot I_{-ny} \cdot I_{y,\alpha} = I'_y \cdot I_{y,n^2\phi(y)} \cdot I_{y,\alpha} \subset R,$$

which are both satisfied because $n(n+1) \geq n^2 \geq 2n$ and $I_{y,\phi(y)} \subset R$. □

By the very definition of $\underline{\mathrm{Proj}}$, the scheme $P^\natural_{(\phi,\psi),\bigstar}$ is covered by open subschemes

$$U_{y,\alpha} := \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[I_y^{-1} \cdot I_{y,\alpha}^{-1} \cdot I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(z-y)}]_{z\in Y,\beta\in\bigstar})$$

relatively affine over A, with $y \in Y$ and $\alpha \in \bigstar$, which are all integral schemes over S. Moreover, for each $z \in Y$, it is clear from the construction that S_z maps $U_{y,\alpha}$ to $U_{z+y,\alpha}$. Therefore, there are only finitely many Y-orbits in the collection $\{U_{y,\alpha}\}_{y\in Y,\alpha\in\bigstar}$, with representatives given by $\{U_{0,\alpha}\}_{\alpha\in\bigstar}$.

Lemma 4.5.1.9. *If Condition 4.5.1.6 is satisfied, then the open subscheme $U_{0,0}$ of $P^\natural_{(\phi,\psi),\bigstar}$ is isomorphic to $G^\natural$.*

Proof. If $I_z \cdot I_{z,\beta} \subset R$ for all $z \in Y$ and $\alpha \in \bigstar$, then we have $U_{0,0} = \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta+\phi(z)}]_{z\in Y,\beta\in\bigstar}) = \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[\mathscr{O}_\beta]_{\beta\in\bigstar}) \cong G^\natural$. □

Lemma 4.5.1.10 (cf. [95, Lem. 1.3]). *Suppose that for every y in $Y - \{0\}$, we are given an integer $n_y > 0$. Then there exist finitely many $y_1,\ldots,y_k \in Y - \{0\}$, and a finite set $Q \subset Y$ such that, for each $z \in Y - Q$,*

$$I_{y_i,\phi(z)} = I_{z,\phi(y_i)} \subset I^{\otimes n_{y_i}}_{y_i,\phi(y_i)} \tag{4.5.1.11}$$

for some y_i, $1 \leq i \leq k$.

Proof. Since Y is finitely generated, all but finitely many discrete valuations $\upsilon \in \Upsilon_1$ satisfy $\upsilon(I_{y,\phi(z)}) = 0$ for all $y,z \in Y$. Hence, by Lemma 4.2.4.2, it suffices to fix a choice of a discrete valuation $\upsilon \in \Upsilon_1$ and verify (4.5.1.11) by evaluating υ.

Note that $B(y,z) := \upsilon(I_{y,\phi(z)})$ defines a positive semidefinite symmetric pairing on $Y \times Y$, which extends by linearity to a positive semidefinite symmetric pairing on $(Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R})$. For each $y \in Y - \{0\}$, let

$$C_y := \left\{ \begin{matrix} z \in Y \underset{\mathbb{Z}}{\otimes} \mathbb{R} : B(y,z) > n_y B(y,y) \text{ for all } i \\ \text{such that } y \notin \mathrm{Rad}(B), \text{ the radical of } B \end{matrix} \right\}.$$

Then C_y is a nonempty convex open subset of $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, and $r \cdot C_y \subset C_y$ if $r \in \mathbb{R}_{\geq 1}$. We claim that $\bigcup_{N=1}^{\infty} \bigcup_{y\in Y-\{0\}} \frac{1}{N} C_y = Y \underset{\mathbb{Z}}{\otimes} \mathbb{R} - \{0\}$.

Since the radicals $\mathrm{Rad}(B)$ are all spanned by elements in Y, they are rationally defined subspaces. Therefore, for each $z \in Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, we can find $y \in Y \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ such that

1. if $z \in \mathrm{Rad}(B)$, then $y \in \mathrm{Rad}(B)$ as well;
2. if $z \notin \mathrm{Rad}(B)$, then $B(y,z) > 0$.

That is, we can approximate each $z \in Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ by an element in $Y \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ in the intersection of those $\mathrm{Rad}(B)$ containing z. On the other hand, for those $\mathrm{Rad}(B)$ not containing z, we have $B(z,z) > 0$ (as in the proof of Lemma 4.5.1.7), and hence $B(y,z) > 0$ when y is close to z. Therefore $N \cdot z \in C_y$ if N is sufficiently large, and hence $z \in \frac{1}{N} C_y$. This proves the claim.

By compactness of the unit sphere in $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ (with respect to the Euclidean norm defined by any basis), there exist finitely many $y_j \in Y - \{0\}$ and integers $N_j > 0$, such that $\bigcup_{j=1}^{l} \frac{1}{N_j} C_{y_j} \supset$ (unit sphere). Then we also have $\bigcup_{j=1}^{l} C_{y_j} \supset \bigcup_{\substack{r \in \mathbb{R} \\ r \geq N := \max_{1\leq j\leq l}(N_j)}} r \cdot$ (unit sphere). Let Q be the set of elements of Y that lie inside the ball of radius $N = \max_{1\leq j\leq l}(N_j)$, which is finite because Y is discrete. Then, for each $z \in Y - Q$, there is a y_j such that $B(z, y_j) \geq n_{y_j} B(y_j, y_j)$, as desired. $\square$

Proposition 4.5.1.12 (cf. [95, Prop. 2.4]). *If Condition 4.5.1.6 is satisfied, then the open subscheme $U_{0,\alpha}$ of $P^\natural$ is of finite type over S for every $\alpha \in \bigstar$.*

Proof. For each $\beta \in \bigstar$ and $z \in Y$ set
$$\mathscr{M}_{z,\beta} := I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(z)}.$$
Then $U_{0,\alpha} = \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[\mathscr{M}_{z,\beta}]_{z\in Y, \beta \in \bigstar})$. We would like to show that it suffices to take finitely many $\mathscr{M}_{z,\beta}$'s as "generators".

For each $y \in Y$, we have the following relations:
$$\begin{aligned}\mathscr{M}_{z,\beta} &= I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(z)} = I_{y+(z-y)} \cdot I_{y+(z-y),\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(y)+\phi(z-y)} \\ &= I_y \cdot I_{z-y} \cdot I_{y,\phi(z-y)} \cdot I_{y,\alpha} \cdot I_{y,\beta-\alpha} \cdot I_{z-y,\beta} \cdot \mathscr{O}_{\phi(y)} \cdot \mathscr{O}_{\beta-\alpha+\phi(z-y)} \\ &= I_{y,\phi(z-y)} \cdot I_{y,\beta-\alpha} \cdot (I_y \cdot I_{y,\alpha} \cdot \mathscr{O}_{\phi(y)}) \cdot (I_{z-y} \cdot I_{z-y,\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(z-y)}) \\ &= I_{y,\phi(z-y)+\beta-\alpha} \cdot \mathscr{M}_{y,\alpha} \cdot \mathscr{M}_{z-y,\beta}.\end{aligned}$$
For each integer n, let us write
$$\begin{aligned}&I_{y,\phi(z-y)+\beta-\alpha} \\ &= I_{y,\phi(z)} \cdot I_{y,\phi(y)}^{\otimes -1} \cdot I_{y,\beta-\alpha} = (I_{y,\phi(z)} \cdot I_{y,\phi(y)}^{\otimes -n}) \cdot (I_{y,\phi(y)}^{\otimes n-2} \cdot I_{y,\beta-\alpha}) \cdot I_{y,\phi(y)}.\end{aligned}$$
By Lemma 4.5.1.7, there is an integer $n_0 > 0$ such that $I_{y,\phi(y)}^{\otimes n_0 - 2} \cdot I_{y,\beta-\alpha} \subset R$ for all y and for all of the finitely many $\beta \in \bigstar$. By Lemma 4.5.1.10, there is a finite subset $\{y_1, \ldots, y_k\}$ of nonzero elements in Y and a finite subset $Q \subset Y$ such that, for each $z \in Y - Q$, there is some y_i such that $I_{y_i,\phi(z)} \cdot I_{y_i,\phi(y_i)}^{\otimes -n_0} \subset R$. By the positivity condition of τ, we have $I_{y_i,\phi(y_i)} \subset I$, because $y_i \neq 0$. Hence, for each $z \in Y - Q$, there is some y_i such that
$$\mathscr{M}_{z,\beta} \subset I \cdot \mathscr{M}_{y_i,\alpha} \cdot \mathscr{M}_{z-y_i,\beta}.$$
We may repeat this process as long as $z - y_i \notin Q$. We claim that this process always stops in a finite number of steps. Then the proposition will follow, because we only need the finitely many generators $\mathscr{M}_{y_i,\alpha}$ and $\mathscr{M}_{q,\beta}$ for $q \in Q$ (instead of all of $\mathscr{M}_{z,\beta}$).

By Condition 4.5.1.6,
$$\mathscr{M}_{z,\beta} = I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta-\alpha+\phi(z)} \subset \mathscr{O}_{\beta-\alpha+\phi(z)}$$
for all $z \in Y$ and $\beta \in \bigstar$. If we have
$$\begin{aligned}\mathscr{M}_{z,\beta} &\subset I \cdot \mathscr{M}_{y_{i_1},\alpha} \cdot \mathscr{M}_{z-y_{i_1},\beta} \subset I^2 \cdot \mathscr{M}_{y_{i_1},\alpha} \cdot \mathscr{M}_{y_{i_2},\alpha} \cdot \mathscr{M}_{z-y_{i_1}-y_{i_2},\beta} \\ &\subset \cdots \subset I^m \cdot \mathscr{M}_{y_{i_1},\alpha} \cdot \mathscr{M}_{y_{i_2},\alpha} \cdot \cdots \cdot \mathscr{M}_{y_{i_m},\alpha} \cdot \mathscr{M}_{z-\sum\limits_{1\leq i\leq m} y_{i_m},\beta},\end{aligned}$$
then
$$\begin{aligned}\mathscr{M}_{z,\beta} &\subset I^m \cdot \mathscr{O}_{\phi(y_{i_1})} \cdot \mathscr{O}_{\phi(y_{i_2})} \cdot \cdots \cdot \mathscr{O}_{\phi(y_{i_m})} \cdot \mathscr{O}_{\beta-\alpha+\phi(z)-\sum\limits_{1\leq i\leq m}\phi(y_{i_m})} \\ &\subset I^m \cdot \mathscr{O}_{\beta-\alpha+\phi(z)}.\end{aligned}$$
If this happens for all $m > 0$, then $\mathscr{M}_{z,\beta} \subset \bigcap_{m=1}^{\infty}(I^m \cdot \mathscr{O}_{\beta-\alpha+\phi(z)}) = 0$, because $\mathscr{O}_{\beta-\alpha+\phi(z)}$ is an invertible sheaf over the abelian scheme A over R, and because R is noetherian; but this is impossible. Therefore the process must stop in a finite number of steps, as claimed. $\square$

Corollary 4.5.1.13. *If Condition 4.5.1.6 is satisfied, then the affine open subscheme $U_{y,\alpha}$ of $P^\natural$ is of finite type over S for all $y \in Y$ and all $\alpha \in \bigstar$.*

Proof. This is because $U_{y,\alpha} = S_y(U_{0,\alpha})$. $\square$

Corollary 4.5.1.14. *Suppose Condition 4.5.1.6 is satisfied. If we take U to be the finite union of those $U_{0,\alpha}$ with α running over elements in $\bigstar$, then U is of finite type over S, and we have $P^\natural = \bigcup_{y\in Y} S_y(U)$. In particular, $P^\natural$ is locally of finite type.*

Proposition 4.5.1.15 (cf. [95, Thm. 2.5] and [42, Ch. III, Prop. 3.3]). *If Condition 4.5.1.6 is satisfied, then the pair $(P^\natural_{(\phi,\psi),\bigstar}, \mathcal{L}^\natural)$ given in Construction 4.5.1.5 is a relatively complete model.*

Proof. So far we have constructed $(P^\natural_{(\phi,\psi),\bigstar}, \mathcal{L}^\natural)$ with all the data 1–5 in Definition 4.5.1.2, and we have verified conditions (i) and (ii) in the definition as well. It only remains to verify condition (iii), namely, the *completeness condition.* That is, for each $\upsilon \in \Upsilon(G^\natural)$ (with center x_υ on A) such that $\upsilon(I) > 0$ and such that, for each $y \in Y$, there exists an integer $n_y > 0$ such that $\upsilon(I_{y,\phi(y)}^{\otimes n_y} \cdot \tilde{x}_\upsilon^*(\mathscr{O}_{\phi(y)})) \geq 0$, we need to show that υ has a center on $P^\natural_{(\phi,\psi),\bigstar}$.

Since $P^\natural_{(\phi,\psi),\bigstar}$ is the union of $U_{y,\alpha}$, each of which is relatively affine over A, the valuation υ has a center on $P^\natural_{(\phi,\psi),\bigstar}$ if it has one on some $\tilde{x}_\upsilon^*(U_{y,\alpha})$. For each $z \in Y$ and $\beta \in \bigstar$, set
$$\mathscr{N}_{z,\beta} := I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta+\phi(z)},$$
so that
$$\tilde{x}_\upsilon^*(U_{y,\alpha}) \cong \mathrm{Spec}(R_\upsilon[\tilde{x}_\upsilon^*(\mathscr{N}_{y,\alpha})^{\otimes -1} \underset{R_\upsilon}{\otimes} \tilde{x}_\upsilon^*(\mathscr{N}_{z,\beta})]_{z\in Y, \beta\in\bigstar}).$$
Then υ has a center on $P^\natural_{(\phi,\psi),\bigstar}$ if $\min_{z\in Y} \upsilon(\tilde{x}_\upsilon^* \mathscr{N}_{z,\beta})$ exists for each of the finitely many $\beta \in \bigstar$. Let us fix a choice of β from now on.

Consider the following relations for each $y \in Y$ (cf. the proof of Proposition 4.5.1.12):
$$\begin{aligned}\mathscr{N}_{z,\beta} &= I_z \cdot I_{z,\beta} \cdot \mathscr{O}_{\beta+\phi(z)} = I_{y+(z-y)} \cdot I_{y+(z-y),\beta} \cdot \mathscr{O}_{\beta+\phi(y)+\phi(z-y)} \\ &= I_y \cdot I_{z-y} \cdot I_{y,\phi(z-y)} \cdot I_{y,\beta} \cdot I_{z-y,\beta} \cdot \mathscr{O}_{\phi(y)} \cdot \mathscr{O}_{\beta+\phi(z-y)} \\ &= I_{y,\phi(z-y)} \cdot I_{y,\beta} \cdot (I_y \cdot \mathscr{O}_{\phi(y)}) \cdot (I_{z-y} \cdot I_{z-y,\beta} \cdot \mathscr{O}_{\beta+\phi(z-y)}) \\ &= I_{y,\phi(z-y)+\beta} \cdot (I_y \cdot \mathscr{O}_{\phi(y)}) \cdot \mathscr{N}_{z-y,\beta}.\end{aligned}$$
For each integer n, let us write
$$\begin{aligned}&I_{y,\phi(z-y)+\beta} \cdot (I_y \cdot \mathscr{O}_{\phi(y)}) \\ &= I_{y,\phi(y)} \cdot (I_{y,\phi(z)} \cdot I_{y,\phi(y)}^{\otimes -2-n-n_y}) \cdot (I_{y,\phi(y)}^{\otimes n} \cdot I_{y,\beta}) \cdot (I_y \cdot I_{y,\phi(y)}^{\otimes n_y} \cdot \mathscr{O}_{\phi(y)}),\end{aligned}$$
where $n_y > 0$ is an integer such that $\upsilon(I_{y,\phi(y)}^{\otimes n_y} \cdot \tilde{x}_\upsilon^*(\mathscr{O}_{\phi(y)})) \geq 0$, which exists by assumption. By Lemma 4.5.1.7, there is an integer $n_0 > 0$ such that $I_{y,\phi(y)}^{\otimes n_0} \cdot I_{y,\beta} \subset R$ for all y. By Lemma 4.5.1.10, there is a finite subset $\{y_1, \ldots, y_k\}$ of $Y - \{0\}$ and a finite subset $Q \subset Y$ such that, for all $z \in Y - Q$, there is some y_i such that $I_{y_i,\phi(z)} \cdot I_{y_i,\phi(y_i)}^{\otimes -2-n_0-n_y} \subset R$. By the positivity condition of τ, we have $I_{y_i,\phi(y_i)} \subset I$

because $y_i \neq 0$. Hence, for each $z \in Y - Q$, there is some y_i such that

$$\mathscr{N}_{z,\beta} \subset I \cdot (I_{y_i} \cdot I_{y_i,\phi(y_i)}^{\otimes n_{y_i}} \cdot \mathscr{O}_{\phi(y_i)}) \cdot \mathscr{N}_{z-y_i,\beta},$$

and so that, by taking the value of υ at $\tilde{x}_\upsilon$,

$$\begin{aligned}\upsilon(\tilde{x}_\upsilon^* \mathscr{N}_{z,\beta}) &\geq \upsilon(I) + \upsilon(I_{y_i} \cdot I_{y_i,\phi(y_i)}^{\otimes n_{y_i}} \cdot \tilde{x}_\upsilon^*(\mathscr{O}_{\phi(y_i)})) + \upsilon(\tilde{x}_\upsilon^* \mathscr{N}_{z-y_i,\beta}) \\ &> \upsilon(\tilde{x}_\upsilon^* \mathscr{N}_{z-y_i,\beta}).\end{aligned}$$

Here we have used $I_y \subset R$ and $\upsilon(R) \geq 0$. This shows that the minimum of $\upsilon(\tilde{x}_\upsilon^* \mathscr{N}_{z,\beta})$ occurs in the finite set Q, and proves the existence of the center of υ on $P^\natural$ (or rather $P_0^\natural := P^\natural \underset{S}{\times} S_0$). □

4.5.2 Construction of the Quotient

Suppose now that we are given a relatively complete model $(P^\natural, \mathcal{L}^\natural)$ of an object $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ in $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split}}(R, I)$. (We do not assume that it satisfies Condition 4.5.1.6.) The goal of this section is to construct a "quotient" of $(G^\natural, \mathcal{L}^\natural)$ by Y in an appropriate sense.

Remark 4.5.2.1. To avoid unnecessary misunderstandings, let us emphasize that *nothing in Sections 4.5.2 and 4.5.3 is due to us.* The arguments in [95] are followed almost verbatim in most steps. (Although [42] has provided brief outlines for necessary modifications from the completely degenerate case to the general case, we have decided to supply more details for ease of understanding.) We hope that the works of Mumford, Faltings, and Chai are so well known that there should be no confusion.

Proposition 4.5.2.2 (cf. [95, Prop. 3.1] and [42, Ch. III, Prop. 4.1])**.** *For each $y \in Y$, we have a canonical embedding $\mathscr{O}_{\phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}] \hookrightarrow \pi_* \mathscr{O}_{P^\natural} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}]$. That is, sections of $\mathscr{O}_{\phi(y)}$ define regular functions on $P^\natural$ over* $\mathrm{Spec}(R[I_{y,\phi(y)}^{\otimes -1}])$.

Proof. By Lemma 4.5.1.7, for each $y \in Y$ and $\chi \in X$, there exists an integer $n > 0$ such that $I_{y,\phi(y)}^{\otimes n} \cdot I_{y,\chi} \subset R$, in which case the section $\tau(y, \chi)$ can be defined over $R[I_{y,\phi(y)}^{\otimes -1}]$. Therefore, $\iota(y)$ extends to an element of $G^\natural(R[I_{y,\phi(y)}^{\otimes -1}])$ for every $y \in Y$.

The translation action $T_{\iota(y)} : P^\natural \xrightarrow{\sim} P^\natural$ and the action $S_y : P^\natural \xrightarrow{\sim} P^\natural$ have to agree whenever they are both defined. Hence it makes sense to compare the $G^\natural$-action $\tilde{T}_{\iota(y)} : T_{\iota(y)}^* \mathscr{N} \xrightarrow{\sim} \mathscr{N}$ on $\mathscr{N}$ with the isomorphism $\tilde{S}_y : S_y^* \mathscr{N} \xrightarrow{\sim} \mathscr{N} \underset{\mathscr{O}_{P^\natural}}{\otimes} \pi^* \mathscr{O}_{\phi(y)}$ deduced from the Y-action $\tilde{S}_y : S_y^* \mathcal{L}^\natural \xrightarrow{\sim} \mathcal{L}^\natural$ on $\mathcal{L}^\natural$ over $R[I_{y,\phi(y)}^{\otimes -1}]$. This gives an isomorphism $\mathscr{N} \xrightarrow{\sim} \mathscr{N} \underset{\mathscr{O}_{P^\natural}}{\otimes} \pi^* \mathscr{O}_{\phi(y)}$ of invertible sheaves over $R[I_{y,\phi(y)}^{\otimes -1}]$, or equivalently an isomorphism $\zeta : \mathscr{O}_{P^\natural} \xrightarrow{\sim} \mathscr{O}_{P^\natural} \underset{\mathscr{O}_{P^\natural}}{\otimes} \pi^* \mathscr{O}_{\phi(y)}$ over $R[I_{y,\phi(y)}^{\otimes -1}]$, both extending the isomorphism $\mathscr{O}_{G^\natural} \xrightarrow{\sim} \mathscr{O}_{G^\natural} \underset{\mathscr{O}_{G^\natural}}{\otimes} \pi^* \mathscr{O}_{\phi(y)}$ by shifting the weights in the decomposition $\pi_* \mathscr{O}_{G^\natural} \cong \underset{\chi \in X}{\oplus} \mathscr{O}_\chi$ into weight subsheaves under the T-action.

The T-action on $P^\natural$ also gives a decomposition $\pi_* \mathscr{O}_{P^\natural} = \underset{\chi \in X}{\oplus} (\pi_* \mathscr{O}_{P^\natural})_\chi$. Since $G^\natural$ is open dense in $P^\natural$, we have a canonical embedding $(\pi_* \mathscr{O}_{P^\natural})_\chi \hookrightarrow \mathscr{O}_\chi$ for each $\chi \in X$. In particular, we have $(\pi_* \mathscr{O}_{P^\natural})_0 = \mathscr{O}_A$ because $\mathscr{O}_A$ is already contained in $\pi_* \mathscr{O}_{P^\natural}$ via the structural morphism $\pi : P^\natural \to A$. The isomorphism ζ above induces a collection of isomorphisms $(\pi_* \mathscr{O}_{P^\natural})_{\chi + \phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}] \xrightarrow{\sim} (\pi_* \mathscr{O}_{P^\natural})_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}]$. By taking $\chi = 0$, we obtain an isomorphism $(\pi_* \mathscr{O}_{P^\natural})_{\phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}] \xrightarrow{\sim} \mathscr{O}_{\phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}]$, whose inverse defines the desired embedding $\mathscr{O}_{\phi(y)} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}] \hookrightarrow \pi_* \mathscr{O}_{P^\natural} \underset{R}{\otimes} R[I_{y,\phi(y)}^{\otimes -1}]$. □

Corollary 4.5.2.3 (cf. [95, Cor. 3.2] and [42, Ch. III, Cor. 4.2])**.** *The open immersion $G^\natural \hookrightarrow P^\natural$ induces an isomorphism $G_\eta^\natural \xrightarrow{\sim} P_\eta^\natural$.*

Proof. Since there is some integer $N \geq 1$ such that $NX \subset \phi(Y)$, Proposition 4.5.2.2 implies that, for every $\chi \in X$, sections of $\mathscr{O}_{\chi,\eta}$ define regular functions on the normalization of $P_\eta^\natural$. Since $G^\natural \cong \underline{\mathrm{Spec}}_{\mathscr{O}_A} \Big(\underset{\chi \in X}{\oplus} \mathscr{O}_\chi \Big)$ is a normal subscheme of $P^\natural$, this forces $G^\natural \hookrightarrow P^\natural$ to be an isomorphism. □

The following technical lemma and its proof are quoted almost verbatim from [95, Lem. 3.4]:

Lemma 4.5.2.4. *Let $f : Z \to Z'$ be a morphism locally of finite type, with Z an irreducible scheme but Z' arbitrary. If f satisfies the valuative criterion for properness for all valuations, then f is proper.*

Proof. The usual valuative criterion (such as, for example, [59, II, 7.3]) would hold if we know that f is of finite type. It suffices to prove that f is quasi-compact. As this is a topological statement, we may replace Z with Z_{red}, and Z' with Z'_{red}. By working locally on the base, we may assume Z' is affine, say $\mathrm{Spec}(A)$. Finally, we may assume that f is dominant. Then A is a subring of the rational function field $K(Z)$ of Z. Let $\mathscr{Z}$ denote Zariski's Riemann surface of $K(Z)/A$, whose underlying set is the set of valuations υ on $K(Z)$ such that $\upsilon(A) \geq 0$ (see [116, p. 110] or [88, p. 73]). By the valuative criterion, every υ has a uniquely determined center on Z. Hence there is a natural map $\pi : \mathscr{Z} \to Z$ taking υ to its center, which is continuous and surjective. Since $\mathscr{Z}$ is a quasi-compact topological space (by [88, Thm. 10.5]), we see that Z is quasi-compact, as desired. □

Proposition 4.5.2.5 (cf. [95, Prop. 3.3])**.** *Every irreducible component of $P_0^\natural := P^\natural \underset{S}{\times} S_0$ is proper over $A_0 := A \underset{S}{\times} S_0$, or equivalently, over S_0.*

Proof. By Lemma 4.5.2.4, it suffices to show that if Z is any component of $P_0^\natural$, and if υ is any valuation of its rational function field $K(Z)$ such that $\upsilon(R_0) \geq 0$, then υ has a center on Z. To show this, let υ_1 be a valuation of $K(G^\natural)$ such that $\upsilon(R) \geq 0$ and such that the center of υ_1 is Z, and let υ_2 be the composite of the valuations υ and υ_1. Then $\upsilon_2(I) > 0$ because Z is a scheme over $\mathrm{Spec}(R_0)$. Let x_{υ_1} (resp. x_{υ_2}) be the center of υ_1 (resp. υ_2) on A, which exists by properness of A. By Proposition 4.5.2.2, for each $y \in Y$ there is an integer $n_y > 0$ such that $\upsilon_1(I_{y,\phi(y)}^{\otimes n_y} \cdot \tilde{x}_{\upsilon_1}^*(\mathscr{O}_{\phi(y)})) \geq 0$. Suppose $y \neq 0$. By the positivity condition of τ, we know that $I_{y,\phi(y)} \subset I$. Since υ_1 has a center on the scheme Z over $\mathrm{Spec}(R_0)$, we may increase n_y and assume that $\upsilon_1(I_{y,\phi(y)}^{\otimes n_y} \cdot \tilde{x}_{\upsilon_1}^*(\mathscr{O}_{\phi(y)})) > 0$, and hence that $\upsilon_2(I_{y,\phi(y)}^{\otimes n_y} \cdot \tilde{x}_{\upsilon_2}^*(\mathscr{O}_{\phi(y)})) > 0$. Otherwise, suppose $y = 0$. Then $\upsilon_2(I_{y,\phi(y)}^{\otimes n} \cdot \tilde{x}_{\upsilon_2}^*(\mathscr{O}_{\phi(y)})) = \upsilon_2(\tilde{x}_{\upsilon_2}^*(\mathscr{O}_A)) \geq 0$ by definition of x_{υ_2}. In either case, υ_2 has a center on $P^\natural$ by the completeness condition (iii) in Definition 4.5.1.2. This implies that υ has a center on Z, as desired. □

Corollary 4.5.2.6 (cf. [95, Cor. 3.5]). *Let $U_0 := U \underset{S}{\times} S_0$, where U is as in 1 of Definition 4.5.1.2. Then the closure $\overline{U_0}$ of U_0 in $P_0^\natural$ is proper over S_0.*

Proof. This is true because U is of finite type over S, and hence U_0 has only finitely many irreducible components. □

Proposition 4.5.2.7 (cf. [95, Prop. 3.6] and [42, Ch. III, Prop. 4.5]). *There is a finite subset $Q \subset Y$ such that $S_y(\overline{U_0}) \cap S_z(\overline{U_0}) = \emptyset$ if $y - z \neq Q$.*

Proof. For each functorial point g of $G^\natural$, we have by assumption the action isomorphism $\tilde{T}_g : T_g^*\mathscr{N} \xrightarrow{\sim} \mathscr{N}$, where $\mathscr{N} = \mathscr{L}^\natural \underset{\mathscr{O}_{P^\natural}}{\otimes} \pi^*\mathscr{M}^{\otimes -1}$. Since T acts trivially on A, each functorial point t of T acts canonically on $\pi^*\mathscr{M}^{\otimes -1}$ and gives an isomorphism $\tilde{T}_t : T_t^*\mathscr{L}^\natural \xrightarrow{\sim} \mathscr{L}^\natural$. On the other hand, for each $y \in Y$ we have by assumption the action isomorphism $\tilde{S}_y : S_y^*\mathscr{L}^\natural \xrightarrow{\sim} \mathscr{L}^\natural$. Therefore it makes sense to compare the two isomorphisms $\tilde{T}_t \circ T^*(\tilde{S}_y) : T_t^* \circ S_y^*(\mathscr{L}^\natural) \xrightarrow{\sim} \mathscr{L}^\natural$ and $\tilde{S}_y \circ S^*(\tilde{T}_t) : S_y^* \circ T_t^*(\mathscr{L}^\natural) \xrightarrow{\sim} \mathscr{L}^\natural$ with the same source and target. We claim that they satisfy the *commutation relation*

$$\tilde{T}_t \circ T^*(\tilde{S}_y) = \phi(y)(t)\, \tilde{S}_y \circ S^*(\tilde{T}_t). \tag{4.5.2.8}$$

It suffices to verify this over $P_\eta^\natural = G_\eta^\natural$. As in Section 4.2.3, étale locally, $\mathscr{L}_\eta^\natural$ can be decomposed into a sum of weight subsheaves $\mathscr{M}_{\chi,\eta}$. Since translation by $\iota(y)$ shifts the weights under the T-action by $\phi(y)$, the claim follows.

Consider the action of $T_0 := T \underset{S}{\times} S_0$ on P_0. Let F be the closed subset of P_0 which is the locus of geometric points fixed by T_0. Since T_0 is a connected solvable linear algebraic group, and since every irreducible component of P_0 is proper (by Proposition 4.5.2.5), Borel's fixed point theorem (see, for example, [112, Thm. 6.2.6]) shows that F intersects every irreducible component of P_0. Since U_0 is of finite type, the set $\overline{U_0} \cap F$ has only finitely many connected components $C_1, \ldots, C_n$. For each $1 \leq i \leq n$, T_0 acts on $\mathscr{L}^\natural|_{C_i}$ via a character χ_i. By (4.5.2.8), T_0 acts on $\mathscr{L}^\natural_{S_y(C_i)}$ via the character $\chi_i + \phi(y)$. Then we can conclude the proof by taking $Q := \phi^{-1}(\{\chi_i - \chi_j : 1 \leq i \leq n, 1 \leq j \leq n\})$. □

Corollary 4.5.2.9 (cf. [95, Cor. 3.7]). *The group Y acts freely on $P_0^\natural$.*

Proof. Since every nontrivial subgroup of Y is infinite, and since $P_0^\natural = \underset{y \in Y}{\cup} S_y(\overline{U_0})$, this follows from Proposition 4.5.2.7. □

Next we would like to show that $P_0^\natural$ is connected. The proof requires some preparation. Since R is I-adically complete, and since R has no idempotents, we know that R_0 has no idempotents either. That is, S_0 is connected. Therefore, $G_0^\natural$ is a connected open subscheme of $P_0^\natural$, and it determines a canonical connected component of $P_0^\natural$.

Let x be an arbitrary point of $P_0^\natural$. Let υ be a discrete valuation of $K(P^\natural) = K(G^\natural)$ (cf. Corollary 4.5.2.3) such that $\upsilon(R) \geq 0$ and has center x. Then υ induces a discrete valuation $\bar{\upsilon}$ of K with valuation ring $R_{\bar{\upsilon}}$. Let $S_{\bar{\upsilon}} := \mathrm{Spec}(R_{\bar{\upsilon}})$ and $P_{\bar{\upsilon}}^\natural := P^\natural \underset{S}{\times} S_{\bar{\upsilon}}$. Then S and $S_{\bar{\upsilon}}$ have the same generic point, and $P^\natural$ and $P_{\bar{\upsilon}}^\natural$ have the same generic fiber. Let $(P_{\bar{\upsilon}}^\natural)'$ be the closure of the generic fiber of $P_{\bar{\upsilon}}^\natural$ in $P_{\bar{\upsilon}}^\natural$. By definition, $(P_{\bar{\upsilon}}^\natural)'$ is an integral scheme with the same rational function field $K(G^\natural)$ as $P^\natural$, and it is locally of finite type over the valuation ring $R_{\bar{\upsilon}}$. Let $(P_{\bar{\upsilon}}^\natural)'_0$ be the fiber of $(P_{\bar{\upsilon}}^\natural)'$ over the closed point of $S_{\bar{\upsilon}}$. Then there is a natural morphism

$$(P_{\bar{\upsilon}}^\natural)'_0 \to P_0^\natural. \tag{4.5.2.10}$$

Lemma 4.5.2.11. *Both x and $G_0^\natural$ meet the image of (4.5.2.10).*

Proof. Let $R_\upsilon \subset K(G^\natural)$ be the valuation ring of υ. By definition of υ, we have a morphism $S_\upsilon := \mathrm{Spec}(R_\upsilon) \to P_{\bar{\upsilon}}^\natural$ taking the generic point of S_υ to the generic fiber of $P_{\bar{\upsilon}}^\natural$. Hence (4.5.2.10) factors through $(P_{\bar{\upsilon}}^\natural)'$. Then x lies in the image of (4.5.2.10) because the image of the closed point of S_υ in $(P_{\bar{\upsilon}}^\natural)'_0$ lies over x. As for $G^\natural$, since $G^\natural$ is smooth over S and $G^\natural \subset P^\natural$, we know that $G_{\bar{\upsilon}}^\natural := G^\natural \underset{S}{\times} S_{\bar{\upsilon}}$ is smooth over $S_{\bar{\upsilon}}$ and $G_{\bar{\upsilon}}^\natural \subset P_{\bar{\upsilon}}^\natural$. Since every point of $G_{\bar{\upsilon}}^\natural$ comes from specialization of a point in the generic fiber, this forces $G_{\bar{\upsilon}}^\natural \subset (P_{\bar{\upsilon}}^\natural)'$. Hence $G_0^\natural$ also meets the image of (4.5.2.10). □

We claim that $(P_{\bar{\upsilon}}^\natural)'$ satisfies a stronger completeness condition than condition (iii) of Definition 4.5.1.2:

Lemma 4.5.2.12 (cf. [95, Lem. 3.9]). *Let ϖ be a generator of the maximal ideal of $R_{\bar{\upsilon}}$, so that $(\varpi) = I \cdot R_{\bar{\upsilon}}$ in $R_{\bar{\upsilon}}$. Let υ' be a valuation of $K(G^\natural)$ such that $\upsilon'(R_{\bar{\upsilon}}) \geq 0$ and $\upsilon'(\varpi) > 0$. Let $x_{\upsilon'}$ be the center of υ' on $A_\upsilon := A \underset{S}{\times} S_\upsilon$, which exists because A_υ is proper over S_υ. Then υ' has a center on $(P_{\bar{\upsilon}}^\natural)'$ if and only if, for all $\chi \in X$, there is some integer $n_\chi > 0$ such that $-n_\chi \upsilon'(\varpi) \leq \upsilon'(\tilde{x}_{\upsilon'}^*(\mathscr{O}_\chi)) \leq n_\chi \upsilon'(\varpi)$.*

Proof. By the original completeness condition (iii) in Definition 4.5.1.2 for $P^\natural$, we know that υ' with $\upsilon'(\varpi) = \upsilon'(I \cdot R_{\bar{\upsilon}}) > 0$ has a center on $P_{\bar{\upsilon}}^\natural$ if, for each $y \in Y$, there exists an integer $n_y > 0$ such that $\upsilon'(I_{y,\phi(y)}^{\otimes n_y} \cdot \tilde{x}_{\upsilon'}^*(\mathscr{O}_{\phi(y)})) \geq 0$. Since $I_{y,\phi(y)} \cdot R_{\bar{\upsilon}} \subset I \cdot R_{\bar{\upsilon}} = (\varpi)$ when $y \neq 0$, and since $\phi(Y)$ has finite index in X, the above condition implies the condition in the statement of the lemma. Since this center necessarily lies in the closure of the generic fiber of $P_{\bar{\upsilon}}^\natural$, we see that the new condition holds for $(P_{\bar{\upsilon}}^\natural)'$.

Conversely, for each $y \in Y$, since $I_{y,\phi(y)} \subset I$ and $I \cdot R_{\bar{\upsilon}} = (\varpi)$ in $R_{\bar{\upsilon}}$, Proposition 4.5.2.2 implies that there is an integer $n_y > 0$ such that both $\upsilon'(\varpi^{n_y} \tilde{x}_{\upsilon'}^*(\mathscr{O}_{\phi(y)})) \geq 0$ and $\upsilon'(\varpi^{n_y} \tilde{x}_{\upsilon'}^*(\mathscr{O}_{-\phi(y)})) \geq 0$. Since $\phi(Y)$ has finite index in X, the analogous statement holds for $\chi \in X$ as well, and we obtain the condition in the lemma, as desired. □

Lemma 4.5.2.13. *The scheme $(P_{\bar{\upsilon}}^\natural)'_0$ is connected.*

Proof. For simplicity, in the proof of this lemma, let us replace R with $R_{\bar{\upsilon}}$, S with $S_{\bar{\upsilon}}$, $P^\natural$ with its integral subscheme $(P_{\bar{\upsilon}}^\natural)'$, and $G^\natural$ with $G_{\bar{\upsilon}}^\natural$. We shall also replace T with $T \underset{S}{\times} S_{\bar{\upsilon}}$ and A with $A_{\bar{\upsilon}}$, so that $G^\natural$ is still the extension of A by T, and so that $P^\natural$ is still a scheme over A. Let us take ϖ, as in Lemma 4.5.2.12, to be any element of R such that (ϖ) defines the maximal ideal of R. Then the old ideal I in the old R is replaced with the new maximal ideal (ϖ) in the new R.

Fix any choice of a basis $\chi_1, \ldots, \chi_r$ of X. For each $n > 0$, set

$$P^{(n)} := \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[\varpi^n \mathscr{O}_{\chi_1}, \varpi^n \mathscr{O}_{-\chi_1}, \ldots, \varpi^n \mathscr{O}_{\chi_r}, \varpi^n \mathscr{O}_{-\chi_r}]).$$

Locally over A, over which each of the $\mathscr{O}_{\chi_i}$, where $1 \leq i \leq r$, becomes principal and generated by some element u_i, we have

$$\begin{aligned} P^{(n)} &\cong \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[\varpi^n u_1, \varpi^n u_1^{-1}, \ldots, \varpi^n u_r, \varpi^n u_r^{-1}]) \\ &\cong \underline{\mathrm{Spec}}_{\mathscr{O}_A}(\mathscr{O}_A[u_1, v_1, \ldots, u_r, v_r]/(u_1 v_1 - \varpi^{2n}, \ldots, u_r v_r - \varpi^{2n})), \end{aligned}$$

where $v_i := u_i^{-1}$ for each $1 \leq i \leq r$. This shows that $P^{(n)}$ is a relative complete intersection in $\mathbb{A}_A^{2r}$ over A, and is smooth over A outside a subset of codimension two. In particular, it is a normal scheme (by Serre's criterion; see [59, IV-2, 5.8.6]). By construction, $P^{(n)}$ is isomorphic to $G^\natural$ when ϖ is invertible. Therefore, $P^{(n)}$ and $P^\natural$ have the same generic fiber isomorphic to that of $G^\natural$, and hence both of them have the same rational function field $K(G^\natural)$. Let $Z^{(n)} \subset P^{(n)} \underset{A}{\times} P^\natural$ be the *join* of this birational correspondence, namely, the closure of the diagonal of the common generic fiber. By Lemma 4.5.2.12, all valuations of $K(G^\natural)$ with a center on $P^{(n)}$ also have a center on P. Since $Z^{(n)}$ is locally of finite type over $P^{(n)}$, by Lemma 4.5.2.4, $Z^{(n)} \to P^{(n)}$ is proper because it satisfies the valuative criterion for properness. By Zariski's connectedness theorem (see [59, III-1, 4.3.1]), this shows that all fibers of $Z^{(n)} \to P^{(n)}$ are connected, because this is the case over the generic fiber. Locally over $A_0 = A \underset{R}{\otimes} (R/(\varpi))$, the closed fiber of $P^{(n)}$ is isomorphic to $\underline{\mathrm{Spec}}_{\mathscr{O}_{A_0}}(\mathscr{O}_{A_0}[u_1, v_1, \ldots, u_r, v_r]/(u_1 v_1, \ldots, u_r v_r))$, which is clearly connected. Therefore the closed fiber of $Z^{(n)}$ is connected. If we set $W_n := \overline{\mathrm{pr}_2(Z_0^{(n)})} \subset P^\natural$, then W_n is connected too.

By Lemma 4.5.2.12, every valuation υ' with a center x on $P_0^\natural$ has a center on $P^{(n)}$ for some $n > 0$. Therefore, for some $n > 0$, x can be lifted to a point of $Z^{(n)}$, or equivalently $x \in W_n$. This shows that $P_0^\natural = \underset{n>0}{\cup} W_n$. By the explicit expressions of the $\mathscr{O}_A$-algebras defining the schemes, there is a morphism from $P^{(m)}$ to $P^{(n)}$ when $n|m$, which is an isomorphism over the generic fiber. In particular, W_n and W_m have nontrivial intersection when $n|m$. This shows that $P_0^\natural = \underset{n>0}{\cup} W_n$ is connected, as desired. □

Proposition 4.5.2.14 (cf. [95, Prop. 3.8]). *The scheme $P_0^\natural$ is connected.*

Proof. Since x is an arbitrary point of $P_0^\natural$, this follows from Lemmas 4.5.2.11 and 4.5.2.13. □

Proposition 4.5.2.15 (cf. [95, Thm. 3.10] and [42, Ch. III, Prop./Def. 4.8]). *For each integer $i \geq 0$, let $P_i^\natural := P^\natural \underset{S}{\times} S_i$. There exists a scheme P_i projective over $S_i = \mathrm{Spec}(R/I^{i+1})$ and an étale surjective morphism $\pi_i : P_i^\natural \to P_i$ such that P_i is the quotient of $P_i^\natural$ as an fpqc sheaf and such that the ample invertible sheaf $\mathcal{L}^\natural \underset{R}{\otimes} R_i$ over $P_i^\natural$ descends to an ample invertible sheaf $\mathcal{L}_i$ over P_i. The schemes P_i fit together as i varies and form a formal scheme P_{for}, and the ample invertible sheaves $\mathcal{L}_i$ also fit together and form a formal ample invertible sheaf $\mathcal{L}_{\mathrm{for}}$ over P_{for}. Hence, by Theorem 2.3.1.4, the pair $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ algebraizes to a pair $(P, \mathcal{L})$, where P is a projective scheme over S, and where $\mathcal{L}$ is an ample invertible sheaf over P.*

Proof. By Proposition 4.5.2.7, there exists an integer $k \geq 1$ such that, for every y, no two distinct points of $S_y(U) \underset{S}{\times} S_i$ are identified with each other under the action of the subgroup $kY \subset Y$ on $P^\natural \underset{S}{\times} S_i$. Thus we can form the quotient

$$\pi_i' : P^\natural \underset{S}{\times} S_i = \underset{y \in Y}{\cup} S_y(U) \underset{S}{\times} S_i \to P_i' := (P^\natural \underset{S}{\times} S_i)/(kY)$$

by gluing the open subschemes $S_y(U) \underset{S}{\times} S_i$ along their overlaps. Since $\mathcal{L}^\natural \underset{R}{\otimes} R_i$ inherits a Y-action from $\mathcal{L}^\natural$, it descends to an invertible sheaf $\mathcal{L}_i'$ over P_i'.

Choose representatives $y_1, \ldots, y_t$ in Y for the cosets in Y/kY. The restriction of π_i' gives a surjection $\overset{t}{\underset{j=1}{\cup}} S_{y_j}(U) \underset{S}{\times} S_i \to P_i'$, and hence a surjection

$$\overset{t}{\underset{j=1}{\cup}} S_{y_j}(\overline{U_0}) \underset{S}{\times} S_i \to P_i'. \tag{4.5.2.16}$$

Since the scheme on the left-hand side of (4.5.2.16) is a finite union of schemes *proper* over S_i by Proposition 4.5.2.5, we see that P_i' is also proper over S_i. The invertible sheaf $\mathcal{L}_i'$ over P_i' pulls back to the restriction of $\mathcal{L}^\natural \underset{R}{\otimes} R_i$ on the left-hand side of (4.5.2.16), which is ample there. Then $\mathcal{L}_i'$ over P_i' is also ample by *Nakai's criterion* (see [70]).

Without going into technical details, let us explain Nakai's criterion as follows: The criterion states that a Cartier divisor over a complete algebraic scheme is ample if and only if it is *arithmetically positive*. When the algebraic scheme in question is a nonsingular variety, this condition means that for each integer $n > 0$, its nth power has strictly positive intersection numbers with all n-dimensional subvarieties. In general, one defines an analogue condition using polynomials defined by the Euler–Poincaré characteristics of the corresponding invertible sheaves. In any case, these conditions can be checked on each irreducible component. Since the two sides of (4.5.2.16) are locally glued from isomorphic irreducible components (without altering the local structures), the conditions are the same for both sides. Hence the ampleness of $\mathcal{L}^\natural \underset{R}{\otimes} R_i$ is equivalent to the ampleness of $\mathcal{L}_i'$, as desired.

Finally, the *finite group Y/kY* acts freely (by Corollary 4.5.2.9) on the *projective scheme P_i'* over S_i, and on the ample sheaf $\mathcal{L}_i'$. Hence, a quotient $P_i = P_i'/(Y/kY)$ exists in the category of projective schemes over S_i (by [39, V, 4.1]), and $\mathcal{L}_i'$ descends to an ample invertible sheaf $\mathcal{L}_i$ over P_i. These projective schemes $(P_i, \mathcal{L}_i)$ over S_i fit together as i varies and form a projective formal scheme $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ over $S_{\mathrm{for}} = \mathrm{Spf}(R, I)$, as in the statement of the proposition. □

Construction 4.5.2.17 (cf. [95, §3, p. 253] and [42, Ch. III, Def. 4.9]). Given the "quotient" $(P, \mathcal{L})$ of $(P^\natural, \mathcal{L}^\natural)$ by Y in Proposition 4.5.2.15, we would like to define an open subscheme G of P such that $(G_\eta, \mathcal{L}_\eta)$ can be interpreted as the "quotient" of $(G_\eta^\natural, \mathcal{L}_\eta^\natural)$ by the action of $\iota(Y)$. Practically, we shall construct the complement C of G in P, with the following steps:

1. $G^{\natural,*} := \underset{y \in Y}{\cup} S_y(G^\natural) \subset P^\natural$ is an open subscheme of $P^\natural$, whose pullback over S_i defines, under the quotient by Y, an open subscheme G_i of P_i. The formation of quotients is compatible among different i's as in the case of P_i's, and we obtain an open formal subscheme G_{for} of P_{for}, which is canonically isomorphic to $G_{\mathrm{for}}^\natural$ because G_i is canonically isomorphic to $G_i^\natural = G^\natural \underset{S}{\times} S_i$ for each i.

2. $C^\natural := (P^\natural - G^{\natural,*})_{\mathrm{red}}$ is a closed reduced subscheme of $P^\natural$, whose pullback over S_i defines, under the quotient by Y, a reduced closed subscheme C_i of P_i, whose underlying topological space coincides with that of $P_i - G_i$. The formation of quotients is again compatible among different i's, and we obtain

a closed formal subscheme C_{for} of P_{for} whose underlying topological space coincides with that of $P_{\text{for}} - G_{\text{for}}$.

3. C_{for} algebraizes to a reduced closed subscheme C of P. Define G to be the open subscheme $P - C$ of P. Then the formal completion G_{for} of G is canonically isomorphic to $G^{\natural}_{\text{for}}$.

We will prove in Section 4.5.3 that G is a group scheme whose group structure is compatible with the one of $G^{\natural}$ via the canonical isomorphism $G_{\text{for}} \cong G^{\natural}_{\text{for}}$. We shall regard G as the "quotient" of $G^{\natural,*}$ under Y, and regard (after taking generic fibers) G_η as the "quotient" of $G^{\natural}_\eta$ by $\iota(Y)$.

Definition 4.5.2.18. *By abuse of language, with the setting as in Proposition 4.5.2.15 and Construction 4.5.2.17, we shall say that $(P, \mathcal{L})$ (resp. $(G, \mathcal{L}) = (G, \mathcal{L}|_G)$, resp. G, resp. G_η) is the "**Mumford quotient**" of $(P^{\natural}, \mathcal{L}^{\natural})$ (resp. $(G^{\natural}, \mathcal{L})$, resp. $G^{\natural}$, resp. $G^{\natural}_\eta$) by Y or $\iota(Y)$. (Though none of these is a genuine quotient in the category of schemes.)*

Let us record some properties of G. Our first claim is that G is smooth over S. This follows from a general criterion provided by Mumford:

Proposition 4.5.2.19 (cf. [95, Prop. 4.1]). *Let $f_j : P_j \to S$, $j = 1, 2$ be two schemes over S such that f_2 is proper. Suppose that there is an étale surjective morphism $p : P_{1,\text{for}} \to P_{2,\text{for}}$ over S_{for} between the formal completions, and suppose that there are two closed subschemes $C_1 \subset P_1$ and $C_2 \subset P_2$ such that*

1. *$P_1 - C_1$ is smooth over S of relative dimension r;*

2. *$C_{1,\text{for}}$ is a formal subscheme of $p^{-1}(C_{2,\text{for}})$.*

Then $P_2 - C_2$ is also smooth over S of relative dimension r.

Proof. First we need to check that $P_2 - C_2$ is flat over S. Let $\mathscr{M} \subset \mathscr{N}$ be two $\mathscr{O}_S$-modules. Consider for $j = 1, 2$ the two kernels $0 \to \mathscr{K}_j \to \mathscr{M} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{P_j} \to \mathscr{N} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{P_j}$. Since $P_1 - C_1$ is flat over S, we have $\operatorname{Supp}(\mathscr{K}_1) \subset C_1$, and hence, for all $x \in P_1$, we have $(\mathscr{K}_1)_x \cdot (\mathscr{I}_{C_1})_x^{\otimes n} = (0)$ for some integer $n > 0$. Taking I-adic completions, we get for $j = 1, 2$ two exact sequences $0 \to \mathscr{K}_{j,\text{for}} \to \mathscr{M} \underset{\mathscr{O}_{S_{\text{for}}}}{\otimes} \mathscr{O}_{P_{j,\text{for}}} \to \mathscr{N} \underset{\mathscr{O}_{S_{\text{for}}}}{\otimes} \mathscr{O}_{P_{j,\text{for}}}$, and it follows that $\mathscr{K}_{1,\text{for}} \cong p^*\mathscr{K}_{2,\text{for}}$ because p is flat. Since $(\mathscr{K}_{1,\text{for}})_x \cdot (\mathscr{I}_{C_{1,\text{for}}})_x^{\otimes n} = (0)$ for all $x \in P_{1,\text{for}}$, and since $C_{1,\text{for}}$ is a formal subscheme of $p^{-1}(C_{2,\text{for}})$, it follows that $(\mathscr{K}_{2,\text{for}})_{p(x)} \cdot (\mathscr{I}_{C_{2,\text{for}}})_{p(x)}^{\otimes n} = (0)$. This implies that, in an open neighborhood of $f_2^{-1}(S_0)$, $\mathscr{K}_2$ is annihilated by $\mathscr{I}_{C_2}$, and hence $\operatorname{Supp}(\mathscr{K}_2) \subset C_2$ in that neighborhood. Since P_2 is proper over S, all closed points of P_2 lie over S_0, and hence $\operatorname{Supp}(\mathscr{K}_2) \subset C_2$ everywhere.

To show that $P_2 - C_2$ is smooth over S, it suffices to show that, in addition to being flat, it is differentially smooth (see [59, IV-4, 16.10.1]). Namely, $\Omega^1_{P_2/S}$ is locally free of rank r outside C_2, and the canonical morphism $\operatorname{Sym}^i_{\mathscr{O}_{P_2}}(\Omega^1_{P_2/S}) \to \operatorname{Gr}^i \mathscr{P}_{P_2/S}$ is an isomorphism outside C_2 for each $i \geq 0$. Since we know these are true for P_1 over S, we deduce in particular, that the following two statements are true at all points x of $f_1^{-1}(S_0)$:

1. For each $h \in (\mathscr{I}_{C_1})_x$, the $\mathscr{O}_{P_1,x}[1/h]$-module $(\Omega^1_{P_1/S})_x[1/h]$ is locally free of rank r.

2. For each $i \geq 0$, the kernel and cokernel of the canonical morphism $\operatorname{Sym}^i_{\mathscr{O}_{P_1}}(\Omega^1_{P_1/S})_x \to \operatorname{Gr}^i \mathscr{P}_{P_1/S,x}$ are annihilated by powers of $(\mathscr{I}_{C_1})_x$.

These two facts imply the corresponding facts for the formal scheme $P_{1,\text{for}}$. Since p is étale, $p^*(\Omega^1_{P_2/S}) \cong \Omega^1_{P_1/S}$ and $p^*(\mathscr{P}_{P_2/S}) \cong \mathscr{P}_{P_1/S}$. Hence the assumption that $C_{1,\text{for}}$ is a formal subscheme of $p^{-1}(C_{2,\text{for}})$ implies the corresponding facts for $P_{2,\text{for}}$, and hence for P_2 at points x of $P_2 \underset{S}{\times} S_0$. Since P_2 is proper over S, they hold everywhere on P_2. This shows that $P_2 - C_2$ is differentially smooth over S as well. $\square$

Corollary 4.5.2.20 (cf. [95, Cor. 4.2]). *The scheme G is smooth over S.*

Proof. Since $G^{\natural}$ and hence $G^{\natural,*}$ are smooth over S, this follows from Proposition 4.5.2.19 with $P_1 = P^{\natural}$, $P_2 = P$, $C_1 = C^{\natural}$, and $C_2 = C$. $\square$

Proposition 4.5.2.21 (cf. [95, Prop. 4.2]). *The scheme P is irreducible.*

Proof. After making base changes to complete discrete valuation rings, we may assume that S is *excellent* (see [87, 34.B]). Then we may replace $P^{\natural}$ with its normalization, and the excellence of S implies that $P^{\natural}_{\text{for}}$ is also normal. This implies that P_{for} and P are also normal. By Proposition 4.5.2.14, $P^{\natural}_0$ and hence $P_0 = P^{\natural}_0/Y$ are connected. Since $P \to S$ is proper, P is also connected. This shows that P is irreducible because it is normal. $\square$

Remark 4.5.2.22. As pointed out in [42, Ch. III, §0], this is the only place in Mumford's original paper [95, Thm. 4.3] where the excellence of the base scheme S is used. Hence we can remove the excellence assumption by making base changes to complete discrete valuation rings.

Proposition 4.5.2.23 (cf. [42, Ch. III, Prop. 4.12] and [95, Cor. 4.9]). *As subschemes of P, we have $G_\eta = P_\eta$. In particular, G_η is proper over η.*

Proof. Since $G^{\natural}_\eta = P^{\natural}_\eta$, there is a nonzero element r of R annihilating $\mathscr{O}_{C^{\natural}}$. Then $\mathscr{O}_C$ is also annihilated by r. In particular, we have $G_\eta = P_\eta$ as well. This shows that G_η is proper over η because P is. $\square$

4.5.3 Functoriality

In this section, we would like to establish the *functoriality* of "Mumford quotients" (see Definition 4.5.2.18), and obtain as a by-product the group structure of each G as in Construction 4.5.2.17.

Definition 4.5.3.1 (cf. [95, Def. 4.4] and [42, Ch. III, Def. 5.1]). *Let $G^{\natural}_1$ be a semi-abelian subscheme of $G^{\natural}$ (i.e., a subgroup scheme of $G^{\natural}$ that is a semi-abelian scheme) such that $G^{\natural}_1$ is the extension of an abelian subscheme A_1 of A by a (necessarily split) subtorus T_1 of T. Let Y_1 be the subgroup $\iota^{-1}(G^{\natural}_{1,\eta})$ of Y. Then we say that $G^{\natural}_1$ is* ***integrable*** *if $\operatorname{rk}_{\mathbb{Z}}(Y_1) = \dim_S(T_1)$.*

Construction 4.5.3.2. Let us start with an inclusion $j^\natural : G_1^\natural \hookrightarrow G^\natural$ without assuming that $G_1^\natural$ is integrable. Let us denote the induced inclusion $Y_1 \hookrightarrow Y$ by j_Y and the induced homomorphism $\iota|_{Y_1} : Y_1 \to G_{1,\eta}^\natural$ by ι_1. Let us denote by $c_1 : X_1 \to A_1^\vee$ the homomorphism giving the extension structure of $G_1^\natural$ and by $\pi_1 : G_1^\natural \to A_1$ the structural morphism. The inclusion $j^\natural$ induces the inclusions $j_T : T_1 \hookrightarrow T$ and $j_A : A_1 \hookrightarrow A$, and hence the surjections $j_X : X \twoheadrightarrow X_1 := \mathbf{X}(T_1)$ and $j_A^\vee : A^\vee \twoheadrightarrow A_1^\vee$, justifying the compatibility $c_1 j_X = j_A^\vee c$. Let $c_1^\vee : Y_1 \to A_1$ be the unique homomorphism extending $\pi_1 \iota_1 : Y_1 \to A_{1,\eta}$ by the properness of A_1, satisfying another compatibility $j_A c_1^\vee = c^\vee j_Y$ by definition. Let $\phi_1 := j_X \phi j_Y : Y_1 \to X_1$ and let $\lambda_{A_1} := j_A^\vee \lambda_A j_A : A_1 \to A_1^\vee$. Then we have the compatibility $c_1 \phi_1 = \lambda_{A_1} c_1^\vee$ induced by the compatibility $c\phi = \lambda_A c^\vee$. If, étale locally, λ_A is induced by some ample invertible sheaf $\mathcal{M}$, then λ_{A_1} is induced by the pullback $\mathcal{M}_{A_1} := j_A^* \mathcal{M}$. Hence λ_{A_1} is a polarization. The homomorphism $\iota_1 : Y_1 \to G_{1,\eta}^\natural$ corresponds to a trivialization $\tau_1 : \mathbf{1}_{Y_1 \times X_1, \eta} \xrightarrow{\sim} (c_1^\vee \times c_1)^* \mathcal{P}_{A_1,\eta}^{\otimes -1}$ of biextensions. For each $y_1 \in Y_1$ and $\chi_1 \in X_1$, where $\chi_1 = j_X(\chi)$ for some element $\chi \in X$, we have $(c_1^\vee(y_1), c_1(\chi_1))^* \mathcal{P}_{A_1,\eta} \cong (c_1^\vee(y_1), c_1 j_X(\chi))^* \mathcal{P}_{A_1,\eta} \cong (c_1^\vee(y_1), j_A^\vee c(\chi))^* \mathcal{P}_{A_1,\eta} \cong (c^\vee j_Y(y_1), c(\chi))^* \mathcal{P}_{A,\eta}$. The isomorphism $\tau_1(y_1, \chi_1) : (c_1^\vee(y_1), c_1(\chi_1))^* \mathcal{P}_{A_1,\eta} \xrightarrow{\sim} \mathscr{O}_\eta$ is by definition the isomorphism $\tau(j_Y(y_1), \chi) : (c^\vee j_Y(y_1), c(\chi))^* \mathcal{P}_{A,\eta} \xrightarrow{\sim} \mathscr{O}_\eta$. Since the positivity of τ_1 is defined by the image of $(c_1^\vee(y_1), c_1(\phi_1(y_1)))^* \mathcal{P}_A$ under $\tau_1(y_1, \phi_1(y_1))$, which is the same as the image of $(c^\vee j_Y(y_1), c(\phi j_Y(y_1)))^* \mathcal{P}_A$ under $\tau(j_Y(y_1), \phi j_Y(y_1))$, the positivity of τ implies the positivity of τ_1, because $\tau_1|_{\mathbf{1}_{Y_1 \times \phi_1(Y_1),\eta}} : \mathbf{1}_{Y_1 \times \phi_1(Y_1),\eta} \xrightarrow{\sim} (c_1^\vee \times c_1)^* \mathcal{P}_{A_1,\eta}^{\otimes -1}$ is the restriction of $\tau|_{Y \times \phi(Y),\eta} : \mathbf{1}_{Y \times \phi(Y),\eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_{A,\eta}^{\otimes -1}$ to $Y_1 \times \phi_1(Y_1)$ (see Definition 4.2.1.10). In particular, ϕ_1 is injective, and $\mathrm{rk}_{\mathbb{Z}}(Y_1) \leq \dim_S(T_1)$. Moreover, $\psi_1 := \psi|_{Y_1} : \mathbf{1}_{Y_1,\eta} \xrightarrow{\sim} \iota_1^*(\mathcal{L}^\natural|_{G_1^\natural})_\eta^{\otimes -1}$ is a cubical trivialization compatible with τ_1 because τ_1 is a restriction of τ.

If the equality $\mathrm{rk}_{\mathbb{Z}}(Y_1) = \dim_S(T_1)$ holds, namely, if $G_1^\natural$ is integrable, then we have everything we need for defining an object in $\mathrm{DD}_{\mathrm{ample}}(R, I)$ or $\mathrm{DD}_{\mathrm{pol}}(R, I)$. Let us record this fact as the following lemma:

Lemma 4.5.3.3. *Suppose we are given an object* $(A, \lambda_A, X, Y, \phi, c, c^\vee, \tau)$ *in* $\mathrm{DD}_{\mathrm{pol}}(R, I)$ *(resp. an object* $(A, X, Y, \phi, c, c^\vee, \mathcal{L}^\natural, \tau, \psi)$ *in* $\mathrm{DD}_{\mathrm{ample}}(R, I)$*, resp. an object* $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ *in* $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split}}(R, I)$*), and suppose* $G_1^\natural$ *is an integrable semi-abelian subscheme of* $G^\natural$*. Then Construction 4.5.3.2 defines an object* $(A_1, \lambda_{A_1}, X_1, Y_1, \phi_1, c_1, c_1^\vee, \tau_1)$ *in* $\mathrm{DD}_{\mathrm{pol}}(R, I)$ *(resp. an object* $(A_1, X_1, Y_1, \phi_1, c_1, c_1^\vee, \mathcal{L}^\natural|_{G_1^\natural}, \tau_1, \psi_1)$ *in* $\mathrm{DD}_{\mathrm{ample}}(R, I)$*, resp. an object* $(A_1, \mathcal{M}_{A_1}, X_1, Y_1, \phi_1, c_1, c_1^\vee, \tau_1, \psi_1)$ *in* $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split}}(R, I)$*).*

Construction 4.5.3.4 (cf. [95, §4, p. 255] and [42, Ch. III, 5.3]). Suppose that $(P^\natural, \mathcal{L}^\natural)$ is a relatively complete model of an object $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ in $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split}}(R, I)$, and $G_1^\natural \hookrightarrow G^\natural$ an integrable semi-abelian subscheme of $G^\natural$. We shall construct a closed subscheme W of P such that the intersection of W with G is an open subscheme G_1 of W such that $G_{1,\mathrm{for}} \cong G_{1,\mathrm{for}}^\natural$, with the following steps:

1. Let $W_1^\natural$ be the scheme-theoretic closure of $G_1^\natural$ in $P^\natural$. Then $W_1^\natural$ is invariant under the restriction of the action of Y to Y_1 as $G_{1,\eta}^\natural$ is. Moreover, $W_1^\natural$ is reduced because $G_1^\natural$ is.
2. Since $W_{1,\mathrm{for}}^\natural$ is invariant under the action of Y_1 (as $W_1^\natural$ is), we set
$$W_{\mathrm{for}}^\natural := \bigcup_{y \in Y} S_y(W_{1,\mathrm{for}}^\natural) = \bigcup_{y \in Y/Y_1} S_y(W_{1,\mathrm{for}}^\natural)$$
as a formal subscheme of $P_{\mathrm{for}}^\natural$. This is *not* fully justified *until* we can show that the right-hand side is a locally finite union.
3. Let $W_{\mathrm{for}} := W_{\mathrm{for}}^\natural / Y$, which is a closed formal subscheme of P_{for}.
4. W_{for} algebraizes to a closed subscheme W of P. Then we take $G_1 := W \cap G$.

To validate Construction 4.5.3.4, we need to show the *local finiteness* of the union in step 2. This requires the integrability condition of $G_1^\natural$:

Proposition 4.5.3.5 (cf. [95, Prop. 4.5] and [42, Ch. III, Prop. 5.4])**.** *Let $G_1^\natural$ be an integrable semi-abelian subscheme of $G^\natural$, and let $W_1^\natural$ be the scheme-theoretic closure of $G_1^\natural$ in $P^\natural$. Then there is a finite subset $Q \subset Y$ such that $(W_1^\natural \underset{S}{\times} S_0) \cap S_y(U_0) = \emptyset$ for all $y \notin Q + Y_1$.*

Proof. Let X' be the kernel of the surjection $X \twoheadrightarrow X_1$, which is the subgroup of characters of T that are identically trivial on T_1. Since $\tau_1(z, \chi) = 1$ for all $\chi \in X'$ and $z \in Y_1$, we have $\phi^{-1}(X') \cap Y_1 = \{0\}$. Therefore, $Y_1 + \phi^{-1}(X')$ has finite index in Y. Let $Y' := \{y \in Y : ny \in Y_1 \text{ for some integer } n \geq 1\}$. Then $Y = Y' \oplus \phi^{-1}(X')$, and the R-submodules $I_{z,\chi}$ of K for $z \in Y$ and $\chi \in X'$ depend (by taking discrete valuations) only on the value of z modulo Y'.

For each nonzero $y \in Y$, choose an integer $n_y > 0$ such that $I_{y,\phi(y)}^{\otimes n_y} \cdot \mathscr{O}_{\phi(y)}$ is contained in $(\pi|_U)_* \mathscr{O}_U$ and is congruent to zero modulo I. This is possible by Proposition 4.5.2.2, because U is of finite type. By Lemma 4.5.1.10, there is a finite subset $\{y_1, \ldots, y_k\}$ of $\phi^{-1}(X')$, and a finite subset Q' of Y, such that, for each $z \in Y$ such that $z \notin Q' + Y'$, there exists some nonzero y_j, $1 \leq j \leq k$, such that $I_{z,\phi(y_j)} \subset I_{y_j,\phi(y_j)}^{\otimes n_{y_j}} \subset I$. Under the translation $S_z : U \xrightarrow{\sim} S_z(U)$, the sections of $\mathscr{O}_\chi|_{S_z(U)}$ correspond to sections of $I_{z,\chi} \cdot \mathscr{O}_\chi|_U$. If $z \notin Q' + Y'$, then sections of $I_{z,\phi(y_j)} \cdot \mathscr{O}_{\phi(y_j)}|_U$ consist of regular functions congruent to zero modulo I. Hence $S_z(U_0)$ and U_0 do not overlap. Now we can conclude the proof by taking Q to be a (finite) set of representatives of $(Q' + Y')/Y_1$. □

Now we are ready for the main result of this section, namely, the functoriality of "Mumford quotients":

Theorem 4.5.3.6 (cf. [95, Thm. 4.6] and [42, Ch. III, Thm. 5.5])**.** *For $j = 1, 2$, let $(A_j, \mathcal{M}_j, X_j, Y_j, \phi_j, c_j, c_j^\vee, \tau_j, \psi_j)$ be an object of $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split}}(R, I)$ with relatively complete model $(P_j^\natural, \mathcal{L}_j^\natural)$, with the "Mumford quotient" $(P_j, \mathcal{L}_j)$ (see Definition 4.5.2.18) containing an open subscheme G_j as in Construction 4.5.2.17. Suppose we have a homomorphism $f_Y : Y_1 \to Y_2$ and a homomorphism $f^\natural : G_1^\natural \to G_2^\natural$ over S such that $\iota_2 \circ f_Y = f^\natural \circ \iota_1$. Then there is a unique homomorphism $f : G_1 \to G_2$ over S whose formal completion $f_{\mathrm{for}} : G_{1,\mathrm{for}}^\natural \to G_{2,\mathrm{for}}^\natural$ is identical to $f_{\mathrm{for}}^\natural : G_{1,\mathrm{for}} \to G_{2,\mathrm{for}}$.*

Proof. Consider the fiber product object
$$(A_1 \underset{S}{\times} A_2, \mathrm{pr}_1^* \mathcal{M}_1 \underset{\mathscr{O}_{A_1 \underset{S}{\times} A_2}}{\otimes} \mathrm{pr}_2^* \mathcal{M}_2, X_1 \times X_2, Y_1 \times Y_2,$$
$$\phi_1 \times \phi_2, c_1 \times c_2, c_1^\vee \times c_2^\vee, \tau_1 \times \tau_2 := \mathrm{pr}_{13}^* \tau + \mathrm{pr}_{24}^* \tau, \psi_1 \times \psi_2)$$

in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$ with relatively complete model

$$(P_1^\natural \underset{S}{\times} P_2^\natural, \mathrm{pr}_1^* \mathcal{L}_1 \underset{\mathscr{O}_{P_1^\natural \underset{S}{\times} P_2^\natural}}{\otimes} \mathrm{pr}_1^* \mathcal{L}_2).$$

Here the definition of $\tau_1 \times \tau_2$ makes sense because

$$\mathcal{P}_{A_1 \underset{S}{\times} A_2} \cong \mathrm{pr}_{13}^* \mathcal{P}_{A_1} \underset{\mathscr{O}_{A_1 \underset{S}{\times} A_2 \underset{S}{\times} A_1^\vee \underset{S}{\times} A_2^\vee}}{\otimes} \mathrm{pr}_{24}^* \mathcal{P}_{A_2}.$$

Let $H^\natural := \mathrm{image}((\mathrm{Id}_{G^\natural}, f^\natural) : G_1^\natural \to G_1^\natural \underset{S}{\times} G_2^\natural)$ be the graph of $f^\natural$, which defines an *integrable* sub-semi-abelian scheme of $G_1^\natural \underset{S}{\times} G_2^\natural$ because $\iota_2 \circ f_Y = f^\natural \circ \iota_1$. As in Construction 4.5.3.4 and Proposition 4.5.3.5, this $H^\natural$ induces a closed subscheme of $G_1 \underset{S}{\times} G_2$ as follows:

1. $W_1^\natural :=$ scheme-theoretic closure of $H^\natural$ in $P_1^\natural \underset{S}{\times} P_2^\natural$.
2. $W_{\mathrm{for}}^\natural := \bigcup_{y \in Y_1 \times Y_2} S_y(W_{1,\mathrm{for}}^\natural)$ as a formal subscheme of $P_{1,\mathrm{for}}^\natural \underset{S_{\mathrm{for}}}{\times} P_{2,\mathrm{for}}^\natural$.
3. $W_{\mathrm{for}} := W_{\mathrm{for}}^\natural/(Y_1 \times Y_2)$ is a formal subscheme of $P_{1,\mathrm{for}} \underset{S_{\mathrm{for}}}{\times} P_{2,\mathrm{for}} = P_{1,\mathrm{for}}^\natural \underset{S_{\mathrm{for}}}{\times} P_{2,\mathrm{for}}^\natural/(Y_1 \times Y_2)$.
4. $H := W \cap (G_1 \underset{S}{\times} G_2)$, where W is the algebraization of W_{for}.

We claim that H defines the graph of a morphism from G_1 to G_2.

First, we need to show that the projection $\mathrm{pr}_1 : W \to P_1$ is smooth of relative dimension zero outside $C_1 := (P_1 - G_1)_{\mathrm{red}}$. This follows by essentially the same argument used in the proof of Proposition 4.5.2.19, as $\mathrm{pr}_1 : W_1^\natural \to P_1^\natural$ is smooth of relative dimension zero outside $C_1^\natural := (P_1^\natural - G_1^\natural)_{\mathrm{red}}$. Locally at every point, $W_{\mathrm{for}}^\natural$ is the formal completion of a finite union $S_{y_1}(W_1^\natural) \cup \cdots \cup S_{y_k}(W_1^\natural)$ for some $y_1, \ldots, y_k \in Y_1 \times Y_2$. Since this is also smooth of relative dimension zero outside $C_1^\natural$, the same is true for $\mathrm{pr}_1 : W_{\mathrm{for}}^\natural \to P_{1,\mathrm{for}}^\natural$. Here, by *smooth outside* $C_1^\natural$, we *do not* mean just smooth at points of $P_{1,\mathrm{for}}^\natural - C_{1,\mathrm{for}}^\natural$. Instead, we mean smoothness in the sense of the two statements in the proof of Proposition 4.5.2.19, namely, smoothness after localizing by the ideal $\mathscr{I}_{C_1^\natural}$. This property descends to smoothness for $\mathrm{pr}_1 : W_{\mathrm{for}} \to P_{1,\mathrm{for}}$, and hence for $\mathrm{pr}_1 : W \to P_1$ as well.

Second, we need to prove that $W \cap (P_1 \underset{S}{\times} C_2) \subset C_1 \underset{S}{\times} C_2$. This follows by descending a stronger ideal-theoretic property on the schemes before quotient: For every finite subset $\{y_1, \ldots, y_k\}$ of $Y_1 \times Y_2$, we have $[S_{y_1}(W_1^\natural) \cup \cdots \cup S_{y_k}(W_1^\natural)] \cap (P_1^\natural \underset{S}{\times} C_2^\natural) \subset C_1^\natural \underset{S}{\times} C_2^\natural$. Hence, over $P_{1,\mathrm{for}}^\natural \underset{S_{\mathrm{for}}}{\times} P_{2,\mathrm{for}}^\natural$, we have $\mathscr{I}_{W_{\mathrm{for}}^\natural} + \mathscr{I}_{P_{1,\mathrm{for}}^\natural \underset{S_{\mathrm{for}}}{\times} C_{2,\mathrm{for}}^\natural} \supset \mathscr{I}^{\otimes N}_{C_{1,\mathrm{for}}^\natural \underset{S_{\mathrm{for}}}{\times} C_{2,\mathrm{for}}^\natural}$ for some integer $N > 0$. This property descends and algebraizes, which shows that $W \cap (P_1 \underset{S}{\times} C_2) \subset C_1 \underset{S}{\times} C_2$. Since $\mathrm{pr}_1 : W \to P_1$ is a proper morphism, its pullback $\mathrm{pr}_1 : H \to G_1$ (under $G_1 \hookrightarrow P_1$, on the target) is also proper.

Combining these two assertions, it follows that $\mathrm{pr}_1 : H \to G_1$ is finite and étale. Since $H_{\mathrm{for}} \cong H_{\mathrm{for}}^\natural$ is the graph of $f_{\mathrm{for}}^\natural : G_{1,\mathrm{for}}^\natural \to G_{2,\mathrm{for}}^\natural$, the morphism $\mathrm{pr}_1 : H \to G_1$ has degree one over S_0. Since G_1 is irreducible, pr_1 has degree 1 everywhere. This proves that H is the graph of a morphism $f : G_1 \to G_2$ over S such that $f_{\mathrm{for}} = f_{\mathrm{for}}^\natural$. Finally, since G_1 is irreducible, such an f is unique because it is determined by f_{for}. $\square$

Corollary 4.5.3.7 (cf. [95, Cor. 4.7] and [42, Ch. III, Cor. 5.6])**.** *The scheme G depends (up to isomorphism) only on $(A, X, Y, c, c^\vee, \tau)$ as an object of $\mathrm{DD}(R,I)$, and is independent of the choice of ϕ, ψ, $\mathcal{M}$, and the relatively complete model $(P^\natural, \mathcal{L}^\natural)$.*

Proof. Let $(A, \mathcal{M}_j, X, Y, \phi_j, c, c^\vee, \tau, \psi_j)$, $j = 1, 2$, be any two tuples in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$ extending the same tuple $(A, X, Y, c, c^\vee, \tau)$ in $\mathrm{DD}(R,I)$. Let $(G_1, \mathcal{L}_1)$ and $(G_2, \mathcal{L}_2)$ be constructed respectively for the two tuples for some choices of relatively complete models. Then, by applying Theorem 4.5.3.6 to the identities $\mathrm{Id}_{G^\natural} : G^\natural \overset{\sim}{\to} G^\natural$ and $\mathrm{Id}_Y : Y \overset{\sim}{\to} Y$, we obtain a canonical isomorphism $G_1 \overset{\sim}{\to} G_2$ inducing $\mathrm{Id}_{G^\natural_{\mathrm{for}}} : G^\natural_{\mathrm{for}} \overset{\sim}{\to} G^\natural_{\mathrm{for}}$, as desired. $\square$

Corollary 4.5.3.8 (cf. [95, Cor. 4.8] and [42, Ch. III, Cor. 5.7])**.** *G is a group scheme over S.*

Proof. By applying Theorem 4.5.3.6 to the multiplication morphisms $m_{G^\natural} : G^\natural \underset{S}{\times} G^\natural \to G^\natural$ and $m_Y : Y \times Y \to Y$ (with other fiber product objects given as in the proof of Theorem 4.5.3.6), we obtain a morphism $m_G : G \underset{S}{\times} G \to G$. By applying Theorem 4.5.3.6 to the inverse isomorphisms $[-1]_{G^\natural} : G^\natural \overset{\sim}{\to} G^\natural$ and $[-1]_Y : Y \overset{\sim}{\to} Y$, we obtain an isomorphism $[-1]_G : G \overset{\sim}{\to} G$. The compatibility relations for m_G and $[-1]_G$ to define a group structure on G are satisfied, because they are satisfied by $(m_{G^\natural}, m_Y)$ and $([-1]_{G^\natural}, [-1]_Y)$ and because of the uniqueness statement in Theorem 4.5.3.6. $\square$

Corollary 4.5.3.9 (cf. [95, Cor. 4.9] and [42, Ch. III, Cor. 5.8])**.** *The scheme G_η is an abelian variety.*

Proof. By Proposition 4.5.2.23, and by Corollaries 4.5.2.20 and 4.5.3.8, G_η is a proper smooth group scheme. Since P is irreducible by Proposition 4.5.2.21, and since we may enlarge the completely discrete valuation ring used in the proof of Proposition 4.5.2.21, $G_\eta = P_\eta$ is geometrically irreducible. Hence G_η is an abelian variety, as desired. $\square$

To prove that G has connected fibers and is indeed a semi-abelian scheme over S, it suffices to have a description of torsion points of G. Let $G^{\natural,*} = \bigcup_{y \in Y} S_y(G^\natural) \subset P^\natural$ as before. For each $y \in Y$, let $\sigma_y : S \to G^{\natural,*}$ be the unique section of $G^{\natural,*}$ over S such that $\sigma_y(\eta) = \iota(y)$. This is nothing but the translation of the identity section $e_{G^\natural} : S \to G^\natural$ under the action S_y on $P^\natural$. For each integer $n \geq 1$, consider the fiber product

$$^{(n)}Z_y^\natural := S \underset{\sigma_y, G^{\natural,*}, [n]}{\times} G^{\natural,*},$$

where $[n] : G^{\natural,*} \to G^{\natural,*}$ is the multiplication by n. For each $y, z \in Y$, the translation S_z induces a canonical isomorphism $^{(n)}Z_y^\natural \overset{\sim}{\to} {}^{(n)}Z_{y+nz}^\natural$. Therefore the disjoint union $\coprod_{y \in Y/nY} {}^{(n)}Z_y^\natural$ is well defined and has the structure of a commutative group scheme over S in a canonical way.

Theorem 4.5.3.10 (cf. [95, Thm. 4.10])**.** *The group scheme $G[n]$ is canonically isomorphic to the group scheme $\coprod\limits_{y \in Y/nY} {}^{(n)}Z_y^\natural$ over S constructed above.*

Proof. Let $\overline{{}^{(n)}Z_y^\natural}$ denote the closure of ${}^{(n)}Z_y^\natural$ in $P^\natural$. By the valuation property of the relatively complete model $P^\natural$, all valuations of $\overline{{}^{(n)}Z_y^\natural}$ have centers on $\overline{{}^{(n)}Z_y^\natural}$. By Lemma 4.5.2.4, this implies that $\overline{{}^{(n)}Z_y^\natural}$ is proper over S. Let $(\overline{{}^{(n)}Z_y^\natural})_{\mathrm{for}}$ be its I-adic completion. We saw in the proofs of Proposition 4.5.3.5 and Theorem 4.5.3.6 that if ${}^{(n)}W_1^\natural$ is the closure in $P^\natural \underset{S}{\times} P^\natural$ of the graph of $x \mapsto nx$, then ${}^{(n)}W^\natural := \bigcup\limits_{y \in Y} S_{(0,y)}({}^{(n)}W^\natural_{1,\mathrm{for}}) \subset P^\natural_{\mathrm{for}} \underset{S_{\mathrm{for}}}{\times} P^\natural_{\mathrm{for}}$ is a locally finite union. Since $\overline{{}^{(n)}Z_y^\natural} \underset{S}{\times} \sigma_y(S) \subset {}^{(n)}W_1^\natural$, and hence $\overline{{}^{(n)}Z_y^\natural} \underset{S}{\times} \sigma_0(S) \subset S_{(0,-y)}({}^{(n)}W_1^\natural)$, it follows from Proposition 4.5.3.5 that $\bigcup\limits_{y \in Y} (\overline{{}^{(n)}Z_y^\natural})_{\mathrm{for}} \subset P^\natural_{\mathrm{for}}$ is a locally finite union, and that

$$\bigcup_{y \in Y} (\overline{{}^{(n)}Z_y^\natural})_{\mathrm{for}} = {}^{(n)}W^\natural_{\mathrm{for}} \cap (P^\natural_{\mathrm{for}} \underset{S_{\mathrm{for}}}{\times} \sigma_0(S_{\mathrm{for}})).$$

Taking compatible quotients by Y and $Y \times Y$, we obtain a closed formal subscheme

$$(\overline{{}^{(n)}Z})_{\mathrm{for}} := \Big[\bigcup_{y \in Y} (\overline{{}^{(n)}Z_y^\natural})_{\mathrm{for}}\Big]/Y = {}^{(n)}W_{\mathrm{for}} \cap (P_{\mathrm{for}} \underset{S_{\mathrm{for}}}{\times} \sigma_0(S_{\mathrm{for}}))$$

of $P^\natural_{\mathrm{for}}$, where ${}^{(n)}W_{\mathrm{for}} = {}^{(n)}W^\natural_{\mathrm{for}}/(Y \times Y)$. It follows that $(\overline{{}^{(n)}Z})_{\mathrm{for}}$ algebraizes to a closed subscheme $\overline{{}^{(n)}Z}$ of P such that $\overline{{}^{(n)}Z} = {}^{(n)}W \cap (P \underset{S}{\times} \sigma_0(S))$, where ${}^{(n)}W$ is the algebraization of ${}^{(n)}W_{\mathrm{for}}$. Hence ${}^{(n)}Z := \overline{{}^{(n)}Z} \cap G$ satisfies ${}^{(n)}Z = {}^{(n)}H \cap (G \underset{S}{\times} \sigma_0(S))$, where ${}^{(n)}H \subset G \underset{S}{\times} G$ is the graph of the homomorphism $x \mapsto nx$. Thus we see that ${}^{(n)}Z$ is the kernel $G[n]$ of the multiplication by n in G.

Now for every *finite* subset $Y_0 \subset Y$, we have a formal morphism

$$\overline{q}_{\mathrm{for}} : \bigcup_{y \in Y_0} (\overline{{}^{(n)}Z_y^\natural})_{\mathrm{for}} \to (\overline{{}^{(n)}Z})_{\mathrm{for}}.$$

Since these formal schemes are the completions of the schemes $\bigcup\limits_{y \in Y_0} \overline{{}^{(n)}Z_y^\natural}$ and $\overline{{}^{(n)}Z}$, which are *proper* over S, this formal morphism $\overline{q}_{\mathrm{for}}$ algebraizes uniquely to a morphism

$$\overline{q} : \bigcup_{y \in Y_0} \overline{{}^{(n)}Z_y^\natural} \to \overline{{}^{(n)}Z^\natural}$$

of schemes (by Theorem 2.3.1.3). Since $C^\natural = (P^\natural - G^\natural)_{\mathrm{red}}$ is the preimage of $C = (P - G)_{\mathrm{red}}$ under the étale morphism $P^\natural \to P$, we see that $\bigcup\limits_{y \in Y_0} (\overline{{}^{(n)}Z_y^\natural})_{\mathrm{for}} \cap C^\natural_{\mathrm{for}}$ is the preimage of $(\overline{{}^{(n)}Z})_{\mathrm{for}} \cap C_{\mathrm{for}}$. Hence $\bigcup\limits_{y \in Y_0} \overline{{}^{(n)}Z_y^\natural} \cap C^\natural$ is the preimage of $\overline{{}^{(n)}Z} \cap C$. Therefore the above morphism restricts to a proper morphism

$$q : \bigcup_{y \in Y_0} {}^{(n)}Z_y^\natural \to {}^{(n)}Z.$$

Since $\overline{q}_{\mathrm{for}} : \bigcup\limits_{y \in Y} (\overline{{}^{(n)}Z_y^\natural})_{\mathrm{for}} \to (\overline{{}^{(n)}Z})_{\mathrm{for}}$ is étale and surjective, for each fixed $y_0 \in Y$, there is a finite subset Y_0 of Y containing y_0 such that $\overline{q}_{\mathrm{for}} : \bigcup\limits_{y \in Y_0} (\overline{{}^{(n)}Z_y^\natural})_{\mathrm{for}} \to (\overline{{}^{(n)}Z})_{\mathrm{for}}$ is étale at all points of $(\overline{{}^{(n)}Z_{y_0}^\natural})_{\mathrm{for}}$, and is surjective. Therefore, its algebraization $\overline{q}$ has the same properties. On the other hand, when we intersect with $G^\natural$, the union $\bigcup\limits_{y \in Y_0} {}^{(n)}Z_y^\natural$ is disjoint. Hence it follows that $q : \bigcup\limits_{y \in Y_0} {}^{(n)}Z_y^\natural \to {}^{(n)}Z$ is étale for every Y_0, and is surjective for Y_0 large enough.

Since S_z maps ${}^{(n)}Z_y^\natural$ isomorphically to ${}^{(n)}Z_{y+nz}^\natural$ for each $y, z \in Y$, the morphism $q : \bigcup\limits_{y \in Y_0} {}^{(n)}Z_y^\natural \to {}^{(n)}Z$ is surjective when Y_0 is a set of coset representatives of Y/nY. By identifying $\bigcup\limits_{y \in Y_0} {}^{(n)}Z_y^\natural$ with $\coprod\limits_{y \in Y/nY} {}^{(n)}Z_y^\natural$, we obtain a natural group scheme structure on $\bigcup\limits_{y \in Y_0} {}^{(n)}Z_y^\natural$, and we see that q has degree one over $\sigma_0(S_0) \subset G_0$. Moreover, over S_0, ${}^{(n)}Z_y^\natural = \emptyset$ unless $y \in nY$, ${}^{(n)}Z_0^\natural$ is the kernel of $[n]$ of the semi-abelian scheme $G_0^\natural$ over S_0, and $q : {}^{(n)}Z_0^\natural \to {}^{(n)}Z$ is the restriction to the kernel of $[n]$ of the canonical isomorphism between $G_0^\natural$ and G_0 over S_0. Thus we see that q has degree one, or equivalently, $q : \coprod\limits_{y \in Y/nY} {}^{(n)}Z_y^\natural \xrightarrow{\sim} {}^{(n)}Z = G[n]$ is an isomorphism. □

Proposition 4.5.3.11 (cf. [42, Ch. III, Prop. 5.10])**.** *For each $s \in S$ and each fixed $y \in Y$ and $n \geq 1$, the following statements are equivalent:*

1. $({}^{(n)}Z_y^\natural)_s \neq \emptyset$.
2. *There exists $z \in Y$ such that $\iota(y - nz)$ extends to a section of $G^\natural$ over* $\mathrm{Spec}(\mathscr{O}_{S,s})$. *(In this case, we write $\iota(y - nz) \in G^\natural(\mathrm{Spec}(\mathscr{O}_{S,s}))$.)*
3. *There exists $z \in Y$ such that $I_{y-nz,\phi(y-nz)} \cdot \mathscr{O}_{S,s} = \mathscr{O}_{S,s}$.*
4. *$y \in Y_s + nY$, where*
   $Y_s := \{z \in Y : I_{z,\phi(z)} \cdot \mathscr{O}_{S,s} = \mathscr{O}_{S,s}\} = \{z \in Y : \iota(z) \in G^\natural(\mathrm{Spec}(\mathscr{O}_{S,s}))\}$.

Proof. Firstly, let us show that 1 and 2 are equivalent. By definition, ${}^{(n)}Z_y^\natural \neq \emptyset$ if there is a subscheme H of $G^\natural$ such that the subscheme $S_z(H)$ of $S_z(G^\natural)$ is mapped to $\sigma_y(s) = S_y(e_{G^\natural})$ under the multiplication $[n] : G^{\natural,*} \to G^{\natural,*}$. Then $S_{nz}([n](H)) = S_y(e_{G^\natural})$, and $S_{y-nz}(e_{G^\natural})$ is a subscheme of $G_s^\natural$. That is, $\iota(y - nz)$ extends to a section of $G^\natural$ over s, and hence over $\mathrm{Spec}(\mathscr{O}_{S,s})$ by smoothness of $G^\natural$. Conversely, if $\iota(y - nz) \in G^\natural(\mathrm{Spec}(\mathscr{O}_{S,s}))$, then $({}^{(n)}Z_y^\natural)_s \neq \emptyset$ because $[n](S_z(G_s^\natural[n])) = S_y(e_{G_s^\natural})$.

Secondly, let us show that 2 and 3 are equivalent. If $w = y - nz$, then $\iota(w) \in G^\natural(\mathrm{Spec}(\mathscr{O}_{S,s}))$ if and only if $I_{w,\chi} \cdot \mathscr{O}_{S,s} = \mathscr{O}_{S,s}$ for all $\chi \in X$, as these $I_{w,\chi}$ are defined by $\tau(w, \chi)$, or equivalently by $\iota(w)$ (see Section 4.2.2 and Definition 4.2.4.6). Therefore, $\iota(w) \in G^\natural(\mathrm{Spec}(\mathscr{O}_{S,s}))$ if and only if $I_{w,\chi} \cdot \mathscr{O}_{S,s} = \mathscr{O}_{S,s}$ for all $\chi \in X$. Since S is normal, to verify equivalences between equalities, we may replace R with a discrete valuation ring of K with valuation υ and center s. Consider the positive semidefinite pairing $B(\,\cdot\,,\,\cdot\,) : (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (X \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}$ defined by $(y, \chi) \mapsto \upsilon(I_{y,\chi})$. As in the proof of Lemma 4.5.1.7, $w \in \mathrm{Rad}(B)$ if and only if $B(w, \phi(w)) = 0$. Therefore, $\iota(w) \in G^\natural(\mathrm{Spec}(\mathscr{O}_{S,s}))$ if and only if $I_{w,\phi(w)} \cdot \mathscr{O}_{S,s} = \mathscr{O}_{S,s}$.

Finally, 4 is just a combination of 2 and 3. □

Corollary 4.5.3.12 (cf. [95, Cor. 4.11] and [42, Ch. III, Cor. 5.11]). *For each integer $n \geq 1$, and for each $s \in S$, there is a natural exact sequence*
$$0 \to G^\natural[n]_s \to G[n]_s \to \tfrac{1}{n}Y_s/Y_s \to 0.$$
After taking limits, we obtain an exact sequence
$$0 \to (G^\natural_s)_{\mathrm{tors}} \to (G_s)_{\mathrm{tors}} \to Y_s \underset{\mathbb{Z}}{\otimes} (\mathbb{Q}/\mathbb{Z}) \to 0.$$

Proof. As we saw in the proofs of Theorem 4.5.3.10 and Proposition 4.5.3.11, $G[n]_s$ is isomorphic to the union of the translations of $G^\natural[n]_s$ under a coset representative of Y_s/nY_s. Hence the corollary follows. □

Corollary 4.5.3.13 (cf. [95, Cor. 4.12] and [42, Ch. III, Cor. 5.12]). *The geometric fibers of G over S are all connected with trivial unipotent radical. That is, G is a semi-abelian scheme over S.*

Proof. This follows from the general fact that if a commutative algebraic group H over an algebraically closed field has the property that the p-primary torsion of H is p-divisible (in the sense of group schemes) and scheme-theoretically dense in H for every prime number p, then H is connected with trivial unipotent radical, and hence is an extension of an abelian variety by a torus. (By reducing to the connected case, this follows from Chevalley's theorem; see [26] and [108], or see [29, Thm. 1.1] for a modern proof.) By Corollary 4.5.3.12, this is exactly the case for G. □

4.5.4 Equivalences and Polarizations

Definition 4.5.4.1. *The category $\mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}}(R,I)$ is the full subcategory of $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$ formed by objects in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$ satisfying Condition 4.5.1.6.*

The constructions in Sections 4.5.2 and 4.5.3, following Mumford and Faltings–Chai, define a functor
$$\begin{aligned}\mathrm{M}^{\mathrm{split},*}_{\mathrm{ample}}(R,I) : \mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}}(R,I) &\to \mathrm{DEG}_{\mathrm{ample}}(R,I) : \\ (A,\mathcal{M},X,Y,\phi,c,c^\vee,\tau,\psi) &\mapsto (G,\mathcal{L}).\end{aligned} \tag{4.5.4.2}$$
The goal of this section is to show that the functor $\mathrm{M}^{\mathrm{split},*}_{\mathrm{ample}}(R,I)$ induces functors
$$\mathrm{M}(R,I) : \mathrm{DD}(R,I) \to \mathrm{DEG}(R,I),$$
$$\mathrm{M}_{\mathrm{ample}}(R,I) : \mathrm{DD}_{\mathrm{ample}}(R,I) \to \mathrm{DEG}_{\mathrm{ample}}(R,I),$$
$$\mathrm{M}_{\mathrm{pol}}(R,I) : \mathrm{DD}_{\mathrm{pol}}(R,I) \to \mathrm{DEG}_{\mathrm{pol}}(R,I),$$
$$\mathrm{M}_{\mathrm{IS}}(R,I) : \mathrm{DD}_{\mathrm{IS}}(R,I) \to \mathrm{DEG}_{\mathrm{IS}}(R,I),$$
compatible with each other in the obvious sense, such that $\mathrm{M}_{\mathrm{ample}}(R,I)$ and $\mathrm{M}_{\mathrm{pol}}(R,I)$ give quasi-inverses of the associations
$$\begin{aligned}\mathrm{F}_{\mathrm{ample}}(R,I) : \mathrm{DEG}_{\mathrm{ample}}(R,I) &\to \mathrm{DD}_{\mathrm{ample}}(R,I) : \\ (G,\mathcal{L}) &\mapsto (A,\underline{X},\underline{Y},\phi,c,c^\vee,\mathcal{L}^\natural,\tau,\psi)\end{aligned}$$
and
$$\begin{aligned}\mathrm{F}_{\mathrm{pol}}(R,I) : \mathrm{DEG}_{\mathrm{pol}}(R,I) &\to \mathrm{DD}_{\mathrm{pol}}(R,I) : \\ (G,\lambda) &\mapsto (A,\lambda_A,\underline{X},\underline{Y},\phi,c,c^\vee,\tau),\end{aligned}$$
described in Theorem 4.2.1.14 and Definition 4.4.8, respectively. Then $\mathrm{M}_{\mathrm{ample}}(R,I)$, $\mathrm{M}_{\mathrm{pol}}(R,I)$, $\mathrm{F}_{\mathrm{ample}}(R,I)$, and $\mathrm{F}_{\mathrm{pol}}(R,I)$ will all be equivalences of categories as claimed in Theorem 4.4.16.

Definition 4.5.4.3. *The category $\mathrm{DD}^{*}_{\mathrm{ample}}(R,I)$ is the full subcategory of $\mathrm{DD}_{\mathrm{ample}}(R,I)$ consisting of objects $(A,\underline{X},\underline{Y},\phi,c,c^\vee,\mathcal{L}^\natural,\tau,\psi)$ over S such that the following is true: Over a finite étale covering $S' = \mathrm{Spec}(R') \to S = \mathrm{Spec}(R)$ (with $I' = \mathrm{rad}(I\cdot R') \subset R'$) where the étale sheaves $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively, so that there exists an ample invertible sheaf $\mathcal{M}$ over $A \underset{R}{\otimes} R'$ such that $\mathcal{L}^\natural \underset{R}{\otimes} R' \cong \pi^*\mathcal{M}$ (by Corollary 3.2.5.7), the tuple $(A,\mathcal{M},\underline{X},\underline{Y},c,c^\vee,\tau,\psi)\underset{R}{\otimes} R'$ defines an object of $\mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}}(R',I')$.*

Proposition 4.5.4.4. *Let $(A,\underline{X},\underline{Y},\phi,c,c^\vee,\mathcal{L}^\natural,\tau,\psi)$ be an object in $\mathrm{DD}^{*}_{\mathrm{ample}}(R,I)$. Then the functor $\mathrm{M}^{\mathrm{split},*}_{\mathrm{ample}}(R,I)$ in (4.5.4.2) extends to a functor*
$$\begin{aligned}\mathrm{M}^{*}_{\mathrm{ample}}(R,I) : \mathrm{DD}^{*}_{\mathrm{ample}}(R,I) &\to \mathrm{DEG}_{\mathrm{ample}}(R,I) : \\ (A,\underline{X},\underline{Y},\phi,c,c^\vee,\mathcal{L}^\natural,\tau,\psi) &\mapsto (G,\mathcal{L})\end{aligned}$$
compatible with finite étale surjective base changes in $S' = \mathrm{Spec}(R') \to S = \mathrm{Spec}(R)$ (with $I' = \mathrm{rad}(I\cdot R') \subset R'$).

Proof. By assumption, there exists a finite étale covering $S' = \mathrm{Spec}(R') \to S = \mathrm{Spec}(R)$ (with $I' = \mathrm{rad}(I\cdot R') \subset R'$) over which both $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively, and over which $\mathcal{L}^\natural \underset{R}{\otimes} R' \cong \pi^*\mathcal{M}'$ for some ample invertible sheaf $\mathcal{M}'$ over $A \underset{R}{\otimes} R'$. Then we obtain an object in $\mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}}(R',I')$ (with this choice of $\mathcal{M}'$), and $\mathrm{M}^{\mathrm{split},*}_{\mathrm{ample}}(R',I')$ gives us an object $(G',\mathcal{L}')$ in $\mathrm{DEG}_{\mathrm{ample}}(R',I')$. Since $\mathcal{L}'$ is ample over G', the pair $(G',\mathcal{L}')$ descends uniquely to a pair $(G,\mathcal{L})$ over S by fpqc descent (see [56, VIII, 7.8]). □

In order to extend the source of the functor $\mathrm{M}^{*}_{\mathrm{ample}}(R,I) : \mathrm{DD}^{*}_{\mathrm{ample}}(R,I) \to \mathrm{DEG}_{\mathrm{ample}}(R,I)$ to the whole category $\mathrm{DD}_{\mathrm{ample}}(R,I)$, we would like to show that $\mathrm{M}^{*}_{\mathrm{ample}}(R,I)$ is compatible with the tensor operation in $\mathrm{DD}_{\mathrm{IS}}(R,I)$ (see Definition 4.4.13):

Lemma 4.5.4.5. *Let $(A,\underline{X},\underline{Y},\phi_1,c,c^\vee,\mathcal{L}^\natural_1,\tau,\psi_1)$ be an object in $\mathrm{DD}_{\mathrm{ample}}(R,I)$. Let $(A,\underline{X},\underline{Y},\phi_2,c,c^\vee,\mathcal{L}^\natural_2,\tau,\psi_2)$ be either an object in $\mathrm{DD}_{\mathrm{ample}}(R,I)$, or an object in $\mathrm{DD}_{\mathrm{IS}}(R,I)$ such that all of ϕ_2, $\mathcal{L}^\natural_2$, and ψ_2 are trivial. Then there is an integer $n_0 > 0$ such that, for every $n \geq n_0$, both the tuples*
$$(A,\underline{X},\underline{Y},(2n+1)\phi_1,c,c^\vee,(\mathcal{L}^\natural_1)^{\otimes n+1} \underset{\mathscr{O}_{G^\natural}}{\otimes} [-1]^*(\mathcal{L}^\natural_1)^{\otimes n},\tau,\psi_1^{n+1}[-1]^*\psi_1^n)$$
and
$$(A,\underline{X},\underline{Y},\phi_2+2n\phi_1,c,c^\vee,\mathcal{L}^\natural_2 \underset{\mathscr{O}_{G^\natural}}{\otimes} (\mathcal{L}^\natural_1)^{\otimes n} \underset{\mathscr{O}_{G^\natural}}{\otimes} [-1]^*(\mathcal{L}^\natural_1)^{\otimes n},\tau,\psi_2\psi_1^n[-1]^*\psi_1^n)$$
define objects in $\mathrm{DD}^{*}_{\mathrm{ample}}(R,I)$.

Proof. This follows from Corollary 4.5.1.8 after making a finite étale surjective base change in S as in Definition 4.5.4.3. □

Construction 4.5.4.6. Let $(A,\underline{X},\underline{Y},f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta)$ be any object in $\mathrm{DD}_{\mathrm{IS}}(R,I)$. By Lemma 4.4.14, there exist $(A,\underline{X},\underline{Y},\phi_1,c,c^\vee,\mathcal{L}^\natural_1,\tau,\psi_1)$ and $(A,\underline{X},\underline{Y},\phi_2,c,c^\vee,\mathcal{L}^\natural_2,\tau,\psi_2)$ in $\mathrm{DD}_{\mathrm{ample}}(R,I)$ such that
$$\begin{aligned}&(A,\underline{X},\underline{Y},f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta)\\ &\cong (A,\underline{X},\underline{Y},\phi_1,c,c^\vee,\mathcal{L}^\natural_1,\tau,\psi_1)\otimes(A,\underline{X},\underline{Y},\phi_2,c,c^\vee,\mathcal{L}^\natural_2,\tau,\psi_2)^{\otimes -1}.\end{aligned}$$
By Lemma 4.5.4.5, there is an integer $n > 0$ such that both the tuples
$$(A,\underline{X},\underline{Y},(2n+1)\phi_1,c,c^\vee,(\mathcal{L}^\natural_1)^{\otimes n+1} \underset{\mathscr{O}_{G^\natural}}{\otimes} [-1]^*(\mathcal{L}^\natural_1)^{\otimes n},\tau,\psi_1^{n+1}[-1]^*\psi_1^n)$$

and

$$(A,\underline{X},\underline{Y},\phi_2+2n\phi_1,c,c^\vee,\mathcal{L}_2^\natural\underset{\mathscr{O}_{G^\natural}}{\otimes}(\mathcal{L}_1^\natural)^{\otimes n}\underset{\mathscr{O}_{G^\natural}}{\otimes}[-1]^*(\mathcal{L}_1^\natural)^{\otimes n},\tau,\psi_2\psi_1^n[-1]^*\psi_1^n)$$

define objects in $\mathrm{DD}^*_{\mathrm{ample}}(R,I)$. For simplicity, let us replace the tuples $(A,\underline{X},\underline{Y},\phi_1,c,c^\vee,\mathcal{L}_1^\natural,\tau,\psi_1)$ and $(A,\underline{X},\underline{Y},\phi_2,c,c^\vee,\mathcal{L}_2^\natural,\tau,\psi_2)$ with these two tuples, respectively. Then the functor $\mathrm{M}^*_{\mathrm{ample}}(R,I)$ defines objects $(G,\mathcal{L}_1)$ and $(G,\mathcal{L}_2)$ in $\mathrm{DEG}_{\mathrm{ample}}(R,I)$ for these two tuples in $\mathrm{DD}^*_{\mathrm{ample}}(R,I)$ (with the same G, by Theorem 4.5.3.6), and defines an object $(G,\mathcal{F}:=\mathcal{L}_1\underset{\mathscr{O}_G}{\otimes}\mathcal{L}_2^{\otimes -1})$ in $\mathrm{DEG}_{\mathrm{IS}}(R,I)$ (see Definition 4.4.13). The point is to show that the assignment

$$(A,\underline{X},\underline{Y},f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta)\mapsto(G,\mathcal{F}) \tag{4.5.4.7}$$

does not depend on the choices.

Lemma 4.5.4.8. *Given any tuple* $(A,\mathcal{M}_1,X,Y,\phi_1,c,c^\vee,\tau,\psi_1)$ *(resp.* $(A,\mathcal{M}_2,X,Y,\phi_2,c,c^\vee,\tau,\psi_2)$*) in* $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$ *that admits a relatively complete model* $(P_1^\natural,\mathcal{L}_1^\natural)$ *(resp.* $(P_2^\natural,\mathcal{L}_2^\natural)$*) extending* $(G^\natural,\mathcal{L}_1^\natural:=\pi_*\mathcal{M}_1)$ *(resp.* $(G^\natural,\mathcal{L}_2^\natural:=\pi_*\mathcal{M}_2)$*), let* $(G,\mathcal{L}_1)$ *(resp.* $(G,\mathcal{L}_2)$*) be the corresponding "Mumford quotient" of* $(G^\natural,\mathcal{L}_1^\natural)$ *(resp.* $(G^\natural,\mathcal{L}_2^\natural)$*) (see Definition 4.5.2.18). Then the tensor product* $(A,\mathcal{M}_1\underset{\mathscr{O}_A}{\otimes}\mathcal{M}_2,X,Y,\phi_1+\phi_2,c,c^\vee,\tau,\psi_1\psi_2)$ *admits a relatively complete model* $(P^\natural,\mathcal{L}^\natural)$ *extending* $(G^\natural,\mathcal{L}^\natural:=\mathcal{L}_1^\natural\underset{\mathscr{O}_{G^\natural}}{\otimes}\mathcal{L}_2^\natural)$*, and* $(G,\mathcal{L}_1\underset{\mathscr{O}_G}{\otimes}\mathcal{L}_2)$ *is (isomorphic to) the "Mumford quotient" of* $(G^\natural,\mathcal{L}^\natural)$*.*

Proof. Consider the diagonal embedding of

$$(A,\mathcal{M}_1\underset{\mathscr{O}_A}{\otimes}\mathcal{M}_2,X,Y,\phi_1+\phi_2,c,c^\vee,\tau,\psi_1\psi_2)$$

into its fiber product with itself (defined as in the proof of Theorem 4.5.3.6). Then a relatively complete model for the fiber product object can be given by $(P_1^\natural\underset{S}{\times}P_2^\natural,\mathrm{pr}_1^*\mathcal{L}_1^\natural\underset{\mathscr{O}_{P_1^\natural\times_S P_2^\natural}}{\otimes}\mathrm{pr}_2^*\mathcal{L}_2^\natural)$, and the "Mumford quotient" of $(G^\natural\underset{S}{\times}G^\natural,\mathrm{pr}_1^*\mathcal{L}_1^\natural\underset{\mathscr{O}_{G^\natural\times_S G^\natural}}{\otimes}\mathrm{pr}_2^*\mathcal{L}_2^\natural)$ is $(G\underset{S}{\times}G,\mathrm{pr}_1^*\mathcal{L}_1\underset{\mathscr{O}_{G\times_S G}}{\otimes}\mathrm{pr}_2^*\mathcal{L}_2)$. By taking the closure $P^\natural$ of the diagonal image of $G^\natural\to G^\natural\underset{S}{\times}G^\natural\hookrightarrow P_1^\natural\underset{S}{\times}P_2^\natural$, and by taking the restriction of $\mathrm{pr}_1^*\mathcal{L}_1^\natural\underset{\mathscr{O}_{P_1^\natural\times_S P_2^\natural}}{\otimes}\mathrm{pr}_2^*\mathcal{L}_2^\natural$ to $P^\natural$, we obtain a relatively complete model of $(A,\mathcal{M}_1\underset{\mathscr{O}_A}{\otimes}\mathcal{M}_2,X,Y,\phi_1+\phi_2,c,c^\vee,\tau,\psi_1\psi_2)$. Since the pullback of $\mathrm{pr}_1^*\mathcal{L}_1\underset{\mathscr{O}_{G\times_S G}}{\otimes}\mathrm{pr}_2^*\mathcal{L}_2$ under the diagonal morphism $G\hookrightarrow G\underset{S}{\times}G$ is $\mathcal{L}_1\underset{\mathscr{O}_G}{\otimes}\mathcal{L}_2$, Theorem 4.5.3.6 implies that the "Mumford quotient" of $(G^\natural,\mathcal{L}^\natural)$ is $(G,\mathcal{L}_1\underset{\mathscr{O}_G}{\otimes}\mathcal{L}_2)$, as desired. □

Corollary 4.5.4.9. *The assignment (4.5.4.7) in Construction 4.5.4.6 is independent of choices, and defines a functor*

$$\mathrm{M}_{\mathrm{IS}}(R,I):\mathrm{DD}_{\mathrm{IS}}(R,I)\to\mathrm{DEG}_{\mathrm{IS}}(R,I)$$

extending $\mathrm{M}^*_{\mathrm{ample}}(R,I):\mathrm{DD}^*_{\mathrm{ample}}(R,I)\to\mathrm{DEG}_{\mathrm{ample}}(R,I)$*. By restriction, it defines a functor*

$$\mathrm{M}_{\mathrm{ample}}(R,I):\mathrm{DD}_{\mathrm{ample}}(R,I)\to\mathrm{DEG}_{\mathrm{ample}}(R,I).$$

By extending and forgetting extra structures, it also defines a functor

$$\mathrm{M}(R,I):\mathrm{DD}(R,I)\to\mathrm{DEG}(R,I).$$

Proof. Since uniqueness can be verified under étale descent, we may assume that both $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively, and that we have four objects $(A,\mathcal{M}_1,X,Y,\phi_1,c,c^\vee,\tau,\psi_1)$, $(A,\mathcal{M}_2,X,Y,\phi_2,c,c^\vee,\tau,\psi_2)$, $(A,\mathcal{M}_1',X,Y,\phi_1',c,c^\vee,\tau,\psi_1')$, and $(A,\mathcal{M}_2',X,Y,\phi_2',c,c^\vee,\tau,\psi_2')$ such that, for $\mathcal{L}_1^\natural:=\pi^*\mathcal{M}_1$, $\mathcal{L}_2^\natural:=\pi^*\mathcal{M}_2$, $\mathcal{L}_1^{\natural\prime}:=\pi^*\mathcal{M}_1'$, and $\mathcal{L}_2^{\natural\prime}:=\pi^*\mathcal{M}_2'$, we have

$$\begin{aligned}&(A,X,Y,f_Y,c,c^\vee,\mathcal{F}^\natural,\tau,\zeta)\\&\cong(A,X,Y,\phi_1,c,c^\vee,\mathcal{L}_1^\natural,\tau,\psi_1)\otimes(A,X,Y,\phi_2,c,c^\vee,\mathcal{L}_2^\natural,\tau,\psi_2)^{\otimes -1}\\&\cong(A,X,Y,\phi_1',c,c^\vee,\mathcal{L}_1^{\natural\prime},\tau,\psi_1')\otimes(A,X,Y,\phi_2',c,c^\vee,\mathcal{L}_2^{\natural\prime},\tau,\psi_2')^{\otimes -1}.\end{aligned}$$

Let $(G,\mathcal{L}_1)$, $(G,\mathcal{L}_2)$, $(G,\mathcal{L}_1')$, and $(G,\mathcal{L}_2')$ be the respective "Mumford quotients" of the four objects in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$ that we have introduced. Then we have two possible definitions of $(G,\mathcal{F})$, which are $(G,\mathcal{L}_1\underset{\mathscr{O}_G}{\otimes}\mathcal{L}_2^{\otimes -1})$ and $(G,\mathcal{L}_1'\underset{\mathscr{O}_G}{\otimes}(\mathcal{L}_2')^{\otimes -1})$. We need to show that $\mathcal{L}_1\underset{\mathscr{O}_G}{\otimes}\mathcal{L}_2^{\otimes -1}\cong\mathcal{L}_1'\underset{\mathscr{O}_G}{\otimes}(\mathcal{L}_2')^{\otimes -1}$, or equivalently $\mathcal{L}_1\underset{\mathscr{O}_G}{\otimes}\mathcal{L}_2'\cong\mathcal{L}_2\underset{\mathscr{O}_G}{\otimes}\mathcal{L}_1'$. This follows from Lemma 4.5.4.8 because we have by assumption an isomorphism of tensor products of objects in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$:

$$\begin{aligned}&(A,X,Y,\phi_1,c,c^\vee,\mathcal{L}_1^\natural,\tau,\psi_1)\otimes(A,X,Y,\phi_2',c,c^\vee,\mathcal{L}_2^{\natural\prime},\tau,\psi_2')\\&\cong(A,X,Y,\phi_2,c,c^\vee,\mathcal{L}_2^\natural,\tau,\psi_2)\otimes(A,X,Y,\phi_1',c,c^\vee,\mathcal{L}_1^{\natural\prime},\tau,\psi_1').\end{aligned}$$

The remaining statements of the corollary are clear. □

Let us construct a functor $\mathrm{M}_{\mathrm{pol}}(R,I):\mathrm{DD}_{\mathrm{pol}}(R,I)\to\mathrm{DEG}_{\mathrm{pol}}(R,I)$ as well. For this purpose, we also need to construct the dual of G using Mumford's construction. Let $(A,\underline{X},\underline{Y},c,c^\vee,\tau)$ be a tuple in $\mathrm{DD}(R,I)$. Let $f_{\underline{Y}\times_S\underline{X}}:\underline{Y}\underset{S}{\times}\underline{X}\xrightarrow{\sim}\underline{X}\underset{S}{\times}\underline{Y}$ be the isomorphism switching the two factors. Let $f_{A\times_S A^\vee}:A\underset{S}{\times}A^\vee\xrightarrow{\sim}A^\vee\underset{S}{\times}A$ be the isomorphism switching the two factors, over which we have an isomorphism $\mathcal{P}_A\xrightarrow{\sim}\mathcal{P}_{A^\vee}$ covering it. Let $\tau^\vee:\mathbf{1}_{\underline{X}\times_S\underline{Y},\eta}\xrightarrow{\sim}(c^\vee\times c)^*\mathcal{P}^{\otimes -1}_{A^\vee,\eta}$ be defined by switching the factors in $\tau:\mathbf{1}_{\underline{Y}\times_S\underline{X},\eta}\xrightarrow{\sim}(c\times c^\vee)^*\mathcal{P}^{\otimes -1}_{A,\eta}$ using the (inverses of the) above-mentioned isomorphisms.

Definition 4.5.4.10. *Let* $(A,\underline{X},\underline{Y},c,c^\vee,\tau)$ *be an object in* $\mathrm{DD}(R,I)$*. We define its* ***dual tuple*** *to be the tuple*

$$(A^\vee,\underline{Y},\underline{X},c^\vee,c,\tau^\vee)$$

(which is not yet known to be a tuple in $\mathrm{DD}(R,I)$ *as we do not know its extendability to an object in* $\mathrm{DD}_{\mathrm{pol}}(R,I)$*).*

Lemma 4.5.4.11. *The dual tuple* $(A^\vee,\underline{Y},\underline{X},c^\vee,c,\tau^\vee)$ *defined above is also an object in* $\mathrm{DD}(R,I)$*. (Then* $\tau^\vee$ *corresponds to a period homomorphism* $\iota^\vee:\mathbf{1}_{\underline{X},\eta}\xrightarrow{\sim}G_\eta^{\vee,\natural}$ *by Lemma 4.2.1.7.)*

Proof. We have to show that there exists

$$(A^\vee,\lambda_{A^\vee},\underline{Y},\underline{X},\phi^\vee,c^\vee,c,\tau^\vee)$$

in $\mathrm{DD}_{\mathrm{pol}}(R,I)$ extending $(A^\vee,\underline{Y},\underline{X},c^\vee,c,\tau^\vee)$, such that $\tau^\vee$ satisfies the positivity condition defined using $\phi^\vee$.

Let $G^\natural$ (resp. $G^{\vee,\natural}$) denote the extension defined by $c : \underline{X} \to A^\vee$ (resp. $c^\vee : \underline{Y} \to (A^\vee)^\vee \cong A$). Let us take any ϕ and λ_A in addition to $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, \tau)$ that (altogether) define an object in $\mathrm{DD}_{\mathrm{pol}}(R, I)$, and let us take an integer $N > 0$ sufficiently large such that there exists a homomorphism $\lambda^{\vee,\natural} : G^{\vee,\natural} \to G^\natural$ such that $\lambda^{\vee,\natural}\lambda^\natural = [N]$ and $\lambda^\natural\lambda^{\vee,\natural} = [N]$ are the multiplications by N on $G^\natural$ and $G^{\vee,\natural}$. (Do not interpret $\lambda^{\vee,\natural}$ as defined by the dual of any homomorphism. The notation simply means we are defining a homomorphism on the dual objects.) Then there exist an embedding $\phi^\vee : \underline{X} \hookrightarrow \underline{Y}$ and an isogeny $\lambda_{A^\vee} : A^\vee \to A$ with the compatibility $c^\vee\phi^\vee = \lambda_{A^\vee} c$ defining $\lambda^{\vee,\natural}$ such that $\phi'\phi = [N]$, $\phi\phi' = [N]$, $\lambda_{A^\vee}\lambda_A = [N]$, and $\lambda_A\lambda_{A^\vee} = [N]$ are the multiplications by N on $\underline{Y}$, $\underline{X}$, A, and $A^\vee$. Then $\lambda_{A^\vee}$ is necessarily a polarization by Corollary 1.3.2.21.

After making a finite étale surjective base change in S such that $\underline{X}$ and $\underline{Y}$ become constant with values X and Y, respectively, the positivity condition is satisfied for τ over $Y \times \phi(Y) \supset \phi'(X) \times \phi(\phi'(X)) = \phi'(X) \times NX$, and hence for $\tau^\vee$ over $NX \times \phi'(X)$, or equivalently over $X \times \phi'(X)$ by the bimultiplicativity. This shows that the tuple $(A^\vee, \lambda_{A^\vee}, \underline{Y}, \underline{X}, \phi^\vee, c^\vee, c, \tau^\vee)$ is in $\mathrm{DD}_{\mathrm{pol}}(R, I)$, as desired. □

By applying the functor $\mathrm{M}(R, I)$ to the dual tuple $(A^\vee, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$ of $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$, we obtain an object G' in $\mathrm{DEG}(R, I)$. We would like to show that this G' is the dual semi-abelian scheme $G^\vee$ of G defined in Theorem 3.4.3.2. For this purpose, we shall construct an birigidified invertible sheaf $\mathcal{P}$ over $G \underset{S}{\times} G'$ such that $\chi(\mathcal{P}_\eta) = \pm 1$. Then $\mathcal{P}_\eta$ sets up a *divisorial correspondence* between G_η and G'_η, as in [94, §8, Prop. 2, and §13, Prop.], making G'_η isomorphic to the dual abelian variety of G_η.

Lemma 4.5.4.12. *Under the switching isomorphism* $f_{A \underset{S}{\times} A^\vee} : A \underset{S}{\times} A^\vee \overset{\sim}{\to} A^\vee \underset{S}{\times} A$, *we have* $\mathcal{D}_2(\mathcal{P}_A) \cong (\mathrm{Id}_{A \underset{S}{\times} A^\vee} \times f_{A \underset{S}{\times} A^\vee})^* \mathcal{P}_{A \underset{S}{\times} A^\vee}$.

Proof. By evaluating $\mathcal{D}_2(\mathcal{P}_A)$ at each functorial point (a_1, a_1', a_2, a_2') of $A \underset{S}{\times} A^\vee \underset{S}{\times} A \underset{S}{\times} A^\vee$, we obtain (using the biextension structure of $\mathcal{P}_A$)

$$\begin{aligned} &\mathcal{P}_A|_{(a_1+a_2, a_1'+a_2')} \underset{\mathscr{O}_S}{\otimes} (\mathcal{P}_A|_{(a_1,a_1')})^{\otimes -1} \underset{\mathscr{O}_S}{\otimes} (\mathcal{P}_A|_{(a_2,a_2')})^{\otimes -1} \\ &\cong \mathcal{P}_A|_{(a_1,a_2')} \underset{\mathscr{O}_S}{\otimes} \mathcal{P}_A|_{(a_2,a_1')} \cong \mathcal{P}_A|_{(a_1,a_2')} \underset{\mathscr{O}_S}{\otimes} \mathcal{P}_{A^\vee}|_{(a_1',a_2)}. \end{aligned}$$

This shows that $\mathcal{D}_2(\mathcal{P}_A) \cong \mathrm{pr}_{14}^* \mathcal{P}_A \underset{\mathscr{O}_{A \underset{S}{\times} A^\vee \underset{S}{\times} A \underset{S}{\times} A^\vee}}{\otimes} \mathrm{pr}_{23}^* \mathcal{P}_{A^\vee}$, which is equivalent to what we need. □

Lemma 4.5.4.13. *Let* $\tau_\times := \mathrm{pr}_{13}^* \tau + \mathrm{pr}_{24}^* \tau^\vee$. *Then* $\mathcal{D}_2(\tau) = \mathrm{pr}_{14}^* \tau + \mathrm{pr}_{23}^* \tau = (\mathrm{Id}_{\underline{Y} \underset{S}{\times} \underline{X}} \times f_{\underline{Y} \underset{S}{\times} \underline{X}})^* \tau_\times$.

Proof. This is because we have

$$\begin{aligned} \mathcal{D}_2(\tau)(y_1, \chi_1, y_2, \chi_2) &= \tau(y_1 + y_2, \chi_1 + \chi_2)\tau(y_1, \chi_1)^{-1}\tau(y_2, \chi_2)^{-1} \\ &= \tau(y_1, \chi_2)\tau(y_2, \chi_1) = \tau(y_1, \chi_2)\tau^\vee(\chi_1, y_2) \end{aligned}$$

at each functorial point $(y_1, \chi_1, y_2, \chi_2)$ of $\underline{Y} \underset{S}{\times} \underline{X} \underset{S}{\times} \underline{Y} \underset{S}{\times} \underline{X}$, □

Construction 4.5.4.14. Let us first construct $(G \underset{S}{\times} G', \mathcal{P})$ as an object in $\mathrm{DEG}_{\mathrm{IS}}(R, I)$ (see Definition 4.4.12). Since the object G (resp. G') in $\mathrm{DEG}(R, I)$ is associated with the object $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ (resp. $(A^\vee, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$) in $\mathrm{DD}(R, I)$, their fiber product $G \underset{S}{\times} G'$ is associated with

$$(A \underset{S}{\times} A^\vee, \underline{X} \underset{S}{\times} \underline{Y}, \underline{Y} \underset{S}{\times} \underline{X}, c \times c^\vee, c^\vee \times c, \tau_\times = \mathrm{pr}_{13}^* \tau + \mathrm{pr}_{24}^* \tau^\vee). \qquad (4.5.4.15)$$

Let $\pi : G^\natural \to A$ and $\pi^\vee : G^{\vee,\natural} \to A^\vee$ denote the structural morphisms. In order to define an object in $\mathrm{DD}_{\mathrm{IS}}(R, I)$ over the object (4.5.4.15) in $\mathrm{DD}(R, I)$, we need

1. the switching isomorphisms $f_{\underline{Y} \underset{S}{\times} \underline{X}} : \underline{Y} \underset{S}{\times} \underline{X} \overset{\sim}{\to} \underline{X} \underset{S}{\times} \underline{Y}$ and $f_{A \underset{S}{\times} A^\vee} : A \underset{S}{\times} A^\vee \overset{\sim}{\to} A^\vee \underset{S}{\times} A$, satisfying the compatibility $(c \times c^\vee) f_{\underline{Y} \underset{S}{\times} \underline{X}} = f_{A \underset{S}{\times} A^\vee}(c^\vee \times c)$;
2. the cubical invertible sheaf $\mathcal{P}^\natural := (\pi \times \pi^\vee)^* \mathcal{P}_A$, compatible with $f_{A \underset{S}{\times} A^\vee}$ by Lemma 4.5.4.12;
3. the cubical trivialization $\psi_{\mathcal{P}} : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X}, \eta} \overset{\sim}{\to} (\iota \times \iota^\vee)^* \mathcal{P}^\natural_\eta$ defined by
$$\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X}, \eta} \overset{\sim}{\to} (c \times c^\vee)^* \mathcal{P}_{A,\eta}(\iota \times \iota^\vee)^*(\pi \times \pi^\vee)^* \mathcal{P}_{A,\eta} \cong (\iota \times \iota^\vee)^* \mathcal{P}^\natural_\eta,$$
which is compatible with $\tau_\times$ by Lemma 4.5.4.13.

Thus $(A \underset{S}{\times} A^\vee, \underline{X} \underset{S}{\times} \underline{Y}, \underline{Y} \underset{S}{\times} \underline{X}, f_{\underline{Y} \underset{S}{\times} \underline{X}}, c, c^\vee, \mathcal{P}^\natural, \tau_\times, \psi_{\mathcal{P}})$ defines an object in $\mathrm{DD}_{\mathrm{IS}}(R, I)$, and we obtain, by applying $\mathrm{M}_{\mathrm{IS}}(R, I)$, an object $(G \underset{S}{\times} G', \mathcal{P})$ in $\mathrm{DEG}_{\mathrm{IS}}(R, I)$.

As explained above, we need $\chi(\mathcal{P}_\eta) = \pm 1$ to show that $\mathcal{P}_\eta$ establishes a divisorial correspondence between G_η and G'_η.

We shall prove a more general result that computes $\chi(\mathcal{F}_\eta)^2$ when $(G, \mathcal{F})$ is associated with some tuple $(A, \underline{X}, \underline{Y}, f_Y, c, c^\vee, \mathcal{F}^\natural, \tau, \zeta)$ in $\mathrm{DD}_{\mathrm{IS}}(R, I)$ under $\mathrm{M}_{\mathrm{IS}}(R, I)$. Since this is a question about equalities, we can always make a finite étale surjective base change in S and assume that both $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively, and that $\mathcal{F} \cong \pi^* \mathcal{N}$ for some invertible sheaf $\mathcal{N}$ over A inducing $f_A : A \to A^\vee$.

Lemma 4.5.4.16. *Let* $\chi \in X$. *For each* $n > 0$, *we have* $I_y \cdot I_{y,\chi} \subset I^n$ *for all but finitely many* $y \in Y$ *(see Definitions 4.2.4.5 and 4.2.4.6).*

Proof. As in Section 4.2.3, let us denote by Υ_1 the set of valuations of K defined by height-one primes of R, and let us denote by Υ_I the set of discrete valuations υ of K having centers on $S_0 = \mathrm{Spec}(R_0)$.

For convenience, let us denote the function $Y \to \mathrm{Inv}(R) : y \mapsto I_y$ (resp. $Y \times X \to \mathrm{Inv}(R) : (y, \chi) \mapsto I_{y,\chi}$) by a (resp. b).

First let us show that $I_y \cdot I_{y,\chi} \subset R$ for all but finitely many $y \notin Q$. Note that $I_y \subset R$ for all but finitely many $y \in Y$. So it suffices to consider the finitely many valuations υ in Υ_1 for which $\upsilon(I_{y,\chi}) < 0$ can happen. As in the proof of Lemma 4.5.1.7, by forming a quotient by the radical of the associated pairing for each υ, we may assume that $\upsilon(b(\,\cdot\,, \phi(\,\cdot\,)))$ is positive definite. Let $\|y\|_\upsilon := \upsilon(b(y, \phi(y)))^{1/2}$ be the associated norm on the real vector space $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, in which we have two lattices X and Y with X embedded in $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ via the embedding $\phi : Y \hookrightarrow X$ with finite cokernel. Since the quadratic function $\upsilon(a(\,\cdot\,))$ has associated bilinear pairing $\upsilon(b(\,\cdot\,, \phi(\,\cdot\,)))$, which is positive semidefinite, there are constants $\kappa_\upsilon^{(2)}, \kappa_\upsilon^{(1)}, \kappa_\upsilon^{(0)} \in \mathbb{R}$, with $\kappa_\upsilon^{(2)} > 0$, such that

$$\upsilon(a(y)) > \kappa_\upsilon^{(2)} \|y\|_\upsilon^2 + \kappa_\upsilon^{(1)} \|y\|_\upsilon + \kappa_\upsilon^{(0)}$$

for all $y \in Y$. Now $\upsilon(b(y,\chi)) > \|y\|_\upsilon \|\chi\|_\upsilon$, and hence we have
$$\upsilon(a(y)) > \kappa_\upsilon^{(2)}\|y\|_\upsilon^2 + (\kappa_\upsilon^{(1)} - \|\chi\|_\upsilon)\|y\|_\upsilon + \kappa_\upsilon^{(0)}$$
with $\kappa_\upsilon^{(2)} > 0$. In particular, $\upsilon(a(y)\cdot b(y,\chi)) \geq 0$ for all but finitely many y. Since the number of υ to consider is finite, by Lemma 4.2.4.2, we have $I_y \cdot I_{y,\chi} \subset R$ for all but finitely many $y \in Y$.

Next let us show that, for each $n > 0$, we have $I_y \cdot I_{y,\chi} \subset I^n$ for all but finitely many $y \in Y$. Let us first exclude those $y \in Y$ such that $I_y \cdot I_{y,\chi} \subset R$ is not true. By the positivity condition for ψ, we have $I_y \subset I^n$ for all but finitely many $y \in Y$. As a result, we only need to consider those finitely many $\upsilon \in \Upsilon_I$ for which $\upsilon(b(y,\chi)) < 0$ can happen. Moreover, for each $\upsilon \in \Upsilon_I$, there are again constants $\kappa_\upsilon^{(2)}, \kappa_\upsilon^{(1)}, \kappa_\upsilon^{(0)} \in \mathbb{R}$, with $\kappa_\upsilon^{(2)} > 0$, such that
$$\upsilon(a(y)) > \kappa_\upsilon^{(2)}\|y\|_\upsilon^2 + \kappa_\upsilon^{(1)}\|y\|_\upsilon + \kappa_\upsilon^{(0)}$$
for all $y \in Y$. Now $\upsilon(b(y,\chi)) > \|y\|_\upsilon \|\chi\|_\upsilon$, and hence we have
$$\upsilon(a(y)) > \kappa_\upsilon^{(2)}\|y\|_\upsilon^2 + (\kappa_\upsilon^{(1)} - \|\chi\|_\upsilon)\|y\|_\upsilon + \kappa_\upsilon^{(0)}$$
with $\kappa_\upsilon^{(2)} > 0$. In particular, $\upsilon(a(y)\cdot b(y,\chi)) \geq n$ for all but finitely many y. Since the number of υ to consider is finite, by Lemma 4.2.4.4, we have $I_y \cdot I_{y,\chi} \subset I^n$ for all but finitely many $y \in Y$, as desired. $\square$

Theorem 4.5.4.17 (cf. [42, Ch. III, Thm. 6.1])**.** *For every object* $(A, \mathcal{N}, X, Y, f_Y, c, c^\vee, \tau, \zeta)$ *such that* $(A, X, Y, f_Y, c, c^\vee, \mathcal{F}^\natural = \pi^*\mathcal{N}, \tau, \zeta)$ *is an object in* $\mathrm{DD}_{\mathrm{IS}}(R,I)$*, we have* $\chi(\mathcal{F}_\eta)^2 = \chi(\mathcal{N}_\eta)^2 \cdot \deg(f_Y^*)^2 = \deg(f_A)\cdot\deg(f_Y^*)^2$*. (The degrees are zero for morphisms that are not finite.)*

Proof. The relation $\chi(\mathcal{N}_\eta)^2 = \deg(f_A)$ follows from the Riemann–Roch theorem for abelian varieties (see [94, §16]). Therefore it suffices to prove the first equality. With the given object $(A, \mathcal{N}, X, Y, f_Y, c, c^\vee, \tau, \zeta)$, there is always an object $(A, \mathcal{M}_0, X, Y, \phi_0, c, c^\vee, \tau, \psi_0)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$ such that for some integer $n_0 > 0$, the tuple $(A, \mathcal{N} \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_0^{\otimes n}, X, Y, f_A + n\phi_0, c, c^\vee, \tau, \zeta\psi_0^n)$ is an object in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$ for all $n \geq n_0$ because we only need the injectivity of $f_A + n\phi_0$ and the relative ampleness of $\mathcal{N} \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_0^{\otimes n}$. This is still true if we replace $\mathcal{M}_0$ with $\mathcal{M}_1 := \mathcal{M}_0 \underset{\mathscr{O}_A}{\otimes} [-1]^*\mathcal{M}_0$, ϕ_0 with $\phi_1 := \phi_0 + [-1]^*\phi_0$, and ψ_0 with $\psi_1 := \psi_0\, [-1]^*\psi_0$. By Corollary 4.5.1.8, there is another $n_1 \geq n_0$ such that $(A, \mathcal{N} \underset{\mathscr{O}_A}{\otimes} \mathcal{M}_1^{\otimes n}, X, Y, f_A + n\phi_1, c, c^\vee, \tau, \zeta\psi_1^n)$ is in $\mathrm{DD}^{\mathrm{split},*}_{\mathrm{ample}}(R,I)$ for all $n \geq n_1$. Let $(G, \mathcal{L}_1)$ be the pair in $\mathrm{DEG}_{\mathrm{ample}}(R,I)$ associated with $(A, X, Y, \phi_1, c, c^\vee, \mathcal{L}_1^\natural = \pi_*\mathcal{M}_1, \tau, \psi_1)$ in $\mathrm{DD}_{\mathrm{ample}}(R,I)$ by $\mathrm{M}_{\mathrm{ample}}(R,I)$. Consider $\chi(\mathcal{F}_\eta \underset{\mathscr{O}_{G,\eta}}{\otimes} \mathcal{L}_{1,\eta}^{\otimes n})^2$ and $\chi(\mathcal{N}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathcal{M}_{1,\eta}^{\otimes n})^2 \cdot \deg(f_Y^* + n\phi_1^*)$, which are both polynomials in n by the Riemann–Roch theorem for abelian varieties (see [94, §16] again). Therefore, if the equality holds for all $n \geq n_1$, then it actually holds for all n, and in particular, $n = 0$. Therefore it suffices to deal with the case that we have an object $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$, with a "Mumford quotient" $(P, \mathcal{L})$ defined by some relatively complete model $(P^\natural, \mathcal{L}^\natural)$ (see Definition 4.5.2.18).

Recall (from Section 4.3) that the Fourier expansion of a section $s \in \Gamma(G, \mathcal{L})$ is obtained by writing s as an I-adically complete sum $\sum_{\chi\in X} \sigma_\chi(s)$ in $\Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) \cong \Gamma(G^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}}) \cong \hat{\bigoplus}_{\chi\in X} \Gamma(A, \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi)$, formed according to the T-action, where $\sigma_\chi(s) \in \Gamma(A, \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi)$. Since $P_\eta = G_\eta$, there is a nonzero element $r \in R$ such that $r \cdot s \in \Gamma(P, \mathcal{L})$. Since $(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}})$ is the quotient of $(P^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}})$ under Y, we have an embedding
$$\Gamma(P, \mathcal{L}) \cong \Gamma(P_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) \cong \Gamma(P^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}})^Y \hookrightarrow \Gamma(P^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}}). \tag{4.5.4.18}$$
Since the union of Y-translations of $G^\natural_{\mathrm{for}}$ form an open dense formal subscheme of $P^\natural_{\mathrm{for}}$, the composition of $\Gamma(P^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}})^Y \hookrightarrow \Gamma(P^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}})$ with the restriction $\Gamma(P^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}}) \to \Gamma(G^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}})$ remains injective. Therefore we have a canonical embedding
$$\Gamma(P, \mathcal{L}) \hookrightarrow \Gamma(G^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}}), \tag{4.5.4.19}$$
from which we obtain the Fourier expansion $r \cdot s = \sum_{\chi\in X} r\cdot\sigma_\chi(s)$, with Y-invariance described by
$$\sigma_{\chi+\phi(y)}(s) = \tilde{S}_y(T^*_{c^\vee(y)}\sigma_\chi(s)) = \psi(y)\tau(y,\chi)\, T^*_{c^\vee(y)}\sigma_\chi(s) \tag{4.5.4.20}$$
(see Construction 4.5.1.5). Let V be the K-subspace of $\Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) \underset{R}{\otimes} K \cong \Gamma(G^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}}) \underset{R}{\otimes} K$ consisting of the (Y-invariant) infinite sums
$$\left\{ \begin{array}{l} \sum_{\chi\in X} \theta_\chi : \theta_\chi \in \Gamma(A_\eta, \mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}), \\ \theta_{\chi+\phi(y)} = \psi(y)\tau(y,\chi)\, T^*_{c^\vee(y)}(\theta_\chi)\ \forall y \end{array} \right\}. \tag{4.5.4.21}$$
Then (4.5.4.19) identifies $\Gamma(G_\eta, \mathcal{L}_\eta) \cong \Gamma(P, \mathcal{L}) \underset{R}{\otimes} K$ as a K-subspace of V.

By Lemma 4.5.4.16, for each $\chi \in X$ and $\theta_\chi \in \Gamma(A_\eta, \mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta})$, there is a nonzero $r \in R$ such that
$$r \sum_{y\in Y} (\psi(y)\tau(y,\chi)\, T^*_{c^\vee(y)}(\theta_\chi)) \tag{4.5.4.22}$$
converges I-adically. That is, for each $n > 0$, the sum (4.5.4.22) has only finitely many nonzero terms modulo I^n (see Section 4.3.1). Since $\dim_K \Gamma(A_\eta, \mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) = \dim_K \Gamma(A_\eta, \mathcal{M}_\eta) = \chi(\mathcal{M}_\eta)$ (for all $\chi \in X$), this shows that $\dim_K V = \chi(\mathcal{M}_\eta)\cdot[X : \phi(Y)] = \chi(\mathcal{M}_\eta)\cdot\deg(f_Y^*)$. On the other hand, since there is an $G^\natural$-invariant open subscheme $U \subset P^\natural$ of finite type over S such that $\bigcup_{y\in Y} S_y(U) = P^\natural$, the convergent sum (4.5.4.22) lies in the image of (4.5.4.18). Since $\chi \in X$ and θ_χ are arbitrary, this shows that $V = \Gamma(G_\eta, \mathcal{L}_\eta)$. Since $\dim_K \Gamma(G_\eta, \mathcal{L}_\eta) = \chi(\mathcal{L}_\eta)$, we obtain the relation $\chi(\mathcal{L}_\eta) = \chi(\mathcal{M}_\eta)\cdot\deg(f_Y^*)$, as desired. $\square$

Corollary 4.5.4.23. *Given any tuple* $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ *in* $\mathrm{DD}(R,I)$ *that defines* G *by* $\mathrm{M}(R,I) : \mathrm{DD}(R,I) \to \mathrm{DEG}(R,I)$*, the dual tuple* $(\lambda_A, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$ *as in Definition 4.5.4.10 defines* $G^\vee$ *by* $\mathrm{M}(R,I)$*.*

Proof. This is because $\deg(f_{\underline{Y} \underset{S}{\times} \underline{X}}) = 1$ and $\deg(f_{A \underset{S}{\times} A^\vee}) = 1$. $\square$

Corollary 4.5.4.24 (of the proof of Theorem 4.5.4.17)**.** *Suppose* $(A, \mathcal{M}, X, Y, \phi, c, c^\vee, \tau, \psi)$ *is an object in* $\mathrm{DD}^{\mathrm{split}}_{\mathrm{ample}}(R,I)$*, with a "Mumford quotient"* $(G, \mathcal{L})$ *defined by some relatively complete model. Then the image of the canonical morphism* $\Gamma(G_\eta, \mathcal{L}_\eta) \hookrightarrow \Gamma(G^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}}) \underset{R}{\otimes} K$ *is the Y-invariant K-subspace of* $\Gamma(G^\natural_{\mathrm{for}}, \mathcal{L}^\natural_{\mathrm{for}}) \underset{R}{\otimes} K$ *described by (4.5.4.21).*

Lemma 4.5.4.25 (see [42, Ch. II, p. 51]). *Suppose that we have a morphism $f : (G_1, \mathcal{L}_1) \to (G_2, \mathcal{L}_2)$ in $\mathrm{DEG}_{\mathrm{ample}}(R, I)$, and suppose that, under the functor $\mathrm{F}_{\mathrm{ample}}(R, I)$ in Theorem 4.2.1.14, f induces an isomorphism in $\mathrm{DD}_{\mathrm{ample}}(R, I)$. Then f is an isomorphism. That is, $\mathrm{F}_{\mathrm{ample}}(R, I)$* ***detects isomorphisms****.*

Proof. For $i = 1, 2$, let $(A_i, \underline{X}_i, \underline{Y}_i, \phi_i, c_i, c_i^\vee, \mathcal{L}_i^\natural, \tau_i, \psi_i)$ be the degeneration data in $\mathrm{DD}_{\mathrm{ample}}(R, I)$ associated with $(G_i, \mathcal{L}_i)$ under $\mathrm{F}_{\mathrm{ample}}(R, I)$. By Lemma 4.5.4.5, up to making a finite étale surjective base change in S, and up to replacing $\mathcal{L}_1$ and $\mathcal{L}_2$ with $\mathcal{L}_1^{\otimes m}$ and $\mathcal{L}_2^{\otimes m}$ for some integer $m > 0$, we may assume that the two degeneration data are induced by objects in $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split},*}(R, I)$, which admit relatively complete models by Proposition 4.5.1.15. By assumption, f induces isomorphisms $f_A : A_1 \xrightarrow{\sim} A_2$, $f_X : \underline{X}_2 \xrightarrow{\sim} \underline{X}_1$, $f_Y : \underline{Y}_1 \xrightarrow{\sim} \underline{Y}_2$, $f^\natural : G_1^\natural \to G_2^\natural$, and $\mathcal{L}_1^\natural \xrightarrow{\sim} (f^\natural)^* \mathcal{L}_2^\natural$, and hence, by Corollary 4.3.4.2, compatible isomorphisms $\Gamma(G_{2,\mathrm{for}}^\natural, (\mathcal{L}_{2,\mathrm{for}}^\natural)^{\otimes n}) \xrightarrow{\sim} \Gamma(G_{1,\mathrm{for}}^\natural, (\mathcal{L}_{1,\mathrm{for}}^\natural)^{\otimes n})$ for all $n \geq 0$. By Corollary 4.5.4.24, the action of $\underline{Y}_1$ (resp. $\underline{Y}_2$) on $(\mathcal{L}_{1,\mathrm{for}}^\natural)^{\otimes n}$ (resp. $(\mathcal{L}_{2,\mathrm{for}}^\natural)^{\otimes n}$) defined by τ_1 and ψ_1 (resp. τ_2 and ψ_2) characterizes the image of the canonical embedding $\Gamma(G_{1,\eta}, (\mathcal{L}_{1,\eta})^{\otimes n}) \hookrightarrow \Gamma(G_{1,\mathrm{for}}^\natural, (\mathcal{L}_{1,\mathrm{for}}^\natural)^{\otimes n}) \underset{R}{\otimes} K$ (resp. $\Gamma(G_{2,\eta}, (\mathcal{L}_{2,\eta})^{\otimes n}) \hookrightarrow \Gamma(G_{2,\mathrm{for}}^\natural, (\mathcal{L}_{2,\mathrm{for}}^\natural)^{\otimes n}) \underset{R}{\otimes} K$). Therefore, the identification between (τ_1, ψ_1) and (τ_2, ψ_2) under the other isomorphisms induces isomorphisms $\Gamma(G_{2,\eta}, \mathcal{L}_{2,\eta}^{\otimes n}) \xrightarrow{\sim} \Gamma(G_{1,\eta}, \mathcal{L}_{1,\eta}^{\otimes n})$ for all $n \geq 0$, which are nothing but the compatible morphisms induced by f_η. Since $\mathcal{L}_{1,\eta}$ and $\mathcal{L}_{2,\eta}$ are ample, this implies that f_η is an isomorphism. Hence, f is also an isomorphism, by Proposition 3.3.1.5. □

Corollary 4.5.4.26. *The functor $\mathrm{F}_{\mathrm{ample}}(R, I) : \mathrm{DEG}_{\mathrm{ample}}(R, I) \to \mathrm{DD}_{\mathrm{ample}}(R, I)$ (given by Theorem 4.2.1.14) is a quasi-inverse of $\mathrm{M}_{\mathrm{ample}}(R, I)$, and hence both $\mathrm{F}_{\mathrm{ample}}(R, I)$ and $\mathrm{M}_{\mathrm{ample}}(R, I)$ are equivalences of categories.*

Proof. By reduction to the case of objects in $\mathrm{DD}_{\mathrm{ample}}^{\mathrm{split},*}(R, I)$ (again), we see in the proof of Theorem 4.5.4.17 that the composition $\mathrm{F}_{\mathrm{ample}}(R, I)\,\mathrm{M}_{\mathrm{ample}}(R, I)$ is canonically isomorphic to the identity. Then $\mathrm{F}_{\mathrm{ample}}(R, I)\,\mathrm{M}_{\mathrm{ample}}(R, I)\,\mathrm{F}_{\mathrm{ample}}(R, I)$ is also canonically isomorphic to $\mathrm{F}_{\mathrm{ample}}(R, I)$. Since $\mathrm{F}_{\mathrm{ample}}(R, I)$ *detects isomorphisms* by Lemma 4.5.4.25, $\mathrm{M}_{\mathrm{ample}}(R, I)\,\mathrm{F}_{\mathrm{ample}}(R, I)$ must be also isomorphic to the identity. This shows that $\mathrm{F}_{\mathrm{ample}}(R, I)$ and $\mathrm{M}_{\mathrm{ample}}(R, I)$ are quasi-inverses of each other, and hence are both equivalences of categories. □

Finally, we are ready to define $\mathrm{M}_{\mathrm{pol}}(R, I)$.

Construction 4.5.4.27. For each $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{pol}}(R, I)$, we obtain a morphism from $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ to $(A^\vee, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$ in $\mathrm{DD}(R, I)$ given by $\lambda_A : A \to A^\vee$, $\phi : \underline{Y} \hookrightarrow \underline{X}$, and $\phi : \underline{Y} \hookrightarrow \underline{X}$. By Corollary 4.5.4.23, the tuple $(A^\vee, \underline{Y}, \underline{X}, c^\vee, c, \tau^\vee)$ defines $G^\vee$ by $\mathrm{M}(R, I)$. As a result, we obtain a homomorphism $\lambda : G \to G^\vee$.

Lemma 4.5.4.28. *The restriction $\lambda_\eta : G_\eta \to G_\eta^\vee$ is a polarization.*

Proof. By Definition 1.3.2.16, we shall verify the condition in Proposition 1.3.2.15 that the pullback $(\mathrm{Id}_{G_\eta}, \lambda_\eta)^* \mathcal{P}$ is ample over G_η. Equivalently, by Lemma 4.2.1.6, we shall verify that $(\mathrm{Id}_G, \lambda)^* \mathcal{P}$ is ample over G.

The morphism $(\mathrm{Id}_G, \lambda) : G \to G \underset{S}{\times} G^\vee$ is given by the morphism

$$(A, \underline{X}, \underline{Y}, c, c^\vee, \tau) \to (A \underset{S}{\times} A^\vee, \underline{X} \underset{S}{\times} \underline{Y}, \underline{Y} \underset{S}{\times} \underline{X}, c \times c^\vee, c^\vee \times c, \tau_\times),$$

where $\tau_\times = \mathrm{pr}_{13}^* \tau \, \mathrm{pr}_{13} + \mathrm{pr}_{24}^* \tau^\vee \, \mathrm{pr}_{24}$ (see Lemma 4.5.4.13), defined by the morphisms $(\mathrm{Id}_A, \lambda_A) : A \to A \underset{S}{\times} A^\vee$, $\mathrm{Id}_{\underline{X}} + \phi : \underline{X} \underset{S}{\times} \underline{Y} \to \underline{X}$, and $(\mathrm{Id}_{\underline{Y}}, \phi) : \underline{Y} \hookrightarrow \underline{Y} \underset{S}{\times} \underline{X}$. On the other hand, in Construction 4.5.4.14, $(G \underset{S}{\times} G^\vee, \mathcal{P})$ is given by

$$(A \underset{S}{\times} A^\vee, \underline{X} \underset{S}{\times} \underline{Y}, \underline{Y} \underset{S}{\times} \underline{X}, f_{\underline{Y} \underset{S}{\times} \underline{X}}, c \times c^\vee, c^\vee \times c, \mathcal{P}^\natural := (\pi \times \pi^\vee)^* \mathcal{P}_A, \tau_\times, \psi_{\mathcal{P}}).$$

The pullback of $f_{\underline{Y} \underset{S}{\times} \underline{X}}$ is the composition $(\mathrm{Id}_{\underline{X}} + \phi)(f_{\underline{Y} \underset{S}{\times} \underline{X}})\phi$, which is simply 2ϕ. Since ϕ is injective, 2ϕ is also injective. The pullback of $(\pi \times \pi^\vee)^* \mathcal{P}_A$ is $\pi^* (\mathrm{Id}_A, \lambda_A)^* \mathcal{P}_A$ because $(\pi \times \pi^\vee)(\mathrm{Id}_{G^\natural}, \lambda^\natural) = (\pi, \lambda^\natural \pi) = (\mathrm{Id}_A, \lambda_A)\pi$. Since λ_A is a polarization, $(\mathrm{Id}_A, \lambda_A)^* \mathcal{P}_A$ is ample, and hence $\pi^* (\mathrm{Id}_A, \lambda_A)^* \mathcal{P}_A$ is also ample. The pullback of $\psi_{\mathcal{P}} = \tau$ is $\psi := (\mathrm{Id}_{\underline{Y}}, \phi)^* \tau$, with the compatibility $\mathcal{D}_2(\psi) = (\mathrm{Id}_{\underline{Y}} \times 2\phi)^* \tau$ given by the bimultiplicativity of the trivialization τ of biextensions. As a result, the pullback object $(A, \underline{X}, \underline{Y}, 2\phi, c, c^\vee, \pi^* (\mathrm{Id}_A, \lambda_A)^* \mathcal{P}_A, \tau, \psi)$, which is a priori an object in $\mathrm{DD}_{\mathrm{IS}}(R, I)$, defines an object in $\mathrm{DD}_{\mathrm{ample}}(R, I)$. Since this object defines $(G, (\mathrm{Id}_G, \lambda)^* \mathcal{P})$ under $\mathrm{M}(R, I)$, we see that $(\mathrm{Id}_G, \lambda)^* \mathcal{P}$ is ample, as desired. □

Corollary 4.5.4.29. *The assignment*

$$(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau) \mapsto (G, \lambda)$$

in Construction 4.5.4.27 defines a functor

$$\mathrm{M}_{\mathrm{pol}}(R, I) : \mathrm{DD}_{\mathrm{pol}}(R, I) \to \mathrm{DEG}_{\mathrm{pol}}(R, I).$$

Corollary 4.5.4.30. *The functors $\mathrm{M}_{\mathrm{ample}}(R, I) : \mathrm{DD}_{\mathrm{ample}}(R, I) \to \mathrm{DEG}_{\mathrm{ample}}(R, I)$, $\mathrm{M}_{\mathrm{pol}}(R, I) : \mathrm{DD}_{\mathrm{pol}}(R, I) \to \mathrm{DEG}_{\mathrm{pol}}(R, I)$, $\mathrm{M}_{\mathrm{IS}}(R, I) : \mathrm{DD}_{\mathrm{IS}}(R, I) \to \mathrm{DEG}_{\mathrm{IS}}(R, I)$, and $\mathrm{M}(R, I) : \mathrm{DD}(R, I) \to \mathrm{DEG}(R, I)$ are compatible with each other under the natural forgetful functors. Moreover, $\mathrm{M}_{\mathrm{ample}}(R, I)$ and $\mathrm{M}_{\mathrm{pol}}(R, I)$ are compatible with the "pullback functor" $\mathrm{DEG}_{\mathrm{pol}}(R, I) \to \mathrm{DEG}_{\mathrm{ample}}(R, I)$ described in Lemma 4.4.4 and the analogous functor $\mathrm{DD}_{\mathrm{pol}}(R, I) \to \mathrm{DD}_{\mathrm{ample}}(R, I)$.*

Proof. The compatibilities with forgetful functors follow from the very constructions of the functors. The compatibilities with the pullback functors follow from the proof of Lemma 4.5.4.28. □

Corollary 4.5.4.31. *The functor $\mathrm{F}_{\mathrm{pol}}(R, I) : \mathrm{DEG}_{\mathrm{pol}}(R, I) \to \mathrm{DD}_{\mathrm{pol}}(R, I)$ (see Definition 4.4.8) is a quasi-inverse of $\mathrm{M}_{\mathrm{pol}}(R, I)$, and hence both $\mathrm{F}_{\mathrm{pol}}(R, I)$ and $\mathrm{M}_{\mathrm{pol}}(R, I)$ are equivalences of categories.*

Proof. This follows from Corollaries 4.5.4.26 and 4.5.4.30. □

4.5.5 Dependence of τ on the Choice of $\mathcal{L}$, Revisited

The major goal of this section is to prove the following:

Proposition 4.5.5.1. *Let (G, λ) be an object in $\mathrm{DEG}_{\mathrm{pol}}(R, I)$ with associated degeneration data $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{pol}}(R, I)$ as in Definition 4.4.8. Then the homomorphism $\iota : \underline{Y}_\eta \to G_\eta^\natural$ corresponding to τ (see Lemma 4.2.1.7) is independent of the choice of λ.*

Lemma 4.5.5.2. *To prove Proposition 4.5.5.1, it suffices to show that the image of ι is independent of the choice of λ.*

Proof. We may replace R with a finite étale extension and assume that $\underline{Y}$ is constant. By Corollary 4.5.3.12 (with $s = \eta$), ι is injective and induces a canonical isomorphism

$\frac{1}{n}Y/Y \xrightarrow{\sim} G[n]_\eta/G^\natural[n]_\eta$ of group schemes for each integer $n \geq 1$ (if we identify Y with the character group of the torus part of $G^{\vee,\natural}$). Thus the conclusion follows from Serre's lemma that no nontrivial root of unity can be congruent to 1 modulo n if $n \geq 3$. $\square$

With the same setting as in Proposition 4.5.5.1, after making a finite étale base change in S if necessary (cf. Corollary 3.2.5.7), suppose moreover that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively. Let $\mathcal{L} := (\mathrm{Id}_G, \lambda)^*\mathcal{P}$ (cf. Lemma 4.2.1.4). Let $\lambda_A : A \to A^\vee$ be the polarization induced by $\lambda^\natural : G^\natural \to G^{\vee,\natural}$, and let $\mathcal{M} := (\mathrm{Id}_A, \lambda_A)^*\mathcal{P}_A$. Then $\mathcal{L}^\natural$ is isomorphic to the pullback of $\mathcal{M}$, and we have $H^0(G^\natural, (\mathcal{L}^\natural)^{\otimes m}) \cong \underset{\chi\in X}{\oplus} H^0(A, \mathcal{M}^{\otimes m} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi)$ for all integers $m \geq 0$, as usual.

Proposition 4.5.5.3 (cf. [42, Ch. III, Prop. 8.1])**.** *With assumptions as above, suppose moreover that R is a complete discrete valuation ring. Then there is a canonical map $p_K : G^\natural(K) \to G(K)$ of sets, independent of the choice of polarizations, realizing $G(K)$ as a quotient of $G^\natural(K)$ under the natural multiplication action of $\iota(Y)$.*

Proof. By Proposition 4.3.4.5, we may replace λ with its multiple by a sufficiently large integer without changing ι, such that a relatively complete model $P^\natural$ of $G^\natural$ exists (by Corollary 4.5.1.8 and Proposition 4.5.1.15).

Consider the canonical morphisms $G^\natural(K) \to P^\natural(K) \leftarrow P^\natural(R)$, where the first map is bijective by Corollary 4.5.2.3. We claim that the second map is also bijective. By separateness, it suffices to show that every K-valued point $x : \mathrm{Spec}(K) \to P^\natural$ extends to an R-valued point, as follows: Choose a valuation of the rational function field $K(P^\natural) \cong K(G^\natural)$ centered at x. By composing this valuation with an extension of the valuation υ of R to K, we obtain a valuation υ_1 satisfying the assumptions in the completeness condition (iii) in Definition 4.5.1.2 (because of the positivity of τ associated with the period homomorphism ι). This shows that υ_1 has a center on $P^\natural$, which shows that x extends to an R-valued point of $P^\natural$, as desired.

On the other hand, we have the canonical morphisms $G(K) \to P(K) \leftarrow P(R)$, where the first map is bijective by Proposition 4.5.2.23, and where the second map is bijective because P is projective (and hence proper) over $S = \mathrm{Spec}(R)$ (cf. Proposition 4.5.2.15).

Since P_{for} is a genuine quotient of $P^\natural_{\mathrm{for}}$ in the category of formal schemes (cf. Proposition 4.5.2.15), we have the canonical isomorphisms $G(K) \cong P(K) \cong P(R) \cong P_{\mathrm{for}}(R) \cong P^\natural_{\mathrm{for}}(R)/Y \cong P^\natural(R)/Y \cong P^\natural(K)/Y \cong G^\natural(K)/\iota(Y)$, defining the desired map $p_K : G^\natural(K) \to G(K)$ realizing $G(K)$ as a quotient of $G^\natural(K)$ under the natural multiplication action of $\iota(Y)$.

The map p_K can be described more explicitly as follows: Evaluation at each $x \in G^\natural(K)$ defines a morphism

$$x^* : \underset{m\geq 0}{\oplus} \Gamma(G^\natural, (\mathcal{L}^\natural)^{\otimes m}) \to \underset{m\geq 0}{\oplus} x^*(\mathcal{L}^\natural)^{\otimes m}.$$

Since $\Gamma(G^\natural_{\mathrm{for}}, (\mathcal{L}^\natural_{\mathrm{for}})^{\otimes m}) \cong \hat{\oplus}_{\chi\in X}\Gamma(A, \mathcal{M}^{\otimes m} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi)$ for every $m \geq 0$ (cf. Section 4.3.1), x^* extends to every subalgebra of $\underset{m\geq 0}{\oplus} \Gamma(G_{\mathrm{for}}, \mathcal{L}^{\otimes m}_{\mathrm{for}}) \cong \underset{m\geq 0}{\oplus} \Gamma(G^\natural_{\mathrm{for}}, (\mathcal{L}^\natural_{\mathrm{for}})^{\otimes m})$ over which the I-adic sums converge. Given the existence of the relatively complete model $P^\natural$ above, x^* defines convergent sums over the image of the canonical morphism $\underset{m\geq 0}{\oplus} \Gamma(G, \mathcal{L}^{\otimes m}) \to \underset{m\geq 0}{\oplus} \Gamma(G_{\mathrm{for}}, (\mathcal{L}_{\mathrm{for}})^{\otimes m})$ (cf. Corollary 4.5.4.24). Hence we obtain a morphism from the point $\mathrm{Proj}(\underset{m\geq 0}{\oplus} x^*(\mathcal{L}^\natural)^{\otimes m})$ to $G_\eta \cong \mathrm{Proj}(\underset{m\geq 0}{\oplus} \Gamma(G_\eta, \mathcal{L}^{\otimes m}_\eta))$, realizing $p_K(x) \in G(K)$. This morphism does not depend on the choice of $P^\natural$, because we have only used the existence of $P^\natural$ for the I-adic convergence of evaluations.

In [106, §1] and [21, §1], the construction of the Raynaud extensions comes equipped with a canonical "rigid analytic quotient" $p_{\mathrm{an}} : G^\natural_{\mathrm{an}} \to G_{\mathrm{an}}$, and the above description (by evaluating sections of powers of an ample invertible sheaf) shows that p_K is the map on K-points induced by p_{an}. In particular, p_K is independent of the choice of λ (or rather $\mathcal{L}$ and $\mathcal{L}^\natural$), as desired. $\square$

Remark 4.5.5.4. The idea of realizing general abelian varieties as rigid analytic quotients has had a long history since [106], generalizing Tate's earlier idea for elliptic curves. Readers who would like to know more about the rigid analytic treatment of such a construction over (not necessarily discrete) complete valuation rings may refer to [115]. The simpler case of quotients of tori can also be found in [45]. We will not explore this idea further in our work, because (for the purpose of construction of compactifications in general) we will need noetherian complete adic rings which are not necessarily local. (The boundary of toroidal compactifications almost always has nontrivial crossings.)

Now Proposition 4.5.5.1 follows from Lemma 4.5.5.2 and Proposition 4.5.5.3, because whether ι is independent of the choice of polarizations is about equalities, which (as in Section 4.3.3) can be checked after making base changes from R to complete discrete valuation rings (under continuous injections).

Corollary 4.5.5.5 (cf. [42, Ch. III, Thm. 7.1])**.** *Let $(G^\natural_1, \iota_1 : \underline{Y}_{1,\eta} \to G^\natural_{1,\eta})$ (resp. $(G^\natural_2, \iota_2 : \underline{Y}_{2,\eta} \to G^\natural_{2,\eta})$) be an object in* $\mathrm{DD}(R, I)$*, with image G_1 (resp. G_2) under* $\mathrm{M}(R, I) : \mathrm{DD}(R, I) \to \mathrm{DEG}(R, I)$ *as in Corollary 4.5.4.9. Suppose there is a homomorphism $f_\eta : G_{1,\eta} \to G_{2,\eta}$ between abelian varieties. Then there is a uniquely determined morphism $(f_Y : \underline{Y}_1 \to \underline{Y}_2, f^\natural : G^\natural_1 \to G^\natural_2)$, satisfying $f^\natural \circ \iota_1 = \iota_2 \circ f_Y$ and hence defining a morphism in* $\mathrm{DD}(R, I)$*, such that $(f_Y, f^\natural)$ defines under* $\mathrm{M}(R, I)$ *the unique homomorphism $f : G_1 \to G_2$ extending f_η (cf. Proposition 3.3.1.5). In particular, the functor* $\mathrm{M}(R, I)$ *is fully faithful and defines an equivalence of categories with quasi-inverse* $\mathrm{F}(R, I) : \mathrm{DEG}(R, I) \to \mathrm{DD}(R, I)$ *induced by* $\mathrm{F}_{\mathrm{ample}}(R, I)$ *(which is well defined by Proposition 4.5.5.1).*

Proof. The unique homomorphism $f : G_1 \to G_2$ extending f_η induces the homomorphism $f^\natural : G^\natural_1 \to G^\natural_2$ by functoriality of Raynaud extensions. The dual isogeny $f^\vee_\eta : G^\vee_{2,\eta} \to G^\vee_{1,\eta}$ induces similarly the homomorphism $G^{\vee,\natural}_2 \to G^{\vee,\natural}_1$, and hence the homomorphism $f_Y : \underline{Y}_2 \to \underline{Y}_1$ on the character groups of torus parts. Thus there is no ambiguity in the choices of f_Y and $f^\natural$. The only question is whether they satisfy $f^\natural \circ \iota_1 = \iota_2 \circ f_Y$.

Consider the fiber product $G := G_1 \underset{S}{\times} G_2$, and consider the isomorphism $h : G \to G$ defined by $(x_1, x_2) \mapsto (x_1, f(x_1) + x_2)$ for all functorial points x_1 of G_1 and x_2 of G_2, respectively. Let $\underline{Y} := \underline{Y}_1 \underset{S}{\times} \underline{Y}_2$, and let $\iota := \iota_1 \times \iota_2 : \underline{Y} \to G^\natural \cong G^\natural_1 \underset{S}{\times} G^\natural_2$. By forming the fiber product of objects in $\mathrm{DD}_{\mathrm{pol}}(R, I)$ extending $(G^\natural_1, \iota_1)$ and $(G^\natural_2, \iota_2)$, respectively, and by Corollary 4.5.4.31, we see that G is isomorphic to $\mathrm{M}(R, I)(G^\natural, \iota)$. Moreover, the morphisms $h_Y : \underline{Y} \to \underline{Y}$ and $h^\natural : G^\natural \to G^\natural$ canonically induced by $h : G \to G$ are given by $h_Y(y_1, y_2) = (y_1, f_Y(y_1) + y_2)$ and $h^\natural(z_1, z_2) = (z_1, f^\natural(z_1) + z_2)$ for all functorial points y_1 of $\underline{Y}_1$, y_2 of $\underline{Y}_2$, z_1 of $G^\natural_1$, and z_2 of $G^\natural_2$.

Let us take any ample invertible sheaf $\mathcal{L}$ over G such that $(G,\mathcal{L})$ is an object of $\mathrm{DEG}_{\mathrm{ample}}(R,I)$. By Proposition 4.5.5.1, both $\mathrm{F}_{\mathrm{ample}}(G,\mathcal{L})$ and $\mathrm{F}_{\mathrm{ample}}(G,h^*\mathcal{L})$ are objects in $\mathrm{DD}_{\mathrm{ample}}(R,I)$ extending (up to isomorphism) the same tuple $(G^\natural,\iota)$, because ι is independent of the choice of polarization. Then Lemma 4.3.4.3 shows that $h^\natural\circ\iota=\iota\circ h_Y$, or equivalently $f^\natural\circ\iota_1=\iota_2\circ f_Y$, as desired. □

Corollary 4.5.5.6 (cf. [42, Ch. III, Cor. 7.2]). *The functor* $\mathrm{M}_{\mathrm{IS}}(R,I)$: $\mathrm{DD}_{\mathrm{IS}}(R,I)\to\mathrm{DEG}_{\mathrm{IS}}(R,I)$ *is an equivalences of categories, with quasi-inverse* $\mathrm{F}_{\mathrm{IS}}(R,I)$: $\mathrm{DEG}_{\mathrm{IS}}(R,I)\to\mathrm{DD}_{\mathrm{IS}}(R,I)$ *induced by* $\mathrm{F}_{\mathrm{ample}}(R,I)$ *by taking tensor products of any choices of objects in* $\mathrm{DEG}_{\mathrm{ample}}(R,I)$ *(which is well defined by Proposition 4.5.5.1).*

Proof. This follows from Corollary 4.5.5.5 and the proof of Corollary 4.5.4.9, because we know from Proposition 4.5.5.1 that the association of τ is independent of the choice of polarizations. □

Remark 4.5.5.7. This finishes the proof of Theorem 4.4.16.

4.5.6 Two-Step Degenerations

Let R be a noetherian normal complete local domain with maximal ideal I. Then R is excellent (see [87, 34.B]), and $S:=\mathrm{Spec}(R)$ fits into the setting of Section 4.1, with generic point $\eta=\mathrm{Spec}(K)$ and special point $s=\mathrm{Spec}(k(s))$.

Suppose $t=\mathrm{Spec}(k(t))$ is any point of S, not necessarily s, whose closure S_1 in S (with its reduced structure) is normal. Note that S_1 contains s because R is local. Let $S_{1,\mathrm{for}}$ be the formal completion of S_1 along s. Let R^1 denote the completion of the localization R_t of R at t, and let $S^1:=\mathrm{Spec}(R^1)$, $K^1:=\mathrm{Frac}(R^1)$, $\eta^1:=\mathrm{Spec}(K^1)$, and $S^1_{/1}:=\mathrm{Spf}(R^1)$. Note that R is naturally a subring of R^1, and K is naturally a subfield of K^1. Let I_1 be the ideal of R defining the closed subscheme S_1 of S. Let $S_{/1}:=\mathrm{Spf}(R,I_1)$. We shall denote by subscripts s, t, or S_1 the pullback of objects to the respective base schemes. However, we shall *never* denote pullbacks to S_1 and $S_{/1}$ by the subscripts 1 and /1.

Both S_1 and S^1 fit into the setting of Section 4.1, and objects in $\mathrm{DEG}_{\mathrm{ample}}(R,I)$ determine semi-abelian schemes (with ample cubical invertible sheaves) over S_1 and S^1 by pullback. Our goal is to analyze these pullbacks using the theory of degeneration. This can be interpreted as analyzing the degeneration in two steps.

Let $(G,\mathcal{L})$ be an object in $\mathrm{DEG}_{\mathrm{ample}}(R,I)$. By Theorem 4.4.16, we know that $(G,\mathcal{L})$ is isomorphic to the image under $\mathrm{M}_{\mathrm{ample}}(R,I)$ of a tuple $(A,\underline{X},\underline{Y},\phi,c,c^\vee,\mathcal{L}^\natural,\tau,\psi)$ in $\mathrm{DD}_{\mathrm{ample}}(R,I)$. Let $\lambda:G\to G^\vee$ and $\lambda_A:A\to A^\vee$ be induced by $\mathcal{L}$ and $\mathcal{L}^\natural$ as in Section 4.2.1. For simplicity, let us assume that the torus parts of fibers of G and $G^\vee$ over s and over t are all *split*, and that $\mathcal{L}^\natural$ descends to some $\mathcal{M}$ over A. (By Remark 3.2.5.6 and Corollary 3.2.5.7, these can be achieved by passing to a finite étale covering of S.)

Let us denote by X, Y, X^1, Y^1 the character groups of the torus parts of G_s, $G_s^\vee$, G_t, and $G_t^\vee$, respectively. By normality of S_1 and by Propositions 3.3.1.5 and 3.3.1.7, the torus part of G_t (resp. $G_t^\vee$) extends uniquely to a subtorus of G_{S_1} (resp. $G^\vee_{S_1}$), and hence we have a canonical surjection $X\twoheadrightarrow X^1$ (resp. $Y\twoheadrightarrow Y^1$), whose kernel we denote by X_1 (resp. Y_1). Let T, $T^\vee$, T_1, $T_1^\vee$, T^1, and $T^{1,\vee}$ denote the tori over S having character groups X, Y, X_1, Y_1, X^1, and Y^1, respectively. Then the torus parts of G_s, $G_s^\vee$, G_t, and $G_t^\vee$ can be identified with T_s, $T_s^\vee$, T_t^1, and $T_t^{1,\vee}$, respectively, the latter two extending to subtori $T^1_{S_1}$ and $T^{1,\vee}_{S_1}$ of G_{S_1} and $G^\vee_{S_1}$, respectively. The quotient $T_{S_1}/T^1_{S_1}$ (resp. $T^\vee_{S_1}/T^{1,\vee}_{S_1}$) can be identified with T_{1,S_1} (resp. $T^\vee_{1,S_1}$). The homomorphism $\phi:Y\to X$ induces homomorphisms $\phi_1:Y_1\to X_1$ and $\phi^1:Y^1\to X^1$, corresponding to homomorphisms between tori over s induced by λ.

The object $(G^1:=G\underset{S}{\times}S^1,\ \mathcal{L}^1:=\mathcal{L}\underset{S}{\times}S^1)$ of $\mathrm{DEG}_{\mathrm{ample}}(R^1)$ corresponds by Theorem 4.4.16 to an object

$$(A^1,X^1,Y^1,\phi^1,c^1,c^{1,\vee},\mathcal{L}^{1,\natural},\tau^1,\psi^1)$$

of $\mathrm{DD}_{\mathrm{ample}}(R^1)$, where X^1, Y^1, and ϕ^1 agree with what we defined above. The abelian scheme A^1 is the unique algebraization of the abelian part $A^1_{/1}$ of $G^1_{/1}:=G\underset{S}{\times}S^1_{/1}$.

By Proposition 3.2.5.4 and Corollary 3.2.5.7, the descent of the invertible sheaf $\mathcal{L}^\natural$ to some $\mathcal{M}$ over A is determined by cubical trivializations of the pullback of $\mathcal{L}_{\mathrm{for}}:=\mathcal{L}\underset{S}{\times}S_{\mathrm{for}}$ (over $G_{\mathrm{for}}:=G\underset{S}{\times}S_{\mathrm{for}}$) to $T_{\mathrm{for}}:=T\underset{S}{\times}S_{\mathrm{for}}$, where $S_{\mathrm{for}}=\mathrm{Spf}(R,I)$. Since all tori involved are split, the restriction of this cubical trivialization to $T^1_{\mathrm{for}}:=T^1\underset{S}{\times}S_{\mathrm{for}}$ partially algebraizes to a cubical trivialization of the pullback of $\mathcal{L}\underset{S}{\times}S_{/1}$ to $T^1\underset{S}{\times}S_{/1}$, which pulls back to a cubical trivialization of the pullback of $\mathcal{L}\underset{S}{\times}S^1_{/1}$ to $T^1\underset{S}{\times}S^1_{/1}$. This last action determines the descent of $\mathcal{L}^{1,\natural}$ to an invertible sheaf $\mathcal{M}^1$ over A^1.

The data c^1, $c^{1,\vee}$, $\mathcal{M}^1$, τ^1, and ψ^1 define (see Definitions 4.2.4.6 and 4.2.4.5) invertible R^1-submodules $I^1_{y^1,\chi^1}$ and $I^1_{y^1}$ in K_1 for each $y^1\in Y^1$ and $\chi^1\in X^1$. Concretely, the invertible R^1-submodule $I^1_{y^1,\chi^1}$ of K^1 is defined by the isomorphism $\tau^1(y^1,\chi^1):(c^{1,\vee}(y^1),c^1(\chi^1))^*\mathcal{P}_{A^1,\eta^1}\overset{\sim}{\to}\mathscr{O}_{S^1,\eta^1}$ and the integral structure of $(c^{1,\vee}(y^1),c^1(\chi^1))^*\mathcal{P}_{A^1,\eta^1}$ given by $(c^{1,\vee}(y^1),c^1(\chi^1))^*\mathcal{P}_{A^1}$. On the other hand, the R^1-submodule $I^1_{y^1}$ of K^1 is defined by the isomorphism $(c^{1,\vee}(y))^*\mathcal{M}_{\eta^1}\overset{\sim}{\to}\mathscr{O}_{S^1,\eta^1}$ and the integral structure of $(c^{1,\vee}(y))^*\mathcal{M}^1_{\eta^1}$ given by $(c^{1,\vee}(y))^*\mathcal{M}^1$. We will show the following proposition below:

Proposition 4.5.6.1. *For each $y\in Y$ and $\chi\in X$ mapped to y^1 and χ^1 under the canonical surjections $Y\twoheadrightarrow Y^1$ and $X\twoheadrightarrow X^1$, respectively, we have $I^1_{y^1,\chi^1}=I_{y,\chi}\cdot R^1$ and $I^1_{y^1}=I_y\cdot R^1$.*

Unless G_t is an abelian scheme, the pullback $(G_{S_1},\mathcal{L}_{S_1})$ does not define an object of $\mathrm{DEG}_{\mathrm{ample}}(R^1)$. Nevertheless, $G_1:=G_{S_1}/T^1_{S_1}$ has an abelian generic fiber (over t). The above-mentioned cubical trivialization of the pullback of $\mathcal{L}\underset{S}{\times}S_{/1}$ to $T^1\underset{S}{\times}S_{/1}$ (used in the descent of $\mathcal{L}^{1,\natural}$ to an invertible sheaf $\mathcal{M}^1$) gives by pullback to S_1 a cubical trivialization of the pullback of $\mathcal{L}_{S_1}$ to $T^1_{S_1}$, which determines (by Proposition 3.2.5.4) a descent of $\mathcal{L}_{S_1}$ to a cubical invertible sheaf $\mathcal{L}_1$ on G_1. By construction, $\mathcal{L}_1$ is compatible with the choices of $\mathcal{M}^1$ and $\mathcal{M}$ in the sense that $\mathcal{M}^1_t\cong\mathcal{L}_{1,t}$ and the pullback of $\mathcal{M}_s$ to $G_{1,s}$ is isomorphic to $\mathcal{L}_{1,t}$.

Then $(G_1,\mathcal{L}_1)$ defines an object of $\mathrm{DEG}_{\mathrm{ample}}(R_1)$, which by Theorem 4.4.16 corresponds to an object

$$(A_1,X_1,Y_1,\phi_1,c_1,c_1^\vee,\mathcal{L}_1^\natural,\tau_1,\psi_1)\tag{4.5.6.2}$$

of $\mathrm{DD}_{\mathrm{ample}}(R_1)$, where X_1, Y_1, and ϕ_1 agree with what we have defined above because $T_{1,s} \cong T_s/T_s^1$ and $T_{1,s}^\vee \cong T_s^\vee/T_s^{1,\vee}$ can be identified with the torus parts of $G_{1,s}$ and $G_{1,s}^\vee$, respectively. Here $G_1^\vee := G_{S_1}^\vee/T_{S_1}^{1,\vee}$, and the pullback $\lambda_{S_1} : G_{S_1} \to G_{S_1}^\vee$ (of the $\lambda : G \to G^\vee$ defined by $\mathcal{L}$) descends to $\lambda_1 : G_1 \to G_1^\vee$. The abelian scheme A_1 over S_1 is the unique algebraization of $A_{1,\mathrm{for}} := A_{S_1} \underset{S_1}{\times} S_{1,\mathrm{for}}$, hence can be identified with A_{S_1}. Since $G_{1,\mathrm{for}} := G_1 \underset{S_1}{\times} S_{1,\mathrm{for}}$ (resp. $G_{1,\mathrm{for}}^\vee := G_1^\vee \underset{S_1}{\times} S_{1,\mathrm{for}}$) is a quotient of $G_{S_{1,\mathrm{for}}} := G_{S_1} \underset{S_1}{\times} S_{1,\mathrm{for}}$ (resp. $G_{S_{1,\mathrm{for}}}^\vee := G_{S_1}^\vee \underset{S_1}{\times} S_{1,\mathrm{for}}$), we see that $c_1 : X_1 \to A_1^\vee$ (resp. $c_1^\vee : Y_1 \to A_1$) is the restriction to X_1 (resp. Y_1) of the pullback of $c : X \to A^\vee$ (resp. $c^\vee : Y \to A$) to S_1. By Proposition 4.5.3.11 and Corollary 4.5.3.12, $\tau : \mathbf{1}_{Y \times X,\eta} \overset{\sim}{\to} (c^\vee \times c)^*\mathcal{P}_{A,\eta}^{\otimes -1}$ extends over a subscheme of S containing t, and the pullback $\tau_t : \mathbf{1}_{Y \times X,t} \overset{\sim}{\to} (c^\vee \times c)^*\mathcal{P}_{A,t}^{\otimes -1}$ makes sense and determines by restriction a trivialization $\tau_1^? : \mathbf{1}_{Y_1 \times X_1,t} \overset{\sim}{\to} (c_1^\vee \times c_1)^*\mathcal{P}_{A_1,t}^{\otimes -1}$. This determines by Lemma 4.2.1.7 a homomorphism $\iota_1^? : Y_{1,t} \to G_{1,t}^\natural$. Similarly, we obtain by extension and restriction a trivialization $\psi_1^? : \mathbf{1}_{Y_1,t} \overset{\sim}{\to} (\iota_1^?)^*(\mathcal{L}_{1,t}^\natural)^{\otimes -1}$. We will show the following proposition later:

Proposition 4.5.6.3. *We have $\tau_1^? = \tau_1$ and $\psi_1^? = \psi_1$.*

Consider (a priori) another object

$$(A, X_1, Y_1, \phi_1, c_2, c_2^\vee, \mathcal{L}_2^\natural, \tau_2, \psi_2) \tag{4.5.6.4}$$

in $\mathrm{DD}_{\mathrm{ample}}(R, I)$. Here A, X_1, Y_1, and ϕ_1 are exactly as above, and we set $c_2 := c|_{X_1}$ and $c_2^\vee := c^\vee|_{Y_1}$. Then we have the Raynaud extension $G_2^\natural$ (resp. $G_2^{\vee,\natural}$) determined by c_2 (resp. $c_2^\vee$), whose torus part is isomorphic to T_1 (resp. $T_1^\vee$). Let $\mathcal{L}_2^\natural$ be the pullback of $\mathcal{M}$ under the structural morphism $G_2^\natural \to A$. The trivialization $\tau : \mathbf{1}_{Y \times X,\eta} \overset{\sim}{\to} (c^\vee \times c)^*\mathcal{P}_{A,\eta}^{\otimes -1}$ determines by restriction a trivialization $\tau_2 : \mathbf{1}_{Y_1 \times X_1,\eta} \overset{\sim}{\to} (c_1^\vee \times c_1)^*\mathcal{P}_{A,\eta}^{\otimes -1}$, and the trivialization $\psi : \mathbf{1}_{Y,\eta} \overset{\sim}{\to} \iota^*(\mathcal{L}_\eta^\natural)^{\otimes -1} \cong (c^\vee)^*(\mathcal{M}_\eta)^{\otimes -1}$ determines by restriction a trivialization $\psi : \mathbf{1}_{Y_1,\eta} \overset{\sim}{\to} \iota_2^*(\mathcal{L}_{2,\eta}^\natural)^{\otimes -1} \cong (c_2^\vee)^*(\mathcal{M}_\eta)^{\otimes -1}$. By Theorem 4.4.16, they define by $\mathrm{M}_{\mathrm{ample}}(R, I)$ an object $(G_2, \mathcal{L}_2)$ in $\mathrm{DEG}_{\mathrm{ample}}(R, I)$. This will be the link between the two steps of degeneration. We will show the following proposition later:

Proposition 4.5.6.5. *There are canonical isomorphisms $G_2 \underset{S}{\times} S_1 \cong G_1$ and $\mathcal{L}_2 \underset{S}{\times} S_1 \cong \mathcal{L}_1$.*

Proof. Since (4.5.6.2) is the pullback of (4.5.6.4) to S_1, we can conclude the proof by taking projective spectra as in the proof of Lemma 4.5.4.25. □

Let us prove the propositions by relating the objects we have defined. The key point is the Fourier expansions of theta functions studied in Section 4.3, which were used to prove Theorems 4.2.1.14 and 4.4.16.

Recall that $S_{/1} = \mathrm{Spf}(R, I_1)$. Let $A_{/1} := G_{S_{/1}}/T_{S_{/1}}^1$. (Then $A_{/1} \underset{S_{/1}}{\times} S_{/1}^1 \cong A_{/1}^1 \cong A^1 \underset{S^1}{\times} S_{/1}^1$. However, $A_{/1} \underset{S_{/1}}{\times} S_1 \cong G_1$, and $A_{/1} \not\cong A \underset{S}{\times} S_{/1}$ in general.) The above-mentioned cubical trivialization of the pullback of $\mathcal{L} \underset{S}{\times} S_{/1}$ to $T^1 \underset{S}{\times} S_{/1}$ determines (by Proposition 3.2.5.4) a descent $\mathcal{M}_{/1}$ of $\mathcal{L} \underset{S}{\times} S_{/1}$ over $A_{/1}$. Let $A_{/1,\mathrm{for}} := A_{/1} \underset{S_{/1}}{\times} S_{\mathrm{for}}$ and $\mathcal{M}_{/1,\mathrm{for}} := \mathcal{M}_{/1} \underset{S_{/1,\mathrm{for}}}{\times} S_{\mathrm{for}}$. We have $A_{/1,\mathrm{for}}/T_{1,\mathrm{for}} \cong A_{\mathrm{for}} := A_{S_{\mathrm{for}}}$, and the pullback of $\mathcal{M}_{\mathrm{for}}$ from A_{for} to $A_{/1,\mathrm{for}}$ is canonically isomorphic to $\mathcal{M}_{/1,\mathrm{for}}$. Now the diagram

$$\begin{tikzcd}
\Gamma(G, \mathcal{L}) \arrow[r] & \Gamma(G \underset{S}{\times} S_{/1}, \mathcal{L} \underset{S}{\times} S_{/1}) \arrow[r] \arrow[d] & \Gamma(G_{\mathrm{for}}, \mathcal{L}_{\mathrm{for}}) \arrow[d] \\
 & \hat{\bigoplus}_{\chi^1 \in X^1} \Gamma(A_{/1}, \mathcal{M}_{/1,\chi^1}) \arrow[r] & \hat{\bigoplus}_{\chi^1 \in X^1} \Gamma(A_{/1,\mathrm{for}}, \mathcal{M}_{/1,\chi^1,\mathrm{for}}) \arrow[d] \\
 & & \hat{\bigoplus}_{\chi \in X} \Gamma(A_{\mathrm{for}}, \mathcal{M}_{\chi,\mathrm{for}})
\end{tikzcd}$$

explains the Fourier expansions of theta functions, with symbolic relations

$$s = \sum_{\chi^1 \in X^1} \sigma_{\chi^1}^1(s) \in \hat{\bigoplus}_{\chi^1 \in X^1} \Gamma(A_{/1}, \mathcal{M}_{/1,\chi^1}) \tag{4.5.6.6}$$

and

$$s = \sum_{\chi \in X} \sigma_\chi(s) \in \hat{\bigoplus}_{\chi \in X} \Gamma(A_{\mathrm{for}}, \mathcal{M}_{\chi,\mathrm{for}}) \cong \hat{\bigoplus}_{\chi \in X} \Gamma(A, \mathcal{M}_\chi) \tag{4.5.6.7}$$

for each $s \in \Gamma(G, \mathcal{L})$. If we choose any homomorphism $X^1 \to X$ splitting $X \twoheadrightarrow X^1 = X/X_1$, then we can make sense of $\mathscr{O}_{\chi^1}$ and $\mathcal{M}_{\chi^1} = \mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi^1}$ over A, and the comparison between (4.5.6.6) and (4.5.6.7) gives

$$\sigma_{\chi^1}^1(s) = \sum_{\chi_1 \in X_1} \sigma_{1,\chi_1}^{\mathcal{M}_{\chi^1}}(\sigma_{\chi^1}^1(s)).$$

By defining $\sigma_{1,\chi} := \sigma_{1,\chi_1}^{\mathcal{M}_{\chi^1}}$ when $\chi = \chi^1 + \chi_1$, we can rewrite this relation as

$$\sigma_{\chi^1}^1(s) = \sum_{\chi \in \chi^1} \sigma_{1,\chi}(\sigma_{\chi^1}^1(s)). \tag{4.5.6.8}$$

This formulation is independent of the choice of the splitting of $X \twoheadrightarrow X^1$. Then we have

$$\sigma_\chi(s) = \sigma_{1,\chi} \circ \sigma_{\chi^1}^1(s). \tag{4.5.6.9}$$

Proof of Proposition 4.5.6.1. By Proposition 4.5.3.11, $I_{y,\phi(y)} \cdot R_t = R_t$ if and only if y lies in Y_1. Then the same linear algebraic technique as in the proof of Lemma 4.5.1.7 shows that $I_{y,\chi}$ is trivial after localizing at a point s if either $y \in Y_1$ or $\chi \in X_1$. As a result, the R_t-submodule $I_{y,\chi} \cdot R_t$ of K depends only on the images y^1 (resp. χ^1) of y (resp. χ) in $Y^1 = Y/Y_1$ (resp. $X^1 = X/X_1$). Since $I_{y+y'} = I_y \cdot I_{y'} \cdot I_{y,y'}$, the R_t-submodule $I_y \cdot R_t$ of K depends only on the image of y^1 of y in Y^1. Since the relation (4.3.1.7) defining τ and ψ (and hence $I_{y,\chi}$ and I_y for each $y \in Y$ and $\chi \in X$) remains valid after making flat base changes, we can define $I_{y,\chi} \cdot R_t$ and $I_y \cdot R_t$ by comparing the morphisms $\sigma_{\chi^1}^1$ in (4.5.6.6). Since the pullback of the relation (4.5.6.6) to $S_{/1}^1 = \mathrm{Spf}(R^1)$ agrees with the corresponding relation

$$s^1 = \sum_{\chi^1 \in X^1} \sigma_{\chi^1}^1(s^1) \in \hat{\bigoplus}_{\chi^1 \in X^1} \Gamma(A_{/1}^1, \mathcal{M}_{/1,\chi^1}^1)$$

over $S_{/1}^1 = \mathrm{Spf}(R^1)$, and since τ^1 and ψ^1 (and hence I_{y^1,χ^1}^1 and $I_{y^1}^1$ for each $y^1 \in Y^1$ and $\chi^1 \in X^1$) are determined by comparison of pullbacks of $\sigma_{\chi^1}^1$'s using the analogue of (4.3.1.7) over R^1, Proposition 4.5.6.1 follows. $\square$

Proof of Proposition 4.5.6.3. The composition

$$\Gamma(G_2,\mathcal{L}_2)\to\Gamma(G_2\underset{S}{\times}S_{/1},\mathcal{L}_2\underset{S}{\times}S_{/1})$$
$$\to\Gamma(G_{2,\mathrm{for}},\mathcal{L}_{2,\mathrm{for}})\to\underset{\chi_1\in X_1}{\hat{\oplus}}\Gamma(A_{\mathrm{for}},\mathcal{M}_{\chi_1,\mathrm{for}})\cong\underset{\chi_1\in X_1}{\hat{\oplus}}\Gamma(A,\mathcal{M}_{\chi_1})$$

of canonical morphisms explains the Fourier expansions for sections in $\Gamma(G_2,\mathcal{L}_2)$. In this case, we obtain a symbolic relation

$$s=\sum_{\chi_1\in X_1}\sigma_{2,\chi_1}(s)\in\underset{\chi_1\in X_1}{\hat{\oplus}}\Gamma(A,\mathcal{M}_{\chi_1}) \tag{4.5.6.10}$$

for each $s\in\Gamma(G_2,\mathcal{L}_2)$, and we know that

$$\psi(y_1)\tau(y_1,\chi_1)\,T^*_{c_1^\vee(y_1)}\circ\sigma_{2,\chi_1}=\sigma_{2,\chi_1+\phi_1(y_1)}, \tag{4.5.6.11}$$

because τ_2 and ψ_2 are defined to be restrictions of τ and ψ. Since $G_2\underset{S}{\times}S_1\cong G_1$ and $\mathcal{L}_2\underset{S}{\times}S_1\cong\mathcal{L}_1$ by Proposition 4.5.6.5, then Proposition 4.5.6.3 follows from comparing (4.5.6.11) with (4.3.1.7) over S_1. $\square$

4.6 Kodaira–Spencer Morphisms

In this section we study how Kodaira–Spencer morphisms for abelian schemes extend to their degenerations. The notation R_i and S_i in the previous sections will no longer have their meanings in this section.

Let us fix a choice of a universal base scheme U in this section. For the moment let us take U to be locally noetherian. Later, in Theorem 4.6.3.43 we will need U to be excellent and normal, mainly for retaining the noetherian normality after passing to completions of étale localizations.

4.6.1 Definition for Semi-Abelian Schemes

Let S be any scheme separated and locally of finite presentation over U such that $\Omega^1_{S/\mathsf{U}}$ is *locally free of finite rank* over $\mathscr{O}_S$. This is the case, for example, when S is separated and smooth over U.

Suppose that we are given a semi-abelian scheme $G^\natural$ of the form $0\to T\to G^\natural\to A\to 0$ over S associated with a homomorphism $c:\underline{X}=\underline{\mathbf{X}}(T)\to A^\vee$. By general argument in Section 2.1.7 applied to the scheme $G^\natural$ smooth over S, we know that there is a Kodaira–Spencer class $\mathsf{KS}_{G^\natural/S/\mathsf{U}}\in H^1(G^\natural,\underline{\mathrm{Der}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\Omega^1_{S/\mathsf{U}})$ describing the deformation of $G^\natural$, which determines a global section of $\underline{H}^1(G^\natural,\underline{\mathrm{Der}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\Omega^1_{S/\mathsf{U}})\cong\underline{H}^1(G^\natural,\mathscr{O}_{G^\natural})\underset{\mathscr{O}_S}{\otimes}\underline{\mathrm{Lie}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\Omega^1_{S/\mathsf{U}}$.

Since $G^\natural\to S$ is a smooth group scheme, by Lemma 2.1.5.11, if we consider $\underline{\mathrm{Lie}}^\vee_{G^\natural/S}:=e^*_{G^\natural}\Omega^1_{G^\natural/S}$ and $\underline{\mathrm{Lie}}_{G^\natural/S}\cong e^*_{G^\natural}\underline{\mathrm{Der}}_{G^\natural/S}$, then they are canonically dual to each other, and $\Omega^1_{G^\natural/S}$ and $\underline{\mathrm{Der}}_{G^\natural/S}$ are canonically isomorphic to their respective pullbacks. However, although $H^1(G^\natural,\underline{\mathrm{Der}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\Omega^1_{S/\mathsf{U}})$ is the space describing the deformation of $G^\natural$ as a scheme smooth over S, there is not enough rigidity (for semi-abelian schemes like $G^\natural$) to force an arbitrary lifting to have a structure of a *commutative group extension* of an abelian scheme by a torus as $G^\natural$ does. Therefore we would like to single out a subgroup of $H^1(G^\natural,\underline{\mathrm{Der}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\Omega^1_{S/\mathsf{U}})$ that does describe the deformation of $G^\natural$ as a commutative group extension.

According to Proposition 3.1.5.1, the liftings of $G^\natural$ as commutative group extensions are the same as the liftings of the pair (A,c), or rather as liftings of T-torsors. Let $S\hookrightarrow\tilde{S}$ be an embedding defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2=0$. The liftings of A as an abelian variety to $\tilde{S}$, if nonempty, is a torsor under the group $H^1(A,\underline{\mathrm{Der}}_{A/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{I})\cong H^1(A,\underline{\mathrm{Lie}}_{A/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{O}_A\underset{\mathscr{O}_S}{\otimes}\mathscr{I})$ (see Propositions 2.1.2.2 and 2.2.2.3, and Lemma 2.1.5.11).

Proposition 4.6.1.1. *Liftings of the pair (A,c) to $\tilde{S}$, if nonempty, is a torsor under the group $H^1(A,\underline{\mathrm{Lie}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{O}_A\underset{\mathscr{O}_S}{\otimes}\mathscr{I})$. Moreover, the forgetful map from the liftings of (A,c) to liftings of A, if the source is nonempty, is equivariant with the canonical morphism*

$$H^1(A,\underline{\mathrm{Lie}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{O}_A\underset{\mathscr{O}_S}{\otimes}\mathscr{I})\to H^1(A,\underline{\mathrm{Der}}_{A/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{I})$$

induced by the canonical morphism $\underline{\mathrm{Lie}}_{G^\natural/S}\to\underline{\mathrm{Lie}}_{A/S}$ given by the structural morphism $G^\natural\to A$ over S.

Proof. Let $(\tilde{A}_i,\tilde{c}_i)$, where $i=1,2$, be any two liftings of (A,c) to $\tilde{S}$. Let $\tilde{G}^\natural_i$ be the extension of $\tilde{A}_i$ by $\tilde{T}:=\mathrm{Hom}_{\tilde{S}}(X,\mathbf{G}_{\mathrm{m},\tilde{S}})$ defined by $\tilde{c}_i$.

Let $r=\dim_S(A)$ and $r'=\dim_S(T)$ (both being the relative dimensions). Take an affine open covering $\{U_\alpha\}_\alpha$ of A such that each U_α is étale over $\mathbb{A}^r_S$. By refining this open covering if necessary, we may assume that $G^\natural$ is trivialized as a T-bundle over each U_α. Then $U_\alpha\underset{S}{\times}T$ is étale over $\mathbb{A}^{r+r'}_S$ because T is an affine open subscheme of $\mathbb{A}^{r'}_S$, and the open coverings $\{U_\alpha\}_\alpha$ and $\{U_\alpha\underset{S}{\times}T\}_\alpha$ are lifted to open coverings of $\tilde{A}_i$ and $\tilde{G}^\natural_i$ over $\tilde{S}$, respectively, for $i=1,2$. Now we can proceed as in the proof of Proposition 2.1.2.2. The difference here is that, instead of considering $\mathrm{Aut}_{\tilde{S}}((\tilde{U}_\alpha\underset{\tilde{S}}{\times}\tilde{T})|_{U_{\alpha\beta}\underset{S}{\times}T},S)\cong H^0(U_{\alpha\beta}\underset{S}{\times}T,\underline{\mathrm{Der}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{I})\cong H^0(U_{\alpha\beta}\underset{S}{\times}T,\underline{\mathrm{Hom}}_{\mathscr{O}_{G^\natural}}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{O}_{G^\natural},\mathscr{I}\underset{\mathscr{O}_S}{\otimes}\mathscr{O}_{G^\natural}))$, which describes the automorphisms of the underlying schemes, we shall consider automorphisms of $\tilde{T}$-torsors (compatible with the action of $\tilde{T}$; cf. Theorem 3.1.1.2), which we denote as $\mathrm{Aut}_{\tilde{T}}((\tilde{U}_\alpha\underset{\tilde{S}}{\times}\tilde{T})|_{U_{\alpha\beta}},T)$ and is isomorphic to $H^0(U_{\alpha\beta},\underline{\mathrm{Hom}}_{\mathscr{O}_A}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{O}_A,\mathscr{I}\underset{\mathscr{O}_S}{\otimes}\mathscr{O}_A))\cong H^0(U_{\alpha\beta},\underline{\mathrm{Lie}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{O}_A\underset{\mathscr{O}_S}{\otimes}\mathscr{I})$. (The remainder of the proof is as in the proof of Proposition 2.1.2.2. The equivariance stated in this proposition follows from the very constructions.) $\square$

If the embedding $S\hookrightarrow\tilde{S}$ is given by the first infinitesimal neighborhood of the diagonal morphism $\Delta:S\to S\underset{\mathsf{U}}{\times}S$, then $\mathscr{I}\cong\Omega^1_{S/\mathsf{U}}$. By pulling back under the two projections, we obtain two liftings $\tilde{G}^\natural_i:=\mathrm{pr}_i^*G^\natural$ of $G^\natural$, and hence an element $\mathsf{KS}_{(A,c)/S/\mathsf{U}}$ of $H^1(A,\underline{\mathrm{Lie}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\mathscr{O}_A\underset{\mathscr{O}_S}{\otimes}\mathscr{I})$ sending $\tilde{G}^\natural_1$ to $\tilde{G}^\natural_2$. The element $\mathsf{KS}_{(A,c)/S/\mathsf{U}}$ defines a global section of $\underline{\mathrm{Lie}}_{A^\vee/S}\underset{\mathscr{O}_S}{\otimes}\underline{\mathrm{Lie}}_{G^\natural/S}\underset{\mathscr{O}_S}{\otimes}\Omega^1_{S/\mathsf{U}}$ by Proposition

4.6.1.1, which by duality can be interpreted as a morphism
$$\underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/S} \to \Omega^1_{S/\mathsf{U}},$$
which we denote by $\mathrm{KS}_{(A,c)/S/\mathsf{U}}$.

Definition 4.6.1.2. *The element* $\mathsf{KS}_{(A,c)/S/\mathsf{U}}$ (*resp. the morphism* $\mathrm{KS}_{(A,c)/S/\mathsf{U}}$) *above is called the* ***Kodaira–Spencer class*** (*resp. the* ***Kodaira–Spencer morphism***) *for* (A,c).

Since the liftings of $G^\natural$ as commutative group scheme extensions to $\tilde{S}$ are also liftings of the underlying smooth scheme, the canonical pullback morphism
$$H^1(A, \underline{\mathrm{Lie}}_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}) \to H^1(G^\natural, \underline{\mathrm{Der}}_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}) \tag{4.6.1.3}$$
sends $\mathsf{KS}_{(A,c)/S/\mathsf{U}}$ (up to a sign convention) to $\mathsf{KS}_{G^\natural/S/\mathsf{U}}$.

Lemma 4.6.1.4. *The canonical pullback morphism $\underline{H}^i(A, \mathscr{O}_A) \to \underline{H}^i(G, \mathscr{O}_G)$ is an embedding for every integer $i \geq 0$.*

Proof. Let us fix any integer $i \geq 0$ as in the statement of the lemma. Denote the structural morphism $G^\natural \to A$ by π. Since π is relatively affine (with fibers étale locally isomorphic to T), so that $R^j\pi_*\mathscr{O}_{G^\natural}$ vanishes for all $j > 0$, the Leray spectral sequence (see [48, Ch. II, Thm. 4.17.1]) shows that $\underline{H}^i(A, \pi_*\mathscr{O}_{G^\natural}) \cong \underline{H}^i(G^\natural, \mathscr{O}_{G^\natural})$. After making a finite étale surjective base change in S that splits T if necessary, we may assume that $\underline{X}$ is constant with value X. Then we have a decomposition $\pi_*\mathscr{O}_{G^\natural} \cong \underset{\chi \in \underline{X}}{\oplus} \mathscr{O}_\chi$, where $\mathscr{O}_\chi$ is the rigidified invertible sheaf corresponding to the point $c(\chi) \in A^\vee$. This gives a corresponding decomposition $\underline{H}^i(A, \pi_*\mathscr{O}_{G^\natural}) \cong \underset{\chi \in \underline{X}}{\oplus} \underline{H}^i(A, \mathscr{O}_\chi)$. In particular, we have a canonical inclusion $\underline{H}^i(A, \mathscr{O}_A) \hookrightarrow \underline{H}^i(G^\natural, \mathscr{O}_{G^\natural})$ corresponding to the term $\mathscr{O}_0 = \mathscr{O}_A$, independent of the trivialization of $\underline{X}$. Hence étale descent applies and shows that the inclusion is defined over S. Moreover, it has to agree with the canonical morphism $\underline{H}^i(A, \mathscr{O}_A) \to \underline{H}^i(G, \mathscr{O}_G)$. In particular, the canonical morphism is injective. □

Proposition 4.6.1.5. *The morphism (4.6.1.3) induces an* ***injection***
$$\underline{\mathrm{Lie}}_{A^\vee/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}} \hookrightarrow \underline{H}^1(G^\natural, \mathscr{O}_{G^\natural}) \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}. \tag{4.6.1.6}$$
The global section of the target defined by $\mathsf{KS}_{G^\natural/S/\mathsf{U}}$ lies in the image of (4.6.1.6), which agrees (up to a sign convention) with the image of the global section of the source defined by $\mathsf{KS}_{(A,c)/S/\mathsf{U}}$.

Proof. Following the proof of Proposition 4.6.1.1, we may assume that S is affine, and apply Lemma 4.6.1.4. □

4.6.2 Definition for Periods

Let S be any scheme separated and locally of finite presentation over U, such that $\Omega^1_{S/\mathsf{U}}$ is *locally free of finite rank* over $\mathscr{O}_S$, and such that all connected components of S are integral. Let $G^\natural$ be an extension of an abelian scheme A by a torus T over S. Let $\underline{X}$ be the character group of T, and let $c : \underline{X} \to A^\vee$ be the homomorphism describing the extension class of $G^\natural$. Let $\underline{Y}$ be the character group of some torus $T^\vee$ having the same dimension as T, and let $c^\vee : \underline{Y} \to A$ be a homomorphism that describes an extension $G^{\vee,\natural}$ of $A^\vee$ by $T^\vee$. Let S_1 be an open dense subscheme of S over which we have a homomorphism $\iota : \underline{Y}_{S_1} \to G^\natural_{S_1}$, such that the composition of ι with the structural morphism $G^\natural_{S_1} \to A_{S_1}$ coincides with the restriction of $c^\vee$ to S_1. We shall investigate how liftings of the pair $(G^\natural, \iota)$ to an embedding $S \hookrightarrow \tilde{S}$ defined by a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$ should be classified, under some suitable additional assumptions.

Let $\tilde{S}_1$ be an open subscheme of $\tilde{S}$ lifting S_1, and let $\mathscr{I}_1 := (\tilde{S}_1 \hookrightarrow \tilde{S})^*\mathscr{I} \cong \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}$ be the ideal defining the embedding $S_1 \hookrightarrow \tilde{S}_1$. The compatible homomorphisms $c^\vee : \underline{Y} \to A$ and $\iota : \underline{Y}_{S_1} \to G^\natural_{S_1}$ can be interpreted as *compatible actions* of $\underline{Y}$ on A and $G^\natural_{S_1}$. Hence the question is about the liftings of actions of $\underline{Y}$. For technical simplicity, let us assume in this section that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively.

Definition 4.6.2.1. *Let H be any scheme with a Y-action over some base scheme Z. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two $\mathscr{O}_H$-modules with Y-actions covering the Y-action on H. A* ***Y-equivariant extension*** *of $\mathcal{E}_1$ by $\mathcal{E}_2$ (over Z) is a sheaf $\mathcal{E}$ of $\mathscr{O}_H$-modules admitting a Y-action (covering the action of Y on H), which fits into an Y-equivariant exact sequence $0 \to \mathcal{E}_2 \to \mathcal{E} \to \mathcal{E}_1 \to 0$. An isomorphism (or equivalence) of Y-equivariant extensions is a Y-equivariant isomorphism of (usual) extensions of $\mathscr{O}_H$-modules. We denote the group of* ***Y-equivariant extension classes*** *of $\mathcal{E}_1$ by $\mathcal{E}_2$ by $\mathrm{Ext}^{1,Y}(\mathcal{E}_1, \mathcal{E}_2)$, and denote the associated sheaf over Z by $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_Z}(\mathcal{E}_1, \mathcal{E}_2)$.*

Lemma 4.6.2.2. *$\underline{H}^i(A, \mathscr{O}_A)^Y = \underline{H}^i(A, \mathscr{O}_A)$ for all $i \geq 0$.*

Proof. By Proposition 2.1.5.14, it suffices to treat the case $i = 1$. By Corollary 2.1.5.9, we know that $\underline{H}^1(A, \mathscr{O}_A) \cong \underline{\mathrm{Lie}}_{A^\vee/S} \cong \underline{\mathrm{Pic}}^0(A/S)$. Since Y acts on A by translations, and since translations act trivially on $\underline{\mathrm{Pic}}^0(A/S)$ by the theorem of the square, we see that Y acts trivially on $\underline{H}^1(A, \mathscr{O}_A)$. □

Corollary 4.6.2.3. $\underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathscr{O}_A, \mathscr{O}_A)^Y \cong \underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathscr{O}_A, \mathscr{O}_A) \cong \underline{\mathrm{Lie}}_{A^\vee/S}$.

Lemma 4.6.2.4. *The forgetful morphism $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_S}(\mathscr{O}_{A_S}, \mathscr{O}_{A_S}) \to \underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathscr{O}_{A_S}, \mathscr{O}_{A_S})$ can be canonically identified with the canonical morphism $\underline{\mathrm{Lie}}_{G^{\vee,\natural}/S} \to \underline{\mathrm{Lie}}_{A^\vee/S}$ induced by the structural morphism $\pi^\vee : G^{\vee,\natural} \to A^\vee$.*

Proof. We may assume that S is affine. Consider the pullback $\mathcal{P}' := (\mathrm{Id} \times \pi^\vee)^*\mathcal{P}_A$ of $\mathcal{P}_A$ to $A \underset{S}{\times} G^{\vee,\natural}$. Let $G^{\vee,\natural}_{(1)}$ (resp. $T^\vee_{(1)}$, resp. $A^\vee_{(1)}$) be the first infinitesimal neighborhood of $e_{G^{\vee,\natural}} : S \to G^{\vee,\natural}$ (resp. $e_{T^\vee} : S \to T^\vee$, resp. $e_{A^\vee} : S \to A^\vee$), and let $S_\varepsilon := \underline{\mathrm{Spec}}_{\mathscr{O}_S}(\mathscr{O}_S[\varepsilon]/(\varepsilon^2))$, which contains S as a subscheme defined by $\varepsilon = 0$. Then we may identify global sections of $\underline{\mathrm{Lie}}_{G^{\vee,\natural}/S}$ with morphisms $v : S_\varepsilon \to G^{\vee,\natural}_{(1)}$ such that $v|_S = e_{G^{\vee,\natural}}$. For each such v, the pullback $(\mathrm{Id} \times v)^*\mathcal{P}'$ over $A \underset{S}{\times} S_\varepsilon$ can be identified with an extension $\mathcal{E}_v$ of $\mathscr{O}_A$ by $\mathscr{O}_A$, because A and $A \underset{S}{\times} S_\varepsilon$ have the same underlying topological spaces. We claim that there is a canonical action of Y on $\mathcal{E}_v$, or rather on $\mathcal{P}'$. Let $\mathscr{O}_y := \mathcal{P}_A|_{\{c^\vee(y)\} \times A^\vee}$. Then the construction of $G^{\vee,\natural}$ from $c^\vee$ shows that $\pi^\vee_*\mathscr{O}_{G^{\vee,\natural}} \cong \underset{y \in Y}{\oplus} \mathscr{O}_y$. By the biextension structure of $\mathcal{P}_A$, we see that $(T_{c^\vee(y)} \times \mathrm{Id}_A)^*\mathcal{P}_A \cong \mathcal{P}_A \underset{\mathscr{O}_{A \times A^\vee}}{\otimes} \mathrm{pr}_2^*\,\mathscr{O}_y$. Therefore there is a canonical isomorphism

$(T_{c^\vee(y)} \times \mathrm{Id}_A)^*\mathcal{P}' \cong \mathcal{P}'$ covering the translation action of Y on the first factor A of $A \underset{S}{\times} G^{\vee,\natural}$, or in other words a canonical action of Y on $\mathcal{P}'$. This proves the claim. It is clear from the proof of the claim that $\mathcal{P}'$ is universal for such a property. As a result, we see that the association $v \mapsto \mathcal{E}_v$ (with its Y-action) gives a canonical isomorphism $\underline{\mathrm{Lie}}_{G^{\vee,\natural}/S} \to \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_S}(\mathscr{O}_A, \mathscr{O}_A)$.

If we replace $\mathcal{P}' = (\mathrm{Id} \times \pi^\vee)^*\mathcal{P}_A$ with $\mathcal{P}_A$ in the above construction, then we obtain the canonical morphism $\underline{\mathrm{Lie}}_{A^\vee/S} \to \underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathscr{O}_A, \mathscr{O}_A)$. This shows that the forgetful morphism $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_S}(\mathscr{O}_{A_S}, \mathscr{O}_{A_S}) \to \underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathscr{O}_{A_S}, \mathscr{O}_{A_S})$ is compatible with $\pi^\vee$, as desired. □

Proposition 4.6.2.5. *Liftings of the tuple $(A_{S_1}, c, c^\vee, \iota)$ to $\tilde{S}_1$, if nonempty, is a torsor under the group*

$$\mathrm{Ext}^{1,Y}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}, \mathscr{I}_1 \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}).$$

Moreover, the forgetful map from the liftings of $(A_{S_1}, c, c^\vee, \iota)$ to the liftings of (A_{S_1}, c), if the source is nonempty, is equivariant with the canonical forgetful morphism

$$\begin{aligned} &\mathrm{Ext}^{1,Y}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}, \mathscr{I}_1 \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}) \\ &\to \mathrm{Ext}^1(\underline{\mathrm{Lie}}^\vee_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}, \mathscr{I}_1 \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}}) \\ &\cong H^1(A_{S_1}, \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{I}_1). \end{aligned}$$

Proof. For simplicity, we may assume that $S = S_1$ is affine. Let $(\tilde{A}_i, \tilde{c}_i, c_i^\vee, \iota_i)$, where $i = 1, 2$, be any two liftings of $(A, c, c^\vee, \iota)$ to $\tilde{S}$. Let $\tilde{G}^\natural_i$ be the extension of $\tilde{A}_i$ by $\tilde{T} := \mathrm{Hom}_{\tilde{S}}(X, \mathbf{G}_{\mathrm{m},\tilde{S}})$ defined by $\tilde{c}_i$.

Take an affine open covering $\{U_\alpha\}_\alpha$ (resp. $\{U_\alpha \underset{S}{\times} T\}_\alpha$) that is lifted to an open covering of $\tilde{A}_i$ (resp. $\tilde{G}^\natural_i$) over $\tilde{S}$, for $i = 1, 2$, as in the proof of Proposition 4.6.1.1. By Lemma 2.1.1.7 and by abuse of notation, let us use the same notation $\tilde{U}_\alpha$ (resp. $\tilde{U}_\alpha \underset{\tilde{S}}{\times} \tilde{T}$) for the lifting of U_α to $\tilde{A}_1$ and $\tilde{A}_2$ (resp. $U_\alpha \underset{S}{\times} T$ to $\tilde{G}^\natural_1$ and $\tilde{G}^\natural_2$) over $\tilde{S}$. As in the proof of Proposition 2.1.2.2, for $i = 1, 2$, the scheme $\tilde{G}^\natural_i$ is given by a collection of gluing isomorphisms $\xi_{\alpha\beta,i} : (\tilde{U}_\alpha \underset{\tilde{S}}{\times} \tilde{T})|_{U_{\alpha\beta}} \overset{\sim}{\to} (\tilde{U}_\beta \underset{\tilde{S}}{\times} \tilde{T})|_{U_{\alpha\beta}}$ such that $\xi_{\alpha\gamma,i} = \xi_{\beta\gamma,i} \circ \xi_{\alpha\beta,i}$, and their difference is (up to a sign convention) measured by the collection of automorphisms $\{\xi^{-1}_{\alpha\beta,2} \circ \xi_{\alpha\beta,1}\}_{\alpha\beta}$, each member lying in $\mathrm{Aut}_{\tilde{T}}((\tilde{U}_\alpha \underset{\tilde{S}}{\times} \tilde{T})|_{U_{\alpha\beta}}, T) \cong H^0(U_{\alpha\beta}, \underline{\mathrm{Hom}}_{\mathscr{O}_A}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A, \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A))$ (as in the proof of Proposition 4.6.1.1), which altogether defines a 1-cocycle in $H^1(A, \underline{\mathrm{Hom}}_{\mathscr{O}_A}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A, \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A))$, or rather a collection of splittings of a global extension $0 \to \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A \to \mathcal{E} \to \underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A \to 0$ over $\{U_\alpha\}_\alpha$.

Let us also consider the difference between ι_1 and ι_2, which can be interpreted as actions of Y. Each of the Y-actions is given by a collection of isomorphisms $\eta(y)_{\alpha,i} : (\tilde{U}_\alpha \underset{\tilde{S}}{\times} \tilde{T}) \overset{\sim}{\to} T_{\iota_i(y)}(\tilde{U}_\alpha \underset{\tilde{S}}{\times} \tilde{T})$, which has to satisfy the compatibility $\eta(y)_{\beta,i} \circ \xi_{\alpha\beta,i} = T^*_{\iota_i(y)}(\xi_{\alpha\beta,i}) \circ \eta(y)_{\alpha,i}$. In other words, the classes of the two 1-cocycles $\{\xi_{\alpha\beta}\}_{\alpha\beta}$ and $\{T_{\iota_i(y)}(\xi_{\alpha\beta})\}_{\alpha\beta}$ differ by a 1-coboundary, and hence they are equivalent. This shows that we only need to consider Y-equivariant extensions $\mathcal{E}$ above. Moreover, the difference between the two actions are (up to a sign convention) measured by a collection of elements $\{\eta(y)^{-1}_{\alpha,2} \circ \eta(y)_{\alpha,1}\}_\alpha$, each member lying in $\mathrm{Hom}_S(Y, \mathrm{Aut}_{\tilde{T}}(\tilde{U}_\alpha \underset{\tilde{S}}{\times} \tilde{T}, T)) \cong H^0(U_\alpha, \underline{\mathrm{Hom}}_{\mathscr{O}_A}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A, \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A))$, which altogether defines a global section of $\mathrm{Hom}_S(Y, \underline{\mathrm{Hom}}_{\mathscr{O}_A}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A, \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A)) \cong \mathrm{Hom}_S(Y, \underline{\mathrm{Hom}}_{\mathscr{O}_S}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S}, \mathscr{I}))$. This corresponds to modifying the Y-action on $\mathcal{E}$ by morphisms from $\underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A$ to $\mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A$ for each Y. On the other hand, every two Y-actions on $\mathcal{E}$ differ by such a difference. Hence we see that the liftings of $(A, c, c^\vee, \iota)$ to $\tilde{S}$ form a torsor under $\mathrm{Ext}^{1,Y}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A, \mathscr{I} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A)$. This is the first statement.

Since forgetful map $(A, c, c^\vee, \iota) \to (A, c)$ corresponds to forgetting the Y-actions on the extensions $\mathcal{E}$ above, the second statement follows by comparing the previous two paragraphs. □

For simplicity, let us call the liftings of the tuple $(A_{S_1}, c, c^\vee, \iota)$ liftings of the pair $(G^\natural_{S_1}, \iota)$, with the understanding that we will only consider liftings of $G^\natural_{S_1}$ that are commutative group scheme extensions of abelian schemes by tori. If the embedding $S \hookrightarrow \tilde{S}$ is given by the first infinitesimal neighborhood of the diagonal morphism $\Delta : S \to S \underset{\mathsf{U}}{\times} S$, then $\mathscr{I} \cong \Omega^1_{S/\mathsf{U}}$, and $\mathscr{I}_1 \cong \Omega^1_{S_1/\mathsf{U}}$. By comparing the pullbacks under the two projections, we obtain an element $\mathsf{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$ of $\mathrm{Ext}^{1,Y}(\underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{A_{S_1}}, \mathscr{I}_1 \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{A_{S_1}})$ sending $\mathrm{pr}_1^*(G^\natural_{S_1}, \iota)$ to $\mathrm{pr}_2^*(G^\natural_{S_1}, \iota)$, by Proposition 4.6.2.5. By Lemma 4.6.2.4, the element $\mathsf{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$ defines a global section of $\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$, which by duality can be interpreted as a morphism

$$\underline{\mathrm{Lie}}^\vee_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}_{S_1}/S_1} \to \Omega^1_{S_1/\mathsf{U}},$$

which we denote by $\mathrm{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$.

Definition 4.6.2.6. *The element $\mathsf{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$ (resp. the morphism $\mathrm{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$) above is called the* ***Kodaira–Spencer class*** *(resp. the* ***Kodaira–Spencer morphism****) for $(G^\natural_{S_1}, \iota)$.*

Remark 4.6.2.7. According to Lemma 4.2.1.7 (or rather its proof in Section 4.2.2), the homomorphism $\iota : Y \to G^\natural_{S_1}$ can be identified with a trivialization $\tau : \mathbf{1}_{(Y \times X)_{S_1}} \overset{\sim}{\to} (c^\vee \times c)^*\mathcal{P}^{-1}_{A_{S_1}}$ of biextensions, and hence a homomorphism $\iota^\vee : X \to G^{\vee,\natural}_{S_1}$ lifting $c : X \to A^\vee_{S_1}$. Then we can apply Proposition 4.6.2.5 to the dual situation of $(A^\vee_{S_1}, c^\vee, c, \iota^\vee)$ as well. The various Kodaira–Spencer classes fit into the natural commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^{\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}} \arrow[r] \arrow[d] & \underline{\mathrm{Lie}}_{A^{\vee}_{S_1}/S_1} \underset{\mathscr{O}_{S}}{\otimes} \underline{\mathrm{Lie}}_{G^{\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}} \arrow[d] \\
\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{A_{S_1}/S} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}} \arrow[r] & \underline{\mathrm{Lie}}_{A^{\vee}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{A_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}
\end{tikzcd}$$

so that

$$\begin{tikzcd}
\mathsf{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}} \arrow[r, mapsto] \arrow[d, mapsto] & \mathsf{KS}_{(A_{S_1},c)/S_1/\mathsf{U}} \arrow[d, mapsto] \\
\mathsf{KS}_{(A^{\vee}_{S_1},c^{\vee})/S_1/\mathsf{U}} \arrow[r, mapsto] & \mathsf{KS}_{A_{S_1}/S_1/\mathsf{U}}
\end{tikzcd}$$

by forgetting structures. (We view $\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{A_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$ as $\underline{\mathrm{Lie}}_{A_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}$ when considering $\mathsf{KS}_{(A^{\vee}_{S_1},c^{\vee})/S_1/\mathsf{U}}$ as its section.)

Suppose that the connected components of S are all integral, so that $\mathscr{O}_S$ is naturally a subsheaf of $(S_1 \hookrightarrow S)_*\mathscr{O}_{S_1}$, or simply $\mathscr{O}_{S_1}$ by abuse of language. If we consider $\underline{\mathrm{Lie}}^{\vee}_{G^{\natural}/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/S}$ as a subsheaf of $(S_1 \hookrightarrow S)_*(\underline{\mathrm{Lie}}^{\vee}_{G^{\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}_{S_1}/S_1})$, and consider the restriction of the Kodaira–Spencer morphism $(S_1 \hookrightarrow S)_*(\mathrm{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}})$ to this subsheaf, then we obtain a morphism

$$\mathrm{KS}_{(G^{\natural},\iota)/S/\mathsf{U}} : \underline{\mathrm{Lie}}^{\vee}_{G^{\natural}/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/S} \to (S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}. \tag{4.6.2.8}$$

For later purposes, it is desirable to replace the target $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$ with some subsheaf of it with a better finiteness property. Since A, c, and $c^{\vee}$ are all defined over the whole base scheme S, the morphisms $\mathrm{KS}_{(A_{S_1},c)/S_1/\mathsf{U}}$, $\mathrm{KS}_{(A^{\vee}_{S_1},c^{\vee})/S_1/\mathsf{U}}$, and $\mathrm{KS}_{A_{S_1}/S_1/\mathsf{U}}$ all extend over S. This shows that the images of both $\underline{\mathrm{Lie}}^{\vee}_{G^{\natural}/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{A^{\vee}/S}$ and $\underline{\mathrm{Lie}}^{\vee}_{A/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/S}$ are contained in the image of $\Omega^1_{S/\mathsf{U}}$ in $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$ (under the canonical morphism). Hence the question is about the target of the induced morphism

$$\underline{\mathrm{Lie}}^{\vee}_{T/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{T^{\vee}/S} \to ((S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}})/\Omega^1_{S/\mathsf{U}}. \tag{4.6.2.9}$$

Since the answer is related only to ι, let us fix a particular lifting of the triple $(A, c, c^{\vee})$ to $\tilde{S}$, and investigate the liftings of ι to $\tilde{S}_1$. (Note that such liftings exist because $G^{\natural}_{S_1}$ is smooth over S_1.)

As already mentioned above, the homomorphism $\iota : \underline{Y}_{S_1} \to G^{\natural}_{S_1}$ is equivalent to a trivialization $\tau : \mathbf{1}_{(Y \times X)_{S_1}} \overset{\sim}{\to} (c^{\vee} \times c)^*\mathcal{P}^{-1}_{A_{S_1}}$ of biextensions, which we can interpret as a collection of isomorphisms $\tau(y,\chi) : \mathscr{O}_{S_1} \overset{\sim}{\to} (c^{\vee}(y), c(\chi))^*\mathcal{P}^{\otimes -1}_{A_{S_1}}$ (satisfying certain bimultiplicative relations) for $y \in Y$ and $\chi \in X$. This collection $\{\tau(y,\chi)\}_{y \in Y, \chi \in X}$ defines a collection of $\mathscr{O}_S$-invertible submodules $I_{y,\chi}$ of $\mathscr{O}_{S_1}$, with $\mathscr{O}_S$-module isomorphisms from $I_{y,\chi}$ to $(c^{\vee}(y), c(\chi))^*\mathcal{P}^{\otimes -1}_A$ induced by $\tau(y,\chi)$ (see Definition 4.2.4.6). If we interpret these as isomorphisms between $\mathbf{G}_{\mathrm{m}}$-torsors over S_1, then over each lifting of $(A, c, c^{\vee})$, the liftings of the collection $\{\tau(y,\chi)\}_{y \in Y, \chi \in X}$ to $\tilde{S}_1$ form a torsor under the group of homomorphisms $Y \otimes X \to \mathscr{I}_1$, which can be identified with

$$\underline{\mathrm{Lie}}_{T^{\vee}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{T_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{I}_1.$$

If the embedding $S \hookrightarrow \tilde{S}$ is the first infinitesimal neighborhood of the diagonal morphism $\Delta : S \to S \underset{\mathsf{U}}{\times} S$, then $\mathscr{I} \cong \Omega^1_{S/\mathsf{U}}$ and $\mathscr{I}_1 \cong \Omega^1_{S_1/\mathsf{U}}$. For each local generator q of $I_{y,\chi}$, which we can interpret as a local section of $\mathbf{G}_{\mathrm{m},S_1}$, the difference between the two pullbacks of the section $\tau(y,\chi)$ is given additively by the local section $dq = \mathrm{pr}_2^*(q) - \mathrm{pr}_1^*(q)$ of $q^*\Omega^1_{\mathbf{G}_{\mathrm{m},S_1}/S_1}$, or rather the local section $d\log(q) := q^{-1}dq$ of $\mathscr{O}_{S_1} \cong e^*\Omega^1_{\mathbf{G}_{\mathrm{m},S_1}/S_1}$. Such *log differentials* define an invertible $\mathscr{O}_S$-submodule $d\log(I_{y,\chi})$ of $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$, independent of the (local) choice of q.

Definition 4.6.2.10. *With the setting as above, the $\mathscr{O}_S$-module $\Omega^1_{S/\mathsf{U}}[d\log\infty]$ is the $\mathscr{O}_S$-submodule of $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$ generated by the image of the canonical morphism $\Omega^1_{S/\mathsf{U}} \to (S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$ and by $d\log(I_{y,\chi})$, for all $y \in Y$ and $\chi \in X$. We call this $\mathscr{O}_S$-module $\Omega^1_{S/\mathsf{U}}[d\log\infty]$ the **sheaf of log 1-differentials** generated by $d\log(I_{y,\chi})$.*

Then the liftings of ι (over each particular lifting of $(A, c, c^{\vee})$) form a torsor under a subgroup of the global sections of

$$\underline{\mathrm{Lie}}_{T^{\vee}/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}_{T/S} \underset{\mathscr{O}_S}{\otimes} \Omega^1_{S/\mathsf{U}}[d\log\infty].$$

Hence we may replace the image $((S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}})/\Omega^1_{S/\mathsf{U}}$ in (4.6.2.9) with $(\Omega^1_{S/\mathsf{U}}[d\log\infty])/\Omega^1_{S/\mathsf{U}}$, and consequently the image $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$ in (4.6.2.8) with $\Omega^1_{S/\mathsf{U}}[d\log\infty]$. To summarize this,

Proposition 4.6.2.11. *With the setting as above, the Kodaira–Spencer morphism*

$$\mathrm{KS}_{(G^{\natural}_{S_1},\iota)/S_1/\mathsf{U}} : \underline{\mathrm{Lie}}^{\vee}_{G^{\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}_{S_1}/S_1} \to \Omega^1_{S_1/\mathsf{U}}$$

for $(G^{\natural}_{S_1}, \iota)$ over S_1 can be extended to a morphism

$$\mathrm{KS}_{(G^{\natural},\iota)/S/\mathsf{U}} : \underline{\mathrm{Lie}}^{\vee}_{G^{\natural}/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/S} \to \Omega^1_{S/\mathsf{U}}[d\log\infty]$$

over S.

Definition 4.6.2.12. *The morphism $\mathrm{KS}_{(G^{\natural},\iota)/S/\mathsf{U}}$ defined above is called the **extended Kodaira–Spencer morphism** for $(G^{\natural}, \iota)$.*

Remark 4.6.2.13. Certainly, the sheaf $\Omega^1_{S/\mathsf{U}}[d\log\infty]$ has a much better meaning in the case when S is smooth over U, when the complement of S_1 in S is a relative Cartier divisor of normal crossings, and when the sheaves $d\log(I_{y,\chi})$ are closely related to the definition of this relative Cartier divisor. (This will be the case when we construct the toroidal compactifications in Chapter 6.) But we do not need these when defining the morphisms.

4.6.3 Compatibility with Mumford's Construction

Let $S = \mathrm{Spec}(R)$ be any affine scheme fitting into the setting of Section 4.1 such that $\widehat{\Omega}^1_{S/\mathsf{U}}$, the completion of $\Omega^1_{S/\mathsf{U}}$ with respect to the topology of R defined by I, is *locally free of finite rank* over $\mathscr{O}_S$ (cf. [59, 0_{IV}, 20.4.9]). Suppose we have an object (G, λ) in $\mathrm{DEG}_{\mathrm{pol}}(R, I)$ mapped to an object $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \tau)$ in $\mathrm{DD}_{\mathrm{pol}}(R, I)$ by $\mathrm{F}_{\mathrm{pol}}(R, I) : \mathrm{DEG}_{\mathrm{pol}}(R, I) \to \mathrm{DD}_{\mathrm{pol}}(R, I)$ (see Definition 4.4.8). Let

$(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$ or equivalently $(G^\natural, \iota : \underline{Y}_\eta \to G^\natural_\eta)$ be the underlying object in $\mathrm{DD}(R, I)$. By Mumford's construction, if we introduce the notion of relatively complete models and pass to the category of formal schemes, we could interpret G as a quotient of $G^\natural$ by the period homomorphism ι (although it is not a quotient in the category of schemes).

Let us suppose that there exists a nonempty open subscheme S_1 of S such that G is an abelian scheme G_{S_1} over S_1, and such that $\iota : \underline{Y}_\eta \to G^\natural_\eta$ extends (necessarily uniquely) to a homomorphism $\underline{Y}_{S_1} \to G^\natural_{S_1}$. Such an open subscheme exists because G_η is an abelian scheme over η, and because of Proposition 4.5.3.11 and Corollary 4.5.3.12. By abuse of notation, let us denote by $\widehat{\Omega}^1_{S_1/\mathsf{U}}$ the pullback of $\widehat{\Omega}^1_{S/\mathsf{U}}$ to the subscheme S_1 of S. Then the canonical morphism $\Omega^1_{S/\mathsf{U}} \to \widehat{\Omega}^1_{S/\mathsf{U}}$ induces a canonical morphism $\Omega^1_{S_1/\mathsf{U}} \to \widehat{\Omega}^1_{S_1/\mathsf{U}}$. Let us also define $\widehat{\Omega}^1_{S/\mathsf{U}}[d\log\infty]$ as in Definition 4.6.2.10, which is the $\mathscr{O}_S$-submodule of $(S_1 \hookrightarrow S)_*\widehat{\Omega}^1_{S_1/\mathsf{U}}$ generated by $\widehat{\Omega}^1_{S/\mathsf{U}}$ and $d\log(I_{y,\chi})$, for all $y \in Y$ and $\chi \in X$. Then we also have a canonical morphism $\Omega^1_{S/\mathsf{U}}[d\log\infty] \to \widehat{\Omega}^1_{S/\mathsf{U}}[d\log\infty]$ compatible with $\Omega^1_{S/\mathsf{U}} \to \widehat{\Omega}^1_{S/\mathsf{U}}$.

Since the $\mathscr{O}_S$-module (resp. $\mathscr{O}_{S_1}$-module) $\widehat{\Omega}^1_{S/\mathsf{U}}$ (resp. $\widehat{\Omega}^1_{S_1/\mathsf{U}}$) is locally free of finite rank, the constructions of Kodaira–Spencer morphisms in Sections 2.1.7, 4.6.1, and 4.6.2 are valid if we use $\widehat{\Omega}^1_{S/\mathsf{U}}$ (resp. $\widehat{\Omega}^1_{S_1/\mathsf{U}}$) instead of $\Omega^1_{S/\mathsf{U}}$ (resp. $\Omega^1_{S_1/\mathsf{U}}$). (The construction of Kodaira–Spencer classes does not require the local freeness of sheaves of differentials such as $\Omega^1_{S/\mathsf{U}}$. Once the Kodaira–Spencer classes are defined, by considering their images under the morphisms induced by canonical morphisms such as $\Omega^1_{S/\mathsf{U}} \to \widehat{\Omega}^1_{S/\mathsf{U}}$, we can translate these images into Kodaira–Spencer morphisms using the local freeness of sheaves such as $\widehat{\Omega}^1_{S/\mathsf{U}}$.) Then, as in Definition 2.1.7.9, G_{S_1} admits a Kodaira–Spencer morphism

$$\mathrm{KS}_{G_{S_1}/S_1/\mathsf{U}} : \underline{\mathrm{Lie}}^\vee_{G_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^\vee_{S_1}/S_1} \to \widehat{\Omega}^1_{S_1/\mathsf{U}}.$$

On the other hand, as in Definition 4.6.2.6, $(G^\natural_{S_1}, \iota)$ defines the morphism

$$\mathrm{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}} : \underline{\mathrm{Lie}}^\vee_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}_{S_1}/S_1} \to \widehat{\Omega}^1_{S_1/\mathsf{U}},$$

which extends to the morphism

$$\mathrm{KS}_{(G^\natural,\iota)/S/\mathsf{U}} : \underline{\mathrm{Lie}}^\vee_{G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/S} \to \widehat{\Omega}^1_{S/\mathsf{U}}[d\log\infty]$$

as in Definition 4.6.2.12. Since $G_{\mathrm{for}} \cong G^\natural_{\mathrm{for}}$ and hence $\underline{\mathrm{Lie}}^\vee_{G_{\mathrm{for}}/S_{\mathrm{for}}} \cong \underline{\mathrm{Lie}}^\vee_{G^\natural_{\mathrm{for}}/S_{\mathrm{for}}}$ over S_{for}, by Theorem 2.3.1.2, there is a canonical isomorphism $\underline{\mathrm{Lie}}^\vee_{G/S} \cong \underline{\mathrm{Lie}}^\vee_{G^\natural/S}$. Similarly, there is a canonical isomorphism $\underline{\mathrm{Lie}}^\vee_{G^\vee/S} \cong \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/S}$. Therefore the two morphisms $\mathrm{KS}_{G_{S_1}/S_1/\mathsf{U}}$ and $\mathrm{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$ have the same source and target, and it is natural to compare them.

Since the identification between two morphisms can be achieved locally in the étale topology, we shall assume until we finish the proof of Theorem 4.6.3.16 that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, for simplicity.

Elements in

$$H^1(G_{S_1}, \underline{\mathrm{Der}}_{G_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}}) \cong \mathrm{Ext}^1(\Omega^1_{G_{S_1}/S_1}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}})$$

parameterize extensions $\mathcal{E}$ of the form

$$0 \to \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}} \to \mathcal{E} \to \Omega^1_{G_{S_1}/S_1} \to 0. \tag{4.6.3.1}$$

By Proposition 2.1.7.3, the Kodaira–Spencer class

$$\mathsf{KS}_{G_{S_1}/S_1/\mathsf{U}} \in H^1(G_{S_1}, \underline{\mathrm{Der}}_{G_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}),$$

interpreted as an element of $\mathrm{Ext}^1(\Omega^1_{G_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}})$, is the extension class of the first exact sequence

$$0 \to \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}} \to \Omega^1_{G_{S_1}/\mathsf{U}} \to \Omega^1_{G_{S_1}/S_1} \to 0 \tag{4.6.3.2}$$

of G_{S_1}. The image of $\mathsf{KS}_{G_{S_1}/S_1/\mathsf{U}}$ under the canonical morphism

$$H^1(G_{S_1}, \underline{\mathrm{Der}}_{G_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}) \to H^1(G_{S_1}, \underline{\mathrm{Der}}_{G_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}})$$

induced by the canonical morphism $\Omega^1_{S_1/\mathsf{U}} \to \widehat{\Omega}^1_{S_1/\mathsf{U}}$, interpreted as an element of $\mathrm{Ext}^1(\Omega^1_{G_{S_1}/S_1}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}})$, is the extension class of the push-out

$$0 \to \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}} \to \widehat{\Omega}^1_{G_{S_1}/\mathsf{U}} \to \Omega^1_{G_{S_1}/S_1} \to 0 \tag{4.6.3.3}$$

of (4.6.3.2) under the canonical morphism $\Omega^1_{S_1/\mathsf{U}} \to \widehat{\Omega}^1_{S_1/\mathsf{U}}$. (Here $\widehat{\Omega}^1_{G_{S_1}/\mathsf{U}}$ is defined as the push-out. Later we can give an alternative identification of $\widehat{\Omega}^1_{G_{S_1}/\mathsf{U}}$ using Mumford's construction.) Similarly, elements in

$$H^1(G^\natural_{S_1}, \underline{\mathrm{Der}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}}) \cong \mathrm{Ext}^1(\Omega^1_{G^\natural_{S_1}/S_1}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})$$

parameterize extensions $\mathcal{E}^\natural$ of the form

$$0 \to \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}} \to \mathcal{E}^\natural \to \Omega^1_{G^\natural_{S_1}/S_1} \to 0; \tag{4.6.3.4}$$

the Kodaira–Spencer class

$$\mathsf{KS}_{G^\natural_{S_1}/S_1/\mathsf{U}} \in H^1(G^\natural_{S_1}, \underline{\mathrm{Der}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}})$$

corresponds by Proposition 2.1.7.3 to the extension class of the first exact sequence

$$0 \to \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}} \to \Omega^1_{G^\natural_{S_1}/\mathsf{U}} \to \Omega^1_{G^\natural_{S_1}/S_1} \to 0 \tag{4.6.3.5}$$

of $G^\natural$, and the image of $\mathsf{KS}_{G^\natural_{S_1}/S_1/\mathsf{U}}$ under the canonical morphism

$$H^1(G^\natural_{S_1}, \underline{\mathrm{Der}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}) \to H^1(G^\natural_{S_1}, \underline{\mathrm{Der}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}})$$

induced by the canonical morphism $\Omega^1_{S_1/\mathsf{U}} \to \widehat{\Omega}^1_{S_1/\mathsf{U}}$ corresponds to the push-out

$$0 \to \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}} \to \widehat{\Omega}^1_{G^\natural_{S_1}/\mathsf{U}} \to \Omega^1_{G^\natural_{S_1}/S_1} \to 0 \tag{4.6.3.6}$$

of (4.6.3.5) under the canonical morphism $\Omega^1_{S_1/\mathsf{U}} \to \widehat{\Omega}^1_{S_1/\mathsf{U}}$.

Since Y acts equivariantly on all three terms in (4.6.3.6), the sequence also defines an Y-equivariant extension class in $\mathrm{Ext}^{1,Y}(\Omega^1_{G^\natural_{S_1}/S_1}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})$.

Remark 4.6.3.7. Assume for the moment (in this remark) that we had the *hypothetical situation* that G_{S_1} is a quotient of $G^\natural_{S_1}$ (in the category of schemes) by the translation action of $\iota : Y \to G^\natural_{S_1}$. Let us denote the quotient morphism by $p : G^\natural_{S_1} \to G_{S_1}$. Assume that p is étale, that the Y-action defined by ι is free, and that there is an integer $k \geq 1$ such that every point of $G^\natural_{S_1}$ admits an affine open neighborhood U such that $\cup_{y \in kY}(\iota(y))(U)$ is a disjoint union. Then we can factor $p : G^\natural_{S_1} \to G_{S_1}$ as a composition $p = p_1 \circ p_2$ of two quotient morphisms, such that

p_2 is a local isomorphism and p_1 is a quotient by a finite group action. Under this hypothesis, there is a canonical morphism

$$H^1(G_{S_1}, \underline{\mathrm{Der}}_{G_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}}) \to H^1(G^\natural_{S_1}, \underline{\mathrm{Der}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \Omega^1_{S_1/\mathsf{U}})$$

defined by pullback, which can be interpreted as the canonical morphism

$$\mathrm{Ext}^1(\Omega^1_{G_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}}) \to \mathrm{Ext}^1(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}}),$$

which pulls back the extension class of $\mathcal{E}$ as in (4.6.3.1) to the extension class of $\mathcal{E}^\natural$ as in (4.6.3.4), and which necessarily pulls back the extension class $\mathsf{KS}_{G_{S_1}/S_1/\mathsf{U}}$ of $\Omega^1_{G_{S_1}/\mathsf{U}}$ to the extension class $\mathsf{KS}_{G^\natural_{S_1}/S_1/\mathsf{U}}$ of $\Omega^1_{G^\natural_{S_1}/\mathsf{U}}$. Moreover, the image is exactly those extension classes that are invariant under the action of Y. Each representative $\mathcal{E}^\natural$ of such an extension class admits a Y-action. Then the theory of descent (for a quotient morphism defined by a group action that is free and moreover a local isomorphism on a subgroup of finite index) implies that there is an equivalence between Y-equivariant extension classes of $\mathcal{E}^\natural$ over $G^\natural_{S_1}$ and the usual extension classes of $\mathcal{E}$ over G_{S_1} that pulls back to $\mathcal{E}^\natural$ over $G^\natural_{S_1}$. Consequently, the canonical pullback morphism

$$\mathrm{Ext}^1(\Omega^1_{G_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}}) \to \mathrm{Ext}^{1,Y}(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})$$

is an isomorphism. Moreover, this isomorphism is compatible with the canonical surjections of the two sides to the Y-invariants $\mathrm{Ext}^1(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})^Y$ in $\mathrm{Ext}^1(\Omega^1_{G^\natural_{S_1}/S_1}, \Omega^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})$, and compatible with the push-outs under the canonical morphism $\Omega^1_{S_1/\mathsf{U}} \to \widehat{\Omega}^1_{S_1/\mathsf{U}}$. (Remark 4.6.3.7 ends here.)

Let us return to reality and resume our setting before Remark 4.6.3.7.

Lemma 4.6.3.8. *For every integer $i \geq 0$, the canonical embedding $\underline{H}^i(A_{S_1}, \mathscr{O}_{A_{S_1}}) \hookrightarrow \underline{H}^i(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}})$ has image in $\underline{H}^i(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}})^Y$. The induced morphism $\underline{H}^i(A_{S_1}, \mathscr{O}_{A_{S_1}}) \to \underline{H}^i(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}})^Y$ is an isomorphism if (the restriction of) τ (to η) satisfies the positivity condition in Definition 4.2.1.10.*

Proof. According to (4.2.3.2), for each $y \in Y$, the translation by $\iota(y)$ on $G^\natural_{S_1}$ is described by the isomorphisms

$$\tau(y,\chi) : T^*_{c^\vee(y)}\mathscr{O}_{\chi,S_1} \cong \mathscr{O}_{\chi,S_1}(c^\vee(y))_{S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{\chi,S_1} \overset{\sim}{\to} \mathscr{O}_{\chi,S_1}.$$

The action of $\iota(y)$ on $\underline{H}^i(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}}) \cong \underset{\chi \in \underline{X}}{\oplus} \underline{H}^i(A_{S_1}, \mathscr{O}_{\chi,S_1})$ (cf. the proof of Lemma 4.6.1.4) is described componentwise by

$$\underline{H}^i(A_{S_1}, \mathscr{O}_{\chi,S_1}) \overset{T^*_{c^\vee(y)}}{\overset{\sim}{\to}} \underline{H}^i(A_{S_1}, T^*_{c^\vee(y)}\mathscr{O}_{\chi,S_1}) \overset{\tau(y,\chi)}{\overset{\sim}{\to}} \underline{H}^i(A_{S_1}, \mathscr{O}_{\chi,S_1}).$$

If $\chi = 0$, then we know that $\tau(y,\chi) = 1$ in the sense of 3 of Lemma 4.3.1.8, namely, it is the canonical structural isomorphism given by translations. Hence, by Lemma 4.6.2.2, the action of Y on $\underline{H}^i(A_{S_1}, \mathscr{O}_{A_{S_1}})$ is trivial. This proves the first statement of the lemma.

Suppose that τ satisfies the positive condition, and that $\chi \neq 0$. Take an integer $N \geq 1$ such that $N\chi \in \phi(Y)$. Let $y \in Y$ be such that $N\chi = \phi(y)$. Then $y \neq 0$, and we know that $I_{y,N\chi} \subset I$. Suppose x is a section of $\underline{H}^i(A_{S_1}, \mathscr{O}_{\chi,S_1})$ invariant under the action of Ny. By multiplying by a nonzero scalar in R, we may assume that $x \in H^i(A, \mathscr{O}_\chi)$. By repeating the action of Ny, we see that $x \in I^k \cdot H^i(A, \mathscr{O}_\chi)$ for all $k \geq 0$. Since $H^i(A, \mathscr{O}_\chi)$ is finitely generated over R, and since R is noetherian and I-adically complete, this implies that $x = 0$. This proves the second statement of the lemma. □

Corollary 4.6.3.9. *The embedding (4.6.1.6) induces a canonical isomorphism*

$$\underline{\mathrm{Lie}}_{A^\vee/S} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}} \cong \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})^Y$$

sending the global section of the source defined by $\mathsf{KS}_{(A_{S_1},c)/S_1/\mathsf{U}}$ to the global section of the target defined by the extension class of $\widehat{\Omega}^1_{G^\natural_{S_1}/\mathsf{U}}$ in (4.6.3.6).

Proof. Consider the composition of the canonical isomorphisms

$$\begin{aligned}
&\underline{\mathrm{Lie}}_{A^\vee_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}} \\
&\cong \underline{H}^1(A_{S_1}, \mathscr{O}_{A_{S_1}}) \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}} \\
&\overset{\sim}{\to} \underline{H}^1(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}/S_1})^Y \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}} \\
&\cong \underline{H}^1(G^\natural_{S_1}, \underline{\mathrm{Der}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}})^Y \\
&\cong \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}})^Y
\end{aligned}$$

(compatible with (4.6.1.6)), where the second isomorphism is the one given by Lemma 4.6.3.8. □

Lemma 4.6.3.10. *The diagram (of canonical morphisms)*

$$\begin{array}{ccc}
\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathscr{O}_{A_{S_1}}, \mathscr{O}_{A_{S_1}}) & \xrightarrow{\text{pullback}} & \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathscr{O}_{G^\natural_{S_1}}, \mathscr{O}_{G^\natural_{S_1}}) \\
\downarrow{\scriptstyle \text{forget}} & & \downarrow{\scriptstyle \text{forget}} \\
\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\mathscr{O}_{A_{S_1}}, \mathscr{O}_{A_{S_1}})^Y & \xrightarrow[\text{pullback}]{} & \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\mathscr{O}_{G^\natural_{S_1}}, \mathscr{O}_{G^\natural_{S_1}})^Y
\end{array} \qquad (4.6.3.11)$$

is commutative, and its horizontal rows are isomorphisms.

Proof. The commutativity follows from their definitions. The bottom row is an isomorphism by Lemma 4.6.3.8. To show that the top row is an isomorphism, we claim that the induced morphism between the kernels of vertical arrows is an isomorphism. The kernel of the vertical arrow on the left-hand side is $\underline{\mathrm{Hom}}_{S_1}(Y, \underline{H}^0(A_{S_1}, \mathscr{O}_{A_{S_1}})^Y)$, while the kernel of the vertical arrow on the right-hand side is $\underline{\mathrm{Hom}}_{S_1}(Y, \underline{H}^0(G^\natural_{S_1}, \underline{\mathrm{Hom}}_{\mathscr{O}_{G^\natural_{S_1}}}(\mathscr{O}_{G^\natural_{S_1}}, \mathscr{O}_{G^\natural_{S_1}})^Y)) \cong \underline{\mathrm{Hom}}_{S_1}(Y, \underline{H}^0(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}})^Y)$. Hence the question is whether the canonical morphism $\underline{H}^0(A_{S_1}, \mathscr{O}_{A_{S_1}})^Y \to \underline{H}^0(G^\natural_{S_1}, \mathscr{O}_{G^\natural_{S_1}})^Y$ defined by pullback is an isomorphism, which again follows from Lemma 4.6.3.8. □

Corollary 4.6.3.12. *$\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathscr{O}_{G^\natural_{S_1}}, \mathscr{O}_{G^\natural_{S_1}})$ is canonically isomorphic to $\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1}$ as an extension of $\underline{\mathrm{Lie}}_{A^\vee_{S_1}/S_1}$ by $\underline{\mathrm{Lie}}_{T^\vee_{S_1}/S_1}$.*

Proof. This is because we have $\underline{\mathrm{Hom}}_{S_1}(Y, \underline{H}^0(A_{S_1}, \mathscr{O}_{A_{S_1}})^Y) \cong \underline{\mathrm{Hom}}_{S_1}(Y, \mathscr{O}_{S_1}) \cong \underline{\mathrm{Lie}}_{T^\vee_{S_1}/S_1}$ in the proof of Lemma 4.6.3.10, and because we have Lemma 4.6.2.4. □

Corollary 4.6.3.13. *The canonical isomorphism*

$$\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}}) \cong \underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}} \quad (4.6.3.14)$$

given by Corollary 4.6.3.12 respects the structures of both sides as extensions of $\underline{\mathrm{Lie}}_{A^\vee_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}}$ *by* $\underline{\mathrm{Lie}}_{T^\vee_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}}$.

Proof. This follows from Proposition 4.6.2.5. □

Proposition 4.6.3.15. *The isomorphism (4.6.3.14) sends the global section on the left-hand side defined by the Y-equivariant extension class of $\widehat{\Omega}^1_{G^\natural_{S_1}/\mathsf{U}}$ in (4.6.3.6) to the global section on the right-hand side defined by the Kodaira–Spencer class* $\mathsf{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$.

Proof. If we reproduce the argument of the proof of Proposition 2.1.7.3 using open coverings of $G^\natural_{S_1}$ as in the proof of Proposition 4.6.2.5, then we see that the isomorphism

$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}}) \cong \underline{\mathrm{Lie}}_{A^\vee_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}}$$

induced by (4.6.3.14) (by forgetting Y-actions of the parameterized objects) sends the global section defined by the Y-equivariant extension class of $\widehat{\Omega}^1_{G^\natural_{S_1}/\mathsf{U}}$ in (4.6.3.6) to the global section defined by the Kodaira–Spencer class $\mathsf{KS}_{(A_{S_1},c)/S_1/\mathsf{U}}$. Let us also take the Y-actions into account. For simplicity of notation, let us assume that $S = S_1$. Let us choose a basis dx_i of $\Omega^1_{G^\natural/S}$ over U_α using coordinates of some $\mathbb{A}^r_S$'s as in the proof of Proposition 2.1.7.3, and write the Y-action on $\Omega^1_{G^\natural/S}$ as $\eta(y)_\alpha : dx_i \mapsto dT_{\iota(y)}(dx_i)$. (These differentials are taken over S.) Take a basis of $\widehat{\Omega}^1_{G^\natural/\mathsf{U}}$ including the dx_i above, which is possible because $\widehat{\Omega}^1_{G^\natural/\mathsf{U}}$ is defined as a push-out. Since the differentials are taken over U, the Y-action might no longer send dx_i to $dT_{\iota(y)}(dx_i)$. The difference is measured by an element in $\underline{\mathrm{Lie}}_{G^\natural/S} \underset{S}{\otimes} \widehat{\Omega}^1_{S/\mathsf{U}}$ for each y. On the other hand, the differentials over U, by definition, can be explicitly obtained by comparing the two pullbacks $\mathrm{pr}_1^*(\eta(y)_\alpha)$ and $\mathrm{pr}_2^*(\eta(y)_\alpha)$ to the first infinitesimal neighborhood $\tilde{S}$ of the diagonal embedding $S \hookrightarrow S \underset{\mathsf{U}}{\times} S$. Up to the morphisms induced by the canonical morphism $\Omega^1_{S/\mathsf{U}} \to \widehat{\Omega}^1_{S/\mathsf{U}}$, this is (up to a sign convention) how we compared the difference between liftings of Y-actions in the proof of Proposition 4.6.2.5. □

Now the question is whether we can find a canonical isomorphism between

$$\underline{\mathrm{Lie}}_{G^\vee_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}} \cong \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G_{S_1}/S_1}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}})$$

and

$$\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}} \cong \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}}),$$

that matches the global section defined by the extension class of $\widehat{\Omega}^1_{G_{S_1}/\mathsf{U}}$ with the global section defined by the Y-equivariant extension class of $\widehat{\Omega}^1_{G^\natural_{S_1}/\mathsf{U}}$. Although G_{S_1} is not a quotient of $G^\natural_{S_1}$ in the category of schemes, Mumford's construction using relatively complete models does realize the formal completion of some proper model P of G_{S_1} as a quotient. This enables us to prove the following:

Theorem 4.6.3.16. *The two morphisms* $\mathrm{KS}_{G_{S_1}/S_1/\mathsf{U}}$ *and* $\mathrm{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$ *are identified with each other under the restriction of the two canonical isomorphisms* $\underline{\mathrm{Lie}}^\vee_{G/S} \cong \underline{\mathrm{Lie}}^\vee_{G^\natural/S}$ *and* $\underline{\mathrm{Lie}}^\vee_{G^\vee/S} \cong \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/S}$ *to S_1. In this case, we can identify* $\mathrm{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$ *with a morphism*

$$\mathrm{KS}_{G/S/\mathsf{U}} : \underline{\mathrm{Lie}}^\vee_{G/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^\vee/S} \to \widehat{\Omega}^1_{S/\mathsf{U}}[d\log D_\infty]$$

extending $\mathrm{KS}_{G_{S_1}/S_1/\mathsf{U}}$, *which we call the* ***extended Kodaira–Spencer morphism*** *for G over S.*

The proof of Theorem 4.6.3.16 will be given after some preparation. Since we are comparing two existing morphisms, we are allowed to prove the theorem after making some finite étale surjective base change. Therefore, we may assume that both $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively. According to Lemma 4.5.4.5 (which uses Corollary 4.5.1.8) and Proposition 4.5.1.15, we know that if we start with an ample invertible sheaf $\mathcal{M}$ over A, pull it back to a cubical invertible sheaf $\mathcal{L}$ over $G^\natural$, and replace it with a sufficiently high tensor power, then we may assume that we have a relatively complete model $(P^\natural, \mathcal{L}^\natural)$ over S for the pair $(G^\natural, \mathcal{L}^\natural)$, together with a Y-action over the whole S extending the one on $G^\natural_\eta$. Let us denote by $(P, \mathcal{L})$ the "Mumford quotient" of $(G^\natural, \mathcal{L}^\natural)$ by Y (see Definition 4.5.2.18). Since $G_{S_1} \to S_1$ is proper, $G_{S_1} = P_{S_1}$.

For $j = 1, 2$, let $\mathcal{E}_j$ be a coherent sheaf over P such that its restriction to G is the pullback of a locally free sheaf $\mathcal{E}^0_j$ of finite rank over S.

Lemma 4.6.3.17. *The canonical morphism*

$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathcal{E}_1, \mathcal{E}_2) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \to \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\mathcal{E}_1 \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}, \mathcal{E}_2 \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}) \quad (4.6.3.18)$$

is an isomorphism.

Proof. Let $\mathcal{E} := \underline{\mathrm{Hom}}_{\mathscr{O}_P}(\mathcal{E}_1, \mathcal{E}_2)$. Since P is separated and of finite type over S, we have a canonical isomorphism

$$H^1(P, \mathcal{E}) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \xrightarrow{\sim} H^1(G_{S_1}, \mathcal{E} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}).$$

By assumptions on the restrictions of $\mathcal{E}_1$ and $\mathcal{E}_2$ to G, the local higher extension classes of $\mathcal{E}_1$ by $\mathcal{E}_2$ (over P) are all supported on $C = (P - G)_{\mathrm{red}}$ (see [52]), which has empty fiber over S_1. As a result, the local-to-global spectral sequence (over P) implies that the kernel and cokernel of the canonical morphism $\underline{H}^1(P, \mathcal{E}) \to \underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathcal{E}_1, \mathcal{E}_2)$ are not supported on S_1. This shows that the canonical morphism (4.6.3.18) is an isomorphism, as desired. □

Let us denote formal completions by the subscript "for" as usual. Since P is proper over S, by Theorem 2.3.1.2, we have a canonical isomorphism

$$\Big[\underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\mathcal{E}_1, \mathcal{E}_2)\Big]_{\mathrm{for}} \xrightarrow{\sim} \underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_{\mathrm{for}}}}(\mathcal{E}_{1,\mathrm{for}}, \mathcal{E}_{2,\mathrm{for}}). \quad (4.6.3.19)$$

(Here we extend Definition 4.6.2.1 naturally to formal schemes.)

Let $p : P^\natural_{\mathrm{for}} \to P_{\mathrm{for}}$ denote the quotient morphism. For $j = 1, 2$, suppose $\mathcal{E}^\natural_j$ is a coherent sheaf over $P^\natural$ with a Y-action covering the Y-action on $P^\natural$, such that its restriction to $G^\natural$ is the pullback of $\mathcal{E}_j$ (the same locally free sheaf over S appeared in the assumption for $\mathcal{E}_j$), and such that $\mathcal{E}^\natural_{j,\mathrm{for}} \cong p^*\mathcal{E}_{j,\mathrm{for}}$.

Lemma 4.6.3.20. *The canonical pullback morphism*
$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_{\mathrm{for}}}}(\mathcal{E}_{1,\mathrm{for}},\mathcal{E}_{2,\mathrm{for}}) \to \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\mathcal{E}^\natural_{1,\mathrm{for}},\mathcal{E}^\natural_{2,\mathrm{for}}) \tag{4.6.3.21}$$
is an isomorphism.

Proof. By construction, P_{for} is a quotient of $P^\natural_{\mathrm{for}}$ (in the category of formal schemes) as in the hypothetical situation in Remark 4.6.3.7 (with the nice properties that the action is free and that the quotient can be realized as a local isomorphism followed by a quotient by finite group action). Hence the lemma follows from the argument in Remark 4.6.3.7. □

On the other hand, since $P^\natural$ is not proper over S in general, we cannot apply Theorem 2.3.1.2 to the canonical morphisms from the formal completion of the cohomology groups of $P^\natural$ to the cohomology groups of the formal completion $P^\natural_{\mathrm{for}}$. Since we only care about Y-invariant classes together with Y-actions on them, and since we only care if there is an isomorphism after we annihilate nonzero torsion elements (which are not supported on S_1), we might circumvent this difficulty by passing to cohomology groups on A, as follows:

Let $G^{\natural,*} = \bigcup\limits_{y\in Y} S_y(G^\natural)$ and $C^\natural = (P^\natural - G^{\natural,*})_{\mathrm{red}}$ be as before, with Y-actions inherited from $P^\natural$. Then $C^\natural$ has empty fiber over S_1, and the formal completion C_{for} of C is the quotient of the formal completion $C^\natural_{\mathrm{for}}$ of $C^\natural$ by Y (see Construction 4.5.2.17). For $j = 1,2$, set $\mathcal{E}^0_{j,P^\natural} := (P^\natural \to S)^*\mathcal{E}^0_j$.

Lemma 4.6.3.22. *The sheaves* $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\mathcal{E}^0_{1,P^\natural,\mathrm{for}},\mathcal{E}^0_{2,P^\natural,\mathrm{for}})$ *and* $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\mathcal{E}^\natural_{1,\mathrm{for}},\mathcal{E}^\natural_{2,\mathrm{for}})$ *over* S_{for} *are coherent (and hence algebraizable). Let us denote the algebraization of a coherent sheaf by the subscript "alg". Then there is a canonical isomorphism*
$$\Big[\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\mathcal{E}^0_{1,P^\natural,\mathrm{for}},\mathcal{E}^0_{2,P^\natural,\mathrm{for}})\Big]_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \to \Big[\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\mathcal{E}^\natural_{1,\mathrm{for}},\mathcal{E}^\natural_{2,\mathrm{for}})\Big]_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \tag{4.6.3.23}$$
(although we do not have morphisms between $\mathcal{E}^\natural_{j,\mathrm{for}}$ *and* $\mathcal{E}^0_{j,P^\natural,\mathrm{for}}$ *for* $j = 1,2$*).*

Proof. The coherence of the two modules in question follows from the facts that the action of a finite index subgroup of Y on P_{for} (and hence $G^{\natural,*}_{\mathrm{for}}$ and C_{for}) is given by local isomorphisms, and that $P^\natural$ is covered by the Y-translations of a scheme of finite type over S. For $j = 1,2$, since the restrictions of $\mathcal{E}^\natural_{j,\mathrm{for}}$ and $\mathcal{E}^0_{j,P^\natural,\mathrm{for}}$ to $G^{\natural,*}_{\mathrm{for}}$ are isomorphic by assumption, their difference is given by local higher cohomology supported on $C^\natural_{\mathrm{for}}$. □

Let us denote the structural morphism $P^\natural \to A$ by π, and denote its formal completion by π_{for}. For $j = 1,2$, set $\mathcal{E}^0_{j,A} := (A \to S)^*\mathcal{E}^0_j$.

Lemma 4.6.3.24. *The kernel and cokernel of the pullback morphism*
$$\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\mathcal{E}^0_{1,A,\mathrm{for}},\mathcal{E}^0_{2,A,\mathrm{for}}) \to \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\mathcal{E}^0_{1,P^\natural,\mathrm{for}},\mathcal{E}^0_{2,P^\natural,\mathrm{for}}) \tag{4.6.3.25}$$
(as coherent sheaves over S*) are not supported on* S_1*.*

Proof. The Leray spectral sequence (see [48, Ch. II, Thm. 4.17.1]) shows that the kernel and cokernel of the canonical morphism $\underline{H}^i(A_{\mathrm{for}},\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}}) \to \underline{H}^i(P^\natural_{\mathrm{for}},\mathscr{O}_{P^\natural_{\mathrm{for}}})$ are subquotients of $\underline{H}^{i-j}(A_{\mathrm{for}},R^j\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})$ with $j > 0$. If we take Y-invariants, then for the same reason as before we obtain coherent modules that algebraize. Hence the kernel and cokernel of
$$\underline{H}^i(A_{\mathrm{for}},\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})^Y \to \underline{H}^i(P^\natural_{\mathrm{for}},\mathscr{O}_{P^\natural_{\mathrm{for}}})^Y \tag{4.6.3.26}$$
(as coherent sheaves over S) are not supported on S_1. On the other hand, the T-action on $P^\natural$ induces a decomposition $\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}} \cong \bigoplus\limits_{\chi\in X}(\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})_\chi$ into weight subsheaves under the T_{for}-action. By construction of $P^\natural$, the Y-action on $P^\natural$ is defined componentwise by the canonical isomorphisms $\tau(y,\chi) : T^*_{c^\vee(y)}(\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})_\chi \cong (\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})_\chi(c^\vee(y)) \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} (\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})_\chi \xrightarrow{\sim} (\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})_\chi$. Accordingly, the action $S(y)^* : \underline{H}^i(A_{\mathrm{for}},\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})^Y \xrightarrow{\sim} \underline{H}^i(A_{\mathrm{for}},\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})^Y$ decomposes componentwise as $\tau(y,\chi) : \underline{H}^i(A_{\mathrm{for}},(\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})_\chi)^Y \xrightarrow{\sim} \underline{H}^i(A_{\mathrm{for}},(\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})_\chi)^Y$. By coherence of $\underline{H}^i(A_{\mathrm{for}},\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})^Y$, and hence the coherence of each component $\underline{H}^i(A_{\mathrm{for}},(\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})_\chi)^Y$, the positivity condition of τ implies, as in the proof of Lemma 4.6.3.8, that
$$\underline{H}^i(A_{\mathrm{for}},\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})^Y \cong \underline{H}^i(A_{\mathrm{for}},(\pi_{\mathrm{for},*}\mathscr{O}_{P^\natural_{\mathrm{for}}})_0)^Y \cong \underline{H}^i(A_{\mathrm{for}},\mathscr{O}_{A_{\mathrm{for}}})^Y. \tag{4.6.3.27}$$
Here the Y-action on A_{for} is induced by the homomorphism $c^\vee : Y \to A$, and we know by Lemma 4.6.2.2 that $\underline{H}^i(A_{\mathrm{for}},\mathscr{O}_{A_{\mathrm{for}}})^Y \cong \underline{H}^i(A_{\mathrm{for}},\mathscr{O}_{A_{\mathrm{for}}})$. Nevertheless, it is suggestive to consider the Y-equivariant extensions $\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\underline{\mathrm{Lie}}^\vee_{G^\natural_{\mathrm{for}}/S_{\mathrm{for}}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{A_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{A_{\mathrm{for}}})$, although it is the same as the ones without Y-equivariant actions. Now we can conclude the proof by combining (4.6.3.26) and (4.6.3.27). □

Since A is proper, there is a canonical isomorphism
$$\Big[\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_S}(\mathcal{E}^0_{1,A},\mathcal{E}^0_{2,A})\Big]_{\mathrm{for}} \to \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\mathcal{E}^0_{1,A,\mathrm{for}},\mathcal{E}^0_{2,A,\mathrm{for}}) \tag{4.6.3.28}$$
(of coherent sheaves over S_{for}). Moreover, there is a canonical isomorphism
$$\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_S}(\mathcal{E}^0_{1,A},\mathcal{E}^0_{2,A}) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \xrightarrow{\sim} \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathcal{E}^0_{1,A} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1},\mathcal{E}^0_{2,A} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}). \tag{4.6.3.29}$$
By Lemma 4.6.3.10, there is a canonical isomorphism
$$\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathcal{E}^0_{1,A} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1},\mathcal{E}^0_{2,A} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}) \xrightarrow{\sim} \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathcal{E}^\natural_1 \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1},\mathcal{E}^\natural_2 \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}). \tag{4.6.3.30}$$
Let us combine the results we have obtained. Denote the algebraization of a coherent sheaf or a morphism by the subscript "alg" (as above).

Proposition 4.6.3.31. *For each* $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}^0_1$, $\mathcal{E}^0_2$, $\mathcal{E}^\natural_1$, *and* $\mathcal{E}^\natural_2$ *as above, there is a canonical isomorphism*
$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\mathcal{E}_1 \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1},\mathcal{E}_2 \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}) \xrightarrow{\sim} \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathcal{E}^\natural_1 \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1},\mathcal{E}^\natural_2 \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}), \tag{4.6.3.32}$$
which can be (canonically) identified with the tensor product of
$$\underline{\mathrm{Hom}}_{\mathscr{O}_{S_1}}(\mathcal{E}^0_1 \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1},\mathcal{E}^0_2 \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1})$$
with (both sides of) the simplest special case
$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\mathscr{O}_{G_{S_1}},\mathscr{O}_{G_{S_1}}) \xrightarrow{\sim} \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathscr{O}_{G^\natural_{S_1}},\mathscr{O}_{G^\natural_{S_1}}) \tag{4.6.3.33}$$
of (4.6.3.32) with $\mathcal{E}_1 = \mathcal{E}_2 = \mathscr{O}_G$, $\mathcal{E}^0_1 = \mathcal{E}^0_2 = \mathscr{O}_S$, *and* $\mathcal{E}^\natural_1 = \mathcal{E}^\natural_2 = \mathscr{O}_{G^\natural}$.

Proof. The morphism (4.6.3.32) is the composition of the canonical isomorphisms $(4.6.3.18)^{-1}$, $(4.6.3.19)_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}$, $(4.6.3.21)_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}$, $(4.6.3.23)^{-1}_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}$,

$(4.6.3.25)^{-1}_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}$, $(4.6.3.28)^{-1}_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}$, (4.6.3.29), and (4.6.3.30). This composition is compatible with the simplest special case because all canonical isomorphisms involved are compatible with tensoring with $\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\mathcal{E}^0_1 \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}, \mathcal{E}^0_2 \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1})$. □

Corollary 4.6.3.34. *We have a canonical isomorphism*

$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_{S_1}}(\Omega^1_{G_{S_1}/S_1}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G_{S_1}}) \xrightarrow{\sim} \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\Omega^1_{G^\natural_{S_1}/S_1}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}}) \quad (4.6.3.35)$$

compatible with the canonical isomorphisms $\underline{\mathrm{Lie}}_{G/S} \cong \underline{\mathrm{Lie}}_{G^\natural/S}$ *and* $\underline{\mathrm{Lie}}_{G^\vee/S} \cong \underline{\mathrm{Lie}}_{G^{\vee,\natural}/S}$.

Proof. The isomorphism (4.6.3.35) is obtained from (4.6.3.32) by substituting $\mathcal{E}_1 = \Omega^1_{P/S}$, $\mathcal{E}_2 = \widehat{\Omega}^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_P$, $\mathcal{E}^0_1 = \underline{\mathrm{Lie}}^\vee_{G/S} \cong \underline{\mathrm{Lie}}^\vee_{G^\natural/S}$, $\mathcal{E}^0_2 = \widehat{\Omega}^1_{S/\mathsf{U}}$, $\mathcal{E}_1 = \Omega^1_{P^\natural/S}$, and $\mathcal{E}_2 = \widehat{\Omega}^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{P^\natural}$. The compatibility with the canonical isomorphism $\underline{\mathrm{Lie}}_{G/S} \cong \underline{\mathrm{Lie}}_{G^\natural/S}$ is already implicit in the validity of the choice of $\mathcal{E}^0_1$. On the other hand, by Proposition 4.6.3.31, (4.6.3.35) can be identified with the tensor product of $\underline{\mathrm{Lie}}_{G_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \widehat{\Omega}^1_{S_1/\mathsf{U}}$ with (4.6.3.33). We claim that (4.6.3.33) can be canonically identified with the canonical isomorphism $\underline{\mathrm{Lie}}_{G^\vee_{S_1}/S_1} \cong \underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1}$. Since the two sides of (4.6.3.33) can be canonically identified with $\underline{\mathrm{Lie}}_{G^\vee_{S_1}/S_1}$ and $\underline{\mathrm{Lie}}_{G^{\vee,\natural}_{S_1}/S_1}$, respectively, as in the proof of Lemma 4.6.2.4 using Poincaré invertible sheaves, the claim follows from the proof of Lemma 4.6.3.20 because $\mathcal{P}$ (over $G \underset{S}{\times} G^\vee$) is the "Mumford quotient" of the pullback of $\mathcal{P}_A$ to $G^\natural \times G^{\vee,\natural}$ as in Construction 4.5.4.14. □

Proof of Theorem 4.6.3.16. Consider the first exact sequence

$$\Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_P \to \Omega^1_{P/\mathsf{U}} \to \Omega^1_{P/S} \to 0 \quad (4.6.3.36)$$

of P over S. The first morphism in (4.6.3.36) is not necessarily injective, but it becomes injective if we pullback the sequence to $\mathscr{O}_{S_1}$. By taking completions with respect to the topology of R defined by I, the exact sequence (4.6.3.36) induces an exact sequence

$$\widehat{\Omega}^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_P \to \widehat{\Omega}^1_{P/\mathsf{U}} \to \Omega^1_{P/S} \to 0, \quad (4.6.3.37)$$

whose pullback to S_1 coincides with the exact sequence (4.6.3.3) and defines an element in

$$\underline{\mathrm{Ext}}^1_{\mathscr{O}_S}(\Omega^1_{P/S}, \widehat{\Omega}^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_P) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}, \quad (4.6.3.38)$$

(cf. (4.6.3.18)).

Similarly, consider the first exact sequence

$$\Omega^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{P^\natural} \to \Omega^1_{P^\natural/\mathsf{U}} \to \Omega^1_{P^\natural/S} \to 0 \quad (4.6.3.39)$$

of $P^\natural$ over S. This has a Y-equivariant structure because the Y-action on $G^\natural_\eta$ extends to the relatively complete model $P^\natural$ over the whole S. By taking completions with respect to the topology of R defined by I, the exact sequence (4.6.3.39) induces an exact sequence

$$\widehat{\Omega}^1_{S/\mathsf{U}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{P^\natural} \to \widehat{\Omega}^1_{P^\natural/\mathsf{U}} \to \Omega^1_{P^\natural/S} \to 0, \quad (4.6.3.40)$$

with an induced Y-equivariant structure. Its pullback to S_{for} defines an element in

$$\Big[\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\Omega^1_{P^\natural_{\mathrm{for}}/S_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}})\Big]_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1} \quad (4.6.3.41)$$

(note that $\widehat{\Omega}^1_{S/\mathsf{U}} \cong (\Omega^1_{S_{\mathrm{for}}/\mathsf{U}})_{\mathrm{alg}}$ by definition), whose image under the canonical isomorphism

$$\Big[\underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_{\mathrm{for}}}}(\Omega^1_{P^\natural_{\mathrm{for}}/S_{\mathrm{for}}}, \Omega^1_{S_{\mathrm{for}}/\mathsf{U}} \underset{\mathscr{O}_{S_{\mathrm{for}}}}{\otimes} \mathscr{O}_{P^\natural_{\mathrm{for}}})\Big]_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}$$

$$\xrightarrow{\sim} \underline{\mathrm{Ext}}^{1,Y}_{\mathscr{O}_{S_1}}(\mathscr{O}_{G^\natural_{S_1}}, \widehat{\Omega}^1_{S_1/\mathsf{U}} \underset{\mathscr{O}_{S_1}}{\otimes} \mathscr{O}_{G^\natural_{S_1}}) \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}_{G^\natural_{S_1}/S_1}$$

(following the second half of the definition of (4.6.3.32), with choices of modules as in the definition of (4.6.3.35)) gives the Y-equivariant extension class of the sequence (4.6.3.6).

By Mumford's construction of P_{for} as a quotient of $P^\natural_{\mathrm{for}}$, the pullback of the exact sequence (4.6.3.37) to S_{for} can be realized as a quotient of the pullback of the extension class of the pullback of the exact sequence (4.6.3.40) to S_{for}. Therefore, the classes they define in (4.6.3.38) and in (4.6.3.41) are identified with each other under the composition of $(4.6.3.19)_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}$ and $(4.6.3.21)_{\mathrm{alg}} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{S_1}$. That is, the extension class of the sequence (4.6.3.3) is identified with the Y-equivariant extension class of the sequence (4.6.3.6) under the isomorphism (4.6.3.35). By duality (using the global sections they defined), we obtain the identification between the Kodaira–Spencer morphisms $\mathrm{KS}_{G_{S_1}/S_1/\mathsf{U}}$ and $\mathrm{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$, as desired. □

Remark 4.6.3.42. The definition of $\mathrm{KS}_{(G^\natural_{S_1},\iota)/S_1/\mathsf{U}}$ in [42, Ch. III, §9] involves some different formulations using universal extensions (see [90] and [89]), which we believe is compatible with ours. Nevertheless, the validity of Theorem 4.6.3.43 below does not depend on the particular definition one takes along the formal completions. Although there is no intentional difference between our Theorem 4.6.3.16 and [42, Ch. III, Thm. 9.4], we do not need to know whether they follow from each other.

Let us globalize Theorem 4.6.3.16 as follows:

Theorem 4.6.3.43. *Let S be an algebraic stack separated and smooth over an excellent normal base scheme U. Let G be a semi-abelian scheme over S. Suppose there is an open dense subalgebraic stack S_1 of S, with complement $D_\infty := S - S_1$ a divisor of normal crossings, such that the restriction G_{S_1} of G to S_1 is an abelian scheme. In this case, there is a semi-abelian scheme $G^\vee$ (up to unique isomorphism) such that the restriction $G^\vee_{S_1}$ of $G^\vee$ to S_1 is the dual abelian scheme of G_{S_1}. Then there is a unique extension of the Kodaira–Spencer morphism*

$$\mathrm{KS}_{G_{S_1}/S_1/\mathsf{U}} : \underline{\mathrm{Lie}}^\vee_{G_{S_1}/S_1} \underset{\mathscr{O}_{S_1}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^\vee_{S_1}/S_1} \to \Omega^1_{S_1/\mathsf{U}}$$

to a morphism

$$\mathrm{KS}_{G/S/\mathsf{U}} : \underline{\mathrm{Lie}}^\vee_{G/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^\vee/S} \to \Omega^1_{S/\mathsf{U}}[d\log D_\infty].$$

Here $\Omega^1_{S/\mathsf{U}}[d\log D_\infty]$ is the sheaf of log 1-differentials, namely, the subsheaf of $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$ generated locally by $\Omega^1_{S/\mathsf{U}}$ and those $d\log q$ where q is a local generator of a component of D_∞.

Proof. By local freeness of $\underline{\mathrm{Lie}}^\vee_{G/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^\vee/S}$ and $\Omega^1_{S/\mathsf{U}}$, and by normality of S, there is always an extension $\underline{\mathrm{Lie}}^\vee_{G/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^\vee/S} \to (S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$. Therefore the question is whether the image of the extension lies in the subsheaf $\Omega^1_{S/\mathsf{U}}[d\log D_\infty]$ of $(S_1 \hookrightarrow S)_*\Omega^1_{S_1/\mathsf{U}}$. Since this question is local in nature, we may replace the smooth algebraic stack S with its completions of étale localizations by affine schemes, which are noetherian and normal by our assumption of excellent normality on U and hence fit into the setting of Section 4.1. Let us also replace S_1 and D_∞ with their corresponding pullbacks. (Then we can consider $\widehat{\Omega}^1_{S/\mathsf{U}}[d\log D_\infty]$ and $\widehat{\Omega}^1_{S_1/\mathsf{U}}$ instead of $\Omega^1_{S/\mathsf{U}}[d\log D_\infty]$ and $\Omega^1_{S_1/\mathsf{U}}$, respectively.) We may also assume that G_{S_1} is equipped with some polarization λ_{S_1} because whether the extension lies in the subsheaf $\widehat{\Omega}^1_{S/\mathsf{U}}[d\log D_\infty]$ of $(S_1 \hookrightarrow S)_*\widehat{\Omega}^1_{S_1/\mathsf{U}}$ (or not) does not depend on this choice of polarization. Then, by Theorem 4.6.3.16, we may take $\mathrm{KS}_{G/S/\mathsf{U}}$ to be the morphism $\mathrm{KS}_{(G^\natural,\iota)/S/\mathsf{U}}$, where $(G^\natural,\iota)$ is the object in $\mathrm{DD}(R,I)$ underlying the degeneration datum $(A,\lambda_A,\underline{X},\underline{Y},\phi,c,c^\vee,\tau)$ in $\mathrm{DD}_{\mathrm{pol}}(R,I)$ associated with (G,λ) in $\mathrm{DEG}_{\mathrm{pol}}(R,I)$ via the functor $\mathrm{F}_{\mathrm{pol}}(R,I)$ in Definition 4.4.8. Since the period homomorphism ι can be defined over S_1, we only need log 1-differentials with poles supported on D_∞, by construction in Section 4.6.2. $\square$

Definition 4.6.3.44. *The extended morphism* $\mathrm{KS}_{G/S/\mathsf{U}}$ *in Theorem 4.6.3.43 is called the* ***extended Kodaira–Spencer morphism*** *for* G *over* S*.*

Chapter 5

Degeneration Data for Additional Structures

In this chapter, we supply a theory of degeneration for abelian varieties with additional structures of PEL-type, based on the theory developed in Chapter 4. The running assumptions and notation in Chapter 4 (see 4.1) will be continued in this chapter without further remark. Moreover, we fix some choices of B, $\mathcal{O}$, and $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ as in Section 1.4.

The main objective is to state and prove Theorem 5.3.1.19, with Theorem 5.3.3.1 and the notion of cusp labels in Section 5.4 as by-products. Technical results worth noting are Propositions 5.1.2.4, 5.2.2.23, 5.2.3.3, and 5.2.3.9, Theorem 5.2.3.14, and Proposition 5.4.3.8. The preparation and proof of Theorem 5.2.3.14 in Sections 5.2.4, 5.2.5 and 5.2.6 form the technical heart of this chapter.

5.1 Data without Level Structures

5.1.1 Data for Endomorphism Structures

Let S be a base scheme satisfying the assumptions in Section 4.1, with generic point η, and let (G, λ) be an object in $\mathrm{DEG}_{\mathrm{pol}}(R, I)$ (see Definition 4.4.2 and Remark 4.4.3).

Suppose moreover that (G_η, λ_η) is equipped with a ring homomorphism $i_\eta : \mathcal{O} \to \mathrm{End}_\eta(G_\eta)$ defining an $\mathcal{O}$-endomorphism structure (with image in $\mathrm{End}_\eta(G_\eta)$; see Definition 1.3.3.1). Recall that our convention is to view G_η as a left $\mathcal{O}$-module (see Remark 1.3.3.3). By Proposition 3.3.1.5, the restriction $\mathrm{End}_S(G) \to \mathrm{End}_\eta(G_\eta)$ is an isomorphism under the noetherian normality assumption on the base scheme S. Therefore, i_η extends (uniquely) to a ring homomorphism $i : \mathcal{O} \to \mathrm{End}_S(G)$.

By functoriality of $\mathrm{M}(R, I)$ in Theorem 4.4.16 (with its quasi-inverse $\mathrm{F}(R, I)$ given in Corollary 4.5.5.5), the $\mathcal{O}$-endomorphism structure $i : \mathcal{O} \to \mathrm{End}_\eta(G_\eta)$ of $(G_\eta, \lambda_\eta) \to \eta$ corresponds to the following data on the tuple $(A, \lambda_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$:

1. A ring homomorphism $i_A : \mathcal{O} \to \mathrm{End}_S(A)$ compatible with λ_A in the sense that it satisfies the *Rosati condition* $i_A(b)^\vee \circ \lambda_A = \lambda_A \circ i_A(b^\star)$ for all $b \in \mathcal{O}$. This defines an $\mathcal{O}$-endomorphism structure of $(A, \lambda_A) \to S$ (with image in $\mathrm{End}_S(A)$).

2. A ring homomorphism $i_T : \mathcal{O} \to \mathrm{End}_S(T)$ giving a left $\mathcal{O}$-module structure of T, which is equivalent to a ring homomorphism $i_{\underline{X}}^{\mathrm{op}} : \mathcal{O}^{\mathrm{op}} \to \mathrm{End}_S(\underline{X})$ giving a right $\mathcal{O}$-module structure of $\underline{X}$, and a ring homomorphism $i_{T^\vee}^{\mathrm{op}} : \mathcal{O}^{\mathrm{op}} \to \mathrm{End}_S(T^\vee)$ giving a right $\mathcal{O}$-module structure of $T^\vee$, which is equivalent to a ring homomorphism $i_{\underline{Y}} : \mathcal{O} \to \mathrm{End}_S(\underline{Y})$ giving a left $\mathcal{O}$-module structure of $\underline{Y}$.

   Note that the $\mathcal{O}$-module structures above satisfy the compatibility ${}^t i_{\underline{X}}^{\mathrm{op}}(b) = i_T(b)$ (resp. ${}^t i_{\underline{Y}}(b) = i_{T^\vee}^{\mathrm{op}}(b)$) under the transposition defined by the canonical pairing between $\underline{X}$ and T (resp. $\underline{Y}$ and $T^\vee$). Hence the two $\mathcal{O}$-module structures on $\underline{Y}$ and $\underline{X}$ make ϕ an *antilinear* $\mathcal{O}$-module morphism, in the sense that $i_{\underline{X}}(b^\star) \circ \phi = \phi \circ i_{\underline{Y}}(b)$ for all $b \in \mathcal{O}$.

   If we view $\underline{X}$ (resp. $T^\vee$) as a left $\mathcal{O}$-module via the ring homomorphism $i_{\underline{X}} : \mathcal{O} \to \mathrm{End}_S(\underline{X})$ (resp. $i_{T^\vee} : \mathcal{O} \to \mathrm{End}_S(T^\vee)$) defined by composing the natural anti-isomorphism $\mathcal{O} \to \mathcal{O}^{\mathrm{op}} : b \mapsto b^\star$ with $i_{\underline{X}}^{\mathrm{op}}$ (resp. $i_{T^\vee}^{\mathrm{op}}$), then we can view $\phi : \underline{Y} \hookrightarrow \underline{X}$ as an $\mathcal{O}$-*equivariant* morphism between *left* $\mathcal{O}$-modules. We will adopt this convention whenever possible.

3. The $\mathcal{O}$-equivariances of $c : \underline{X} \to A^\vee$ and $c^\vee : \underline{Y} \to A$. Here we endow $A^\vee$ with a left $\mathcal{O}$-module structure by $i_{A^\vee} : \mathcal{O} \to \mathrm{End}_S(A^\vee)$ defined by $i_{A^\vee}(b) := i_A(b^\star)^\vee$ for every $b \in \mathcal{O}$. Alternatively, we may define a natural right $\mathcal{O}$-module structure $i_{A^\vee}^{\mathrm{op}} : \mathcal{O}^{\mathrm{op}} \to \mathrm{End}_S(A^\vee)$ on $A^\vee$ by $i_{A^\vee}^{\mathrm{op}}(b) := i_A(b)^\vee$, and then set $i_{A^\vee}$ to be the composition of the natural anti-isomorphism $\mathcal{O} \to \mathcal{O}^{\mathrm{op}}$ with $i_{A^\vee}^{\mathrm{op}}$.

   (The data so far, together with the compatibility $\lambda_A c^\vee = c\phi$, correspond to a ring homomorphism $i^\natural : \mathcal{O} \to \mathrm{End}_S(G^\natural)$ compatible with $\lambda^\natural : G^\natural \to G^{\vee,\natural}$ in the sense that $i^\natural(b)^\vee \circ \lambda^\natural = \lambda^\natural \circ i^\natural(b^\star)$ for all $b \in \mathcal{O}$.)

4. The $\mathcal{O}$-equivariance of the *period homomorphism* $\iota : \underline{Y} \to G_\eta^\natural$. This is equivalent to the condition for the trivialization $\tau : \mathbf{1}_{\underline{Y} \underset{S}{\times} \underline{X}, \eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_{A,\eta}^{\otimes -1}$ of biextensions that
$$(i_{\underline{Y}}(b) \times \mathrm{Id}_{\underline{X}})^* \tau = (\mathrm{Id}_{\underline{Y}} \times i_{\underline{X}}^{\mathrm{op}}(b))^* \tau = (\mathrm{Id}_{\underline{Y}} \times i_{\underline{X}}(b^\star))^* \tau$$
for all $b \in \mathcal{O}$ (cf. the proofs of Lemma 4.3.4.3 and Corollary 4.5.5.5), which makes sense because $(f \times \mathrm{Id}_{A^\vee})^* \mathcal{P}_A \cong (\mathrm{Id}_A \times f^\vee)^* \mathcal{P}_A$ for all $f \in \mathrm{End}_S(A)$ (by Lemma 1.3.2.10), and because c and $c^\vee$ are $\mathcal{O}$-equivariant.

Lemma 5.1.1.1. *Over a finite étale covering of S trivializing the étale sheaves $\underline{X}$ and $\underline{Y}$, their respective values X and Y are $\mathcal{O}$-lattices with their* (*left*) *$\mathcal{O}$-module structure.*

Proof. It suffices to know that X and Y are $\mathbb{Z}$-lattices (by Definition 1.1.1.22), which is the case because T and $T^\vee$ are tori (by Definition 3.1.1.5). □

Definition 5.1.1.2. *With assumptions as in Section 4.1, the category $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}(R,I)$ has objects of the form (G,λ,i) (over S), where the pair (G,λ) defines an object in $\mathrm{DEG}_{\mathrm{pol}}(R,I)$, and where $i:\mathcal{O}\to\mathrm{End}_S(G)$ defines by restriction an $\mathcal{O}$-structure $i_\eta:\mathcal{O}\to\mathrm{End}_\eta(G_\eta)$ of (G_η,λ_η). (By Proposition 3.3.1.5, the restriction homomorphism $\mathrm{End}_S(G)\to\mathrm{End}_\eta(G_\eta)$ is an isomorphism.)*

Definition 5.1.1.3. *With assumptions as in Section 4.1, the category $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}(R,I)$ has objects of the form $(A,\lambda_A,i_A,\underline{X},\underline{Y},\phi,c,c^\vee,\tau)$, with $(A,\lambda_A,\underline{X},\underline{Y},\phi,c,c^\vee,\tau)$ defining an object in $\mathrm{DD}_{\mathrm{pol}}(R,I)$, and with additional $\mathcal{O}$-structure compatibilities described as follows:*

1. *$i_A:\mathcal{O}\to\mathrm{End}_S(A)$ defines an $\mathcal{O}$-structure for (A,λ_A).*
2. *The étale sheaves $\underline{X}$ and $\underline{Y}$ are equipped with ring homomorphisms $i_{\underline{X}}:\mathcal{O}\to\mathrm{End}_S(\underline{X})$ and $i_{\underline{Y}}:\mathcal{O}\to\mathrm{End}_S(\underline{Y})$, respectively, making them étale sheaves of $\mathcal{O}$-lattices of the same $\mathcal{O}$-multirank (see Definition 1.2.1.21). The embedding $\phi:\underline{Y}\hookrightarrow\underline{X}$ is $\mathcal{O}$-equivariant with respect to these $\mathcal{O}$-module structures.*
3. *The homomorphisms $c:\underline{X}\to A^\vee$ and $c^\vee:\underline{Y}\to A$ are both $\mathcal{O}$-equivariant as morphisms between $\mathcal{O}$-modules.*
4. *The trivialization $\tau:\mathbf{1}_{\underline{Y}\underset{S}{\times}\underline{X},\eta}\xrightarrow{\sim}(c^\vee\times c)^*\mathcal{P}_{A,\eta}^{\otimes -1}$ satisfies the compatibility $(i_{\underline{Y}}(b)\times\mathrm{Id}_{\underline{X}})^*\tau=(\mathrm{Id}_{\underline{Y}}\times i_{\underline{X}}(b^\star))^*\tau$ for all $b\in\mathcal{O}$, which gives rise to the $\mathcal{O}$-equivariance of the **period homomorphism** $\iota:\underline{Y}\to G_\eta^\natural$.*

Then our result in this section can be summarized as follows:

Theorem 5.1.1.4. *There is an equivalence of categories*

$$\mathrm{M}_{\mathrm{PE},\mathcal{O}}(R,I):\mathrm{DD}_{\mathrm{PE},\mathcal{O}}(R,I)\to\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}(R,I):$$
$$(A,\lambda_A,i_A,\underline{X},\underline{Y},\phi,c,c^\vee,\tau)\mapsto(G,\lambda,i).$$

5.1.2 Data for Lie Algebra Conditions

In this section we continue with the setting in Section 5.1.1. Although it might be natural to treat the Lie algebra conditions as part of the endomorphism structures, they should be viewed as the de Rham analogue of the level structures defined by Tate modules.

Let V_0 and V_0^c be the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-module defined by $(L,\langle\cdot,\cdot\rangle,h)$ as in (1.2.5.1), with signatures $(p_\tau)_\tau$ and $(q_\tau)_\tau$ (see Definition 1.2.5.2), respectively, satisfying $p_\tau=q_{\tau\circ c}$.

Lemma 5.1.2.1. *With assumptions as above, let W be an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$-module such that $W\underset{\mathbb{R}}{\otimes}\mathbb{C}$ has $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-multirank $(r_\tau)_\tau$ (see Definition 1.2.1.25 and Lemma 1.2.1.31). Then there is a totally isotropic embedding $W\underset{\mathbb{Z}}{\otimes}\mathbb{R}\hookrightarrow L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$-modules if we have $p_\tau\geq r_\tau$ and $q_\tau\geq r_\tau$ for all $\tau:F\to\mathbb{C}$.*

Proof. For each homomorphism $\tau:F\to\mathbb{C}$, let W_τ be the unique simple $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-module on which F acts by τ. Let $(m_\tau)_\tau$ be the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-multirank of $L\underset{\mathbb{Z}}{\otimes}\mathbb{C}$. Then we have decompositions $L\underset{\mathbb{Z}}{\otimes}\mathbb{C}\cong\underset{\tau:F\to\mathbb{C}}{\oplus}W_\tau^{\oplus m_\tau}$, $V_0\cong\underset{\tau:F\to\mathbb{C}}{\oplus}W_\tau^{\oplus p_\tau}$, $V_0^c\cong\underset{\tau:F\to\mathbb{C}}{\oplus}W_\tau^{\oplus q_\tau}$, and $W\cong\underset{\tau:F\to\mathbb{C}}{\oplus}W_\tau^{\oplus r_\tau}$ of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-modules.

We say that $\tau:F\to\mathbb{C}$ is real if $\tau(F)\subset\mathbb{R}$, and complex otherwise. Let $W_{[\tau]_c}:=W_\tau$ if τ is real, and let $W_{[\tau]_c}:=W_\tau\oplus W_{\tau\circ c}$ if τ is complex. Then we have a decomposition $L\underset{\mathbb{Z}}{\otimes}\mathbb{C}\cong\underset{[\tau]_c}{\oplus}W_{[\tau]_c}^{\oplus m_{[\tau]_c}}$, where $[\tau]_c$ denotes a $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-orbit of morphisms $F\to\mathbb{C}$, and $m_{[\tau]_c}=m_\tau$ for every representative τ of $[\tau]_c$. Similarly, we have $W\underset{\mathbb{R}}{\otimes}\mathbb{C}\cong\underset{[\tau]_c}{\oplus}W_{[\tau]_c}^{\oplus r_{[\tau]_c}}$, where $r_{[\tau]_c}=r_\tau$ for every representative τ of $[\tau]_c$.

If $p_\tau\geq r_\tau$ and $q_\tau=p_{\tau\circ c}\geq r_\tau$ for every $\tau:F\to\mathbb{C}$, then by comparison of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-multiranks there exists an embedding $\varepsilon_1:W\underset{\mathbb{R}}{\otimes}\mathbb{C}\hookrightarrow V_0$ of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-modules. Note that V_0 carries not only the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-action, but also the $\mathbb{C}$-action induced by $h:\mathbb{C}\to\mathrm{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})$. By taking the complex conjugation with respect to the $\mathbb{C}$-action induced by h (while keeping the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-action the same), we obtain a second embedding $\varepsilon_2:W\underset{\mathbb{R}}{\otimes}\mathbb{C}\hookrightarrow V_0^c$ of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-modules. Then we define $\varepsilon:W\underset{\mathbb{R}}{\otimes}\mathbb{C}\hookrightarrow L\underset{\mathbb{Z}}{\otimes}\mathbb{C}\cong V_0\oplus V_0^c$ by $x\mapsto\varepsilon(x)=\varepsilon_1(x)+\varepsilon_2(x)$ using the decomposition (1.2.5.1). Since $h(z)$ acts by $1\otimes z$ (resp. by $1\otimes z^c$) on V_0 (resp. V_0^c) by definition, the complex conjugation with respect to the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-action also interchanges ε_1 and ε_2. Thus ε induces an embedding $W\underset{\mathbb{Z}}{\otimes}\mathbb{R}\hookrightarrow L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$-modules.

By definition of h (see Definition 1.2.1.2), for each $x,y\in W\underset{\mathbb{R}}{\otimes}\mathbb{C}$, we have the symmetry

$$\langle\varepsilon(x),h(\sqrt{-1})\varepsilon(y)\rangle=\langle\varepsilon(y),h(\sqrt{-1})\varepsilon(x)\rangle.$$

By definition of V_0 and V_0^c (see (1.2.5.1)), we have

$$\begin{aligned}\langle\varepsilon(x),h(\sqrt{-1})\varepsilon(y)\rangle&=\langle\varepsilon_1(x)+\varepsilon_2(x),\varepsilon_1((1\otimes\sqrt{-1})y)+\varepsilon_2(-(1\otimes\sqrt{-1})y)\rangle\\&=\sqrt{-1}(-\langle\varepsilon_1(x),\varepsilon_2(y)\rangle+\langle\varepsilon_1(y),\varepsilon_2(x)\rangle)\end{aligned}$$

and hence the antisymmetry

$$\langle\varepsilon(x),h(\sqrt{-1})\varepsilon(y)\rangle=-\langle\varepsilon(y),h(\sqrt{-1})\varepsilon(x)\rangle.$$

These show that they must be all zero, and hence ε defines a totally isotropic embedding, as desired. ∎

Proposition 5.1.2.2. *With assumptions as above, let X be an $\mathcal{O}$-lattice such that $X\underset{\mathbb{Z}}{\otimes}\mathbb{C}$ has $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-multirank $(r_\tau)_\tau$. Suppose there exists a totally isotropic embedding $\mathrm{Hom}_{\mathbb{R}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{R},\mathbb{R}(1))\hookrightarrow L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$-modules, where X is the $\mathcal{O}$-lattice given by the value of $\underline{X}$ over some geometric point over η. Let us denote the image of this embedding by $\mathsf{Z}_{-2,\mathbb{R}}$, and denote its annihilator by $\mathsf{Z}_{-1,\mathbb{R}}$. Let $\mathsf{Z}_{0,\mathbb{R}}$ be $L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ and $\mathsf{Z}_{-3,\mathbb{R}}$ be 0. Then we have a symplectic admissible filtration $\mathsf{Z}_{\mathbb{R}}:=\{\mathsf{Z}_{-i,\mathbb{R}}\}$ on $L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ (by construction). The pairing $\langle\cdot,\cdot\rangle$ and the polarization h induce a pairing $\langle\cdot,\cdot\rangle_{11,\mathbb{R}}$ and a polarization h_{-1} on $\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{Z}}:=\mathsf{Z}_{-1,\mathbb{R}}/\mathsf{Z}_{-2,\mathbb{R}}$. The isomorphism class of the polarized symplectic $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$-module $(\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{Z}},\langle\cdot,\cdot\rangle_{11,\mathbb{R}},h_{-1})$ is independent of the choice of the embedding $\mathrm{Hom}_{\mathbb{R}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{R},\mathbb{R}(1))\hookrightarrow L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$, and has the same reflex field F_0 (see Definition 1.2.5.4) as $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle,h)$ does. Moreover, we have $p_\tau\geq r_\tau$ and $q_\tau\geq r_\tau$ for all $\tau:F\to\mathbb{C}$, and the signatures of $(\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{Z}},\langle\cdot,\cdot\rangle_{11,\mathbb{R}},h_{-1})$ are $(p_\tau-r_\tau,q_\tau-r_\tau)_\tau$.*

Proof. Note that h is determined by $h(\sqrt{-1})$, and the positive definiteness of $\frac{1}{2\pi\sqrt{-1}} \circ \langle\,\cdot\,, h(\sqrt{-1})\,\cdot\,\rangle$ (see Definition 1.2.1.2) implies (essentially by $\mathbb{R}$-dimension counting) that the composition of canonical morphisms $h(\sqrt{-1})(\mathsf{Z}_{-2,\mathbb{R}}) \hookrightarrow L \underset{\mathbb{Z}}{\otimes} \mathbb{R} \twoheadrightarrow \mathrm{Gr}^{\mathsf{Z}}_{0,\mathbb{R}} = \mathsf{Z}_{0,\mathbb{R}}/\mathsf{Z}_{-1,\mathbb{R}}$ (of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-modules) is an isomorphism. In particular, $\mathsf{Z}_{-2,\mathbb{R}} \cap h(\sqrt{-1})(\mathsf{Z}_{-2,\mathbb{R}}) = \{0\}$. If we set $\mathsf{Z}_{-2,h(\mathbb{C})} := \mathsf{Z}_{-2,\mathbb{R}} \oplus h(\sqrt{-1})(\mathsf{Z}_{-2,\mathbb{R}})$, then $\mathsf{Z}_{-2,h(\mathbb{C})}$ embeds as an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-submodule of $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, and taking the annihilator of $\mathsf{Z}_{-2,h(\mathbb{C})}$ under $\langle\,\cdot\,,\,\cdot\,\rangle$ defines an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-equivariant orthogonal direct sum

$$(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle) \cong (\mathsf{Z}_{-2,h(\mathbb{C})}, \langle\,\cdot\,,\,\cdot\,\rangle|_{\mathsf{Z}_{-2,h(\mathbb{C})}}) \overset{\perp}{\oplus} (\mathsf{Z}^{\perp}_{-2,h(\mathbb{C})}, \langle\,\cdot\,,\,\cdot\,\rangle|_{\mathsf{Z}^{\perp}_{-2,h(\mathbb{C})}}) \quad (5.1.2.3)$$

respected by the action of $\mathbb{C}$ under h. Since $\mathsf{Z}^{\perp}_{-2,h(\mathbb{C})} \subset \mathsf{Z}^{\perp}_{-2,\mathbb{R}} = \mathsf{Z}_{-1,\mathbb{R}}$, we obtain a morphism $\mathsf{Z}^{\perp}_{-2,h(\mathbb{C})} \to \mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}} = \mathsf{Z}_{-1,\mathbb{R}}/\mathsf{Z}_{-2,\mathbb{R}}$, which (again, by $\mathbb{R}$-dimension counting) underlies a symplectic isomorphism

$$(\mathsf{Z}^{\perp}_{-2,h(\mathbb{C})}, \langle\,\cdot\,,\,\cdot\,\rangle|_{\mathsf{Z}^{\perp}_{-2,h(\mathbb{C})}}) \xrightarrow{\sim} (\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}}).$$

Then the restriction of h to $\mathsf{Z}^{\perp}_{-2,h(\mathbb{C})}$ induces the desired polarization h_{-1} of $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}})$. Since $(\mathsf{Z}_{-2,h(\mathbb{C})}, \langle\,\cdot\,,\,\cdot\,\rangle|_{\mathsf{Z}_{-2,h(\mathbb{C})}}, h|_{\mathsf{Z}_{-2,h(\mathbb{C})}})$ has signatures $(r_\tau, r_\tau)_\tau$ and reflex field $\mathbb{Q}$ (essentially by definition), the decomposition (5.1.2.3) shows that $p_\tau \geq r_\tau$ and $q_\tau \geq r_\tau$ for all $\tau : F \to \mathbb{C}$, that the signatures of $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}}, h_{-1})$ are $(p_\tau - r_\tau, q_\tau - r_\tau)_\tau$, and that $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}}, h_{-1})$ has the same reflex field F_0 (see Definition 1.2.5.4) as $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ does.

The choice of the embedding $\mathrm{Hom}_{\mathbb{R}}(X \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \mathbb{R}(1)) \hookrightarrow L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ is immaterial, because every two choices of the embedding will determine orthogonal direct sums as in (5.1.2.3) whose right-hand sides are isomorphic symplectic $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-modules by comparison of signatures. Then they can be mapped to each other by some element of $\mathrm{G}(\mathbb{R})$ acting on the left-hand side $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle)$, matching the data $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}}, h_{-1})$ on the middle graded pieces. □

Now let us assume the setting of Section 4.1, and assume moreover that the generic point η of the base scheme S is defined over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, where F_0 is defined by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ as in Definition 1.2.5.4. (We do not need the whole scheme S to be defined over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$.)

Proposition 5.1.2.4. *With assumptions as above, suppose (G, λ, i) is an object in $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}(R, I)$, with associated degeneration datum $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}(R, I)$. Then, for $(G_\eta, \lambda_\eta, i_\eta)$ to satisfy the Lie algebra condition defined by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ (see Definition 1.3.4.1), it is necessary that there exists a totally isotropic embedding $\mathrm{Hom}_{\mathbb{R}}(X \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \mathbb{R}(1)) \hookrightarrow L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-modules, where X is the $\mathcal{O}$-lattice given by the value of $\underline{X}$ over some geometric point over η. Moreover, suppose that such a totally isotropic embedding does exist. Then, for $(G_\eta, \lambda_\eta, i_\eta)$ to satisfy the Lie algebra condition defined by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$, it is both necessary and sufficient that $(A_\eta, \lambda_{A,\eta}, i_{A,\eta})$ satisfies the Lie algebra condition defined by $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}}, h_{-1})$.*

Proof. Let $(r_\tau)_\tau$ be the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-multirank of $X \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ (see Definition 1.2.1.21). By Lemma 5.1.2.1, to show that there is a totally isotropic embedding $\mathrm{Hom}_{\mathbb{R}}(X \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \mathbb{R}(1)) \hookrightarrow L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-modules, we have to show $p_\tau \geq r_\tau$ and $q_\tau \geq r_\tau$ for all $\tau : F \to \mathbb{C}$.

By assumption above that $\eta = \mathrm{Spec}(K)$ is defined over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, the characteristic p of K is unramified in F. Suppose the morphism $\eta \to \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ factors through $\eta \to \mathrm{Spec}(k) \to \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ for some field k, which we assume to be finite when $p > 0$. Let K^{sep} denote a separable closure of K, and let k^{sep} denote the separable closure of k in K^{sep}. According to Lemma 1.1.3.4, finitely generated projective $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \Lambda$-modules M admit decompositions $M \cong M_\tau^{\oplus m_\tau}$. If we tensor any such decomposition with k^{sep}, then we obtain a decomposition compatible with the classification of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$-modules as in Proposition 1.1.2.20. On the other hand, if we tensor any such decomposition with a separable closure $\mathrm{Frac}(\Lambda)^{\mathrm{sep}}$ of $\mathrm{Frac}(\Lambda)$, we obtain a decomposition compatible with the classification of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules if we embed $\mathrm{Frac}(\Lambda)^{\mathrm{sep}}$ into $\mathbb{C}$. When $p > 0$, this can be achieved by choosing an auxiliary isomorphism $\mathbb{C} \cong \mathbb{C}_p$, which has the effect of matching $\mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{C})$ and $\mathrm{Hom}_{\mathbb{F}_p}(\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} \mathbb{F}_p, \bar{\mathbb{F}}_p)$ because p is unramified in F. Then we can talk about the signatures of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} k^{\mathrm{sep}}$-modules and $\mathcal{O} \underset{\mathbb{Z}}{\otimes} K^{\mathrm{sep}}$-modules as if they were $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules.

Let F_0' and L_0 be chosen as in Lemma 1.2.5.9, with F_0' unramified at p when $p > 0$. Suppose that $\mathrm{Lie}_{G_\eta/\eta}$ satisfies the Lie algebra condition defined by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$. Then we have an isomorphism $\mathrm{Lie}_{G_\eta/\eta} \underset{K}{\otimes} K^{\mathrm{sep}} \cong L_0 \underset{\mathcal{O}_{F_0'}}{\otimes} K^{\mathrm{sep}}$ of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} K^{\mathrm{sep}}$-modules by Proposition 1.1.2.20, and hence $\mathrm{Lie}_{G_\eta/\eta} \underset{K}{\otimes} K^{\mathrm{sep}}$ has the same signatures $(p_\tau)_\tau$ as V_0 by the above identification.

The (separable) polarization $\lambda_\eta : G_\eta \to G_\eta^\vee$ defines an $\mathcal{O}$-equivariant isomorphism $d\lambda_\eta : \mathrm{Lie}_{G_\eta/\eta} \xrightarrow{\sim} \mathrm{Lie}_{G_\eta^\vee/\eta}$. The action of $\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} K$ on $\mathrm{Lie}_{G_\eta/\eta}$ and on $\mathrm{Lie}^\vee_{G_\eta^\vee/\eta}$ differ by the restriction of the involution $\star$ to $\mathcal{O}_F$ (which is compatible with the complex conjugation under any homomorphism $\mathcal{O}_F \hookrightarrow F \xrightarrow{\tau} \mathbb{C}$). Therefore, if $\mathrm{Lie}_{G_\eta/\eta} \underset{K}{\otimes} K^{\mathrm{sep}}$ has signatures $(p_\tau)_\tau$, then $\mathrm{Lie}^\vee_{G_\eta^\vee/\eta} \underset{K}{\otimes} K^{\mathrm{sep}}$ has signatures $(q_\tau)_\tau$ because $q_\tau = p_{\tau\circ c}$.

Let X and Y denote the values of $\underline{X}$ and $\underline{Y}$ over the geometric point $\bar{\eta} := \mathrm{Spec}(K^{\mathrm{sep}})$ over η. (This redefines X up to an isomorphism of $\mathcal{O}$-modules, but doing so is harmless for our purpose.) Since there is an $\mathcal{O}$-equivariant embedding $\phi : Y \hookrightarrow X$ of finite index, $X \underset{\mathbb{Z}}{\otimes} \mathbb{R} \cong Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ as $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-modules. By definition, we have canonical isomorphisms $T_{\bar{\eta}} \cong \mathrm{Hom}_{\bar{\eta}}(X, \mathbf{G}_{\mathrm{m},\bar{\eta}})$ and $T^\vee_{\bar{\eta}} \cong \mathrm{Hom}_{\bar{\eta}}(Y, \mathbf{G}_{\mathrm{m},\bar{\eta}})$ with compatible $\mathcal{O}$-actions, and hence there are canonical isomorphisms $\mathrm{Lie}_{T_{\bar{\eta}}/\bar{\eta}} \cong \mathrm{Hom}(X, K^{\mathrm{sep}})$ and $\mathrm{Lie}_{T^\vee_{\bar{\eta}}/\bar{\eta}} \cong \mathrm{Hom}(Y, K^{\mathrm{sep}})$ as $\mathcal{O} \underset{\mathbb{Z}}{\otimes} K^{\mathrm{sep}}$-modules. As a result, the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} K^{\mathrm{sep}}$-modules $\mathrm{Lie}_{T_{\bar{\eta}}/\bar{\eta}} \cong \mathrm{Lie}_{T_\eta/\eta} \underset{K}{\otimes} K^{\mathrm{sep}}$, $\mathrm{Lie}_{T^\vee_{\bar{\eta}}/\bar{\eta}} \cong \mathrm{Lie}_{T^\vee_\eta/\eta} \underset{K}{\otimes} K^{\mathrm{sep}}$, and hence $\mathrm{Lie}^\vee_{T^\vee_\eta/\eta} \underset{K}{\otimes} K^{\mathrm{sep}}$ all have the same signatures $(r_\tau)_\tau$ as $X \underset{\mathbb{Z}}{\otimes} \mathbb{C}$.

By Theorem 2.3.1.2, we have a canonical isomorphism $\underline{\mathrm{Lie}}_{G/S} \cong \underline{\mathrm{Lie}}_{G^\natural/S}$, because $G_{\mathrm{for}} \cong G^\natural_{\mathrm{for}}$ and hence $\underline{\mathrm{Lie}}_{G_{\mathrm{for}}/S_{\mathrm{for}}} \cong \underline{\mathrm{Lie}}_{G^\natural_{\mathrm{for}}/S_{\mathrm{for}}}$ over S_{for}. Similarly, we have a canonical isomorphism $\underline{\mathrm{Lie}}_{G^\vee/S} \cong \underline{\mathrm{Lie}}_{G^{\vee,\natural}/S}$. Both of the canonical isomorphisms are $\mathcal{O}\underset{\mathbb{Z}}{\otimes}K$-equivariant, by the functoriality in Theorem 2.3.1.2. Then $p_\tau \geq r_\tau$ (resp. $q_\tau \geq r_\tau$) for all $\tau : F \to \mathbb{C}$ because $\mathrm{Lie}_{T_\eta/\eta}$ (resp. $\mathrm{Lie}^\vee_{T^\vee_\eta/\eta}$) is a $\mathcal{O}\underset{\mathbb{Z}}{\otimes}K$-subquotient of $\mathrm{Lie}_{G^\natural_\eta/\eta}$ (resp. $\mathrm{Lie}^\vee_{G^{\vee,\natural}_\eta/\eta}$).

Since $\mathrm{Lie}_{T_\eta/\eta}\underset{K}{\otimes}K^{\mathrm{sep}} \cong \mathrm{Hom}(X, K^{\mathrm{sep}})$ as $\mathcal{O}\underset{\mathbb{Z}}{\otimes}K^{\mathrm{sep}}$-modules, $\mathrm{Lie}_{T_\eta/\eta}$ satisfies the Lie algebra condition defined by $(\mathsf{Z}_{-2,h(\mathbb{C})}, \langle\,\cdot\,,\,\cdot\,\rangle|_{\mathsf{Z}_{-2,h(\mathbb{C})}}, h|_{\mathsf{Z}_{-2,h(\mathbb{C})}})$ in the sense that $\mathrm{Det}_{\mathcal{O}|\underline{\mathrm{Lie}}_{T_\eta/\eta}}$ agrees with the image of $\mathrm{Det}_{\mathcal{O}|\mathsf{Z}_{-2,\mathbb{R}}} = \mathrm{Det}_{\mathcal{O}|\mathrm{Hom}_{\mathbb{R}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{R},\mathbb{R}(1))} = \mathrm{Det}_{\mathcal{O}|\mathrm{Hom}(X,\mathbb{Z}_{(\Box)})}$ under the structural homomorphism $\mathbb{Z}_{(\Box)} \to K$ (cf. Definition 1.3.4.1). Then the exact sequence $0 \to \mathrm{Lie}_{T_\eta/\eta} \to \mathrm{Lie}_{G^\natural_\eta/\eta} \to \mathrm{Lie}_{A_\eta/\eta} \to 0$ of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}K$-modules and the orthogonal direct sum (5.1.2.3) show that $(G_\eta, \lambda_\eta, i_\eta)$ satisfies the Lie algebra condition defined by $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ if and only if $(A_\eta, \lambda_{A,\eta}, i_{A,\eta})$ satisfies the Lie algebra condition defined by $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}}, h_{-1})$, as desired. $\Box$

Definition 5.1.2.5. *With assumptions as in the paragraph preceding Proposition 5.1.2.4, the category* $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle,h)}(R, I)$ *has objects of the form* (G, λ, i) *(over* $S = \mathrm{Spec}(R)$*), each defining an object in* $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}(R, I)$ *such that* $(G_\eta, \lambda_\eta, i_\eta)$ *satisfies the Lie algebra condition defined by* $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ *(see Definition 1.3.4.1).*

Definition 5.1.2.6. *With assumptions as in the paragraph preceding Proposition 5.1.2.4, the category* $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle,h)}(R, I)$ *has objects of the form* $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$*, each tuple defining an object in* $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}(R, I)$*, such that there exists a totally isotropic embedding* $\mathrm{Hom}_{\mathbb{R}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \mathbb{R}(1)) \hookrightarrow L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ *of* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$*-modules, where* X *is the value of* $\underline{X}$ *over some geometric point over* η*, and such that* $(A_\eta, \lambda_{A,\eta}, i_{A,\eta})$ *satisfies the Lie algebra condition defined by* $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}}, h_{-1})$ *(see Definition 1.3.4.1 and Proposition 5.1.2.4).*

Then Theorem 5.1.1.4 can be strengthened as follows:

Theorem 5.1.2.7. *There is an equivalence of categories*

$$\mathrm{M}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle,h)}(R, I) :$$
$$\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle,h)}(R, I) \to \mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle,h)}(R, I) :$$
$$(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau) \mapsto (G, \lambda, i).$$

5.2 Data for Principal Level Structures

5.2.1 The Setting for This Section

Let G be defined by $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ as in Definition 1.2.1.6, and let $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^\Box)$ be an open compact subgroup. Let the moduli problem $\mathsf{M}_{\mathcal{H}}$ be defined over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ as in Definition 1.4.1.4.

For technical reasons regarding the existence of splittings of filtrations, we shall assume that L satisfies Condition 1.4.3.10 (see Lemma 5.2.2.4 below). Practically, this means we might have to replace L with a larger lattice L' such that the action of $\mathcal{O}$ extends to a maximal order $\mathcal{O}'$ containing $\mathcal{O}$. By Corollary 1.4.3.8, although this assumption does impose a restriction on the order $\mathcal{O}$ and the $\mathcal{O}$-lattice L that we could work with, it does not affect our purpose of studying and compactifying the moduli problem $\mathsf{M}_{\mathcal{H}}$ if $\mathcal{H}$ can still be chosen to be contained in $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ under this assumption (see Remark 1.4.3.9).

With the same setting as in Section 4.1, assume moreover that the generic point $\eta = \mathrm{Spec}(K)$ of the base scheme $S = \mathrm{Spec}(R)$ is defined over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. (We do not need the whole scheme S to be defined over $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$.) Fix a choice of a geometric point $\bar{\eta} = \mathrm{Spec}(\bar{K})$ over η. Let $K \hookrightarrow \tilde{K}$ be any finite separable subextension of $K \hookrightarrow \bar{K}$ defining a finite étale morphism $\tilde{\eta} = \mathrm{Spec}(\tilde{K}) \to \eta = \mathrm{Spec}(K)$. In this case, the inclusion $\tilde{K} \hookrightarrow \bar{K}$ allows us to lift $\bar{\eta}$ canonically to a geometric point over $\tilde{\eta}$. Recall that we have the following:

Lemma 5.2.1.1. *Let* R_1 *be any noetherian normal domain with field of fractions* K_1*. Suppose* K_2 *is a finite separable extension of* K_1*, and let* R_2 *be the integral closure of* R_1 *in* K_2*. Then* R_2 *is a finite* R_1*-module. In particular,* R_2 *is again noetherian.*

This convenient fact can be found, for example, in [88, §33, Lem. 1] or [41, Prop. 13.14]. (For general extensions $K_1 \hookrightarrow K_2$, it may not be true that R_2 is noetherian. Nevertheless, it is true if R_1 is excellent or, more generally, Nagata. See [87, §§31–34] for discussions on this.)

5.2.2 Analysis of Principal Level Structures

In this section we study the construction of (principal) level-n structures using the theory of degeneration, assuming the theory in Section 5.1.2 for Lie algebra conditions.

With the setting as in Section 5.2.1, consider any triple (G, λ, i) defining an object in $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle,h)}(R, I)$. By Theorem 5.1.2.7, we have an associated degeneration datum $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle,h)}(R, I)$. Assume for simplicity that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively. Then X and Y are $\mathcal{O}$-lattices of the same $\mathcal{O}$-multirank (see Definition 1.2.1.21), as we saw in Lemma 5.1.1.1. Let $\phi : Y \hookrightarrow X$ be the $\mathcal{O}$-equivariant embedding defined by λ. We know that the homomorphism $\lambda : G \to G^\vee$ induces a homomorphism between Raynaud extensions $\lambda^\natural : G^\natural \to G^{\vee,\natural}$, and induces a polarization $\lambda_A : A \to A^\vee$ (of the abelian part) because λ extends a polarization $\lambda_\eta : G_\eta \to G^\vee_\eta$.

By Corollary 4.5.3.12, with the general theorem of orthogonality in mind (see, for example, [57, IX, 2.4] and [93, IV, 2.4], or Theorem 3.4.2.4), the structure of $G[n]_\eta$ can be described as follows:

Proposition 5.2.2.1. *With the setting as above, we have a canonical* $\mathcal{O}$*-equivariant exact sequence of finite étale group schemes*

$$0 \to G^\natural[n]_\eta \to G[n]_\eta \to \tfrac{1}{n}Y/Y \to 0$$

over $\mathrm{Spec}(R)$*, which induces, by taking the limit over* n *with* $\Box \nmid n$*, an exact sequence*

$$0 \to \mathrm{T}^\Box\, G^\natural_{\bar{\eta}} \to \mathrm{T}^\Box\, G_{\bar{\eta}} \to Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box \to 0$$

of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$*-modules. Moreover, we have a canonical* $\mathcal{O}$*-equivariant exact sequence of finite étale group schemes*

$$0 \to T[n] \to G^\natural[n] \to A[n] \to 0$$

over $\mathrm{Spec}(R)$*, which induces, by taking the limit over* n *with* $\Box \nmid n$*, an exact sequence*

$$0 \to \mathrm{T}^\Box\, T_{\bar{\eta}} \to \mathrm{T}^\Box\, G^\natural_{\bar{\eta}} \to \mathrm{T}^\Box\, A_{\bar{\eta}} \to 0$$

of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$-modules. Under the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}$, the submodules $\mathrm{T}^\Box T_{\tilde{\eta}}$ and $\mathrm{T}^\Box G^\natural_{\tilde{\eta}}$ of $\mathrm{T}^\Box G_{\tilde{\eta}}$ are identified with the annihilators of each other, which induce the λ_A-Weil pairing e^{λ_A} on $\mathrm{T}^\Box A_{\tilde{\eta}}$, and a pairing

$$\mathrm{e}^\phi : \mathrm{T}^\Box T_{\tilde{\eta}} \times (Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box) \to \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\tilde{\eta}}$$

which is the canonical one

$$\begin{aligned}\mathrm{T}^\Box T_{\tilde{\eta}} \times (Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box) &\overset{\text{can.}}{\underset{\sim}{\to}} \underline{\mathrm{Hom}}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\tilde{\eta}}) \times Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box \\ &\overset{\mathrm{Id}\times\phi}{\to} \underline{\mathrm{Hom}}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\tilde{\eta}}) \times X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box \overset{\text{can.}}{\to} \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\tilde{\eta}}\end{aligned}$$

with the sign convention that $\mathrm{e}^\phi(t,y) = t(\phi(y)) = (\phi(y))(t)$ for all $t \in \mathrm{T}^\Box T_{\tilde{\eta}}$ and $y \in Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$. (This is compatible with the sign convention that we will adopt later in Sections 5.2.4 and 5.2.6.)

Now let $\tilde{\eta} \to \eta$ be any finite étale morphism defined by a field extension as in Section 5.2.1. (The reason to consider such étale localizations of η is for the applicability of the theory to the study of level structures that are not principal, as defined in Section 1.3.7.)

Suppose there is a level-n structure of $(G_{\tilde{\eta}}, \lambda_{\tilde{\eta}}, i_{\tilde{\eta}})$ of type $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle)$ (defined over $\tilde{\eta}$; see Definition 1.3.6.2). By Lemma 1.3.6.5, this means we have an $\mathcal{O}$-equivariant isomorphism $\alpha_n : L/nL \overset{\sim}{\to} G[n]_{\tilde{\eta}}$ that can be lifted (noncanonically) to an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$-equivariant symplectic isomorphism $\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box \overset{\sim}{\to} \mathrm{T}^\Box G_{\tilde{\eta}}$, together with an isomorphism $\nu(\alpha_n) : \hat{\mathbb{Z}}^\Box(1) \overset{\sim}{\to} \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\tilde{\eta}}$ (of $\hat{\mathbb{Z}}^\Box$-modules), which carry the chosen pairing $\langle\cdot,\cdot\rangle$ on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$ to the λ_η-Weil pairing. Then for each choice of the lifting $\hat{\alpha}$, the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$-module filtration

$$0 \subset \mathrm{T}^\Box T_{\tilde{\eta}} \subset \mathrm{T}^\Box G^\natural_{\tilde{\eta}} \subset \mathrm{T}^\Box G_{\tilde{\eta}}$$

on $\mathrm{T}^\Box G[n]_{\tilde{\eta}}$ (described in Proposition 5.2.2.1) induces an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$-module filtration

$$0 \subset \mathtt{Z}_{-2} \subset \mathtt{Z}_{-1} \subset \mathtt{Z}_0 := L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$$

on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$, together with isomorphisms (of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$-modules)

$$\mathrm{Gr}^{\mathtt{Z}}_{-2} := \mathtt{Z}_{-2} \overset{\sim}{\to} \mathrm{T}^\Box T_{\tilde{\eta}},$$
$$\mathrm{Gr}^{\mathtt{Z}}_{-1} := \mathtt{Z}_{-1}/\mathtt{Z}_{-2} \overset{\sim}{\to} \mathrm{T}^\Box A_{\tilde{\eta}},$$
$$\mathrm{Gr}^{\mathtt{Z}}_{0} := \mathtt{Z}_{0}/\mathtt{Z}_{-1} \overset{\sim}{\to} Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$$

on the graded pieces. Here the first isomorphism can be given more precisely by the composition

$$\mathrm{Gr}^{\mathtt{Z}}_{-2} \overset{\sim}{\to} \underline{\mathrm{Hom}}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \hat{\mathbb{Z}}^\Box(1)) \overset{\sim}{\to} \underline{\mathrm{Hom}}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\tilde{\eta}}) \overset{\sim}{\to} \mathrm{T}^\Box T_{\tilde{\eta}},$$

in which the first is the essential datum, in which the second is given by the isomorphism $\nu(\hat{\alpha}) : \hat{\mathbb{Z}}^\Box(1) \overset{\sim}{\to} \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\tilde{\eta}}$ given by $\hat{\alpha}$, and in which the third is canonical. Different choices of the lifting $\hat{\alpha}$ might induce different filtrations on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$, but the reductions modulo n are the same. Moreover, the isomorphism between the filtrations is *symplectic*. Namely, $\mathtt{Z}_{-2}$ and $\mathtt{Z}_{-1}$ are the annihilators of each other under the pairing $\langle\cdot,\cdot\rangle$ of $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$ (as $\mathrm{T}^\Box T_{\tilde{\eta}}$ and $\mathrm{T}^\Box G^\natural_{\tilde{\eta}}$ are), and the induced isomorphisms

$$\mathrm{Gr}^{\mathtt{Z}}_{-2} \times \mathrm{Gr}^{\mathtt{Z}}_{0} \overset{\sim}{\to} \mathrm{T}^\Box T_{\tilde{\eta}} \times (Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$$

and

$$\mathrm{Gr}^{\mathtt{Z}}_{-1} \times \mathrm{Gr}^{\mathtt{Z}}_{-1} \overset{\sim}{\to} \mathrm{T}^\Box A_{\tilde{\eta}} \times \mathrm{T}^\Box A_{\tilde{\eta}}$$

on the graded pieces respect the pairings on both sides under the same unique isomorphism $\nu(\hat{\alpha}) : \hat{\mathbb{Z}}^\Box(1) \overset{\sim}{\to} \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\tilde{\eta}}$ given by $\hat{\alpha}$, induced respectively by the pairing $\langle\cdot,\cdot\rangle$ on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$ and the λ-Weil pairing on $\mathrm{T}^\Box G_{\tilde{\eta}}$.

On the other hand, having isomorphisms on the graded pieces alone is not sufficient for recovering the isomorphism between the whole modules. Let us introduce some noncanonical choices in this setting, namely, splittings of the underlying modules (see Section 1.2.6).

Now suppose that we are given some $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$-module filtration

$$0 \subset \mathtt{Z}_{-2} \subset \mathtt{Z}_{-1} \subset \mathtt{Z}_0 = L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box.$$

It does not make sense to consider arbitrary such filtrations, as the filtrations on $\mathrm{T}^\Box G_{\tilde{\eta}}$ do satisfy some special conditions.

Lemma 5.2.2.2. *Every filtration $\mathtt{Z} = \{\mathtt{Z}_{-i}\}_i$ coming from $\mathrm{T}^\Box G_{\tilde{\eta}}$ as above is* ***integrable*** *and* ***symplectic*** *(see Definitions 1.2.6.2 and 1.2.6.8).*

Proof. Let $\mathtt{Z}$ be such a filtration. The fact that $\mathtt{Z}$ is symplectic follows from Proposition 5.2.2.1 and the explanation above. Let us denote by $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\cdot,\cdot\rangle_{11})$ the induced symplectic $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$-module. By Propositions 5.1.2.4 and 5.1.2.2, we also have a filtration $\mathtt{Z}_\mathbb{R}$ on $L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$, with an induced polarized symplectic $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$-module $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}}, \langle\cdot,\cdot\rangle_{11,\mathbb{R}}, h_{-1})$. Let us show that $\mathtt{Z}$ is integrable. By the construction above, it is clear that $\mathrm{Gr}^{\mathtt{Z}}_{-2}$ and $\mathrm{Gr}^{\mathtt{Z}}_{0}$ are integrable. Therefore it remains to show that $\mathrm{Gr}^{\mathtt{Z}}_{-1}$ is integrable. Consider the *abelian part* $(A_{\tilde{\eta}}, \lambda_{A_{\tilde{\eta}}}, i_{i,\tilde{\eta}}, \varphi_{-1,n})$ (over $\tilde{\eta}$) of the data we have, which defines a point of the smooth moduli problem defined by $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}}, \langle\cdot,\cdot\rangle_{11,\mathbb{R}}, h_{-1})$ and $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\cdot,\cdot\rangle_{11})$. Then, as explained in Remark 1.4.3.14, there exists (noncanonically) a PEL-type $\mathcal{O}$-lattice $(L^{\mathtt{Z}}, \langle\cdot,\cdot\rangle^{\mathtt{Z}}, h^{\mathtt{Z}})$ as in Definition 1.2.1.3 such that $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\cdot,\cdot\rangle_{11}) \cong (L^{\mathtt{Z}}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle^{\mathtt{Z}})$ (over $\hat{\mathbb{Z}}^\Box$) and $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}}, \langle\cdot,\cdot\rangle_{11,\mathbb{R}}, h_{-1}) \cong (L^{\mathtt{Z}}\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \langle\cdot,\cdot\rangle^{\mathtt{Z}}, h^{\mathtt{Z}})$ (over $\mathbb{R}$). In particular, $\mathrm{Gr}^{\mathtt{Z}}_{-1}$ is integrable, as desired. □

As a preparation for Lemma 5.2.2.4 below, let us include the following:

Lemma 5.2.2.3. *Let C be any finite-dimensional semisimple algebra over $\mathbb{Q}$, and let C' be a finitely generated $\mathbb{Z}$-subalgebra of C. Then C' is generated as a $\mathbb{Z}$-algebra by finitely many elements in $C' \cap C^\times$.*

Proof. First we claim that, for each element $c \in C'$ but $c \notin C^\times$, there exists an integer $n_c \in \mathbb{Z}$ (depending on c) such that $c + n_c \in C^\times$. To justify this claim, consider elements of C as endomorphisms of the complex vector space $C\underset{\mathbb{Q}}{\otimes}\mathbb{C}$. Then an element c is invertible if all its finitely many eigenvalues are nonzero, and the claim follows because addition of an integer n increases all the eigenvalues by n. Now that we have the claim, by adding integers to a set of generators of C' over $\mathbb{Z}$, we may assume that they are all in $C' \cap C^\times$, as desired. □

By Corollary 1.2.6.5, we know that an integrable filtration is automatically split when $\mathcal{O}$ is *maximal*. For general $\mathcal{O}$ (which might not be maximal), we have the following:

Lemma 5.2.2.4. *Under the assumption that the PEL-type $\mathcal{O}$-lattice $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ satisfies Condition 1.4.3.10, every $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$-module filtration $\mathtt{Z}$ on $\mathtt{Z}_0 = L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ that can be realized as a pullback of the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$-module filtration $0 \subset \mathrm{T}^{\Box}\,T_{\bar{\eta}} \subset \mathrm{T}^{\Box}\,G^{\natural}_{\bar{\eta}} \subset \mathrm{T}^{\Box}\,G_{\bar{\eta}}$ by an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$-equivariant symplectic isomorphism $\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box}\,G_{\bar{\eta}}$ is necessarily **split**.*

Proof. Condition 1.4.3.10 means that the action of $\mathcal{O}$ on L extends to an action of some maximal order $\mathcal{O}'$ in B containing $\mathcal{O}$. By Lemma 5.2.2.3, $\mathcal{O}'$ is generated as a $\mathbb{Z}$-algebra by $\mathcal{O}'\cap B^{\times}$. By Proposition 1.1.1.21, there exists an integer $m \geq 1$, with no prime factors other than those of Disc, such that $m\mathcal{O}' \subset \mathcal{O}$. Hence elements of $\mathcal{O}' \cap B^{\times}$ define $\mathbb{Z}^{\times}_{(\Box)}$-isogenies. By Lemma 1.3.5.2, such $\mathbb{Z}^{\times}_{(\Box)}$-isogenies are determined by their induced morphisms on $\mathrm{V}^{\Box}\,G_{\bar{\eta}}$, or rather their induced morphisms on $L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}$ via the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}$-equivariant symplectic isomorphism $L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box}\,G_{\bar{\eta}}$ induced by $\hat{\alpha}$. Since the action of $\mathcal{O}$ on L extends to an action of $\mathcal{O}'$ on L, we see that the morphisms induced by elements of $\mathcal{O}' \cap B^{\times}$ on $\mathrm{V}^{\Box}\,G_{\bar{\eta}}$ map $\mathrm{T}^{\Box}\,G_{\bar{\eta}}$ to itself. This shows that elements of $\mathcal{O}' \cap B^{\times}$ define endomorphisms of $G_{\bar{\eta}}$, or rather $G_{\eta'}$ for some point $\eta' = \mathrm{Spec}(K')$ finite étale over $\tilde{\eta}$. As a result, all elements of $\mathcal{O}'$ define endomorphisms of $G_{\eta'}$.

Let R' be the normalization of R in K'. By Lemma 5.2.1.1, R' with $I' := \mathrm{rad}(I \cdot R') \subset R'$ satisfies the requirements in Section 4.1. Let $S' = \mathrm{Spec}(R')$, and let $G' = G \underset{S}{\times} S'$. By Proposition 3.3.1.5, the endomorphisms of $G_{\eta'}$ defined by $\mathcal{O}'$ extend to endomorphisms of G' over S', which also induce endomorphisms of $(G')^{\natural} := G^{\natural} \underset{S}{\times} S'$ (resp. $T' := T \underset{S}{\times} S'$) by the functoriality of Raynaud extensions. Thus, the action of $\mathcal{O}'$ on $\mathrm{T}^{\Box}\,G_{\bar{\eta}}$ maps $\mathrm{T}^{\Box}\,G^{\natural}_{\bar{\eta}}$ (resp. $\mathrm{T}^{\Box}\,T_{\bar{\eta}}$) to itself.

Now $0 \subset \mathrm{T}^{\Box}\,T_{\bar{\eta}} \subset \mathrm{T}^{\Box}\,G^{\natural}_{\bar{\eta}} \subset \mathrm{T}^{\Box}\,G_{\bar{\eta}}$ is an integrable filtration of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$-modules (respected by the action of $\mathcal{O}'$), with each of the graded pieces of the form $M\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ for some $\mathcal{O}$-lattices M. Writing $\hat{\mathbb{Z}}^{\Box} = (\prod\limits_{p|\mathrm{Disc}} \mathbb{Z}_p) \times (\prod\limits_{\Box\nmid p\nmid\mathrm{Disc}} \mathbb{Z}_p)$, we see that $\mathcal{O} \underset{\mathbb{Z}}{\otimes} (\prod\limits_{\Box\nmid p\nmid\mathrm{Disc}} \mathbb{Z}_p) = \mathcal{O}' \underset{\mathbb{Z}}{\otimes} (\prod\limits_{\Box\nmid p\nmid\mathrm{Disc}} \mathbb{Z}_p)$. In particular, $M \underset{\mathbb{Z}}{\otimes} (\prod\limits_{\Box\nmid p\nmid\mathrm{Disc}} \mathbb{Z}_p)$ is projective also as an $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} (\prod\limits_{\Box\nmid p\nmid\mathrm{Disc}} \mathbb{Z}_p)$-module. On the other hand, for each of the finitely many $p \mid \mathrm{Disc}$, the $\mathbb{Z}_p$-module $M \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is torsion-free. By Definition 1.1.1.22, $M \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is an $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-lattice, because of its induced $\mathcal{O}'$-action. By Proposition 1.1.1.23, $M \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$ is projective as an $\mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_p$-module. Putting the *finite* product together again, we see that the filtration is projective (see Definition 1.2.6.2), and is therefore split (by Lemma 1.2.6.4). Since $\mathcal{O}'\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ contains $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ as a subring, the splitting is $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$-equivariant as well, as desired. $\Box$

Corollary 5.2.2.5 (of the proof of Lemma 5.2.2.4)**.** *Under the assumption that the PEL-type $\mathcal{O}$-lattice $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h)$ satisfies Condition 1.4.3.10, suppose that the action of $\mathcal{O}$ on L extends to an action of some maximal order $\mathcal{O}'$ in B containing $\mathcal{O}$, and that there exists at least one $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$-equivariant symplectic isomorphism $\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box} \xrightarrow{\sim} \mathrm{T}^{\Box}\,G_{\bar{\eta}}$. Then the actions of $\mathcal{O}$ on T, A, and Y all extend to actions of $\mathcal{O}'$ compatible with $c : X \to A^{\vee}$ and $c^{\vee} : Y \to A$. (Here the $\mathcal{O}'$-action on T (resp. A) is defined (in Section 5.1.1) to be equivalent to the $(\mathcal{O}')^{\star}$-action on X (resp. $A^{\vee}$), where $(\mathcal{O}')^{\star}$ is the image of $\mathcal{O}'$ under the anti-isomorphism $^{\star} : B \xrightarrow{\sim} B^{\mathrm{op}}$.)*

As a result, for the purpose of studying level structures, we only need to consider $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$-module filtrations $\mathtt{Z}$ on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ that are integrable, symplectic, and *split*. Recall that (in Definition 1.2.6.6) a filtration on an integrable module is called *admissible* if it is both integrable and split. For simplicity, in what follows, we shall often suppress modifiers such as "$\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$-equivariant" or "of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$-modules" for admissible filtrations, their splittings, and various related objects.

Definition 5.2.2.6. *The **$\mathcal{O}$-multirank** of a symplectic admissible filtration $\mathtt{Z}$ on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ is the $\mathcal{O}$-multirank (see Definition 1.2.1.25) of $\mathtt{Z}_{-2}$ as an integrable $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$-module.*

Let us investigate the possible splittings of an admissible filtration on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$. Let us define $\mathrm{Gr}^{\mathtt{Z}}_{-i} := \mathtt{Z}_{-i}/\mathtt{Z}_{-i-1}$ as before.

If there is a first splitting, then we obtain a direct sum decomposition
$$\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} := \mathrm{Gr}^{\mathtt{Z}}_{-2} \oplus \mathrm{Gr}^{\mathtt{Z}}_{-1} \oplus \mathrm{Gr}^{\mathtt{Z}}_{0} \xrightarrow{\sim} L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}.$$
The pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ can thus be expressed in *matrix form* as
$$\begin{pmatrix} & & \langle\,\cdot\,,\,\cdot\,\rangle_{20} \\ & \langle\,\cdot\,,\,\cdot\,\rangle_{11} & \langle\,\cdot\,,\,\cdot\,\rangle_{10} \\ \langle\,\cdot\,,\,\cdot\,\rangle_{02} & \langle\,\cdot\,,\,\cdot\,\rangle_{01} & \langle\,\cdot\,,\,\cdot\,\rangle_{00} \end{pmatrix},$$
where the pairings
$$\langle\,\cdot\,,\,\cdot\,\rangle_{ij} : \mathrm{Gr}^{\mathtt{Z}}_{-i} \times \mathrm{Gr}^{\mathtt{Z}}_{-j} \to \hat{\mathbb{Z}}^{\Box}(1)$$
satisfy $\langle\,\cdot\,,\,\cdot\,\rangle_{ij} = -\,{}^t\langle\,\cdot\,,\,\cdot\,\rangle^{\star}_{ji}$ for all i and j. Namely, they satisfy $\langle x, by\rangle_{ij} = \langle b^{\star}x, y\rangle_{ij} = -\langle y, b^{\star}x\rangle_{ji}$ for all $x \in \mathrm{Gr}^{\mathtt{Z}}_{-i}$, $y \in \mathrm{Gr}^{\mathtt{Z}}_{-j}$, and $b \in \mathcal{O}$. Here we have nothing on the three upper-left blocks because $\mathtt{Z}_{-2}$ and $\mathtt{Z}_{-1}$ are the annihilators of each other.

If there is a second splitting $\hat{\delta}' : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$, then there is a *change of basis* $\hat{\mathtt{z}} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{Z}}$ such that $\hat{\delta}' = \hat{\delta} \circ \hat{\mathtt{z}}$. In matrix form, we can write
$$\hat{\mathtt{z}} = \begin{pmatrix} 1 & \mathtt{z}_{21} & \mathtt{z}_{20} \\ & 1 & \mathtt{z}_{10} \\ & & 1 \end{pmatrix},$$
where
$$\mathtt{z}_{ij} : \mathrm{Gr}^{\mathtt{Z}}_{-j} \to \mathrm{Gr}^{\mathtt{Z}}_{-i}$$
are morphisms between the graded pieces. The matrix of the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$

using the second splitting can be expressed by

$$\begin{pmatrix} & & \langle\,\cdot\,,\,\cdot\,\rangle'_{20} \\ & \langle\,\cdot\,,\,\cdot\,\rangle'_{11} & \langle\,\cdot\,,\,\cdot\,\rangle'_{10} \\ \langle\,\cdot\,,\,\cdot\,\rangle'_{02} & \langle\,\cdot\,,\,\cdot\,\rangle'_{01} & \langle\,\cdot\,,\,\cdot\,\rangle'_{00} \end{pmatrix}$$
$$:= \begin{pmatrix} 1 & & \\ {}^t\mathtt{z}^\star_{21} & 1 & \\ {}^t\mathtt{z}^\star_{20} & {}^t\mathtt{z}^\star_{10} & 1 \end{pmatrix} \begin{pmatrix} & & \langle\,\cdot\,,\,\cdot\,\rangle_{20} \\ & \langle\,\cdot\,,\,\cdot\,\rangle_{11} & \langle\,\cdot\,,\,\cdot\,\rangle_{10} \\ \langle\,\cdot\,,\,\cdot\,\rangle_{02} & \langle\,\cdot\,,\,\cdot\,\rangle_{01} & \langle\,\cdot\,,\,\cdot\,\rangle_{00} \end{pmatrix} \begin{pmatrix} 1 & \mathtt{z}_{21} & \mathtt{z}_{20} \\ & 1 & \mathtt{z}_{10} \\ & & 1 \end{pmatrix},$$

where

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle'_{20} &= \langle\,\cdot\,,\,\cdot\,\rangle_{20}, \\ \langle\,\cdot\,,\,\cdot\,\rangle'_{11} &= \langle\,\cdot\,,\,\cdot\,\rangle_{11}, \\ \langle\,\cdot\,,\,\cdot\,\rangle'_{10} &= \langle\,\cdot\,,\,\cdot\,\rangle_{10} + {}^t\mathtt{z}^\star_{21}\langle\,\cdot\,,\,\cdot\,\rangle_{20} + \langle\,\cdot\,,\,\cdot\,\rangle_{11}\mathtt{z}_{10}, \\ \langle\,\cdot\,,\,\cdot\,\rangle'_{00} &= \langle\,\cdot\,,\,\cdot\,\rangle_{00} + ({}^t\mathtt{z}^\star_{20}\langle\,\cdot\,,\,\cdot\,\rangle_{20} - {}^t\langle\,\cdot\,,\,\cdot\,\rangle^\star_{20}\mathtt{z}_{20}) \\ &\quad + ({}^t\mathtt{z}^\star_{10}\langle\,\cdot\,,\,\cdot\,\rangle_{10} - {}^t\langle\,\cdot\,,\,\cdot\,\rangle^\star_{10}\mathtt{z}_{10}) + {}^t\mathtt{z}^\star_{10}\langle\,\cdot\,,\,\cdot\,\rangle_{11}\mathtt{z}_{10}, \end{aligned}$$

and where

$$\begin{aligned} {}^t\mathtt{z}^\star_{ki}\langle x,y\rangle_{kj} &= \langle \mathtt{z}_{ki}(x),y\rangle_{kj}, \\ \langle x,y\rangle_{ik}\mathtt{z}_{kj} &= \langle x,\mathtt{z}_{kj}(y)\rangle_{ik}, \\ {}^t\mathtt{z}^\star_{ki}\langle x,y\rangle_{kl}\mathtt{z}_{lj} &= \langle \mathtt{z}_{ki}(x),\mathtt{z}_{lj}(y)\rangle_{kl} \end{aligned}$$

for all $x \in \mathrm{Gr}^{\mathtt{Z}}_{-i}$ and $y \in \mathrm{Gr}^{\mathtt{Z}}_{-j}$. The notation thus designed then satisfies the symbolic relation

$${}^t({}^t\mathtt{z}^\star_{ki}\langle\,\cdot\,,\,\cdot\,\rangle_{kj})^\star = {}^t\langle\,\cdot\,,\,\cdot\,\rangle^\star_{kj}\mathtt{z}_{ki} = -\langle\,\cdot\,,\,\cdot\,\rangle_{jk}\mathtt{z}_{ki}.$$

Definition 5.2.2.7. *Two pairs of pairings* $(\langle\,\cdot\,,\,\cdot\,\rangle_{10},\langle\,\cdot\,,\,\cdot\,\rangle_{00})$ *and* $(\langle\,\cdot\,,\,\cdot\,\rangle'_{10},\langle\,\cdot\,,\,\cdot\,\rangle'_{00})$ *as above are equivalent under* $(\langle\,\cdot\,,\,\cdot\,\rangle_{20},\langle\,\cdot\,,\,\cdot\,\rangle_{11})$*, denoted*

$$(\langle\,\cdot\,,\,\cdot\,\rangle_{10},\langle\,\cdot\,,\,\cdot\,\rangle_{00}) \sim (\langle\,\cdot\,,\,\cdot\,\rangle'_{10},\langle\,\cdot\,,\,\cdot\,\rangle'_{00}),$$

if there are some $\mathtt{z}_{21}$*,* $\mathtt{z}_{10}$*, and* $\mathtt{z}_{20}$ *such that*

$$\begin{aligned} \langle\,\cdot\,,\,\cdot\,\rangle'_{10} &= \langle\,\cdot\,,\,\cdot\,\rangle_{10} + {}^t\mathtt{z}^\star_{21}\langle\,\cdot\,,\,\cdot\,\rangle_{20} + \langle\,\cdot\,,\,\cdot\,\rangle_{11}\mathtt{z}_{10}, \\ \langle\,\cdot\,,\,\cdot\,\rangle'_{00} &= \langle\,\cdot\,,\,\cdot\,\rangle_{00} + ({}^t\mathtt{z}^\star_{20}\langle\,\cdot\,,\,\cdot\,\rangle_{20} - {}^t\langle\,\cdot\,,\,\cdot\,\rangle^\star_{20}\mathtt{z}_{20}) \\ &\quad + ({}^t\mathtt{z}^\star_{10}\langle\,\cdot\,,\,\cdot\,\rangle_{10} - {}^t\langle\,\cdot\,,\,\cdot\,\rangle^\star_{10}\mathtt{z}_{10}) + {}^t\mathtt{z}^\star_{10}\langle\,\cdot\,,\,\cdot\,\rangle_{11}\mathtt{z}_{10}. \end{aligned}$$

Thus, $\langle\,\cdot\,,\,\cdot\,\rangle_{20}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{11}$ are independent of the splitting $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$, while $\langle\,\cdot\,,\,\cdot\,\rangle_{10}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{00}$ are well defined only up to equivalence.

Now suppose that we are given any splitting $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$. By reduction modulo n, we obtain an admissible filtration

$$0 \subset \mathtt{Z}_{-2,n} \subset \mathtt{Z}_{-1,n} \subset \mathtt{Z}_{0,n} := L/nL,$$

where $\mathtt{Z}_{-i,n} := \mathtt{Z}_{-i}/n\mathtt{Z}_{-i}$, with graded pieces $\mathrm{Gr}^{\mathtt{Z}}_{-i,n} := \mathtt{Z}_{-i,n}/\mathtt{Z}_{-i-1,n} \cong \mathrm{Gr}^{\mathtt{Z}}_{-i}/n\,\mathrm{Gr}^{\mathtt{Z}}_{-i}$. We shall always follow the convention that $-i \leq 0$ and $n \geq 1$, so that there is no ambiguity when we write

$$\delta_n : \mathrm{Gr}^{\mathtt{Z}}_n := \mathrm{Gr}^{\mathtt{Z}}_{-2,n} \oplus \mathrm{Gr}^{\mathtt{Z}}_{-1,n} \oplus \mathrm{Gr}^{\mathtt{Z}}_{0,n} \xrightarrow{\sim} L/nL$$

as the reduction modulo n of $\hat{\delta}$.

Remark 5.2.2.8. To keep the information after reduction modulo n, we shall equip the filtration $\mathtt{Z}_n := \{\mathtt{Z}_{-i,n}\}_i$ with the notion of $\mathcal{O}$-multiranks, given by the $\mathcal{O}$-multiranks (see Definition 5.2.2.6) of the (admissible) filtrations $\mathtt{Z}$ we started with. Therefore, even if $n = 1$, in which case L/nL is trivial, there might still be different filtrations $\mathtt{Z}_n$ on L/nL because there might be different multiranks. This convention will be tacitly assumed in all our arguments.

Definition 5.2.2.9. *An admissible filtration*

$$0 \subset \mathtt{Z}_{-2,n} \subset \mathtt{Z}_{-1,n} \subset \mathtt{Z}_{0,n} = L/nL$$

on L/nL *of a prescribed* $\mathcal{O}$*-multirank is called* ***symplectic-liftable*** *if it is the reduction modulo* n *of some symplectic admissible filtration*

$$0 \subset \mathtt{Z}_{-2} \subset \mathtt{Z}_{-1} \subset \mathtt{Z}_0 = L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$$

(see Definitions 1.2.6.6 and 1.2.6.8) of the prescribed $\mathcal{O}$*-multirank. Equivalently, a symplectic-liftable filtration* $\mathtt{Z}_n$ *on* L/nL *is an equivalence class of symplectic admissible filtrations* $\mathtt{Z}$ *on* $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$*, where two symplectic admissible filtrations* $\mathtt{Z}$ *and* $\mathtt{Z}'$ *are defined to be equivalent if their* $\mathcal{O}$*-multiranks are the same, and if their reductions modulo* n *are the same.*

Definition 5.2.2.10. *A splitting* $\delta_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} L/nL$ *for a symplectic-liftable filtration* $\{\mathtt{Z}_{-i,n}\}_i$ *on* L/nL *is called* ***liftable*** *if it is the reduction modulo* n *of some splitting* $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$.

On the other hand, we would like to perform the same analysis for $\mathrm{T}^\square\, G_{\bar{\eta}}$. In particular, we shall investigate the possible splittings of the filtration

$$0 \subset \mathrm{T}^\square\, T_{\bar{\eta}} \subset \mathrm{T}^\square\, G^\natural_{\bar{\eta}} \subset \mathrm{T}^\square\, G_{\bar{\eta}}$$

(when it is admissible). In analogous notation, let us define

$$\mathtt{W}_{-2} := \mathrm{T}^\square\, T_{\bar{\eta}}, \qquad \mathtt{W}_{-1} := \mathrm{T}^\square\, G^\natural_{\bar{\eta}}, \qquad \mathtt{W}_0 := \mathrm{T}^\square\, G_{\bar{\eta}},$$
$$\mathrm{Gr}^{\mathtt{W}}_{-i} := \mathtt{W}_{-i}/\mathtt{W}_{-i-1}, \qquad \mathrm{Gr}^{\mathtt{W}} := \mathrm{Gr}^{\mathtt{W}}_{-2} \oplus \mathrm{Gr}^{\mathtt{W}}_{-1} \oplus \mathrm{Gr}^{\mathtt{W}}_0$$

as in the case of $\mathrm{Gr}^{\mathtt{Z}}$. Then we know that $\mathrm{Gr}^{\mathtt{W}}_{-2} = \mathrm{T}^\square\, T_{\bar{\eta}}$, $\mathrm{Gr}^{\mathtt{W}}_{-1} \cong \mathrm{T}^\square\, A_{\bar{\eta}}$, and $\mathrm{Gr}^{\mathtt{W}}_0 \cong Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$. A splitting of this filtration corresponds to an isomorphism

$$\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \cong \mathrm{T}^\square\, T_{\bar{\eta}} \oplus \mathrm{T}^\square\, A_{\bar{\eta}} \oplus (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square) \xrightarrow{\sim} \mathrm{T}^\square\, G_{\bar{\eta}}.$$

Let us denote the multiplication and inversion in $\mathrm{T}^\square\, \mathbf{G}_{\mathrm{m},\bar{\eta}}$ *additively* by the notation $+$ and $-$ when we talk about matrix entries. Then the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}$ on $\mathrm{T}^\square\, G_{\bar{\eta}}$ can be expressed in *matrix form* as

$$\begin{pmatrix} & & \mathrm{e}_{20} \\ & \mathrm{e}_{11} & \mathrm{e}_{10} \\ \mathrm{e}_{02} & \mathrm{e}_{01} & \mathrm{e}_{00} \end{pmatrix}$$

where the pairings

$$\mathrm{e}_{ij} : \mathrm{Gr}^{\mathtt{W}}_{-i} \times \mathrm{Gr}^{\mathtt{W}}_{-j} \to \mathrm{T}^\square\, \mathbf{G}_{\mathrm{m},\bar{\eta}}$$

satisfy $\mathrm{e}_{ij} = -{}^t\mathrm{e}^\star_{ji}$ for all i and j. Namely, they satisfy $\mathrm{e}_{ij}(x,by) = \mathrm{e}_{ij}(b^\star x,y) = -\mathrm{e}_{ji}(y,b^\star x)$ for all $x \in \mathrm{Gr}^{\mathtt{W}}_{-i}$, $y \in \mathrm{Gr}^{\mathtt{W}}_{-j}$, and $b \in \mathcal{O}$. Here we have nothing on the three upper-left blocks because $\mathtt{W}_{-2}$ and $\mathtt{W}_{-1}$ are the annihilators of each other. Note that $\mathrm{e}_{20} = \mathrm{e}^\phi$ and $\mathrm{e}_{11} = \mathrm{e}^{\lambda_A}$.

If there is now a second splitting $\hat{\varsigma}' : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^\square\, G_{\bar{\eta}}$, then there is a *change of basis* $\hat{\mathtt{w}} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ such that $\hat{\varsigma}' = \hat{\varsigma} \circ \hat{\mathtt{w}}$. In matrix form, we have

$$\hat{\mathtt{w}} = \begin{pmatrix} 1 & \mathtt{w}_{21} & \mathtt{w}_{20} \\ & 1 & \mathtt{w}_{10} \\ & & 1 \end{pmatrix},$$

where

$$\mathtt{w}_{ij} : \mathrm{Gr}^{\mathtt{W}}_{-j} \to \mathrm{Gr}^{\mathtt{W}}_{-i}$$

are morphisms between the graded pieces. The matrix of the pairing $\mathrm{e}^{\lambda_\eta}$ on $\mathrm{T}^\square\, G_{\bar{\eta}}$ using the second splitting can be expressed by

$$\begin{pmatrix} & & \mathrm{e}'_{20} \\ & \mathrm{e}'_{11} & \mathrm{e}'_{10} \\ \mathrm{e}'_{02} & \mathrm{e}'_{01} & \mathrm{e}'_{00} \end{pmatrix} := \begin{pmatrix} 1 & & \\ {}^t\mathtt{w}^\star_{21} & 1 & \\ {}^t\mathtt{w}^\star_{20} & {}^t\mathtt{w}^\star_{10} & 1 \end{pmatrix} \begin{pmatrix} & & \mathrm{e}_{20} \\ & \mathrm{e}_{11} & \mathrm{e}_{10} \\ \mathrm{e}_{02} & \mathrm{e}_{01} & \mathrm{e}_{00} \end{pmatrix} \begin{pmatrix} 1 & \mathtt{w}_{21} & \mathtt{w}_{20} \\ & 1 & \mathtt{w}_{10} \\ & & 1 \end{pmatrix},$$

where

$$\begin{aligned} e'_{20} &= e_{20} = e^{\phi}, \\ e'_{11} &= e_{11} = e^{\lambda_A}, \\ e'_{10} &= e_{10} + {}^t\mathtt{w}^\star_{21} e_{20} + e_{11}\mathtt{w}_{10}, \\ e'_{00} &= e_{00} + ({}^t\mathtt{w}^\star_{20} e_{20} - {}^t e^\star_{20}) + ({}^t\mathtt{w}^\star_{10} e_{10} - {}^t e^\star_{10}) + {}^t\mathtt{w}^\star_{10} e_{11}\mathtt{w}_{10}, \end{aligned}$$

and the terms such as ${}^t\mathtt{w}^\star_{21} e_{20}$ are interpreted in the same way as before.

Definition 5.2.2.11. *Two pairs of pairings* (e_{10}, e_{00}) *and* (e'_{10}, e'_{00}) *as above are equivalent under* $(e_{20}, e_{11}) = (e^{\phi}, e^{\lambda_A})$*, denoted*

$$(e_{10}, e_{00}) \sim (e'_{10}, e'_{00}),$$

if there are some $\mathtt{w}_{21}$*,* $\mathtt{w}_{10}$*, and* $\mathtt{w}_{20}$ *such that*

$$\begin{aligned} e'_{10} &= e_{10} + {}^t\mathtt{w}^\star_{21} e_{20} + e_{11}\mathtt{w}_{10}, \\ e'_{00} &= e_{00} + ({}^t\mathtt{w}^\star_{20} e_{20} - {}^t e^\star_{20}) + ({}^t\mathtt{w}^\star_{10} e_{10} - {}^t e^\star_{10}) + {}^t\mathtt{w}^\star_{10} e_{11}\mathtt{w}_{10}. \end{aligned}$$

Thus, $e_{20} = e^{\phi}$ and $e_{11} = e^{\lambda_A}$ are independent of the splitting $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$, while e_{10} and e_{00} are well defined only up to equivalence.

Let us denote the reduction modulo n of the filtration $\{\mathtt{W}_{-i}\}_i$ by $\{\mathtt{W}_{-i,n}\}_i$. Then we have the filtration

$$0 \subset \mathtt{W}_{-2,n} = T[n]_{\eta} \subset \mathtt{W}_{-1,n} = G^{\natural}_{\eta} \subset \mathtt{W}_{0,n} = G[n]_{\eta},$$

with $\mathrm{Gr}^{\mathtt{W}}_{-i,n} := \mathtt{W}_{-i,n}/\mathtt{W}_{-i-1,n}$ and $\mathrm{Gr}^{\mathtt{W}}_n := \mathrm{Gr}^{\mathtt{W}}_{-2,n} \oplus \mathrm{Gr}^{\mathtt{W}}_{-1,n} \oplus \mathrm{Gr}^{\mathtt{W}}_{0,n}$ as in the case of $\mathrm{Gr}^{\mathtt{Z}}_n$. By abuse of notation, we shall denote the pullbacks of the objects $\mathtt{W}_{-i,n}$ to $\tilde{\eta}$ or $\bar{\eta}$ by the same symbols. We write

$$\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n \xrightarrow{\sim} G[n]_{\bar{\eta}}$$

as the reduction modulo n of $\hat{\varsigma}$. If it is defined over $\tilde{\eta}$, then we also denote it by $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n \xrightarrow{\sim} G[n]_{\tilde{\eta}}$.

Definition 5.2.2.12. *A splitting* $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ *for the filtration* $\{\mathtt{W}_{-i,n}\}_i$ *on* $G[n]_{\tilde{\eta}}$ *is called* ***liftable*** *if it is the reduction modulo* n *of some splitting* $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$.

Now suppose that we have a level-n structure $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ defined over $\tilde{\eta}$, which by definition can be lifted to some $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$-equivariant symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$. The filtration $\mathtt{W} = \{\mathtt{W}_{-i}\}_i$ on $\mathrm{T}^{\square} G_{\bar{\eta}}$ induces a symplectic admissible filtration $\mathtt{Z} = \{\mathtt{Z}_{-i}\}_i$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$ by $\hat{\alpha}$, with isomorphisms $\mathrm{Gr}_{-i}(\hat{\alpha}) : \mathrm{Gr}^{\mathtt{Z}}_{-i} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_{-i}$ on the graded pieces. Let $\mathrm{Gr}(\hat{\alpha}) := \bigoplus_i \mathrm{Gr}_{-i}(\hat{\alpha})$. A splitting $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$ determines (and is determined by) a splitting of $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$ by the relation $\hat{\delta} = \hat{\alpha}^{-1} \circ \hat{\varsigma} \circ \mathrm{Gr}(\hat{\alpha})$.

If the λ_{η}-Weil pairing $e^{\lambda_{\eta}}$ on $\mathrm{T}^{\square} G_{\bar{\eta}}$ and the symplectic pairing $\langle \cdot, \cdot \rangle$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$ are given under the splittings $\hat{\delta}$ and $\hat{\delta}$ in matrix forms

$$\begin{pmatrix} & & \langle \cdot, \cdot \rangle_{20} \\ & \langle \cdot, \cdot \rangle_{11} & \langle \cdot, \cdot \rangle_{10} \\ \langle \cdot, \cdot \rangle_{02} & \langle \cdot, \cdot \rangle_{01} & \langle \cdot, \cdot \rangle_{00} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} & & e_{20} \\ & e_{11} & e_{10} \\ e_{02} & e_{01} & e_{00} \end{pmatrix},$$

respectively, then we would like to match e_{ij} with $\langle \cdot, \cdot \rangle_{ij}$ under $\mathrm{Gr}(\hat{\alpha})$.

Definition 5.2.2.13. *Given splittings* $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$ *and* $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$*, a graded isomorphism* $\hat{f} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ *is a* ***symplectic isomorphism*** *with respect to* $\hat{\delta}$ *and* $\hat{\varsigma}$ *if* $\hat{\varsigma} \circ \hat{f} \circ \hat{\delta}^{-1} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$ *is a symplectic isomorphism. (Note that* $\hat{f}$ *is equipped with an isomorphism* $\nu(\hat{f}) : \hat{\mathbb{Z}}^{\square}(1) \xrightarrow{\sim} \mathrm{T}^{\square} \mathbf{G}_{\mathrm{m},\bar{\eta}}$*, as in Definition 1.1.4.8.)*

By definition, the $\mathrm{Gr}(\hat{\alpha})$ constructed above is symplectic. We shall view $\hat{\delta}$ and $\hat{\varsigma}$ as symplectic isomorphisms by setting the *similitudes* to be ones (i.e., the identity homomorphisms). Then the relation $\hat{\delta} = \hat{\alpha}^{-1} \circ \hat{\varsigma} \circ \mathrm{Gr}(\hat{\alpha})$ above is an identity of symplectic isomorphisms.

Lemma 5.2.2.14. *Suppose we are given splittings* $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$ *and* $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square} \mathbf{G}_{\mathrm{m},\bar{\eta}}$*, so that we have the induced pairings* $\langle \cdot, \cdot \rangle_{ij}$ *and* e_{ij} *defined on the graded pieces. Then a graded isomorphism* $\hat{f} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ *(with an isomorphism* $\nu(\hat{f}) : \hat{\mathbb{Z}}^{\square}(1) \xrightarrow{\sim} \mathrm{T}^{\square} \mathbf{G}_{\mathrm{m},\bar{\eta}}$*) defines a symplectic isomorphism with respect to* $\hat{\delta}$ *and* $\hat{\varsigma}$ *in the sense of Definition 5.2.2.13 if and only if* $\hat{f}^*(e_{ij}) = \nu(\hat{f}) \circ \langle \cdot, \cdot \rangle_{ij}$ *for every* i *and* j.

Definition 5.2.2.15. *A triple* $(\hat{\delta}, \hat{\varsigma}, \hat{f})$ *as in Definition 5.2.2.13 such that* $\hat{f}$ *is symplectic with respect to* $\hat{\delta}$ *and* $\hat{\varsigma}$ *is called a* ***symplectic triple****.*

The filtration $\mathtt{Z} = \{\mathtt{Z}_{-i}\}_i$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$ defines by reduction modulo n a filtration $\mathtt{Z}_n = \{\mathtt{Z}_{-i,n}\}_i$ on L/nL, which depends on α_n but not on the choice of the lifting $\hat{\alpha}$ of α_n. The isomorphisms $\mathrm{Gr}_{-i}(\hat{\alpha}) : \mathrm{Gr}^{\mathtt{Z}}_{-i} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_{-i}$ induced by reduction modulo n isomorphisms $\mathrm{Gr}_{-i,n}(\alpha_n) : \mathrm{Gr}^{\mathtt{Z}}_{-i,n} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_{-i,n}$ on the graded pieces, and a symplectic isomorphism $\mathrm{Gr}_n(\alpha_n) := \bigoplus_i \mathrm{Gr}_{-i,n} : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_n$, all of which depend on α_n but not on the choice of $\hat{\alpha}$. This symplectic isomorphism $\mathrm{Gr}_n(\alpha_n)$ is the reduction modulo n of the above symplectic isomorphism $\mathrm{Gr}(\hat{\alpha})$.

Definition 5.2.2.16. *Suppose we are given a symplectic-liftable filtration* $\{\mathtt{Z}_{-i,n}\}_i$ *on* $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$*, and liftable splittings* $\delta_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} L/nL$ *and* $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ *(defined over* $\tilde{\eta}$*). A graded symplectic isomorphism* $f_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_n$ *defined over* $\tilde{\eta}$ *is called* ***symplectic-liftable*** *if there are splittings* $\hat{\delta} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$ *and* $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$ *lifting* δ_n *and* ς_n*, respectively, such that* f_n *is the reduction modulo* n *of a graded isomorphism* $\hat{f} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ *that is symplectic with respect to* $\hat{\delta}$ *and* $\hat{\varsigma}$ *(i.e.,* $\hat{\varsigma} \circ \hat{f} \circ \hat{\delta}^{-1}$ *is a symplectic isomorphism), and such that* $\nu(f_n)$ *is the reduction modulo* n *of* $\nu(\hat{f})$*.*

For simplicity, we shall call a symplectic-liftable graded symplectic isomorphism a symplectic-liftable graded isomorphism, omitting the second appearance of symplectic.

By definition, the $\mathrm{Gr}_n(\alpha_n)$ constructed above is *symplectic-liftable*.

Conversely, suppose we are given a symplectic-liftable filtration $\{\mathtt{Z}_{-i,n}\}_i$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$, and liftable splittings $\delta_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} L/nL$ and $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n \xrightarrow{\sim} G[n]_{\tilde{\eta}}$. Suppose we are given a *symplectic-liftable* graded isomorphism $f_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_n$ defined over $\tilde{\eta}$. Then there are splittings $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$ and $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\bar{\eta}}$, together with a graded isomorphism $\hat{f} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ lifting f_n, which is symplectic with respect to

$\hat{\delta}$ and $\hat{\varsigma}$. From these we can produce a symplectic isomorphism $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ (defined over $\tilde{\eta}$) by setting $\alpha_n := \varsigma_n \circ f_n \circ \delta_n^{-1}$, which is symplectic-liftable because it is the reduction modulo n of the symplectic isomorphism $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\tilde{\eta}}$ defined by setting $\hat{\alpha} := \hat{\varsigma} \circ \hat{f} \circ \hat{\delta}^{-1}$. In other words, α_n is a (principal) level-n structure (of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}, \langle \cdot, \cdot \rangle)$). (The $\mathcal{O}$-equivariance is implicitly assumed in the above analysis; see the paragraph preceding Definition 5.2.2.6.)

Definition 5.2.2.17. *A triple $(\delta_n, \varsigma_n, f_n)$ defined over $\tilde{\eta}$ as in Definition 5.2.2.16 such that f_n is symplectic-liftable with respect to δ_n and ς_n is called a **symplectic-liftable triple**.*

If we start with a level-n structure $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$, then we can recover α_n from $\mathrm{Gr}_n(\alpha_n)$ by applying the above process to $f_n = \mathrm{Gr}_n(\alpha_n)$.

We may obtain different level-n structures α_n if we intentionally modify the splittings. Indeed, this is equivalent to having a change of basis $\mathtt{z}_n : \mathrm{Gr}_n^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}_n^{\mathtt{Z}}$ that is the reduction modulo n of some change of basis $\hat{\mathtt{z}} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{Z}}$ given by a matrix of the form $\begin{pmatrix} 1 & \mathtt{z}_{21} & \mathtt{z}_{20} \\ & 1 & \mathtt{z}_{10} \\ & & 1 \end{pmatrix}$. Then $\alpha_n = \varsigma_n \circ f_n \circ \delta_n^{-1}$ is replaced with $\alpha'_n = \varsigma_n \circ f_n \circ \mathtt{z}_n^{-1} \circ \delta_n^{-1}$.

Definition 5.2.2.18. *Two symplectic triples $(\hat{\delta}, \hat{\varsigma}, \hat{f})$ and $(\hat{\delta}', \hat{\varsigma}', \hat{f}')$ as in Definition 5.2.2.15 are **equivalent** if $\hat{f} \circ \hat{\mathtt{z}} = \hat{\mathtt{w}} \circ \hat{f}'$, where $\hat{\mathtt{z}}$ is the change of basis such that $\hat{\delta}' = \hat{\delta} \circ \hat{\mathtt{z}}$, and where $\hat{\mathtt{w}}$ is the change of basis such that $\hat{\varsigma}' = \hat{\varsigma} \circ \hat{\mathtt{w}}$. (Then necessarily $\hat{\varsigma} \circ \hat{f} \circ \hat{\delta}^{-1} = \hat{\varsigma}' \circ \hat{f}' \circ (\hat{\delta}')^{-1}$.)*

Definition 5.2.2.19. *A change of basis $\mathtt{z}_n : \mathrm{Gr}_n^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}_n^{\mathtt{Z}}$ of the form $\mathtt{z}_n = \begin{pmatrix} 1 & \mathtt{z}_{21,n} & \mathtt{z}_{20,n} \\ & 1 & \mathtt{z}_{10,n} \\ & & 1 \end{pmatrix}$ is called **liftable** if it is the reduction modulo n of some change of basis $\hat{\mathtt{z}} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{Z}}$ of the form $\hat{\mathtt{z}} = \begin{pmatrix} 1 & \mathtt{z}_{21} & \mathtt{z}_{20} \\ & 1 & \mathtt{z}_{10} \\ & & 1 \end{pmatrix}$.*

Definition 5.2.2.20. *A change of basis $\mathtt{w}_n : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} \mathrm{Gr}_n^{\mathtt{W}}$ of the form $\mathtt{w}_n = \begin{pmatrix} 1 & \mathtt{w}_{21,n} & \mathtt{w}_{20,n} \\ & 1 & \mathtt{w}_{10,n} \\ & & 1 \end{pmatrix}$ is called **liftable** if it is the reduction modulo n of some change of basis $\hat{\mathtt{w}} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}$ of the form $\hat{\mathtt{w}} = \begin{pmatrix} 1 & \mathtt{w}_{21} & \mathtt{w}_{20} \\ & 1 & \mathtt{w}_{10} \\ & & 1 \end{pmatrix}$.*

Definition 5.2.2.21. *Two symplectic-liftable triples $(\delta_n, \varsigma_n, f_n)$ and $(\delta'_n, \varsigma'_n, f'_n)$ as in Definition 5.2.2.17 are said to be **equivalent** if $f_n \circ \mathtt{z}_n = \mathtt{w}_n \circ f'_n$, where $\mathtt{z}_n$ is the liftable change of basis such that $\delta'_n = \delta_n \circ \mathtt{z}_n$, and where $\mathtt{w}_n$ is the liftable change of basis such that $\varsigma'_n = \varsigma_n \circ \mathtt{w}_n$. Then necessarily $\varsigma_n \circ f_n \circ \delta_n^{-1} = \varsigma'_n \circ f'_n \circ (\delta'_n)^{-1}$.*

We can summarize our analysis in this section as follows:

Proposition 5.2.2.22. *Suppose that we are given a symplectic admissible filtration $\mathtt{Z} := \{\mathtt{Z}_{-i}\}_i$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$. Then the ($\mathcal{O} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}$-equivariant) symplectic isomorphisms $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\tilde{\eta}}$ matching the filtration $\mathtt{Z}$ with the filtration $\mathtt{W}$ are in bijection with equivalence classes of symplectic triples $(\hat{\delta}, \hat{\varsigma}, \hat{f})$ as in Definition 5.2.2.18.*

Proposition 5.2.2.23. *Suppose that we are given a symplectic-liftable admissible filtration $\mathtt{Z}_n := \{\mathtt{Z}_{-i,n}\}_i$ on L/nL in the sense of Definition 5.2.2.9. Then the level-n structures $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}, \langle \cdot, \cdot \rangle)$ (as in Definition 1.3.6.2) matching the filtration $\mathtt{Z}_n$ with the filtration $\mathtt{W}_n$ are in bijection with equivalence classes of symplectic-liftable triples $(\delta_n, \varsigma_n, f_n)$ (defined over $\tilde{\eta}$) as in Definition 5.2.2.21.*

5.2.3 Analysis of Splittings for $G[n]_\eta$

Retaining the setting of Section 5.2.1, suppose that we have a triple (G, λ, i) defining an object in $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}}, (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h)}(R, I)$, which by Theorem 5.1.1.4 corresponds to a tuple $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}}, (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h)}(R, I)$. For simplicity, let us continue to assume that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively.

Let $\tilde{\eta} \to \eta$ be a finite étale morphism defined by a field extension as in Section 5.2.2. We would like to study level-n structures $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ of $(G_{\tilde{\eta}}, \lambda_{\tilde{\eta}}, i_{\tilde{\eta}}, \alpha_n)$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}, \langle \cdot, \cdot \rangle)$ as in Definition 1.3.6.2, under the additional assumption that (G, λ, i) defines an object in $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}}, (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h)}(R, I)$. Based on the analysis in Section 5.2.2, we would like to study splittings $\varsigma_n : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ that are liftable to splittings $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square} G_{\tilde{\eta}}$. Let us first proceed without the liftability condition. (For the purpose of studying the splittings ς_n and $\hat{\varsigma}$, it suffices to proceed with the assumption that (G, λ, i) defines an object in $\mathrm{DEG}_{\mathrm{PE}, \mathcal{O}}(R, I)$, not necessarily in $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}}, (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h)}(R, I)$.)

To split the filtration
$$0 \subset \mathtt{W}_{-2,n} = T[n]_{\tilde{\eta}} \subset \mathtt{W}_{-1,n} = G^\natural[n]_{\tilde{\eta}} \subset \mathtt{W}_{0,n} = G[n]_{\tilde{\eta}},$$
we need to split both the surjections $G^\natural[n]_{\tilde{\eta}} \twoheadrightarrow A[n]_{\tilde{\eta}}$ and $G[n]_{\tilde{\eta}} \twoheadrightarrow \frac{1}{n}Y/Y$.

If we have a splitting for the surjection $G^\natural[n]_{\tilde{\eta}} \twoheadrightarrow A[n]_{\tilde{\eta}}$, then the image of the splitting gives a closed subgroup scheme of $G^\natural[n]_{\tilde{\eta}}$, which we again denote by $A[n]_{\tilde{\eta}}$. Thus this splitting defines an isogeny $G^\natural_{\tilde{\eta}} \twoheadrightarrow G^{\natural\,\prime}_{\tilde{\eta}} := G^\natural_{\tilde{\eta}}/A[n]_{\tilde{\eta}}$. The subgroup scheme $T_{\tilde{\eta}}$ of $G^\natural_{\tilde{\eta}}$ embeds into a subgroup scheme $T'_{\tilde{\eta}}$ of $G^{\natural\,\prime}_{\tilde{\eta}}$, because $T_{\tilde{\eta}} \cap A[n]_{\tilde{\eta}} = 0$. Hence we have the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] & T_{\tilde{\eta}} \arrow[r] \arrow[d, "\wr"] & G^\natural_{\tilde{\eta}} \arrow[r] \arrow[d] & A_{\tilde{\eta}} \arrow[r] \arrow[d, "\mathrm{mod}\ A[n]_{\tilde{\eta}}"] & 0 \\
0 \arrow[r] & T'_{\tilde{\eta}} \arrow[r] & G^{\natural\,\prime}_{\tilde{\eta}} \arrow[r] & A_{\tilde{\eta}} \arrow[r] & 0
\end{tikzcd}$$

We can complete this to a diagram

$$\begin{tikzcd} 0 \arrow[r] & T_{\tilde\eta} \arrow[r] \arrow[d, "\wr"] & G^\natural_{\tilde\eta} \arrow[r] \arrow[d] & A_{\tilde\eta} \arrow[r] \arrow[d, "\mathrm{mod}\ A[n]_{\tilde\eta}"] & 0 \\ 0 \arrow[r] & T'_{\tilde\eta} \arrow[r] \arrow[d, "\mathrm{mod}\ T'_{\tilde\eta}[n]"'] & {G^\natural_{\tilde\eta}}' \arrow[r] \arrow[d] & A_{\tilde\eta} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T_{\tilde\eta} \arrow[r] & G^\natural_{\tilde\eta} \arrow[r] & A_{\tilde\eta} \arrow[r] & 0 \end{tikzcd}$$

in which every composition of two vertical arrows is the multiplication by n. Therefore, finding a splitting of $G^\natural[n]_{\tilde\eta} \twoheadrightarrow A[n]_{\tilde\eta}$ is equivalent to finding an isogeny ${G^\natural_{\tilde\eta}}' \twoheadrightarrow G^\natural_{\tilde\eta}$ of the following form.

$$\begin{tikzcd} 0 \arrow[r] & T'_{\tilde\eta} \arrow[r] \arrow[d, "\mathrm{mod}\ T'_{\tilde\eta}[n]"'] & {G^\natural_{\tilde\eta}}' \arrow[r] \arrow[d] & A_{\tilde\eta} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T_{\tilde\eta} \arrow[r] & G^\natural_{\tilde\eta} \arrow[r] & A_{\tilde\eta} \arrow[r] & 0 \end{tikzcd}$$

Since the surjection $T'_{\tilde\eta} \twoheadrightarrow T_{\tilde\eta}$ is the dual of the inclusion $X \hookrightarrow \frac{1}{n}X$, by Proposition 3.1.5.1, isogenies ${G^\natural_{\tilde\eta}}' \to G^\natural_{\tilde\eta}$ of the above form are equivalent to liftings $c_n : \frac{1}{n}X \to A^\vee_{\tilde\eta}$ over $\tilde\eta$ of the homomorphism $c : X \to A^\vee$ defining the extension structure of $0 \to T \to G^\natural \to A \to 0$. Since all the homomorphisms we consider above are $\mathcal{O}$-equivariant, the lifting c_n is also $\mathcal{O}$-equivariant by functoriality of Proposition 3.1.5.1. To summarize the above investigation,

Lemma 5.2.3.1. *With the setting as above, splittings of $G^\natural[n]_{\tilde\eta} \twoheadrightarrow A[n]_{\tilde\eta}$ correspond bijectively to liftings $c_n : \frac{1}{n}X \to A^\vee_{\tilde\eta}$ of $c : X \to A^\vee$ over $\tilde\eta$.*

Next let us study splittings for the surjection $G[n]_{\tilde\eta} \twoheadrightarrow \frac{1}{n}Y/Y$, which form a torsor under the group scheme $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y/Y, G^\natural[n]_{\tilde\eta})$.

Let $\tilde S = \mathrm{Spec}(\tilde R) \twoheadrightarrow S = \mathrm{Spec}(R)$ be the normalization of S over $\tilde\eta \to \eta$, which is noetherian and normal by Lemma 5.2.1.1. Let $\tilde I := \mathrm{rad}(I \cdot \tilde R) \subset \tilde R$. For technical simplicity, let us replace $\tilde R$ with a discrete valuation ring of $\mathrm{Frac}(\tilde R)$ containing $\tilde R$ and with a center on $\mathrm{Spec}(\tilde R/\tilde I)$, which is possible by Lemma 4.2.4.3, and replace $\tilde S$ accordingly. (The generic point $\tilde\eta = \mathrm{Spec}(\mathrm{Frac}(\tilde R))$ of $\tilde S$ is unchanged.)

Let $(A', \lambda_{A'}, i_{A'}, X', Y', \phi', c', (c^\vee)', \tau')$ be any tuple defined as follows:

1. $A' := A_{\tilde S}$, $i_{A'}$ is canonically induced by i_A, and $\lambda_{A'}$ is the pullback of the composition $A \overset{[n]}{\to} A \overset{\lambda_A}{\to} A^\vee \overset{[n]}{\to} A^\vee$ to $\tilde S$, which is simply $n^2\lambda_{A,\tilde S}$.

2. $X' := X$, $Y' := \frac{1}{n}Y$, and $\phi' : Y' \to X'$ is the pullback of the composition $\frac{1}{n}Y \overset{[n]}{\underset{\sim}{\to}} Y \overset{\phi}{\to} X \overset{[n]}{\to} X$ to $\tilde S$, which (by abuse of notation) is the pullback of $n^2\phi_n : \frac{1}{n}Y \to nX \hookrightarrow X$ if we extend ϕ naturally to $\phi_n : \frac{1}{n}Y \to \frac{1}{n}X$.

3. $c' := c_{\tilde S}$, and $(c^\vee)' : Y' = \frac{1}{n}Y \to A' = A_{\tilde S}$ is the unique extension of some lifting $c^\vee_n : \frac{1}{n}Y \to A'_{\tilde\eta} = A_{\tilde\eta}$ of $c^\vee_{\tilde\eta} : Y \to A_{\tilde\eta}$. Such a unique extension exists because $\tilde R$ is a discrete valuation ring, and because $A_{\tilde S} \to \tilde S$ (as the pullback of $A \to S$) is proper. Note that we have the compatibility $\lambda_{A'}(c^\vee)' = c'\phi'$ because
$$(n^2\lambda_{A,\tilde\eta})(c^\vee_n(\tfrac{1}{n}y)) = n\lambda_{A,\tilde\eta}(c^\vee(y)) = nc_{\tilde\eta}(\phi(y)) = c_{\tilde\eta}(n^2\phi_n(\tfrac{1}{n}y))$$
for all $y \in Y$.

4. $\tau' : \mathbf{1}_{\frac{1}{n}Y \times X,\tilde\eta} \overset{\sim}{\to} (c^\vee_n \times c)^*\mathcal{P}^{\otimes -1}_{A,\tilde\eta}$ is a trivialization of biextensions satisfying $\tau'|_{\mathbf{1}_{Y \times X,\tilde\eta}} = \tau_{\tilde\eta}$. By the same convention as c_n and $c^\vee_n$, we will also denote τ' by τ_n and consider it as a lifting of $\tau_{\tilde\eta}$.

   Note that we have the symmetry $\tau_n(\frac{1}{n}y_1, \phi'(\frac{1}{n}y_2)) = \tau_n(\frac{1}{n}y_2, \phi'(\frac{1}{n}y_1))$ for all $y_1, y_2 \in Y$, because
$$\begin{aligned}\tau_n(\tfrac{1}{n}y_1, n^2\phi_n(\tfrac{1}{n}y_2)) &= \tau_n(y_1, \phi(y_2)) = \tau(y_1, \phi(y_2)) \\ &= \tau(y_2, \phi(y_1)) = \tau_n(y_2, \phi(y_1)) = \tau_n(\tfrac{1}{n}y_2, n^2\phi_n(\tfrac{1}{n}y_1)).\end{aligned}$$
   It is convenient to replace the relation
$$\tau_n(\tfrac{1}{n}y_1, n^2\phi_n(\tfrac{1}{n}y_2)) = \tau_n(\tfrac{1}{n}y_2, n^2\phi_n(\tfrac{1}{n}y_1))$$
   for all $y_1, y_2 \in Y$ with the equivalent relation
$$\tau_n(y_1, \phi(y_2)) = \tau_n(y_2, \phi(y_1))$$
   for all $y_1, y_2 \in Y$, whose validity is more transparent because the restriction of τ_n to $\mathbf{1}_{Y \times X,\tilde\eta}$ is $\tau_{\tilde\eta}$.

   We require moreover that τ_n satisfies the compatibility $\tau_n(b\frac{1}{n}y, \chi) = \tau_n(\frac{1}{n}y, b^\star\chi)$ for all $\frac{1}{n}y \in \frac{1}{n}Y$, $\chi \in X$, and $b \in \mathcal{O}$.

Then $(A', \lambda_{A'}, i_{A'}, X', Y', \phi', c', (c^\vee)', \tau')$ defines an object in $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}(\tilde R, \tilde I)$.

Each pair of liftings $(c^\vee_{\tilde\eta}, \tau_n)$ as above corresponds to a period homomorphism $\iota_n : \frac{1}{n}Y \to G^\natural_{\tilde\eta}$ lifting the pullback $\iota_{\tilde\eta}$ of $\iota_\eta : Y_\eta \to G^\natural_\eta$ to $\tilde\eta$, such that $\iota_n|_Y = \iota_{\tilde\eta}$. Therefore the tuples $(A', \lambda_{A'}, i_{A'}, X', Y', \phi', c', (c^\vee)', \tau')$ as above form a torsor under the group scheme $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y/Y, G^\natural[n]_{\tilde\eta})$, the same one we saw above.

By Theorem 5.1.1.4, each tuple $(A', \lambda_{A'}, i_{A'}, X', \underline{Y}', \phi', c', (c^\vee)', \tau')$ as above defines an object (G', λ', i') of $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}(\tilde R, \tilde I)$. The morphisms $A_{\tilde S} \overset{\sim}{\to} A' = A_{\tilde S} \overset{[n]}{\to} A_{\tilde S}$, $X \overset{[n]}{\to} X' \overset{=}{\underset{\sim}{\to}} X$, and $\underline{Y}_{\tilde S} \hookrightarrow \underline{Y}' = \frac{1}{n}\underline{Y}_{\tilde S} \overset{[n]}{\underset{\sim}{\to}} \underline{Y}_{\tilde S}$ define morphisms $(A_{\tilde S}, X, \underline{Y}_{\tilde S}, c_{\tilde S}, c^\vee_{\tilde S}, \tau_{\tilde\eta}) \to (A', X', \underline{Y}', c', (c^\vee)', \tau') \to (A_{\tilde S}, X, \underline{Y}_{\tilde S}, c_{\tilde S}, c^\vee_{\tilde S}, \tau_{\tilde\eta})$ in $\mathrm{DD}(\tilde R, \tilde I)$ (see Definition 4.4.10), which by Theorem 4.4.16 induce homomorphisms $G_{\tilde S} \to G' \to G_{\tilde S}$ defining morphisms in $\mathrm{DEG}(\tilde R, \tilde I)$ (which are $\mathcal{O}$-equivariant by functoriality), whose composition is nothing but the multiplication by n on $G_{\tilde S}$. This shows that both the homomorphisms $G_{\tilde S} \to G'$ and $G' \to G_{\tilde S}$ are isogenies (see Definition 1.3.1.9) with quasi-finite flat kernels (see Lemma 1.3.1.11) annihilated by multiplication by n.

The first isogeny $G_{\tilde S} \to G'$ induces an isomorphism $G^\natural_{\tilde S} \overset{\sim}{\to} (G')^\natural$ between the Raynaud extensions because it is defined by $A_{\tilde S} \overset{\sim}{\to} A' = A_{\tilde S}$ and $\underline{Y}_{\tilde S} \hookrightarrow \underline{Y}'$, and because $c' := c_{\tilde S}$. Since we have a canonical short exact sequence $0 \to G^\natural[n]_{\tilde\eta} \to G[n]_{\tilde\eta} \to \frac{1}{n}Y/Y \to 0$, this shows that the kernel of $G_{\tilde S} \to G'$ is isomorphic to $\frac{1}{n}Y/Y$ and defines a splitting for the surjection $G[n]_{\tilde\eta} \twoheadrightarrow \frac{1}{n}Y/Y$.

By the proofs of Theorem 4.5.3.10 and Corollary 4.5.3.12, the assignment to each lifting $\iota_n : \frac{1}{n}Y \to G^\natural_{\tilde\eta}$ of $\iota_{\tilde\eta}$ a splitting for the surjection $G[n]_{\tilde\eta} \twoheadrightarrow \frac{1}{n}Y/Y$ is equivariant with the actions of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y/Y, G^{\natural}[n]_{\tilde{\eta}})$. Thus we have obtained the following lemma:

Lemma 5.2.3.2. *With the setting as above, ($\mathcal{O}$-equivariant) splittings of $G[n]_{\tilde{\eta}} \twoheadrightarrow \frac{1}{n}Y/Y$ correspond bijectively to ($\mathcal{O}$-equivariant) liftings $\iota_n : \frac{1}{n}Y \to G^{\natural}_{\tilde{\eta}}$ of $\iota : Y \to G^{\natural}_{\tilde{\eta}}$, and hence bijectively to liftings $(c_n^\vee, \tau_n)$ of $(c^\vee, \tau)$ over $\tilde{\eta}$, in the sense that $c_n^\vee : \frac{1}{n}Y \to A_{\tilde{\eta}}$ and $\tau_n : \mathbf{1}_{\frac{1}{n}Y \times X, \tilde{\eta}} \xrightarrow{\sim} (c_n^\vee \times c_{\tilde{\eta}})^* \mathcal{P}_{A,\tilde{\eta}}^{\otimes -1}$ respect the $\mathcal{O}$-structures as $c^\vee$ and τ do, and satisfy $c_n^\vee|_Y = c_{\tilde{\eta}}^\vee$ and $\tau_n|_{\mathbf{1}_{Y \times X, \tilde{\eta}}} = \tau_{\tilde{\eta}}$, respectively. In this case, τ_n satisfies the symmetry condition $\tau_n(y_1, \phi(y_2)) = \tau_n(y_2, \phi(y_1))$ for all $y_1, y_2 \in Y$.*

Proposition 5.2.3.3. *Let $(A, \lambda_A, i_A, X, \underline{Y}, \phi, c, c^\vee, \tau)$ be a tuple in $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}(R, I)$ corresponding to a triple (G, λ, i) in $\mathrm{DEG}_{\mathrm{PE},\mathcal{O}}(R, I)$ via Theorem 5.1.1.4. Assume for simplicity that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively. Let $\{\mathtt{W}_{-i,n}\}_i$ denote the filtration $0 \subset \mathtt{W}_{-2,n} = T[n]_{\tilde{\eta}} \subset \mathtt{W}_{-1,n} = G^{\natural}[n]_{\tilde{\eta}} \subset \mathtt{W}_{0,n} = G[n]_{\tilde{\eta}}$ on $G[n]_{\tilde{\eta}}$, with graded pieces $\mathrm{Gr}^{\mathtt{W}}_{-i,n} := \mathtt{W}_{-i,n}/\mathtt{W}_{-i-1,n}$. Then splittings $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n = \bigoplus_i \mathrm{Gr}^{\mathtt{W}}_{-i,n} \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ of the filtration are in bijection with triples $(c_n, c_n^\vee, \tau_n)$ lifting $(c, c^\vee, \tau)$ over $\tilde{\eta}$, in the sense that the homomorphisms $c_n : \frac{1}{n}X \to A^\vee_{\tilde{\eta}}$, $c_n^\vee : \frac{1}{n}Y \to A_{\tilde{\eta}}$, and $\tau_n : \mathbf{1}_{\frac{1}{n}Y \times X, \tilde{\eta}} \xrightarrow{\sim} (c_n^\vee \times c_{\tilde{\eta}})^* \mathcal{P}_{A,\tilde{\eta}}^{\otimes -1}$ respect the $\mathcal{O}$-structures as c, $c^\vee$, and τ do, and satisfy $c_n|_X = c_{\tilde{\eta}}$, $c_n^\vee|_Y = c^\vee_{\tilde{\eta}}$, and $\tau_n|_{\mathbf{1}_{Y \times X, \tilde{\eta}}} = \tau_{\tilde{\eta}}$, respectively.*

Definition 5.2.3.4. *A triple $(c_n, c_n^\vee, \tau_n)$ as in Proposition 5.2.3.3 is called **liftable** if it is liftable to some $(c_m, c_m^\vee, \tau_m)$ over $\bar{\eta}$ as in Proposition 5.2.3.3 for all m such that $n|m$ and $\Box \nmid m$. We shall write a compatible system of such liftings $\{(c_m, c_m^\vee, \tau_m)\}_{n|m, \Box\nmid m}$ symbolically as a triple $(\hat{c}, \hat{c}^\vee, \hat{\tau})$, where the homomorphisms are written as $\hat{c} : X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box \to A^\vee_{\tilde{\eta}}$, as $\hat{c}^\vee : Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box \to A_{\tilde{\eta}}$, and as $\hat{\tau} : \mathbf{1}_{(Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \times Y, \tilde{\eta}} \xrightarrow{\sim} (\hat{c}^\vee \times c_{\tilde{\eta}})^* \mathcal{P}_{A,\tilde{\eta}}^{\otimes -1}$, respectively.*

Corollary 5.2.3.5. *With the setting as in Proposition 5.2.3.3, ς_n is liftable (see Definition 5.2.2.12) if and only if the triple $(c_n, c_n^\vee, \tau_n)$ is liftable (see Definition 5.2.3.4). In this case, a lifting $\hat{\varsigma}$ corresponds to a symbolic triple $(\hat{c}, \hat{c}^\vee, \hat{\tau})$ as in Definition 5.2.3.4.*

It is natural to ask whether a lifting $(c_n, c_n^\vee, \tau_n)$ (over $\tilde{\eta}$) determines the original triple $(c, c^\vee, \tau)$. To answer this, let us introduce some new notation:

Definition 5.2.3.6. *Let U be a scheme. Let $\underline{N}$ be any étale sheaf of left $\mathcal{O}$-modules that becomes constant over a finite étale covering of U. Let (Z, λ_Z) be any polarized abelian scheme over U with left $\mathcal{O}$-module structure given by some $i_Z : \mathcal{O} \to \mathrm{End}_U(Z)$. Then we denote by $\underline{\mathrm{Hom}}_{\mathcal{O}}(\underline{N}, Z)$ the (commutative) group functor of $\mathcal{O}$-equivariant group homomorphisms from the group functor $\underline{N}$ to the group functor Z.*

Lemma 5.2.3.7. *Suppose that W is a commutative proper group scheme of finite presentation over U. Suppose that W_0 is an abelian subscheme of W (i.e., a subgroup scheme that is an abelian scheme), and that there is an integer $m \geq 1$ invertible in $\mathscr{O}_U$ such that multiplication by m defines a homomorphism $[m] : W \to W$ with schematic image a (closed) subscheme of W_0 and such that $W[m]$, the m-torsion subgroup scheme of W, is finite étale over U. Then, for every geometric point $\bar{s} \to U$, the fiber $(W_0)_{\bar{s}}$ is the connected component of $W_{\bar{s}}$ containing the identity section. The quotient group functor W/W_0 is representable by a commutative finite étale group scheme E. The group $\pi_0(W_{\bar{s}})$ of connected components can be canonically identified with the $\bar{s}$-valued points of $E_{\bar{s}}$.*

Proof. Since W is commutative and since W_0 is (fppf locally) m-divisible as an abelian scheme, the condition that $[m]$ sends W to W_0 shows that W/W_0 can be identified with the quotient of $W[m]$ by $W_0[m] = W_0 \cap W[m]$. Since m is invertible in $\mathscr{O}_U$, both $W[m]$ and $W_0[m]$ are finite étale, and hence the quotient W/W_0 can be representable by a finite étale group scheme. The statements on the identity components and group of connected components of geometric fibers are obvious. $\Box$

Definition 5.2.3.8. *Suppose W_0 is an abelian subscheme of a proper group scheme W over a base scheme U, such that for every geometric point $\bar{s} \to U$, the fiber $(W_0)_{\bar{s}}$ is the connected component of $W_{\bar{s}}$ containing the identity section. Then we say that W_0 is the **fiberwise geometric identity component** of W, and denote it by W°. (By [59, IV-2, 4.5.13], it is also correct to say that W_0 is the **fiberwise identity component**, without the term **geometric**; cf. Remark 1.3.1.2.)*

*Suppose the quotient group functor W/W_0 is representable by a finite group scheme E. Then we say that E is the **group scheme of fiberwise geometric connected components**, and denote it by $\underline{\pi}_0(W/U)$.*

By Lemma 5.2.3.7, the finite group scheme $\underline{\pi}_0(W/U)$ is defined and is finite étale over U if W is commutative and if there is an integer $m \geq 1$ invertible in $\mathscr{O}_U$ such that multiplication by m defines a homomorphism $[m] : W \to W$ with schematic image a (closed) subscheme of W_0 and such that $W[m]$ is finite étale over U.

Proposition 5.2.3.9. *With the setting as in Definition 5.2.3.6, suppose $\underline{N}$ is constant with value some finitely generated $\mathcal{O}$-module N. Then the following are true:*

1. *The group functor $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ is representable by a proper subgroup scheme of an abelian scheme over U.*
2. *If N is torsion with number of elements prime to the residue characteristics of U, then $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ is finite étale over U.*
3. *If N is projective as an $\mathcal{O}$-module, then $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ is representable by an abelian scheme over U.*
4. *If N is an $\mathcal{O}$-lattice, and if the residue characteristics of U are unramified in $\mathcal{O}$, then $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ is representable by a proper smooth group scheme over U which is an extension of a (commutative) finite étale group scheme, whose rank has no prime factors other than those of the discriminant $\mathrm{Disc} = \mathrm{Disc}_{\mathcal{O}/\mathbb{Z}}$, by an abelian scheme over U.*

   Following Definition 5.2.3.8, we shall say that $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)$ is the extension of the finite étale group scheme $\underline{\pi}_0(\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)/U)$ by the abelian scheme $\underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z)^\circ$ over U.

This is called *Serre's construction*. (Definition 5.2.3.6, Lemma 5.2.3.7, Definition 5.2.3.8, and Proposition 5.2.3.9 all generalize naturally to the case when U is an algebraic stack.)

Proof. Since $\mathcal{O}$ is (left) noetherian (see, for example, [107, Cor. 2.10]), and since N is finitely generated, there is a *free resolution*
$$\mathcal{O}^{\oplus r_1} \to \mathcal{O}^{\oplus r_0} \to N \to 0$$
for some integers $r_0, r_1 \geq 0$. By taking $\underline{\mathrm{Hom}}_{\mathcal{O}}(\,\cdot\,, Z)$, we obtain an exact sequence
$$0 \to \underline{\mathrm{Hom}}_{\mathcal{O}}(N, Z) \to Z^{r_0} \to Z^{r_1} \tag{5.2.3.10}$$

(of fppf sheaves) over U, where Z^{r_0} (resp. Z^{r_1}) stands for the fiber products of r_0 (resp. r_1) copies of Z over U, which shows that $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ is representable because it is the kernel of the homomorphism $Z^{r_0} \to Z^{r_1}$ between abelian schemes in (5.2.3.10).

To show that $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ is proper over U, note that the first homomorphism in (5.2.3.10) is a closed immersion because Z^s is separated over U, and every closed subscheme of Z^r is proper over U. This proves 1 of Proposition 5.2.3.9.

Suppose N is torsion with number of elements prime to the residue characteristics of U. Let $m \geq 1$ be an integer that is invertible in $\mathscr{O}_U$ and annihilates every element in N. Then $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ is isomorphic to the closed subscheme $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z[m])$ of the finite *étale* group scheme $\underline{\mathrm{Hom}}(N,Z[m])$ over U. Over an étale covering of U over which $Z[m]$ with its $\mathcal{O}$-action becomes a constant group scheme, the condition of compatibility with $\mathcal{O}$-actions is both open and closed on the pullback of $\underline{\mathrm{Hom}}(N,Z[m])$. This shows that $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ is finite étale over U, and proves 2 of Proposition 5.2.3.9.

If N is projective, then it is, in particular, flat (by [107, Cor. 2.16]). This is the same for its dual (right) $\mathcal{O}$-module $N^\vee$. Hence, for every embedding $U \hookrightarrow \tilde{U}$ defined by an ideal $\mathscr{I}$ such that $\mathscr{I}^2 = 0$, the surjectivity of the morphism $Z(\tilde{U}) \to Z(U)$ of $\mathcal{O}$-modules implies the surjectivity of the morphism $(N^\vee \underset{\mathcal{O}}{\otimes} Z)(\tilde{U}) \cong N^\vee \underset{\mathcal{O}}{\otimes} Z(\tilde{U}) \to (N^\vee \underset{\mathcal{O}}{\otimes} Z)(U) \cong N^\vee \underset{\mathcal{O}}{\otimes} Z(U)$. This shows that $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z) \to U$ is formally smooth, and hence smooth because it is (locally) of finite presentation (see [59, IV-4, 17.3.1 and 17.5.2]). Moreover, since N is projective, there exists some projective $\mathcal{O}$-module N' such that $N \oplus N' \cong \mathcal{O}^{\oplus r}$ for some $r \geq 0$. Then we have $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z) \underset{U}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(N',Z) \cong Z^r$, which shows that the geometric fibers of $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z) \to U$ are connected. Hence we see by definition that $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ is an abelian scheme over U. This proves 3 of Proposition 5.2.3.9.

Finally, suppose that N is an $\mathcal{O}$-lattice, and that the residue characteristics of U are unramified in $\mathcal{O}$.

Let $\mathcal{O}'$ be any maximal order in B containing $\mathcal{O}$. By Proposition 1.1.1.21, there exists an integer $m \geq 1$, with no prime factors other than those of Disc, such that $m\mathcal{O}' \subset \mathcal{O}$. Therefore, there is an isogeny $Z \twoheadrightarrow Z'$ with kernel a subgroup scheme of $Z[m]$ such that the action of $\mathcal{O}$ on Z induces an action of $\mathcal{O}'$ on Z'. In this case, there is also a canonical isogeny $Z' \twoheadrightarrow Z$ whose pre- and postcompositions with the previous isogeny $Z \twoheadrightarrow Z'$ are multiplications by m on Z and Z', respectively. Let N' be the $\mathcal{O}'$-span of N in $N \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. Since N' is the $\mathcal{O}'$-space of N, the canonical isogeny $Z \twoheadrightarrow Z'$ induces a canonical homomorphism $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z) \to \underline{\mathrm{Hom}}_{\mathcal{O}'}(N',Z')$. On the other hand, the canonical isogeny $Z' \twoheadrightarrow Z$ above induces a canonical homomorphism $\underline{\mathrm{Hom}}_{\mathcal{O}'}(N',Z') \to \underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$, whose pre- and postcomposition with the previous canonical homomorphism $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z) \to \underline{\mathrm{Hom}}_{\mathcal{O}'}(N',Z')$ is nothing but the multiplications by m on $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ and $\underline{\mathrm{Hom}}_{\mathcal{O}'}(N',Z')$, respectively. As usual, we denote by $[m]$ all such multiplications by m.

Since $\mathcal{O}'$ is maximal, N is projective as an $\mathcal{O}'$-module by Proposition 1.1.1.23. By 3 of Proposition 5.2.3.9 proved above, we know that $\underline{\mathrm{Hom}}_{\mathcal{O}'}(N',Z')$ is an abelian scheme. Since $[m] : \underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ factors as the composition of canonical homomorphisms $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z) \to \underline{\mathrm{Hom}}_{\mathcal{O}'}(N',Z') \to \underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$, this shows that the schematic image of $[m] : \underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)$ is an abelian scheme. On the other hand, by working over an étale covering of U over which $Z[m]$ with its $\mathcal{O}$-action becomes a constant group scheme (as in the proof of 2 of Proposition 5.2.3.9 above), we see that $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)[m] \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z[m])$ is finite étale (of rank dividing a power of m) over U. Hence, by Lemma 5.2.3.7, we see that both $\underline{\pi}_0(\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)/U)$ and $\underline{\mathrm{Hom}}_{\mathcal{O}}(N,Z)^\circ$ are defined with the desired properties. $\square$

Let us return to the context preceding Definition 5.2.3.6. As a consequence of Proposition 5.2.3.9, we see that group functors such as $\underline{\mathrm{Hom}}_{\mathcal{O}}(X,A^\vee)$ and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y,A)$ are representable by *proper schemes* over S (which are smooth over each subscheme of S over which the discriminant Disc is invertible). In particular,

Corollary 5.2.3.11. *The morphism c (resp. $c^\vee$) is determined (as a unique extension) by its pullback c_η (resp. $c^\vee_\eta$) to η by noetherian normality of S, and hence by c_n (resp. $c^\vee_n$) if $c_n|_X$ (resp. $c^\vee_n|_Y$) descends to η.*

Note that $\lambda_A c^\vee = c\phi$ implies that $(\lambda_{A,\tilde{\eta}} c^\vee_n - c_n\phi_n)(\frac{1}{n}y)$ is n-torsion in $A^\vee_{\tilde{\eta}}$ for all $y \in Y$. Also, the relation $\tau_n(y_1,\phi(y_2)) = \tau_n(y_2,\phi(y_1))$ implies that $\tau_n(\frac{1}{n}y_1,\phi(y_2))\tau_n(\frac{1}{n}y_2,\phi(y_1))^{-1}$ is n-torsion in $\mathbf{G}_{\mathrm{m},\tilde{\eta}}$ for all $y_1,y_2 \in Y$. Here the comparison between $\tau_n(\frac{1}{n}y_1,\phi(y_2))$ and $\tau_n(\frac{1}{n}y_2,\phi(y_1))^{-1}$ makes sense because we have a canonical isomorphism

$$\begin{aligned}\mathcal{P}_{A,\tilde{\eta}}|_{(c^\vee_n(\frac{1}{n}y_1),c\phi(y_2))} &\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{D}_2(\mathcal{M}^{\otimes n}_{\tilde{\eta}})|_{(c^\vee_n(\frac{1}{n}y_1),c^\vee_n(\frac{1}{n}y_2))} \\ &\overset{\text{sym.}}{\xrightarrow{\sim}} \mathcal{D}_2(\mathcal{M}^{\otimes n}_{\tilde{\eta}})|_{(c^\vee_n(\frac{1}{n}y_2),c^\vee_n(\frac{1}{n}y_1))} \overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{P}_{A,\tilde{\eta}}|_{(c^\vee_n(\frac{1}{n}y_2),c\phi(y_1))}.\end{aligned}$$

Let us record this observation as follows:

Lemma 5.2.3.12. *With the setting as in Proposition 5.2.3.3, each splitting $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ that corresponds to a lifting $(c_n,c^\vee_n,\tau_n)$ of $(c,c^\vee,\tau)$ over $\tilde{\eta}$ defines two pairings:*

1. *The first pairing*
$$\mathtt{d}_{10,n} : \mathrm{Gr}^{\mathtt{W}}_{-1,n} \times \mathrm{Gr}^{\mathtt{W}}_{0,n} \cong A[n]_{\tilde{\eta}} \times (\tfrac{1}{n}Y/Y) \to \boldsymbol{\mu}_{n,\tilde{\eta}}$$
sends $(a,\frac{1}{n}y)$ to $\mathrm{e}_{A[n]}(a,(\lambda_{A,\tilde{\eta}}c^\vee_n - c_n\phi_n)(\frac{1}{n}y))$, where
$$\mathrm{e}_{A[n]} : A[n] \times A^\vee[n] \to \boldsymbol{\mu}_{n,S}$$
is the canonical pairing between $A[n]$ and $A^\vee[n]$. The pairing $\mathrm{e}_{A[n]}$ allows us to identify $A^\vee[n]$ with $\underline{\mathrm{Hom}}_S(A[n],\mathbf{G}_{\mathrm{m},S}) \cong \underline{\mathrm{Hom}}_S(A[n],\boldsymbol{\mu}_{n,S})$, the Cartier dual of $A[n]$. (For simplicity, we use the same notation for the pullback of $\mathrm{e}_{A[n]}$ to $\tilde{\eta}$.)

2. *The second pairing*
$$\mathtt{d}_{00,n} : \mathrm{Gr}^{\mathtt{W}}_{0,n} \times \mathrm{Gr}^{\mathtt{W}}_{0,n} \cong (\tfrac{1}{n}Y/Y) \times (\tfrac{1}{n}Y/Y) \to \boldsymbol{\mu}_{n,\tilde{\eta}}$$
sends $(\frac{1}{n}y_1,\frac{1}{n}y_2)$ to $\tau_n(\frac{1}{n}y_1,\phi(y_2))\tau_n(\frac{1}{n}y_2,\phi(y_1))^{-1}$.

Then $-{}^t\mathtt{d}^\star_{00,n} = \mathtt{d}_{00,n}$ (in additive notation, as in Section 5.2.2).

Corollary 5.2.3.13. *With the setting as in Corollary 5.2.3.5, each splitting $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \xrightarrow{\sim} \mathrm{T}^{\square}\, G_{\tilde{\eta}}$ that corresponds to a lifting $(\hat{c},\hat{c}^\vee,\hat{\tau}) = \{(c_m,c^\vee_m,\tau_m)\}_{n|m,\square\nmid m}$ of $(c,c^\vee,\tau)$ defines two pairings:*

1. *The first pairing*
$$\mathtt{d}_{10} : \mathrm{Gr}^{\mathtt{W}}_{-1} \times \mathrm{Gr}^{\mathtt{W}}_0 \cong \mathrm{T}^{\square}\, A_{\tilde{\eta}} \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}) \to \mathrm{T}^{\square}\, \mathbf{G}_{\mathrm{m},\tilde{\eta}}$$
is defined by the pairings $\{\mathtt{d}_{10,m}\}_{n|m,\square\nmid m}$. We can interpret this pairing as defined by sending (a,y) to $\mathrm{e}_A(a,(\lambda_{A,\tilde{\eta}}\hat{c}^\vee - \widehat{c\phi})(y))$, where
$$\mathrm{e}_A : \mathrm{T}^{\square}\, A_{\tilde{\eta}} \times \mathrm{T}^{\square}\, A^\vee_{\tilde{\eta}} \to \mathrm{T}^{\square}\, \mathbf{G}_{\mathrm{m},\tilde{\eta}}$$
is the canonical pairing defined by $\{\mathrm{e}_{A[m]}\}_{\square\nmid m}$.

2. *The second pairing*
$$\mathtt{d}_{00} : \mathrm{Gr}_0^{\mathtt{W}} \times \mathrm{Gr}_0^{\mathtt{W}} \cong (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\square}) \to \mathbf{G}_{\mathrm{m},\bar{\eta}}$$
is defined by the pairings $\{\mathtt{d}_{00,m}\}_{n|m,\square\nmid m}$.

Then $-{}^t\mathtt{d}_{00}^{\star} = \mathtt{d}_{00}$ *(in additive notation, as in the context of Section 5.2.2).*

On the other hand, the splitting $\varsigma_n : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} G[n]_{\bar{\eta}}$ allows us to write the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}$ on $G[n]_{\bar{\eta}}$ in matrix form
$$\begin{pmatrix} & & \mathrm{e}_{20,n} \\ & \mathrm{e}_{11,n} & \mathrm{e}_{10,n} \\ \mathrm{e}_{02,n} & \mathrm{e}_{01,n} & \mathrm{e}_{00,n} \end{pmatrix},$$
where
$$\mathrm{e}_{ij,n} : \mathrm{Gr}_{-i,n}^{\mathtt{W}} \times \mathrm{Gr}_{-j,n}^{\mathtt{W}} \to \boldsymbol{\mu}_{n,\bar{\eta}}$$
are the pairings induced by ς_n. We know that $\mathrm{e}_{20,n} = \mathrm{e}^{\phi}$ and $\mathrm{e}_{11,n} = \mathrm{e}^{\lambda_A}$ (in their natural modulo-n versions). The point is to identify $\mathrm{e}_{10,n}$ and $\mathrm{e}_{00,n}$ with something we can parameterize.

The following comparison is the key to the generalization of Faltings and Chai's theory:

Theorem 5.2.3.14. *With the setting as above, we have* $\mathrm{e}_{10,n} = \mathtt{d}_{10,n}$ *and* $\mathrm{e}_{00,n} = \mathtt{d}_{00,n}$.

Corollary 5.2.3.15. *With the setting as above, we have* $\mathrm{e}_{10} = \mathtt{d}_{10}$ *and* $\mathrm{e}_{00} = \mathtt{d}_{00}$.

5.2.4 Weil Pairings in General

Before we present the proof of Theorem 5.2.3.14, let us review the calculation of Weil pairings in general. (In particular, let us make clear our choice of sign conventions.)

In this section, we shall assume that we are given an abelian scheme A over an arbitrary base scheme S. Whenever possible, we shall intentionally confuse the notion of $\mathbf{G}_{\mathrm{m}}$-torsors and invertible sheaves (see Corollary 3.1.2.14). In particular, we shall use $\mathscr{O}_A$ to denote both the trivial invertible sheaf and the trivial $\mathbf{G}_{\mathrm{m}}$-torsor over A. Moreover, $\mathscr{O}_A$ will also be used to mean the structural sheaf of A. We hope that the convenience of such an abuse of notation will outweigh the confusion it incurs.

Let us denote by $\mathcal{P}_A$ the Poincaré biextension of $A \underset{S}{\times} A^\vee$ by $\mathbf{G}_{\mathrm{m},S}$, and denote by $\mathcal{N}_h$ the invertible sheaf $\mathcal{P}_A|_{A \underset{S}{\times} \{h\}}$ for each $h : S \to A^\vee$. (Here we are using an abuse of notation alluded to above.) Then the tautological rigidification $\mathcal{P}_A|_{e \underset{S}{\times} A^\vee} \xrightarrow{\sim} \mathscr{O}_{A^\vee}$ gives a rigidification $\mathcal{N}_h(e) \xrightarrow{\sim} \mathscr{O}_{A^\vee}(h) \xrightarrow{\sim} \mathscr{O}_S$. The last isomorphism $\mathscr{O}_{A^\vee}(h) := h^*\mathscr{O}_{A^\vee} \xrightarrow{\sim} \mathscr{O}_S$ is the tautological one given by the structural morphism $h^* : \mathscr{O}_{A^\vee} \to \mathscr{O}_S$ of $h : S \to A^\vee$.

Let us first explain how to calculate the canonical perfect pairing
$$\mathrm{e}_{A[n]} : A[n] \times A^\vee[n] \to \boldsymbol{\mu}_{n,S}$$
for each integer $n \geq 1$. More generally, suppose that we have an isogeny $f : A \to A'$ with kernel K. Suppose that the dual isogeny $f^\vee : (A')^\vee \to A^\vee$ has kernel $K^\vee$. Then the kernels K and $K^\vee$ are related by a canonical perfect pairing
$$\mathrm{e}_K : K \times K^\vee \to \mathbf{G}_{\mathrm{m},S},$$
and $\mathrm{e}_{A[n]}$ is the special case of e_K with $K = A[n]$. We shall explain how to calculate e_K for general K.

For each point $h \in K^\vee$, the invertible sheaf $\mathcal{N}_h := \mathcal{P}_{A'}|_{A' \underset{S}{\times} \{h\}}$ satisfies
$$f^*\mathcal{N}_h = f^*(\mathcal{P}_{A'}|_{A' \underset{S}{\times} \{h\}}) \overset{\text{can.}}{\xrightarrow{\sim}} ((f \times \mathrm{Id}_{(A')^\vee})^*\mathcal{P}_{A'})|_{A \underset{S}{\times} \{h\}}$$
$$\overset{\text{can.}}{\xrightarrow{\sim}} ((\mathrm{Id}_A \times f^\vee)^*\mathcal{P}_A)|_{A \underset{S}{\times} \{h\}} \overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{P}_A|_{A \underset{S}{\times} \{f^\vee(h)\}} \overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{P}_A|_{A \underset{S}{\times} \{e\}} \overset{\text{rig.}}{\xrightarrow{\sim}} \mathscr{O}_A,$$
because of the canonical isomorphism $(f \times \mathrm{Id}_{(A')^\vee})^*\mathcal{P}_{A'} \overset{\text{can.}}{\xrightarrow{\sim}} (\mathrm{Id}_A \times f^\vee)^*\mathcal{P}_A$ given by Lemma 1.3.2.10, and because of the rigidification along the first factor of the Poincaré biextension. On the other hand, we shall interpret $f : A \to A'$ as identifying A' with the quotient A/K of A by the finite flat group scheme K of finite presentation. Therefore, by the theory of descent, the isomorphism $f^*\mathcal{N}_h \xrightarrow{\sim} \mathscr{O}_A$ corresponds to the descent datum on $\mathscr{O}_A$ describing $\mathcal{N}$ as a descended form of $\mathscr{O}_A$. The descent datum is given by an action of K on $\mathscr{O}_A$, which is a compatible collection of isomorphisms
$$\kappa(a) : T_a^*\mathscr{O}_A \xrightarrow{\sim} \mathscr{O}_A$$
for each $a \in K$. On the other hand, the structural morphism of the translation isomorphism $T_a : A \xrightarrow{\sim} A$ gives another isomorphism
$$\mathrm{str.}(a) : T_a^*\mathscr{O}_A \xrightarrow{\sim} \mathscr{O}_A.$$
Since A is an abelian scheme, which satisfies Assumption 3.1.2.7, the two isomorphisms can only differ in the rigidifications. We shall redefine $\kappa(a)$ (resp. $\mathrm{str.}(a)$) to be the isomorphism $\mathscr{O}_A(a) \xrightarrow{\sim} \mathscr{O}_S$ inducing the original $\kappa(a)$ (resp. $\mathrm{str.}(a)$) above by composing $T_a^*\mathscr{O}_A \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_A \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a) \xrightarrow{\sim} \mathscr{O}_A \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S \cong \mathscr{O}_A$. Their difference is a section
$$\tilde{h}(a) := \mathrm{str.}(a) \circ \kappa(a)^{-1} : \mathscr{O}_S \xrightarrow{\sim} \mathscr{O}_S,$$
which (with varying $a \in K$) defines a morphism
$$\tilde{h} : K \to \mathbf{G}_{\mathrm{m},S}.$$
The multiplicative structure of A gives a commutative diagram as follows.
$$\begin{tikzcd} \mathscr{O}_A(a+a') \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\mathrm{str.}(a+a')"', "\wr"] & \mathscr{O}_A(a) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a') \arrow[d, "\wr"', "\mathrm{str.}(a)\otimes\mathrm{str.}(a')"] \\ \mathscr{O}_S \arrow[r, "\sim", "\text{can.}"'] & \mathscr{O}_S \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S \end{tikzcd} \tag{5.2.4.1}$$
The compatibility of the action of K, which compares
$$T_{a+a'}^*\mathscr{O}_A \overset{\text{can.}}{\xrightarrow{\sim}} T_a^*(\mathscr{O}_A \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a')) \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_A \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a') \xrightarrow{\sim} \mathscr{O}_A$$
using $\kappa(a) : \mathscr{O}_A(a) \xrightarrow{\sim} \mathscr{O}_S$ and $\kappa(a') : \mathscr{O}_A(a') \xrightarrow{\sim} \mathscr{O}_S$ with
$$T_{a+a'}^*\mathscr{O}_A \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_A \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a+a') \xrightarrow{\sim} \mathscr{O}_S$$
using $\kappa(a+a') : \mathscr{O}_A(a+a') \xrightarrow{\sim} \mathscr{O}_A$ (resp. $\mathrm{str.}(a+a') : \mathscr{O}_A(a+a') \xrightarrow{\sim} \mathscr{O}_A$), gives another commutative diagram:
$$\begin{tikzcd} \mathscr{O}_A(a+a') \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\kappa(a+a')"', "\wr"] & \mathscr{O}_A(a) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_A(a') \arrow[d, "\wr"', "\kappa(a)\otimes\kappa(a')"] \\ \mathscr{O}_S \arrow[r, "\sim", "\text{can.}"'] & \mathscr{O}_S \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S \end{tikzcd} \tag{5.2.4.2}$$

The comparison of (5.2.4.1) and (5.2.4.2) shows that the morphism $\tilde{h}$ is a *homomorphism*. That is, $\tilde{h}$ is an element of $\underline{\mathrm{Hom}}_S(K, \mathbf{G}_{\mathrm{m},S})$, the Cartier dual of K. This sets up an identification between $h \in K^\vee$ and $\tilde{h} \in \underline{\mathrm{Hom}}_S(K, \mathbf{G}_{\mathrm{m},S})$, which defines an isomorphism $K^\vee \overset{\sim}{\to} \underline{\mathrm{Hom}}_S(K, \mathbf{G}_{\mathrm{m},S})$. Hence we have a perfect pairing

$$\mathrm{e}_K : K \times K^\vee \to \mathbf{G}_{\mathrm{m},S}.$$

This is essentially the argument of [94, §15, proof of Thm. 1].

Let us summarize the above argument as follows:

Lemma 5.2.4.3. *The diagram*

$$\begin{tikzcd}
((f \times \mathrm{Id}_{(A')^\vee})^* \mathcal{P}_{A'})|_{(a,h)} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] \arrow[dr, "\text{can.}", "\sim"'] & \mathcal{P}_{A'}|_{(f(a),h)} \arrow[r, equal] & \mathcal{P}_{A'}|_{(e,h)} \arrow[r, "\text{rig.}", "\sim"'] & \mathscr{O}_S \arrow[d, equal] \\
((\mathrm{Id}_A \times f^\vee)^* \mathcal{P}_A)|_{(a,h)} \arrow[d, "\text{can.}"', "\wr"] & (f^*(\mathcal{N}_h))(a) \arrow[rr, "\sim"] \arrow[d, "\text{can.}"', "\wr"] & & \mathscr{O}_S \arrow[d, "\wr"', "\tilde{h}(a) = \mathrm{e}_K(a,h)"] \\
\mathcal{P}_A|_{(a,f^\vee(h))} \arrow[d, equal] & \mathscr{O}_A(a) \arrow[rr, "\sim", "\text{str.}(a)"'] \arrow[urr, "\kappa(a)", "\sim"'] & & \mathscr{O}_S \arrow[d, equal] \\
\mathcal{P}_A|_{(a,e)} \arrow[rrr, "\sim", "\text{rig.}"'] \arrow[ur, "\sim", "\text{can.}"'] & & & \mathscr{O}_S
\end{tikzcd}$$

is commutative.

Proof. The key point is to observe that the action morphism $\kappa(a) : T_a^* \mathscr{O}_A \overset{\sim}{\to} \mathscr{O}_A$ is the composition of $T_a^* \mathscr{O}_A \overset{\text{can.}}{\underset{\sim}{\to}} T_a^* f^* \mathcal{N}_h \overset{\text{can.}}{\underset{\sim}{\to}} f^* T_{f(a)}^* \mathcal{N}_h = f^* T_e^* \mathcal{N}_h = f^* \mathcal{N}_h \overset{\text{can.}}{\underset{\sim}{\to}} \mathscr{O}_A$. If we pullback under the identity section $e : S \to A$, and compose with the rigidification $\mathscr{O}_A(e) \overset{\sim}{\to} \mathscr{O}_S$, then we see that $\kappa(a) : \mathscr{O}_A(a) \overset{\sim}{\to} \mathscr{O}_A(e) \overset{\text{rig.}}{\underset{\sim}{\to}} \mathscr{O}_S$ is the composition $\mathscr{O}_A(a) \overset{\sim}{\to} (f^*(\mathcal{N}_h))(a) \overset{\sim}{\to} \mathcal{N}_h(f(a)) = \mathcal{P}_{A'}|_{(f(a),h)} = \mathcal{P}_{A'}|_{(e,h)} \overset{\text{rig.}}{\underset{\sim}{\to}} \mathscr{O}_S$, which is exactly the upper half of the diagram. The commutativity of the remainder of the diagram is clear. $\square$

This diagram shows that, in order to compute $\mathrm{e}_K(a,h)$, it is not necessary to know the isomorphisms $\kappa(a)$ and $\text{str.}(a)$. The following proposition is the essential point we need:

Proposition 5.2.4.4. *The canonical pairing*

$$\mathrm{e}_K : K \times K^\vee \to \mathbf{G}_{\mathrm{m},S}$$

gives, for each $a \in K$ and $h \in K^\vee$, the isomorphism $\mathrm{e}_K(a,h)$ which makes the diagram

$$\begin{tikzcd}
((f \times \mathrm{Id}_{(A')^\vee})^* \mathcal{P}_{A'})|_{(a,h)} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & \mathcal{P}_{A'}|_{(f(a),h)} \arrow[r, equal] & \mathcal{P}_{A'}|_{(e,h)} \arrow[r, "\text{rig.}", "\sim"'] & \mathscr{O}_S \arrow[d, "\mathrm{e}_K(a,h)"', "\wr"] \\
((\mathrm{Id}_A \times f^\vee)^* \mathcal{P}_A)|_{(a,h)} \arrow[r, "\sim", "\text{can.}"'] & \mathcal{P}_A|_{(a,f^\vee(h))} \arrow[r, equal] & \mathcal{P}_A|_{(a,e)} \arrow[r, "\sim", "\text{rig.}"'] & \mathscr{O}_S
\end{tikzcd}$$

commutative. That is, $\mathrm{e}_K(a,h)$ measures the difference between the two rigidifications of the biextension $((f \times \mathrm{Id}_{(A')^\vee})^* \mathcal{P}_{A'})|_{(a,h)} \cong ((\mathrm{Id}_A \times f^\vee)^* \mathcal{P}_A)|_{(a,h)}$.

Now suppose that we are given a polarization $\lambda_A : A \to A^\vee$ of A (see Definition 1.3.2.16). Then the λ_A-Weil pairing e^{λ_A} (on $A[n]$) is defined to be the composition

$$A[n] \times A[n] \overset{\mathrm{Id}_A \times \lambda_A}{\to} A[n] \times A^\vee[n] \overset{\mathrm{e}_{A[n]}}{\to} \boldsymbol{\mu}_{n,S}.$$

That is, $\mathrm{e}^{\lambda_A}(a, a') = \mathrm{e}_{A[n]}(a, \lambda_A(a'))$ for all $a, a' \in A[n]$.

Suppose that (after an étale localization if necessary) λ_A is of the form $\lambda_{\mathcal{M}}$ for some invertible sheaf $\mathcal{M}$ over A (relatively ample over S) (see Construction 1.3.2.7 and Proposition 1.3.2.15). Then $(\mathrm{Id}_A \times \lambda_A)^* \mathcal{P}_A \cong \mathcal{D}_2(\mathcal{M})$, and we have the following corollary:

Corollary 5.2.4.5. *The λ_A-Weil pairing*

$$\mathrm{e}^{\lambda_A} : A[n] \times A[n] \to \boldsymbol{\mu}_{n,S}$$

gives, for each $a, a' \in A[n]$, the isomorphism $\mathrm{e}^{\lambda_A}(a,a')$ which makes the diagram

$$\begin{tikzcd}
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, equal] & (([n]_A \times \lambda_A)^* \mathcal{P}_A)|_{(a,a')} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & \mathcal{P}_A|_{(e,\lambda_A(a'))} \arrow[r, "\text{rig.}", "\sim"'] & \mathscr{O}_S \arrow[d, "\mathrm{e}^{\lambda_A}(a,a')"', "\wr"] \\
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')} \arrow[r, "\sim", "\text{can.}"'] & ((\mathrm{Id}_A \times n\lambda_A)^* \mathcal{P}_A)|_{(a,a')} \arrow[r, "\sim", "\text{can.}"'] & \mathcal{P}_A|_{(a,e)} \arrow[r, "\sim", "\text{rig.}"'] & \mathscr{O}_S
\end{tikzcd}$$

commutative. That is, $\mathrm{e}^{\lambda_A}(a,a')$ measures the difference between the two rigidifications of the biextension $\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')}$.

Now let us relate this calculation to the so-called *Riemann form* defined by an invertible sheave $\mathcal{M}$ over A relatively ample over S. Suppose $K = K(\mathcal{M}) := \ker(\lambda_{\mathcal{M}})$ is defined as in (3.2.4.1). Then, for each $a \in K$, we have an isomorphism $\mathcal{D}_2(\mathcal{M})|_{A \times \{a\}} \overset{\text{can.}}{\underset{\sim}{\to}} \mathcal{P}_A|_{A \underset{S}{\times} \{e\}} \overset{\text{rig.}}{\underset{\sim}{\to}} \mathscr{O}_A$ given by one of the rigidifications of the Poincaré biextension. This gives a canonical isomorphism $T_a^* \mathcal{M} \overset{\text{can.}}{\underset{\sim}{\to}} \mathcal{M} \underset{\mathscr{O}_S}{\otimes} \mathcal{M}(a)$ as usual. Therefore, each section $\tilde{a} \in \mathcal{M}(a)$, or rather $\tilde{a} : \mathscr{O}_S \overset{\sim}{\to} \mathcal{M}(a)$, gives an isomorphism $\tilde{a}^{-1} : \mathcal{M}(a) \overset{\sim}{\to} \mathscr{O}_S$, and hence an isomorphism $\tilde{a}^{-1} : T_a^* \mathcal{M} \overset{\sim}{\to} \mathcal{M}$. Let $\mathcal{G}(\mathcal{M}) := \mathcal{M}|_K$. Then $\tilde{a}$ defines by restriction an isomorphism $\tilde{a}^{-1} : T_a^* \mathcal{G}(\mathcal{M}) \overset{\sim}{\to} \mathcal{G}(\mathcal{M})$, and hence a group structure on $\mathcal{G}(\mathcal{M})$ covering the group structure on K.

For each $a, a' \in K$, $\tilde{a} \in \mathcal{M}(a)$, and $\tilde{a}' \in \mathcal{M}(a')$, the group structure defines a composition $\tilde{a} * \tilde{a}' \in \mathcal{M}(a + a')$ such that its *inverse*

$$(\tilde{a} * \tilde{a}')^{-1} : T_{a+a'}^* \mathcal{M} \overset{\text{can.}}{\underset{\sim}{\to}} \mathcal{M} \underset{\mathscr{O}_S}{\otimes} \mathcal{M}(a + a') \overset{\sim}{\to} \mathcal{M}$$

makes the diagram

$$\begin{tikzcd}
T_a^* T_{a'}^* \mathcal{M} \arrow[r, "T_a^*((\tilde{a}')^{-1})", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & T_a^* \mathcal{M} \arrow[d, "\wr"', "\tilde{a}^{-1}"] \\
T_{a+a'}^* \mathcal{M} \arrow[r, "\sim", "(\tilde{a}' * \tilde{a})^{-1}"'] & \mathcal{M}
\end{tikzcd}$$

commutative. In other words, we define

$$(\tilde{a} * \tilde{a}')^{-1} := \tilde{a}^{-1} \circ T_a^*((\tilde{a}')^{-1}).$$

In alternative language, the isomorphism

$$\mathcal{D}_2(\mathcal{M})|_{(a,a')} \overset{\text{can.}}{\underset{\sim}{\to}} \mathcal{P}_A|_{(a,\lambda_{\mathcal{M}}(a'))} = \mathcal{P}_A|_{(a,e)} \overset{\text{rig.}}{\underset{\sim}{\to}} \mathscr{O}_S$$

gives an isomorphism $\mathcal{M}(a+a')\otimes\mathcal{M}(a)^{\otimes -1}\otimes\mathcal{M}(a')^{\otimes -1}\xrightarrow{\sim}\mathscr{O}_S$, or equivalently an isomorphism

$$\mathcal{M}(a)^{\otimes -1}\otimes\mathcal{M}(a')^{\otimes -1}\xrightarrow{\sim}\mathcal{M}(a+a')^{\otimes -1},$$

which sends $\tilde{a}^{-1}\otimes(\tilde{a}')^{-1}$ to $(\tilde{a}*\tilde{a}')^{-1}$ according to our definition.

Definition 5.2.4.6. *The* ***Riemann form***

$$\mathrm{e}^{\mathcal{M}}:K\times K\to\mathbf{G}_{\mathrm{m},S}$$

is defined by setting $\mathrm{e}^{\mathcal{M}}(a,a')$ *to be the difference between the two sections* $\tilde{a}*\tilde{a}'$ *and* $\tilde{a}'*\tilde{a}$ *of* $\mathcal{M}(a+a')$*, for each* $\tilde{a}\in\mathcal{M}(a)$ *and* $\tilde{a}'\in\mathcal{M}(a')$*, such that the diagram*

$$\begin{tikzcd}
\mathscr{O}_S \arrow[r, "\tilde{a}*\tilde{a}'", "\sim"'] \arrow[d, equal] & \mathcal{M}(a+a') \\
\mathscr{O}_S \arrow[r, "\sim", "\tilde{a}'*\tilde{a}"'] & \mathcal{M}(a+a') \arrow[u, "\wr"', "\mathrm{e}^{\mathcal{M}}(a,a')"]
\end{tikzcd}$$

is commutative, or equivalently such that the diagram

$$\begin{tikzcd}
T_a^*T_{a'}^*\mathcal{M} \arrow[r, "T_a^*((\tilde{a}')^{-1})", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & T_a^*\mathcal{M} \arrow[r, "\tilde{a}^{-1}", "\sim"'] & \mathcal{M} \\
T_{a'}^*T_a^*\mathcal{M} \arrow[r, "\sim", "T_{a'}^*(\tilde{a}^{-1})"'] & T_{a'}^*\mathcal{M} \arrow[r, "\sim", "(\tilde{a}')^{-1}"'] & \mathcal{M} \arrow[u, "\wr"', "\mathrm{e}^{\mathcal{M}}(a,a')"]
\end{tikzcd}$$

is commutative. (Note that $\mathrm{e}^{\mathcal{M}}$ *is alternating by definition.)*

The construction above of $\tilde{a}*\tilde{a}'$ using one of the rigidifications of $\mathcal{P}_A$ implies the following:

Proposition 5.2.4.7. *The Riemann form*

$$\mathrm{e}^{\mathcal{M}}:K\times K\to\mathbf{G}_{\mathrm{m},S}$$

gives, for each $a,a'\in K$*, the isomorphism* $\mathrm{e}^{\mathcal{M}}(a,a')$ *which makes the diagram*

$$\begin{tikzcd}
\mathcal{D}_2(\mathcal{M})|_{(a,a')} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, leftrightarrow, "\text{sym.}"', "\wr"] & \mathcal{P}_A|_{(a,e)} \arrow[r, "\text{rig.}", "\sim"'] & \mathscr{O}_S \\
\mathcal{D}_2(\mathcal{M})|_{(a',a)} \arrow[r, "\sim", "\text{can.}"'] & \mathcal{P}_A|_{(a',e)} \arrow[r, "\sim", "\text{rig.}"'] & \mathscr{O}_S \arrow[u, "\wr"', "\mathrm{e}^{\mathcal{M}}(a,a')"]
\end{tikzcd}$$

commutative. That is, $\mathrm{e}^{\mathcal{M}}(a,a')$ *measures the difference between the two rigidifications of the biextension* $\mathcal{D}_2(\mathcal{M})|_{(a,a')}\cong\mathcal{D}_2(\mathcal{M})|_{(a',a)}$*. (Here the symmetry isomorphism is the canonical one as in Lemma 3.2.2.1.)*

The same argument applies when we replace $\mathcal{M}$ with $\mathcal{M}^{\otimes n}$. Suppose moreover that $a\in A[n]$ and $a'\in K(\mathcal{M}^{\otimes n})=[n]^{-1}K(\mathcal{M})=\lambda_{\mathcal{M}}^{-1}(A^\vee[n])$. Then the diagram

$$\begin{tikzcd}
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, leftrightarrow, "\text{sym.}"', "\wr"] & \mathcal{P}_A|_{(a,e)} \arrow[r, "\text{rig.}", "\sim"'] & \mathscr{O}_S \\
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a',a)} \arrow[r, "\sim", "\text{can.}"'] & \mathcal{P}_A|_{(a',e)} \arrow[r, "\sim", "\text{rig.}"'] & \mathscr{O}_S \arrow[u, "\wr"', "\mathrm{e}^{\mathcal{M}^{\otimes n}}(a,a')"]
\end{tikzcd}$$

is commutative. On the other hand, for formal reasons, the diagram

$$\begin{tikzcd}
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a',a)} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, leftrightarrow, "\text{sym.}"', "\wr"] & ((\mathrm{Id}_A\times n\lambda_{\mathcal{M}})^*\mathcal{P}_A)|_{(a',a)} \arrow[r, "\text{can.}", "\sim"'] & \mathcal{P}_A|_{(a',e)} \arrow[r, "\text{rig.}", "\sim"'] \arrow[d, leftrightarrow, "\text{sym.}"', "\wr"] & \mathscr{O}_S \arrow[d, equal] \\
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')} \arrow[r, "\sim", "\text{can.}"'] & (([n]_A\times\lambda_{\mathcal{M}})^*\mathcal{P}_A)|_{(a,a')} \arrow[r, "\sim", "\text{can.}"'] & \mathcal{P}_A|_{(e,\lambda_{\mathcal{M}}(a'))} \arrow[r, "\sim", "\text{rig.}"'] & \mathscr{O}_S
\end{tikzcd}$$

is also commutative. Comparing the diagrams, we obtain the following:

Corollary 5.2.4.8. *The restriction of the Riemann form*

$$\mathrm{e}^{\mathcal{M}^{\otimes n}}:K(\mathcal{M}^{\otimes n})\times K(\mathcal{M}^{\otimes n})\to\mathbf{G}_{\mathrm{m},S}$$

to $A[n]\times K(\mathcal{M}^{\otimes n})$ *gives, for each* $a\in A[n]$ *and* $a'\in\times K(\mathcal{M}^{\otimes n})$*, the isomorphism* $\mathrm{e}^{\mathcal{M}^{\otimes n}}(a,a')$ *which makes the diagram*

$$\begin{tikzcd}
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, equal] & \mathcal{P}_A|_{(e,\lambda_{\mathcal{M}}(a'))} \arrow[r, "\text{rig.}", "\sim"'] & \mathscr{O}_S \arrow[d, "\wr"', "\mathrm{e}^{\mathcal{M}^{\otimes n}}(a,a')"] \\
\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')} \arrow[r, "\sim", "\text{can.}"'] & \mathcal{P}_A|_{(a,e)} \arrow[r, "\sim", "\text{rig.}"'] & \mathscr{O}_S
\end{tikzcd}$$

commutative. That is, $\mathrm{e}^{\mathcal{M}^{\otimes n}}(a,a')$ *measures the difference between the two rigidifications of the biextension* $\mathcal{D}_2(\mathcal{M}^{\otimes n})|_{(a,a')}$*.*

Comparing Corollary 5.2.4.8 with Corollary 5.2.4.5, we arrive at the following important formula (cf. [94, §23, p. 228, (5)]):

Proposition 5.2.4.9. *If* $\mathcal{M}$ *is an invertible sheaf over* A *relatively ample over* S*, then*

$$\mathrm{e}^{\lambda_{\mathcal{M}}}(a,a')=\mathrm{e}_{A[n]}(a,\lambda_{\mathcal{M}}(a'))=\mathrm{e}^{\mathcal{M}^{\otimes n}}(a,a')$$

for all $a,a'\in A[n]$*.*

Since every polarization λ_A is étale locally of the form $\lambda_{\mathcal{M}}$ for some invertible sheaf $\mathcal{M}$ over A relatively ample over S (by Definition 1.3.2.16 and Proposition 1.3.2.15), we have the following:

Corollary 5.2.4.10. *The* λ_A*-Weil pairing* e^{λ_A} *is alternating for every polarization* λ_A*.*

(This is the same argument used in [94, §23].)

Remark 5.2.4.11. Although Proposition 5.2.4.9 is the main tool people use for calculating e^{λ_A}, the realization of the Weil pairings or Riemann forms as differences between rigidifications of pullbacks of the Poincaré biextension will be crucial for us in later arguments.

5.2.5 Splittings of $G[n]_\eta$ in Terms of Sheaves of Algebras

Now let us return to the context of Section 5.2.3, with the additional (harmless) assumption that $\tilde{\eta}=\eta$ and $\tilde{S}=S$ for simplicity of notation.

As we saw in Section 5.2.3, the splitting $\varsigma_n:\mathrm{Gr}_n^{\mathtt{W}}\xrightarrow{\sim}G[n]_\eta$ can be described by a triple $(c_n,c_n^\vee,\tau_n)$, where $c_n:\frac{1}{n}X\to A_\eta^\vee$ corresponds to a splitting of $G^\natural[n]_\eta\twoheadrightarrow A[n]_\eta$, and where the pair $(c_n^\vee,\tau_n)$, being equivalent to a lifting $\iota_n:\frac{1}{n}Y\to G_\eta^\natural$, corresponds to a splitting of $G[n]_\eta\twoheadrightarrow\frac{1}{n}Y/Y$.

Let us first describe the splitting $G^\natural[n]_\eta \twoheadrightarrow A[n]_\eta$ given by c_n. For simplicity, we shall assume that c_n extends to a homomorphism $\frac{1}{n}X \to A^\vee$ over S, which we still denote by c_n, and describe instead the corresponding splitting $G^\natural[n] \twoheadrightarrow A[n]$. (We can achieve this by replacing S with the spectrum of a discrete valuation ring, without changing η, as in the proof of Lemma 5.2.3.2.) By a convention we have adopted since Chapter 4 (see, in particular, Section 4.2.2), we shall identify $\mathscr{O}_{G^\natural}$ with its push-forward under the affine morphism $\pi : G^\natural \to A$, and write $\mathscr{O}_{G^\natural}$ as a sum $\mathscr{O}_{G^\natural} \cong \underset{\chi\in X}{\oplus} \mathscr{O}_\chi$ of weight subsheaves under the T-action. (In what follows, we shall adopt such an abuse of notation for other push-forwards under affine morphisms, without further remark.) Then the homomorphism $c : X \to A^\vee$ is defined by the isomorphism $\mathscr{O}_\chi \cong \mathscr{N}_{c(\chi)} := \mathscr{P}_A|_{A \underset{S}{\times} \{c(\chi)\}}$ in $\underline{\mathrm{Pic}}^0_e(A/S)$ (respecting rigidifications) for every $\chi \in X$.

In terms of relatively affine group schemes over $A[n]$, we have a short exact sequence

$$0 \longrightarrow G^\natural[n] \longrightarrow G^\natural|_{A[n]} \xrightarrow{\text{“[n]”}} [n]^*_A(G^\natural|_{\{e\}}) \longrightarrow 0$$

(in which e means the identity section of A, not of $G^\natural$). In terms of $\mathscr{O}_{A[n]}$-algebras, $\mathscr{O}_{G^\natural[n]}$ is given by the cokernel of the injection

$$[n]^*_A(\underset{\chi\in X}{\oplus} \mathscr{O}_\chi|_{e_A}) \cong \underset{\chi\in X}{\oplus}([n]^*_A\mathscr{O}_\chi)|_{A[n]} \hookrightarrow \underset{\chi\in X}{\oplus} \mathscr{O}_\chi|_{A[n]}.$$

Let us write this as a short exact sequence

$$0 \to \underset{\chi\in X}{\oplus}([n]^*_A\mathscr{O}_\chi)|_{A[n]} \to \underset{\chi\in X}{\oplus} \mathscr{O}_\chi|_{A[n]} \to \underset{\bar\chi\in X/nX}{\oplus} \mathscr{O}_{\bar\chi} \to 0.$$

The point is the isomorphism

$$\mathscr{O}_{n\chi}|_{A[n]} \overset{\text{can.}}{\overset{\sim}{\to}} ([n]^*\mathscr{O}_\chi)|_{A[n]} \overset{[n]^*(\text{rig.})}{\overset{\sim}{\to}} ([n]^*\mathscr{O}_A)|_{A[n]} \overset{\text{can.}}{\overset{\sim}{\to}} \mathscr{O}_A|_{A[n]},$$

where rig. can be identified with the composition rig. : $\mathscr{O}_\chi(e) \overset{\text{rig.}}{\overset{\sim}{\to}} \mathscr{O}_S \overset{\text{str.}}{\overset{\sim}{\to}} \mathscr{O}_A(e)$. Hence there is an isomorphism

$$\mathscr{O}_{\chi+n\chi'}|_{A[n]} \overset{\text{can.}}{\overset{\sim}{\to}} \mathscr{O}_\chi|_{A[n]} \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_{n\chi'}|_{A[n]} \overset{\sim}{\to} \mathscr{O}_\chi|_{A[n]}$$

for each $\chi, \chi' \in X$. If we take a representative χ for each class $\bar\chi$ of X/nX, and define $\mathscr{O}_{\bar\chi}$ to be $\mathscr{O}_\chi|_{A[n]}$, then the algebra structure of $\mathscr{O}_{G^\natural}$ given by $\mathscr{O}_\chi \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_{\chi'} \overset{\text{can.}}{\overset{\sim}{\to}} \mathscr{O}_{\chi+\chi'}$ induces isomorphisms $\mathscr{O}_{\bar\chi} \underset{\mathscr{O}_{A[n]}}{\otimes} \mathscr{O}_{\bar\chi'} \overset{\text{can.}}{\overset{\sim}{\to}} \mathscr{O}_{\bar\chi+\bar\chi'}$ giving the algebra structure of $\underset{\bar\chi\in X/nX}{\oplus} \mathscr{O}_{\bar\chi}$. This gives a *realization* $\underset{\bar\chi\in X/nX}{\oplus} \mathscr{O}_{\bar\chi}$ of the $\mathscr{O}_{A[n]}$-algebra $\mathscr{O}_{G^\natural[n]}$ (based on our choices of representatives of X/nX), which is unique up to unique isomorphism.

Similarly, suppose $G^\natural_n$ is the group scheme defined by $c_n : \frac{1}{n}X \to A^\vee$, which can be given in terms of $\mathscr{O}_A$-algebras by $\mathscr{O}_{G^\natural_n} \cong \underset{\frac{1}{n}\chi\in\frac{1}{n}X}{\oplus} \mathscr{O}_{\frac{1}{n}\chi}$. Then the subgroup scheme $G^\natural_n[n]$ of $G^\natural_n$ can be realized via the short exact sequence

$$0 \to \underset{\frac{1}{n}\chi\in\frac{1}{n}X}{\oplus} [n]^*_A\mathscr{O}_{\frac{1}{n}\chi}|_{A[n]} \to \underset{\frac{1}{n}\chi\in\frac{1}{n}X}{\oplus} \mathscr{O}_{\frac{1}{n}\chi}|_{A[n]} \to \underset{\frac{1}{n}\bar\chi\in\frac{1}{n}X/X}{\oplus} \mathscr{O}_{\frac{1}{n}\bar\chi} \to 0$$

as $\underset{\frac{1}{n}\bar\chi\in\frac{1}{n}X/X}{\oplus} \mathscr{O}_{\frac{1}{n}\bar\chi}$, where $\mathscr{O}_{\frac{1}{n}\bar\chi}$ is defined as above using the sheaves $\mathscr{O}_{\frac{1}{n}\chi}|_{A[n]}$ defined by representatives $\frac{1}{n}\chi$ of $\frac{1}{n}\bar\chi$.

Note that there is a structural morphism $\mathscr{O}_A|_{A[n]} \overset{\sim}{\to} \mathscr{O}_{\frac{1}{n}\bar 0} \hookrightarrow \underset{\frac{1}{n}\bar\chi\in\frac{1}{n}X/X}{\oplus} \mathscr{O}_{\frac{1}{n}\bar\chi}$, because there is an isomorphism from $\mathscr{O}_A|_{A[n]} \cong \mathscr{O}_{\frac{1}{n}0}|_{A[n]}$ to $\mathscr{O}_{\frac{1}{n}\bar 0} = \mathscr{O}_{\frac{1}{n}\chi_0}|_{A[n]}$, for whatever representative $\frac{1}{n}\chi_0$ we choose for $\frac{1}{n}\bar 0 \in \frac{1}{n}X/X$. (Here we are using clumsy notation such as $\mathscr{O}_{\frac{1}{n}\bar 0}$ to avoid identification with $\mathscr{O}_{\bar 0}$. They should not be confused.)

Now the natural diagram

$$\begin{tikzcd}
0 \arrow[r] & G^\natural_n[n] \arrow[r] \arrow[d] & G^\natural_n|_{A[n]} \arrow[r] \arrow[d] & {[n]^*(G^\natural_n|_{\{e\}})} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & G^\natural[n] \arrow[r] & G^\natural|_{A[n]} \arrow[r] & {[n]^*(G^\natural|_{\{e\}})} \arrow[r] & 0
\end{tikzcd}$$

corresponds to the natural diagram

$$\begin{tikzcd}
0 \arrow[r] & \underset{\chi\in X}{\oplus}[n]^*_A\mathscr{O}_\chi|_{A[n]} \arrow[r] \arrow[d] & \underset{\chi\in X}{\oplus}\mathscr{O}_\chi|_{A[n]} \arrow[r] \arrow[d] & \underset{\bar\chi\in X/nX}{\oplus}\mathscr{O}_{\bar\chi} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \underset{\frac{1}{n}\chi\in\frac{1}{n}X}{\oplus}[n]^*_A\mathscr{O}_{\frac{1}{n}\chi}|_{A[n]} \arrow[r] & \underset{\frac{1}{n}\chi\in\frac{1}{n}X}{\oplus}\mathscr{O}_{\frac{1}{n}\chi}|_{A[n]} \arrow[r] & \underset{\frac{1}{n}\bar\chi\in\frac{1}{n}X/X}{\oplus}\mathscr{O}_{\frac{1}{n}\bar\chi} \arrow[r] & 0
\end{tikzcd}$$

of sheaves. Since the image of $\underset{\chi\in X}{\oplus}\mathscr{O}_\chi|_{A[n]}$ lies in $\mathscr{O}_{\frac{1}{n}\bar 0}$ inside $\underset{\frac{1}{n}\bar\chi\in\frac{1}{n}X/X}{\oplus}\mathscr{O}_{\frac{1}{n}\bar\chi}$, the induced morphism $\underset{\bar\chi\in X/nX}{\oplus}\mathscr{O}_{\bar\chi} \to \underset{\frac{1}{n}\bar\chi\in\frac{1}{n}X/X}{\oplus}\mathscr{O}_{\frac{1}{n}\bar\chi}$ factors through $\underset{\bar\chi\in X/nX}{\oplus}\mathscr{O}_{\bar\chi} \twoheadrightarrow \mathscr{O}_{\frac{1}{n}\bar 0} \hookrightarrow \underset{\frac{1}{n}\bar\chi\in\frac{1}{n}X/X}{\oplus}\mathscr{O}_{\frac{1}{n}\bar\chi}$. However, this shows that the structural morphism $\mathscr{O}_A|_{A[n]} \overset{\sim}{\to} \mathscr{O}_{\bar 0} \hookrightarrow \underset{\bar\chi\in X/nX}{\oplus}\mathscr{O}_{\bar\chi}$ for $G[n] \twoheadrightarrow A[n]$ has a right inverse given by $\underset{\bar\chi\in X/nX}{\oplus}\mathscr{O}_{\bar\chi} \twoheadrightarrow \mathscr{O}_{\frac{1}{n}\bar 0} \overset{\sim}{\to} \mathscr{O}_A|_{A[n]}$. In other words, $G^\natural[n] \twoheadrightarrow A[n]$ splits. In particular, since every $\mathscr{O}_{\bar\chi}$ is isomorphic to $\mathscr{O}_{\bar 0}$ because it is also mapped isomorphically to $\mathscr{O}_{\frac{1}{n}\bar 0} \overset{\sim}{\to} \mathscr{O}_A|_{A[n]}$, the splitting defines an isomorphism

$$\underset{\bar\chi\in X/nX}{\oplus}\mathscr{O}_{\bar\chi} \overset{\sim}{\to} \mathscr{O}_A|_{A[n]} \underset{\mathscr{O}_S}{\otimes} (\underset{\bar\chi\in X/nX}{\oplus}\mathscr{O}_{S,\bar\chi}),$$

where $\mathscr{O}_{S,\bar\chi}$ is just a copy of $\mathscr{O}_S$ with the prescribed weight $\bar\chi$ under the action of $T[n]$. In other words, $\underset{\bar\chi\in X/nX}{\oplus}\mathscr{O}_{S,\bar\chi}$ is a realization of $\mathscr{O}_{T[n]}$ as an $\mathscr{O}_S$-algebra. This corresponds to the isomorphism $A[n] \underset{S}{\times} T[n] \overset{\sim}{\to} G[n]$ defined by the splitting, because the homomorphism $T[n] \hookrightarrow G|_{\{e\}}$ is given by the rigidification isomorphisms $\mathscr{O}_\chi(e) \overset{\sim}{\to} \mathscr{O}_S$ respected by the isomorphisms $\mathscr{O}_{\frac{1}{n}\bar 0} \overset{\sim}{\to} \mathscr{O}_A|_{A[n]}$ above.

For each point $a \in A[n]$, the splitting gives a morphism $S \to G^\natural$, which can be described in terms of sheaves of algebras by a surjection

$$\mathscr{O}_{G^\natural}(a) \cong \underset{\chi\in X}{\oplus}\mathscr{O}_\chi(a) \to \mathscr{O}_S$$

whose restriction to each $\mathscr{O}_\chi(a)$ is given by the isomorphism

$$\begin{aligned}\mathscr{O}_\chi(a) &\overset{\text{can.}}{\overset{\sim}{\to}} \mathscr{P}_A|_{(a,c(\chi))} = \mathscr{P}_A|_{(a,nc_n(\frac{1}{n}\chi))} \\ &\overset{\text{can.}}{\overset{\sim}{\to}} \mathscr{P}_A|_{(na,c_n(\frac{1}{n}\chi))} = \mathscr{P}_A|_{(e,c_n(\frac{1}{n}\chi))} \overset{\text{rig.}}{\overset{\sim}{\to}} \mathscr{O}_S.\end{aligned}$$

Let us denote this isomorphism by $r(a, c_n(\frac{1}{n}\chi))$, to signify the choice of c_n involved in this definition.

Every lifting $c'_n : \frac{1}{n}X \to A^\vee$ of c is necessarily of the form $c'_n = c_n + d_n$ for some homomorphism $d_n : \frac{1}{n}X \to A^\vee[n]$, or rather $d_n : \frac{1}{n}X/X \to A^\vee[n]$, because we need to have $c'_n|_X = c_n|_X = c$. Let us investigate the effect of such a modification.

The splitting defined by $c'_n = c_n + d_n$ is defined by
$$\underset{\chi\in X}{\oplus} r(a, c'_n(\tfrac{1}{n}\chi)) : \mathscr{O}_{G^\natural}(a) \cong \underset{\chi\in X}{\oplus} \mathscr{O}_\chi(a) \to \mathscr{O}_S$$
with morphisms $r(a, c'_n(\frac{1}{n}\chi)) : \mathscr{O}_\chi(a) \overset{\sim}{\to} \mathscr{O}_S$ defined by
$$\mathscr{O}_\chi(a) \overset{\text{can.}}{\underset{\sim}{\to}} \mathcal{P}_A|_{(a,c(\chi))} = \mathcal{P}_A|_{(a,n(c_n+d_n)(\frac{1}{n}\chi))}$$
$$\overset{\text{can.}}{\underset{\sim}{\to}} \mathcal{P}_A|_{(na,c_n(\frac{1}{n}\chi)+d_n(\frac{1}{n}\chi))} = \mathcal{P}_A|_{(e,c_n(\frac{1}{n}\chi)+d_n(\frac{1}{n}\chi))} \overset{\text{rig.}}{\underset{\sim}{\to}} \mathscr{O}_S.$$

We can compare $r(a, c_n(\frac{1}{n}\chi))$ and $r(a, c_n(\frac{1}{n}\chi) + d_n(\frac{1}{n}\chi))$ using the commutative diagram
$$\begin{tikzcd}
\mathcal{P}_A|_{(a,c(\chi))} \arrow[r, "\text{can.}", "\sim"'] & \mathscr{O}_\chi(a) \arrow[d, "r(a,c_n(\frac{1}{n}\chi)+d_n(\frac{1}{n}\chi))", "\wr"'] \\
\mathcal{P}_A|_{(e,c_n(\frac{1}{n}\chi)+d_n(\frac{1}{n}\chi))} \arrow[u, "\text{can.}", "\wr"'] \arrow[r, "\text{rig.}", "\sim"'] & \mathscr{O}_S \\
\mathcal{P}_A|_{(e,c_n(\frac{1}{n}\chi))} \underset{\mathscr{O}_S}{\otimes} \mathcal{P}_A|_{(e,d_n(\frac{1}{n}\chi))} \arrow[u, "\text{can.}", "\wr"'] \arrow[d, "\text{can.}", "\wr"'] \arrow[r, "\text{rig.}\otimes\text{rig.}", "\sim"'] & \mathscr{O}_S \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S \arrow[u, "\text{can.}"', "\wr"] \\
\mathcal{P}_A|_{(a,c(\chi))} \underset{\mathscr{O}_S}{\otimes} \mathcal{P}_A|_{(a,e)} \arrow[r, "\sim", "\text{can.}\otimes\text{rig.}"'] & \mathscr{O}_\chi(a) \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_S \arrow[u, "r(a,c_n(\frac{1}{n}\chi))\otimes \mathrm{e}_{A[n]}(a,d_n(\frac{1}{n}\chi))^{-1}", "\wr"']
\end{tikzcd}$$
in which we have used Proposition 5.2.4.4 to find the correct isomorphism $\mathrm{e}_{A[n]}(a, d_n(\frac{1}{n}\chi))$. As a result, we obtain the symbolic relation
$$r(a, c_n(\tfrac{1}{n}\chi) + d_n(\tfrac{1}{n}\chi)) = r(a, c_n(\tfrac{1}{n}\chi))\mathrm{e}_{A[n]}(a, d_n(\tfrac{1}{n}\chi))^{-1}.$$

On the other hand, consider the canonical isomorphism
$$\underline{\mathrm{Hom}}_S(\tfrac{1}{n}X, A^\vee[n]) \cong \underline{\mathrm{Hom}}_S(A[n], T_n[n]) \cong \underline{\mathrm{Hom}}_S(A[n], T[n]),$$
which we denote by $d_n \mapsto {}^t d_n$, with sign convention determined by the relation $\chi({}^t d_n(a)) = \mathrm{e}_{A[n]}(a, d_n(\frac{1}{n}\chi))$ for all $\chi \in X$ and $a \in A[n]$. Then, by definition of χ, the multiplication by ${}^t d_n(a)$ is given by multiplication by $\chi({}^t d_n(a)) = \mathrm{e}_{A[n]}(a, d_n(\frac{1}{n}\chi))$ on $\mathscr{O}_\chi$. As a result, the modified splitting
$$r(\chi, c_n(\tfrac{1}{n}\chi) + d_n(\tfrac{1}{n}\chi)) : \mathscr{O}_\chi(a) \overset{\sim}{\to} \mathscr{O}_S$$
can be interpreted as a composition
$$\mathscr{O}_\chi(a) \overset{\chi(-\,{}^t d_n(a))}{\underset{\sim}{\to}} \mathscr{O}_\chi(a) \overset{r(\chi,c_n(\frac{1}{n}\chi))}{\underset{\sim}{\to}} \mathscr{O}_S,$$
which is the same as multiplying the section $S \to G^\natural$ defined by c_n by ${}^t d_n(a) \in T[n]$. This completes the picture of splittings produced by c_n.

We will not need a description (in terms of sheaves of algebras) of the splitting $G[n]_\eta \twoheadrightarrow \frac{1}{n}Y/Y$ described by c_n and τ_n. What we will need is a description of the lifting $\iota_n : \frac{1}{n}Y \to G^\natural_\eta$ of $\iota : Y \to G^\natural_\eta$, which is given by the isomorphisms
$$\tau_n(\tfrac{1}{n}y, \chi) : \mathscr{O}_\chi(c_n^\vee(\tfrac{1}{n}y))_\eta \overset{\sim}{\to} \mathscr{O}_{S,\eta}$$
for $\chi \in X$ that altogether form the algebra homomorphism
$$\iota_n(\tfrac{1}{n}y)^* := \underset{\chi\in X}{\oplus} \tau_n(\tfrac{1}{n}y, \chi) : \mathscr{O}_{G^\natural}(c_n^\vee(\tfrac{1}{n}y))_\eta \cong \underset{\chi\in X}{\oplus} \mathscr{O}_\chi(c_n^\vee(\tfrac{1}{n}y))_\eta \to \mathscr{O}_{S,\eta}$$
of $\iota_n(\frac{1}{n}y) : \eta \to G^\natural_\eta$. This lifts the structure homomorphism
$$\iota(y)^* := \underset{\chi\in X}{\oplus} \tau(y, \chi) : \mathscr{O}_{G^\natural}(c^\vee(y))_\eta \cong \underset{\chi\in X}{\oplus} \mathscr{O}_\chi(c^\vee(y))_\eta \to \mathscr{O}_{S,\eta}$$
of $\iota(y) : \eta \to G^\natural_\eta$ when we restrict to the subgroup Y of $\frac{1}{n}Y$.

To make the picture complete, we would like to see what happens when we have different liftings $((c_n^\vee)', \tau'_n)$ of $(c_\eta^\vee, \tau)$. As in the case of c_n, every lifting $(c_n^\vee)'$ of $c_\eta^\vee$ is necessarily of the form $(c_n^\vee)' = c_n^\vee + d_n^\vee$ for some homomorphism $d_n^\vee : \frac{1}{n}Y \to A[n]_\eta$, or rather $d_n^\vee : \frac{1}{n}Y/Y \to A[n]_\eta$. Using the splitting of $G^\natural[n] \twoheadrightarrow A[n]$ defined by c_n, which defines an isomorphism $A[n]_\eta \oplus T[n]_\eta \overset{\sim}{\to} G^\natural[n]_\eta$, every lifting ι'_n of ι is given by the difference $\iota'_n - \iota_n : \frac{1}{n}Y \to G^\natural[n]_\eta$ covering the difference $(c_n^\vee)' - c_n^\vee = d_n^\vee : \frac{1}{n}Y$ in the $A[n]_\eta$ component. Therefore the essential new information is the difference of $\iota'_n - \iota_n$ in the $T[n]_\eta$ component, given by a homomorphism $e_n : \frac{1}{n}Y \to T[n]_\eta$, or rather $e_n : \frac{1}{n}Y/Y \to T[n]_\eta$.

The multiplication morphism on $G^\natural$, which covers the multiplication morphism on A, is given by isomorphisms
$$m_A^* \mathscr{O}_\chi \overset{\sim}{\to} \mathrm{pr}_1^* \mathscr{O}_\chi \otimes \mathrm{pr}_2^* \mathscr{O}_\chi$$
for $\chi \in X$ given by the theorem of the square (see Section 3.1.4). In particular, for each $a \in A[n]$, the translation by a on $G^\natural$ (using the splitting) defined by c_n is given by the compositions of isomorphisms
$$T_a^* \mathscr{O}_\chi \overset{\text{can.}}{\underset{\sim}{\to}} \mathscr{O}_\chi \underset{\mathscr{O}_S}{\otimes} \mathscr{O}_\chi(a) \overset{r(a,c_n(\frac{1}{n}\chi))}{\underset{\sim}{\to}} \mathscr{O}_\chi$$
for $\chi \in X$, which we shall also denote by $r(a, c_n(\frac{1}{n}\chi))$ when the context is clear.

As a result, the translation $\iota'_n(\frac{1}{n}(y))$ of $\iota_n(\frac{1}{n}y)$ by $(c_n^\vee(\frac{1}{n}y), e_n(\frac{1}{n}y))$ in $A[n]_\eta \times T[n]_\eta \overset{\sim}{\to} G^\natural[n]_\eta$ corresponds to the isomorphisms
$$\tau'_n(\tfrac{1}{n}y, \chi) : \mathscr{O}_\chi(c_n^\vee(\tfrac{1}{n}y))_\eta \overset{\sim}{\to} \mathscr{O}_{S,\eta}$$
for $\chi \in X$, each of which is as defined by the dotted arrow in the diagram
$$\begin{tikzcd}
\mathscr{O}_\chi(c_n^\vee(\frac{1}{n}y) + d_n^\vee(\frac{1}{n}y))_\eta \arrow[r, "\chi(e_n(\frac{1}{n}y))", "\sim"'] \arrow[ddd, dotted, "\tau'_n(\frac{1}{n}y,\chi)"', "\wr"] & \mathscr{O}_\chi(c_n^\vee(\frac{1}{n}y) + d_n^\vee(\frac{1}{n}y))_\eta \arrow[d, "\text{can.}", "\wr"'] \\
& \mathscr{O}_\chi(c_n^\vee(\frac{1}{n}y))_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(d_n^\vee(\frac{1}{n}y))_\eta \arrow[dd, "\tau_n(\frac{1}{n}y,\chi)\otimes r(d_n^\vee(\frac{1}{n}y),c_n(\frac{1}{n}\chi))", "\wr"'] \\
& \\
\mathscr{O}_{S,\eta} & \mathscr{O}_{S,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{S,\eta} \arrow[l, "\sim"', "\text{can.}"]
\end{tikzcd} \tag{5.2.5.1}$$
by the composition of the other arrows. Symbolically, we can write this as
$$\tau'_n(\tfrac{1}{n}y, \chi) = \tau_n(\tfrac{1}{n}y, \chi) r(d_n^\vee(\tfrac{1}{n}y), c_n(\tfrac{1}{n}\chi))\chi(e_n(\tfrac{1}{n}y)).$$

5.2.6 Weil Pairings for $G[n]_\eta$ via Splittings

Let us continue with the notation and assumptions in Section 5.2.5. In particular, we shall retain the assumptions that $\tilde{\eta} = \eta$ and $\tilde{S} = S$ for simplicity. The goal of the section is to compute the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}$ on $G[n]_\eta$ and prove Theorem 5.2.3.14. Although the argument is elementary in nature, it is the technical heart of this chapter.

By étale localization if necessary, let us assume moreover that the tuple $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}(R, I)$ is a *split object*, in the sense that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively, and that λ is induced by some ample invertible sheaf $\mathcal{L}$ over G such that $\mathcal{L}^\natural$ is the pullback (via $\pi : G^\natural \to A$) of some ample invertible sheaf $\mathcal{M}$ over A. This assumption is harmless for the calculation of Weil pairings.

By Proposition 5.2.4.9, we can compute $\mathrm{e}^{\lambda_\eta}$ on $G[n]_\eta$ using the Riemann form defined by $\mathcal{L}_\eta^{\otimes n}$. Ideally, for two points g_1 and g_2 of $G[n]_\eta$, we shall find sections of $\mathcal{L}_\eta^{\otimes n}(g_1)$ and $\mathcal{L}_\eta^{\otimes n}(g_2)$, which are unique up to constants in $\mathbf{G}_{\mathrm{m},\eta}$ and can be realized as isomorphisms $T_{g_1}^*\mathcal{L}_\eta^{\otimes n} \xrightarrow{\sim} \mathcal{L}_\eta^{\otimes n}$ and $T_{g_2}^*\mathcal{L}_\eta^{\otimes n} \xrightarrow{\sim} \mathcal{L}_\eta^{\otimes n}$, where T_{g_1} and T_{g_2} are translations on G. Then the difference between the two compositions $T_{g_1}^* T_{g_2}^* \mathcal{L}_\eta^{\otimes n} \xrightarrow{\sim} T_{g_1}^*\mathcal{L}_\eta^{\otimes n} \xrightarrow{\sim} \mathcal{L}_\eta^{\otimes n}$ and $T_{g_1}^* T_{g_2}^* \mathcal{L}_\eta^{\otimes n} \xrightarrow{\sim} T_{g_2}^*\mathcal{L}_\eta^{\otimes n} \xrightarrow{\sim} \mathcal{L}_\eta^{\otimes n}$ gives us the constant $\mathrm{e}^{\lambda_\eta}(g_1, g_2)$ in $\mathbf{G}_{\mathrm{m},\eta}$. Note that this constant can be found by comparing the effects of the isomorphisms on any of the nonzero global sections. To proceed further, let us make use of the full invertible sheaf $\mathcal{L}$ over S (extending $\mathcal{L}_\eta$ over η), and pass to the formal completion $\mathcal{L}_{\mathrm{for}} \cong \mathcal{L}_{\mathrm{for}}^\natural$ over $G_{\mathrm{for}} \cong G_{\mathrm{for}}^\natural$. Then the sections of $\Gamma(G_\eta, \mathcal{L}_\eta)$ can be realized using its Fourier expansions (as defined in Section 4.3) as elements in $\Gamma(G_{\mathrm{for}}^\natural, \mathcal{L}_{\mathrm{for}}) \underset{R}{\otimes} K$.

Recall that we have decomposed $\mathscr{O}_{G^\natural}$ as an $\mathscr{O}_A$-algebra as $\underset{\chi \in X}{\oplus} \mathscr{O}_\chi$, so that sections of $\mathscr{O}_{G^\natural,\mathrm{for}}$ can be realized as the I-adically convergent sums in $\hat{\oplus}_{\chi \in X} \mathscr{O}_{\chi,\mathrm{for}}$. By fixing the choice of $\mathcal{M}$ over A such that $\mathcal{L}^\natural = \pi^*\mathcal{M}$ (via $\pi : G^\natural \to A$), we can decompose $\mathcal{L}^\natural$ as an $\mathscr{O}_A$-module as $\underset{\chi \in X}{\oplus} (\mathcal{M} \underset{\mathscr{O}_A}{\otimes} \mathscr{O}_\chi)$, and so that sections of $\mathcal{L}_{\mathrm{for}}^\natural$ are written as the I-adically convergent sums in $\hat{\oplus}_{\chi \in X} (\mathcal{M}_{\mathrm{for}} \underset{\mathscr{O}_{A,\mathrm{for}}}{\otimes} \mathscr{O}_{\chi,\mathrm{for}})$. To compute the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}$ using Riemann forms for $g_1, g_2 \in G[n]_\eta$, we shall replace the isomorphisms $T_{g_1}^*\mathcal{L}_\eta^{\otimes n} \xrightarrow{\sim} \mathcal{L}_\eta^{\otimes n}$ and $T_{g_2}^*\mathcal{L}_\eta^{\otimes n} \xrightarrow{\sim} \mathcal{L}_\eta^{\otimes n}$ with suitable isomorphisms $T_{g_1^\natural}^*(\mathcal{L}_\eta^\natural)^{\otimes n} \xrightarrow{\sim} (\mathcal{L}_\eta^\natural)^{\otimes n}$ and $T_{g_2^\natural}^*(\mathcal{L}_\eta^\natural)^{\otimes n} \xrightarrow{\sim} (\mathcal{L}_\eta^\natural)^{\otimes n}$, with choices of $g_1^\natural, g_2^\natural \in G_\eta^\natural$ to be made clear later, so that the pullback of these isomorphisms to their formal completions are compatible with the formation of Fourier expansions. Since we have a splitting $\mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} G[n]_\eta$, it suffices to calculate the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}$ between pairs consisting of elements in $T[n]_\eta$, $A[n]_\eta$, and $\frac{1}{n}Y/Y$ only.

The choices of $g_i^\natural$ can be made as follows: For points of $T[n]_\eta$ and $A[n]_\eta$, they are already identified with points in $G^\natural[n]_\eta$ under the splitting. For points of $\frac{1}{n}Y/Y$, we take any representatives of them in $\frac{1}{n}Y$, and embed them into $G_\eta^\natural$ using the given lifting $\iota_n : \frac{1}{n}Y \hookrightarrow G_\eta^\natural$ of $\iota : Y \hookrightarrow G_\eta^\natural$. Note that the action of any of these elements can be described in terms of its effect on the weight subsheaves $\mathcal{M}_\eta^{\otimes n} \otimes \mathscr{O}_{\chi,\eta}$, and the resulting pairing $\mathrm{e}^{\lambda_\eta}(g_1, g_2)$, as a constant in $\mathbf{G}_{\mathrm{m},\eta}$, can be seen on any of the weight subsheaves we consider. Therefore we can disregard the notion of I-adic sums and formal completions from now on, and focus only on the weight subsheaves $\mathcal{M} \otimes \mathscr{O}_\chi$ as part of the sections of $\mathcal{L}_\eta^\natural$.

Let us summarize the information we have at this point:

1. The Y-action on $\mathcal{L}_\eta^\natural$ is given by the isomorphisms
$$\begin{aligned} &T_{c^\vee(y)}^*(\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \\ &\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}(c^\vee(y))_\eta \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(c^\vee(y))_\eta \\ &\overset{\psi(y)\tau(y,\chi)}{\xrightarrow{\sim}} \mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta} \end{aligned}$$
covering the translation by $\iota(y)$ given by the isomorphisms
$$T_{c^\vee(y)}^*\mathscr{O}_{\chi,\eta} \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(c^\vee(y))_\eta \overset{\tau(y,\chi)}{\xrightarrow{\sim}} \mathscr{O}_{\chi,\eta}$$
for $y \in Y$ and $\chi \in X$.
2. If we tensor n copies of $\mathcal{L}_\eta^\natural$ together, then we obtain the isomorphisms
$$\begin{aligned} &(\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi_1,\eta}) \underset{\mathscr{O}_{A,\eta}}{\otimes} (\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi_2,\eta}) \underset{\mathscr{O}_{A,\eta}}{\otimes} \cdots \underset{\mathscr{O}_{A,\eta}}{\otimes} (\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi_n,\eta}) \\ &\qquad \xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi_1+\chi_2+\cdots+\chi_n,\eta} \end{aligned}$$
for $\chi_1, \chi_2, \ldots, \chi_n \in X$, with Y-action given by
$$\psi(y)^n\tau(y,\chi) : T_{c^\vee(y)}^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(ny),\eta}$$
for $y \in Y$ and $\chi \in X$.
3. The translation by $T[n]_\eta$ on $G_\eta^\natural$ is given by the isomorphisms
$$\chi(t) : \mathscr{O}_{\chi,\eta} \xrightarrow{\sim} \mathscr{O}_{\chi,\eta}$$
for $t \in T[n]_\eta$ and $\chi \in X$.
4. The translation by $A[n]_\eta$ on $G_\eta^\natural$, using the splitting defined by c_n, is given by the isomorphisms
$$r(a, c_n(\tfrac{1}{n}\chi)) : T_a^*\mathscr{O}_{\chi,\eta} \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a) \xrightarrow{\sim} \mathscr{O}_{\chi,\eta}$$
for $a \in A[n]_\eta$ and $\chi \in X$.
5. The translation by $\frac{1}{n}Y$ on $G_\eta^\natural$, using the period homomorphism $\iota_n : \frac{1}{n}Y \to G_\eta^\natural$, is given by the isomorphisms
$$\tau_n(y,\chi) : T_{c_n^\vee(\frac{1}{n}y)}^*\mathscr{O}_{\chi,\eta} \overset{\text{can.}}{\xrightarrow{\sim}} \mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_\chi(c_n^\vee(\tfrac{1}{n}y))_\eta \xrightarrow{\sim} \mathscr{O}_{\chi,\eta}$$
for $y \in Y$ and $\chi \in X$.

We would like to compute $\mathrm{e}^{\lambda_\eta}(t, t')$, $\mathrm{e}^{\lambda_\eta}(t, a)$, $\mathrm{e}^{\lambda_\eta}(t, \frac{1}{n}y)$, $\mathrm{e}^{\lambda_\eta}(a, a')$, $\mathrm{e}^{\lambda_\eta}(a, \frac{1}{n}y)$, and $\mathrm{e}^{\lambda_\eta}(\frac{1}{n}y, \frac{1}{n}y')$ for each $t, t' \in T[n]_\eta$, $a, a' \in A[n]_\eta$, and $\frac{1}{n}y, \frac{1}{n}y' \in \frac{1}{n}Y$. The reason to include the pairings on $A[n]_\eta \times A[n]_\eta$ and on $T[n]_\eta \times \frac{1}{n}Y/Y$ is to make sure that the sign conventions in Proposition 5.2.2.1, Theorem 5.2.3.14, and Section 5.2.4 are compatible with each other.

To do this, we need to choose isomorphisms of the form $T_{g^\natural}^*(\mathcal{L}_\eta^\natural)^{\otimes n} \xrightarrow{\sim} (\mathcal{L}_\eta^\natural)^{\otimes n}$ covering the translation morphisms $T_{g^\natural}$ on $G^\natural$. Let us give the choices we need in each case:

1. For each $t \in T[n]_\eta$, we consider the isomorphisms
$$\chi(t) : \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta},$$

which is the pullback to η of the action of $T[n]$ on $\mathcal{L}^\natural$ which makes the cubical $\mathbf{G}_{\mathrm{m}}$-torsor $\mathcal{L}^\natural$ over $G^\natural$ descend to $\mathcal{M}$ over A.

2. For each $a \in A[n]_\eta$, which in particular, satisfies $a \in K(\mathcal{M}_\eta^{\otimes n})$, we consider isomorphisms
$$\begin{aligned} T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) &\overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a) \\ &\overset{\tilde{a}^{-1}\otimes r(a,c_n(\frac{1}{n}\chi))}{\xrightarrow{\sim}} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}, \end{aligned}$$
where $\tilde{a}^{-1} : \mathcal{M}_\eta^{\otimes n}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}$ is given by some section $\tilde{a} \in \mathcal{M}_\eta^{\otimes n}(a)$.

3. For each $\frac{1}{n}y \in \frac{1}{n}Y$, we need to make some additional choices.

   Note that
$$\begin{aligned} \psi(y)^n &= \psi(y)^{n-2}\psi(y)\psi(y) = \psi(y)^{n-2}\psi(2y)\tau(y,\phi(y))^{-1} \\ &= \psi(y)^{n-3}\psi(y)\psi(2y)\tau(y,\phi(y))^{-1} \\ &= \psi(y)^{n-3}\psi(3y)\tau(y,\phi(2y))^{-1}\tau(y,\psi(y))^{-1} \\ &= \cdots = \psi(ny)\tau(y,\phi(\tfrac{1}{2}(n-1)ny))^{-1} \end{aligned}$$
under the pullback to η of the composition of canonical isomorphisms
$$\begin{aligned} (c^\vee)^*\mathcal{M}^{\otimes n} &\xrightarrow{\sim} (nc^\vee)^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (\mathrm{Id}_Y, \tfrac{1}{2}(n-1)n\phi)^*(c^\vee\times c)^*\mathcal{P}_A^{\otimes -1} \\ &\overset{\text{can.}}{\xrightarrow{\sim}} (nc^\vee)^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (c^\vee, \tfrac{1}{2}(n-1)nc\phi)^*\mathcal{P}_A^{\otimes -1} \\ &= (nc^\vee)^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (c^\vee, \tfrac{1}{2}(n-1)n\lambda_A c^\vee)^*\mathcal{P}_A^{\otimes -1} \\ &\overset{\text{can.}}{\xrightarrow{\sim}} (nc^\vee)^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (c^\vee, \tfrac{1}{2}(n-1)nc^\vee)^*(\mathrm{Id}_A\times\lambda_A)^*\mathcal{P}_A^{\otimes -1} \\ &\overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee)^*([n]_A^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (\mathrm{Id}_A, [\tfrac{1}{2}(n-1)n]_A)^*\mathcal{D}_2(\mathcal{M})^{\otimes -1}). \end{aligned}$$
This composition is the pullback under $c^\vee$ of the canonical isomorphism
$$\mathcal{M}^{\otimes n} \overset{\text{can.}}{\xrightarrow{\sim}} [n]_A^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (\mathrm{Id}_A, [\tfrac{1}{2}(n-1)n]_A)^*\mathcal{D}_2(\mathcal{M})^{\otimes -1}$$
given by repeated application of the canonical isomorphisms
$$\begin{aligned} [m_1]^*\mathcal{M}\otimes[m_2]^*\mathcal{M} &\overset{\text{can.}}{\xrightarrow{\sim}} [m_1+m_2]^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (m_1,m_2)^*\mathcal{D}_2(\mathcal{M})^{\otimes -1} \\ &\overset{\text{can.}}{\xrightarrow{\sim}} [m_1+m_2]^*\mathcal{M} \underset{\mathscr{O}_A}{\otimes} (1,m_1m_2)^*\mathcal{D}_2(\mathcal{M})^{\otimes -1}. \end{aligned}$$
(The upshot is that there is nowhere in these isomorphisms that we use any of the two rigidifications of $\mathcal{P}_A$.)

   If we pullback the canonical isomorphisms (over η) by $c_n^\vee$, then we get
$$\begin{aligned} (c_n^\vee)^*\mathcal{M}_\eta^{\otimes n} &\overset{\text{can.}}{\xrightarrow{\sim}} (c_n^\vee)^*([n]_A^*\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} (\mathrm{Id}_A, [\tfrac{1}{2}(n-1)n]_A)^*\mathcal{D}_2(\mathcal{M}_\eta)^{\otimes -1}) \\ &\overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee)^*\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} (c_n^\vee, \tfrac{1}{2}(n-1)nc_n^\vee)^*\mathcal{D}_2(\mathcal{M}_\eta)^{\otimes -1}. \end{aligned}$$
Let us set $\epsilon = 1$ when n is odd and $\epsilon = 2$ when n is even. Then the above composition can be rewritten as
$$(c_n^\vee)^*\mathcal{M}_\eta^{\otimes n} \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee)^*\mathcal{M}_\eta \underset{\mathscr{O}_{A,\eta}}{\otimes} (c_n^\vee, \tfrac{1}{2}(n-1)\epsilon c_\epsilon^\vee)^*\mathcal{D}_2(\mathcal{M}_\eta)^{\otimes -1}.$$
Certainly, it would be desirable if the pullback
$$\begin{aligned} \tau_n\circ(\mathrm{Id}_Y\times\phi) : \mathbf{1}_{\frac{1}{n}Y\times Y,\eta} &\xrightarrow{\sim} (c_n^\vee, c\phi)^*\mathcal{P}_{A,\eta}^{\otimes -1} \\ &= (c_n^\vee, \lambda_A c^\vee)^*\mathcal{P}_{A,\eta}^{\otimes -1} \overset{\text{can.}}{\xrightarrow{\sim}} (c_n^\vee, c^\vee)^*\mathcal{D}_2(\mathcal{M}_\eta)^{\otimes -1} \end{aligned}$$
is liftable to some trivialization
$$\tilde{\tau}_{n,\epsilon} : \mathbf{1}_{\frac{1}{n}Y\times\frac{1}{\epsilon}Y} \xrightarrow{\sim} (c_n^\vee, c_\epsilon^\vee)^*\mathcal{D}_2(\mathcal{M}_\eta)^{\otimes -1},$$
where $c_\epsilon^\vee = c_n^\vee|_{\frac{1}{\epsilon}Y} : \frac{1}{\epsilon}Y \to A$ is the restriction. Then we can define ψ_n by
$$\psi_n(\tfrac{1}{n}y) = \psi(y)\tilde{\tau}_{n,\epsilon}(\tfrac{1}{n}y, \tfrac{n-1}{2}y)^{-1},$$
where $\psi_n(\frac{1}{n}y)$ is interpreted as a section of $\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\frac{1}{n}y))^{\otimes -1}$. Note that we have, in particular,
$$\psi_n(y) = \psi(y)^n$$
for every $y \in Y$. Since Y is finitely generated, this is always true after a finite étale localization (over η). Therefore, we may assume that all the $\psi_n(\frac{1}{n}y)$'s that we need exist and have been chosen. The value of the λ_η-Weil pairing is invariant under étale localizations and independent of the choices.

   We shall consider the isomorphisms
$$\psi_n(\tfrac{1}{n}y)\tau_n(\tfrac{1}{n}y,\chi) : T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n}\otimes\mathscr{O}_{\chi,\eta}) \to \mathcal{M}_\eta^{\otimes n}\otimes\mathscr{O}_{\chi+\phi(y),\eta},$$
which covers the translation by $\iota_n(\frac{1}{n}Y)$ on $G_\eta^\natural$.

Note that we do not check that these isomorphisms commute with the Y-action on $(\mathcal{L}_\eta^\natural)^{\otimes n}$ defined by ψ. This statement will be a by-product (see Proposition 5.2.6.3 below) when we compute the commutators between these isomorphisms, because the restriction of ψ_n to Y can always be chosen to be ψ^n, and other choices only differ by an element in $\mathbf{G}_{\mathrm{m},\eta}$.

Let us start with the computation of pairings involving $T[n]_\eta$:

Proposition 5.2.6.1. *Suppose $t, t' \in T[n]_\eta$. Then $\mathrm{e}^{\lambda_\eta}(t,t') = 1$ (as in Proposition 5.2.2.1).*

Proof. Since we have an action of (the commutative group scheme) $T[n]$ on $\mathcal{L}^\natural$, the commutator between $\chi(t)$ and $\chi(t')$ is always trivial. □

Proposition 5.2.6.2. *Suppose $t \in T[n]_\eta$ and $a \in A[n]_\eta$. Then $\mathrm{e}^{\lambda_\eta}(t,a) = 1$ (as in Proposition 5.2.2.1).*

Proof. Using the splitting defined by c_n, we have $\mathrm{e}^{\lambda_\eta}(t,a) = 1$ because the diagram
$$\begin{tikzcd} T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \arrow[r, "\tilde{a}^{-1}\otimes r(a,c_n(\frac{1}{n}\chi))", "\sim"'] \arrow[d, "\chi(t)"', "\wr"] & \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \arrow[d, "\chi(t)", "\wr"'] \\ T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \arrow[r, "\sim", "\tilde{a}^{-1}\otimes r(a,c_n(\frac{1}{n}\chi))"'] & \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \end{tikzcd}$$
is commutative. □

Proposition 5.2.6.3. *Suppose $t \in T[n]_\eta$ and $\frac{1}{n}y \in \frac{1}{n}Y$. Then $\mathrm{e}^{\lambda_\eta}(t,y) = (\phi(y))(t)$ (as in Proposition 5.2.2.1). In particular, the $T[n]_\eta$-action commutes with the Y-action on $(\mathcal{L}_\eta^\natural)^{\otimes n}$ (regardless of the choice of ψ_n above).*

Proof. Since the diagram

$$\begin{tikzcd}
T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \arrow[r, "\psi_n(\frac{1}{n}y)\tau_n(\frac{1}{n}y,\chi)", "\sim"'] \arrow[dd, "\chi(t)"', "\wr"] & \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta} \arrow[d, "\wr"', "(\chi+\phi(y))(t)"] \\
 & \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta} \\
T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \arrow[r, "\sim", "\psi_n(\frac{1}{n}y)\tau_n(\frac{1}{n}y,\chi)"'] & \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta} \arrow[u, "\wr", "(\phi(y))(t)"']
\end{tikzcd}$$

is commutative, by comparing this with the sign convention in Definition 5.2.4.6, and by Proposition 5.2.4.9, we see that $\mathrm{e}^{\lambda_\eta}(t,y) = (\phi(y))(t)$. (This is the same sign convention that we used in Proposition 5.2.2.1.) $\square$

Let us calculate those pairings involving $A[n]_\eta$:

Proposition 5.2.6.4. *Suppose $a, a' \in A[n]_\eta$. Then $\mathrm{e}^{\lambda_\eta}(a,a') = \mathrm{e}^{\lambda_A}(a,a')$ (as in Proposition 5.2.2.1).*

Proof. Choose sections $\tilde{a} \in \mathcal{M}_\eta^{\otimes n}(a)$ and $\tilde{a}' \in \mathcal{M}_\eta^{\otimes n}(a')$, which define isomorphisms $\tilde{a}^{-1} : \mathcal{M}_\eta^{\otimes n}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}$ and $(\tilde{a}')^{-1} : \mathcal{M}_\eta^{\otimes n}(a') \xrightarrow{\sim} \mathscr{O}_{S,\eta}$, respectively.

Let us analyze the first combination

$$T_a^* T_{a'}^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \xrightarrow{\sim} T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}.$$

More precisely, this isomorphism is the composition of the following canonical isomorphisms

$$\begin{aligned}
& T_a^* T_{a'}^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \\
& \overset{\text{can.}}{\xrightarrow{\sim}} T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a') \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a')) \\
& \overset{\text{can.}}{\xrightarrow{\sim}} T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a') \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a') \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{A,\eta}(a) \\
& \overset{\text{can.}}{\xrightarrow{\sim}} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta} \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a) \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a) \\
& \qquad \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathcal{M}_\eta^{\otimes n}(a') \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}(a') \underset{\mathscr{O}_{S,\eta}}{\otimes} \mathscr{O}_{A,\eta}(a)
\end{aligned}$$

with the isomorphisms

$$\begin{aligned}
& \tilde{a}^{-1} : \mathcal{M}_\eta^{\otimes n}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}, \\
& r(a, c_n(\tfrac{1}{n}\chi)) : \mathscr{O}_{\chi,\eta}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}, \\
& (\tilde{a}')^{-1} : \mathcal{M}_\eta^{\otimes n}(a') \xrightarrow{\sim} \mathscr{O}_{S,\eta}, \\
& r(a', c_n(\tfrac{1}{n}\chi)) : \mathscr{O}_{\chi,\eta}(a') \xrightarrow{\sim} \mathscr{O}_{S,\eta}, \\
& \mathrm{str.}(a) : \mathscr{O}_{A,\eta}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}.
\end{aligned}$$

The second combination

$$T_{a'}^* T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \xrightarrow{\sim} T_{a'}^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) \xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}$$

can be described analogously by switching the roles of a and a'. Therefore, the essential difference between the two combinations is given by the difference between the two isomorphisms $\mathrm{str.}(a)$ and $\mathrm{str.}(a')$, and the diagram

$$\begin{tikzcd}
\mathscr{O}_{A,\eta}(a) \arrow[r, "\text{can.}", "\sim"'] & \mathcal{D}_2(\mathcal{M}_\eta)|_{(a,e)} \arrow[r, "\text{rig.}", "\sim"'] \arrow[d, leftrightarrow, "\text{sym.}"', "\wr"] & \mathscr{O}_{S,\eta} \\
\mathscr{O}_{A,\eta}(a') \arrow[r, "\sim", "\text{can.}"'] & \mathcal{D}_2(\mathcal{M}_\eta)|_{(a',e)} \arrow[r, "\sim", "\text{rig.}"'] & \mathscr{O}_{S,\eta} \arrow[u, "\wr"', "\mathrm{e}^{\lambda_\eta}(a,a')"]
\end{tikzcd}$$

is commutative. (Here the isomorphism sym. in the middle is induced by the symmetry of $\mathcal{D}_2(\mathcal{M}_\eta^{\otimes n})$.) Comparing this with Corollary 5.2.4.5, we see that $\mathrm{e}^{\lambda_\eta}(a,a') = \mathrm{e}^{\lambda_A}(a,a')$, as desired. $\square$

Proposition 5.2.6.5. *Suppose $a \in A[n]_\eta$ and $\frac{1}{n}y \in \frac{1}{n}Y$. Then $\mathrm{e}^{\lambda_\eta}(a, \frac{1}{n}y) = \mathrm{e}_{A[n]}(a, b_n(\frac{1}{n}y))$, where $b_n := \lambda_A c_n^\vee - c_n\phi_n$ as in Lemma 5.2.3.12. (This proves the first part of Theorem 5.2.3.14.)*

Proof. Choose a section $\tilde{a} \in \mathcal{M}_\eta^{\otimes n}(a)$ that defines an isomorphism $\tilde{a}^{-1} : \mathcal{M}_\eta^{\otimes n}(a) \xrightarrow{\sim} \mathscr{O}_{S,\eta}$. On the other hand, we have $\psi_n(\frac{1}{n}y) \in \mathcal{M}_\eta^{\otimes n}(c_n^\vee(\frac{1}{n}y))^{\otimes -1}$ chosen above.

Let us analyze the first combination

$$\begin{aligned}
T_a^* T_{c_n^\vee(\frac{1}{n}y)}^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi,\eta}) & \xrightarrow{\sim} T_a^*(\mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta}) \\
& \xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n} \underset{\mathscr{O}_{A,\eta}}{\otimes} \mathscr{O}_{\chi+\phi(y),\eta}.
\end{aligned}$$

More precisely, this isomorphism is the composition of the following canonical isomorphisms

$$
\begin{aligned}
&T_a^*T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta})\\
&\overset{\text{can.}}{\overset{\sim}{\to}} T_a^*(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\phi(y),\eta}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y)))\\
&\overset{\text{can.}}{\overset{\sim}{\to}} T_a^*(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta})\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\phi(y),\eta}\\
&\qquad\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\phi(y),\eta}(a)\\
&\overset{\text{can.}}{\overset{\sim}{\to}} \mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(a)\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(a)\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\phi(y),\eta}\\
&\qquad\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\phi(y),\eta}(a)\\
&\overset{\text{can.}}{\overset{\sim}{\to}} \mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y),\eta}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(a)\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(a)\\
&\qquad\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\phi(y),\eta}(a)
\end{aligned}
$$

with the isomorphisms

$$
\begin{aligned}
&\tilde{a}^{-1}:\mathcal{M}_\eta^{\otimes n}(a)\overset{\sim}{\to}\mathscr{O}_{S,\eta},\\
&r(a,c_n(\tfrac{1}{n}\chi)):\mathscr{O}_{\chi,\eta}(a)\overset{\sim}{\to}\mathscr{O}_{S,\eta},\\
&\psi_n(\tfrac{1}{n}y):\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\overset{\sim}{\to}\mathscr{O}_{S,\eta},\\
&\tau_n(\tfrac{1}{n}y,\chi):\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))\overset{\sim}{\to}\mathscr{O}_{S,\eta},\\
&r(a,c_n(\tfrac{1}{n}\phi(y))):\mathscr{O}_{\phi(y),\eta}(a)\overset{\sim}{\to}\mathscr{O}_{S,\eta}.
\end{aligned}
$$

On the other hand, the second combination

$$
\begin{aligned}
T^*_{c_n^\vee(\frac{1}{n}y)}T_a^*(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta})&\overset{\sim}{\to}T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta})\\
&\overset{\sim}{\to}\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y),\eta}
\end{aligned}
$$

is the composition of the following canonical isomorphisms

$$
\begin{aligned}
&T^*_{c_n^\vee(\frac{1}{n}y)}T_a^*(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta})\\
&\overset{\text{can.}}{\overset{\sim}{\to}} T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(a)\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(a))\\
&\overset{\text{can.}}{\overset{\sim}{\to}} T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta})\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(a)\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(a)\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{A,\eta}(c_n^\vee(\tfrac{1}{n}y))\\
&\overset{\text{can.}}{\overset{\sim}{\to}} \mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y)}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))\\
&\qquad\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(a)\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(a)\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{A,\eta}(c_n^\vee(\tfrac{1}{n}y))
\end{aligned}
$$

with the isomorphisms

$$
\begin{aligned}
&\psi_n(\tfrac{1}{n}y):\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\overset{\sim}{\to}\mathscr{O}_{S,\eta},\\
&\tau_n(\tfrac{1}{n}y,\chi):\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))\overset{\sim}{\to}\mathscr{O}_{S,\eta},\\
&\tilde{a}^{-1}:\mathcal{M}_\eta^{\otimes n}(a)\overset{\sim}{\to}\mathscr{O}_{S,\eta},\\
&r(a,c_n(\tfrac{1}{n}\chi)):\mathscr{O}_{\chi,\eta}(a)\overset{\sim}{\to}\mathscr{O}_{S,\eta},\\
&\text{str.}(c_n^\vee(\tfrac{1}{n}y)):\mathscr{O}_{A,\eta}(c_n^\vee(\tfrac{1}{n}y))\overset{\sim}{\to}\mathscr{O}_{S,\eta}.
\end{aligned}
$$

The essential difference between these two combinations is given by the difference between

$$
\begin{aligned}
\mathcal{D}_2(\mathcal{M}_\eta^{\otimes n})|_{(a,c_n^\vee(\frac{1}{n}y))}&\overset{\text{can.}}{\overset{\sim}{\to}}\mathcal{D}_2(\mathcal{M}_\eta)|_{(a,c^\vee(y))}\overset{\text{can.}}{\overset{\sim}{\to}}\mathcal{P}_{A,\eta}|_{(a,\lambda_A c^\vee(y))}\\
&=\mathcal{P}_{A,\eta}|_{(a,c\phi(y))}\overset{\text{can.}}{\overset{\sim}{\to}}\mathscr{O}_{\phi(y),\eta}(a)\overset{r(a,c_n(\frac{1}{n}\phi(y)))}{\overset{\sim}{\to}}\mathscr{O}_{S,\eta}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{D}_2(\mathcal{M}_\eta^{\otimes n})|_{(c_n^\vee(\frac{1}{n}y),a)}&\overset{\text{can.}}{\overset{\sim}{\to}}\mathcal{D}_2(\mathcal{M}_\eta)|_{(c_n^\vee(\frac{1}{n}y),na)}=\mathcal{D}_2(\mathcal{M}_\eta)|_{(c_n^\vee(\frac{1}{n}y),e)}\\
&\overset{\text{can.}}{\overset{\sim}{\to}}\mathcal{P}_{A,\eta}|_{(c_n^\vee(\frac{1}{n}y),e)}\overset{\text{rig.}}{\overset{\sim}{\to}}\mathscr{O}_{A,\eta}(c_n^\vee(\tfrac{1}{n}y))\overset{\text{str.}}{\overset{\sim}{\to}}\mathscr{O}_{S,\eta}.
\end{aligned}
$$

(Here str. is the same isomorphism as $\text{str.}(c_n^\vee(\frac{1}{n}y))$ above.) The last part of the composition

$$\mathcal{P}_{A,\eta}|_{(c_n^\vee(\frac{1}{n}y),e)}\overset{\text{rig.}}{\overset{\sim}{\to}}\mathscr{O}_{A,\eta}(c_n^\vee(\tfrac{1}{n}y))\overset{\text{str.}}{\overset{\sim}{\to}}\mathscr{O}_{S,\eta}$$

can be interpreted alternatively as

$$\mathcal{P}_{A,\eta}|_{(c_n^\vee(\frac{1}{n}y),e)}\overset{\text{sym.}}{\overset{\sim}{\to}}\mathcal{P}_{A,\eta}|_{(e,\lambda_A c_n^\vee(\frac{1}{n}y))}\overset{\text{rig.}}{\overset{\sim}{\to}}\mathscr{O}_{S,\eta},$$

and hence the diagram

$$
\begin{tikzcd}
\mathscr{O}_{\phi(y),\eta}(a) \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & \mathcal{P}_{A,\eta}|_{(a,c\phi(y))} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, "\text{sym.}"', "\wr"] & \mathcal{P}_{A,\eta}|_{(e,c_n\phi_n(\frac{1}{n}y))} \arrow[r, "\text{rig.}", "\sim"'] & \mathscr{O}_{S,\eta} \\
\mathscr{O}_{A,\eta}(c_n^\vee(\frac{1}{n}y)) \arrow[r, "\sim", "\text{can.}"'] & \mathcal{P}_{A,\eta}|_{(c_n^\vee(\frac{1}{n}y),e)} \arrow[r, "\sim", "\text{sym.}"'] & \mathcal{P}_{A,\eta}|_{(e,\lambda_A c_n^\vee(\frac{1}{n}y))} \arrow[r, "\sim", "\text{rig.}"'] & \mathscr{O}_{S,\eta} \arrow[u, "\mathrm{e}^{\lambda_\eta}(a,\frac{1}{n}y)"', "\wr"]
\end{tikzcd}
$$

is commutative. There are some obvious redundancy in the diagram. By considering $b_n=\lambda_A c_n^\vee-c_n\phi_n$, the commutativity of above diagram implies that the diagram

$$
\begin{tikzcd}
 & \mathcal{P}_{A,\eta}|_{(e,-\lambda_A c_n^\vee(\frac{1}{n}y))}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{P}_{A,\eta}|_{(e,b_n(\frac{1}{n}y))} \arrow[d, "\wr"', "\text{rig.}\otimes\text{rig.}"] \\
\mathcal{P}^{\otimes n}_{A,\eta}|_{(a,-\lambda_A c_n^\vee(\frac{1}{n}y))}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{P}^{\otimes n}_{A,\eta}|_{(a,b_n(\frac{1}{n}y))} \arrow[ur, bend left, "\text{can.}", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & \mathscr{O}_{S,\eta}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{S,\eta} \arrow[d, "\wr"', "\mathrm{Id}\otimes\mathrm{e}^{\lambda_\eta}(a,\frac{1}{n}y)"] \\
\mathcal{P}_{A,\eta}|_{(e,-\lambda_A c_n^\vee(\frac{1}{n}y))}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{P}_{A,\eta}|_{(a,e)} \arrow[r, "\sim", "\text{rig.}\otimes\text{rig.}"'] & \mathscr{O}_{S,\eta}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{S,\eta}
\end{tikzcd}
$$

is commutative. After removing canonical isomorphisms, the essential content is that the diagram

$$\begin{tikzcd}
\mathcal{P}_{A,\eta}|_{(e,b_n(\frac{1}{n}y))} \arrow[r, "\text{rig.}", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & \mathscr{O}_{S,\eta} \arrow[d, "\mathrm{e}^{\lambda_\eta}(a,\frac{1}{n}y)", "\wr"'] \\
\mathcal{P}_{A,\eta}|_{(a,e)} \arrow[r, "\sim", "\text{can.}"'] & \mathscr{O}_{S,\eta}
\end{tikzcd}$$

is commutative. Comparing this with Proposition 5.2.4.4, we obtain $\mathrm{e}^{\lambda_\eta}(a,\frac{1}{n}y) = \mathrm{e}_{A[n]}(a,b_n(\frac{1}{n}y))$, as desired. $\square$

Finally, let us calculate the pairing on $(\frac{1}{n}Y/Y)\times(\frac{1}{n}Y/Y)$:

Proposition 5.2.6.6. *Suppose $\frac{1}{n}y,\frac{1}{n}y'\in\frac{1}{n}Y/Y$. Then (symbolically)*

$$\mathrm{e}^{\lambda_\eta}(\tfrac{1}{n}y,\tfrac{1}{n}y') = \tau_n(\tfrac{1}{n}y,\phi(y'))\tau_n(\tfrac{1}{n}y',\phi(y))^{-1}. \tag{5.2.6.7}$$

(*This proves the second part of Theorem 5.2.3.14.*)

Proof. Let us analyze the first combination

$$\begin{aligned} T^*_{c_n^\vee(\frac{1}{n}y)}T^*_{c_n^\vee(\frac{1}{n}y')}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}) &\xrightarrow{\sim} T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y'),\eta}) \\ &\xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y)+\phi(y'),\eta}. \end{aligned}$$

More precisely, this isomorphism is the composition of the following canonical isomorphisms

$$\begin{aligned}
& T^*_{c_n^\vee(\frac{1}{n}y)}T^*_{c_n^\vee(\frac{1}{n}y')}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}) \\
\overset{\text{can.}}{\xrightarrow{\sim}}\ & T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\phi(y'),\eta} \\
& \quad \underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y'))) \\
\overset{\text{can.}}{\xrightarrow{\sim}}\ & T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta})\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\phi(y'),\eta} \\
& \quad \underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\phi(y'),\eta}(c_n^\vee(\tfrac{1}{n}y)) \\
\overset{\text{can.}}{\xrightarrow{\sim}}\ & \mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y),\eta}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\phi(y'),\eta} \\
& \quad \underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\phi(y'),\eta}(c_n^\vee(\tfrac{1}{n}y)) \\
\overset{\text{can.}}{\xrightarrow{\sim}}\ & \mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y)+\phi(y'),\eta}\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y)) \\
& \quad \underset{\mathscr{O}_{S,\eta}}{\otimes}\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y'))\underset{\mathscr{O}_{S,\eta}}{\otimes}\mathscr{O}_{\phi(y'),\eta}(c_n^\vee(\tfrac{1}{n}y))
\end{aligned}$$

with the isomorphisms

$$\begin{aligned}
&\psi_n(\tfrac{1}{n}y):\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y))\xrightarrow{\sim}\mathscr{O}_{S,\eta}, \\
&\tau_n(\tfrac{1}{n}y,\chi):\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y))\xrightarrow{\sim}\mathscr{O}_{S,\eta}, \\
&\psi_n(\tfrac{1}{n}y'):\mathcal{M}_\eta^{\otimes n}(c_n^\vee(\tfrac{1}{n}y'))\xrightarrow{\sim}\mathscr{O}_{S,\eta}, \\
&\tau_n(\tfrac{1}{n}y',\chi):\mathscr{O}_{\chi,\eta}(c_n^\vee(\tfrac{1}{n}y'))\xrightarrow{\sim}\mathscr{O}_{S,\eta}, \\
&\tau_n(\tfrac{1}{n}y,\phi(y')):\mathscr{O}_{\phi(y'),\eta}(c_n^\vee(\tfrac{1}{n}y))\xrightarrow{\sim}\mathscr{O}_{S,\eta}.
\end{aligned}$$

As in the case of $\mathrm{e}^{\lambda_\eta}(a,a')$, the second combination

$$\begin{aligned} T^*_{c_n^\vee(\frac{1}{n}y')}T^*_{c_n^\vee(\frac{1}{n}y)}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi,\eta}) &\xrightarrow{\sim} T^*_{c_n^\vee(\frac{1}{n}y')}(\mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y),\eta}) \\ &\xrightarrow{\sim} \mathcal{M}_\eta^{\otimes n}\underset{\mathscr{O}_{A,\eta}}{\otimes}\mathscr{O}_{\chi+\phi(y)+\phi(y'),\eta}. \end{aligned}$$

can be described analogously by switching the roles of y and y'. Therefore, the essential difference between the two combinations is given by the difference between the two isomorphisms $\tau_n(\frac{1}{n}y,\phi(y'))$ and $\tau_n(\frac{1}{n}y',\phi(y))$, and the diagram

$$\begin{tikzcd}
\mathcal{D}_2(\mathcal{M}_\eta^{\otimes n})|_{(c_n^\vee(\frac{1}{n}y),c_n^\vee(\frac{1}{n}y'))} \arrow[r, "\text{can.}", "\sim"'] \arrow[d, leftrightarrow, "\text{sym.}"', "\wr"] & \mathcal{P}_{A,\eta}|_{(c_n^\vee(\frac{1}{n}y),c\phi(y'))} \arrow[r, "\tau_n(\frac{1}{n}y,\phi(y'))", "\sim"'] \arrow[d, "\text{can.}"', "\wr"] & \mathscr{O}_{S,\eta} \\
\mathcal{D}_2(\mathcal{M}_\eta^{\otimes n})|_{(c_n^\vee(\frac{1}{n}y'),c_n^\vee(\frac{1}{n}y))} \arrow[r, "\sim", "\text{can.}"'] & \mathcal{P}_{A,\eta}|_{(c_n^\vee(\frac{1}{n}y'),c\phi(y))} \arrow[r, "\sim", "\tau_n(\frac{1}{n}y',\phi(y))"'] & \mathscr{O}_{S,\eta} \arrow[u, "\mathrm{e}^{\lambda_\eta}(\frac{1}{n}y,\frac{1}{n}y')"', "\wr"]
\end{tikzcd}$$

is commutative. This implies the symbolic relation (5.2.6.7), as desired. $\square$

Now Theorem 5.2.3.14 follows from the combination of Propositions 5.2.6.5 and 5.2.6.6.

5.2.7 Construction of Principal Level Structures

With the same setting as in Section 5.2.1, assume that we have a tuple $(A,\lambda_A,i_A,\underline{X},\underline{Y},\phi,c,c^\vee,\tau)$ in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle,h)}(R,I)$. Then we know by Theorem 5.1.2.7 that there is an object (G,λ,i) in $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle,h)}(R,I)$ corresponding to the tuple above via $\mathrm{M}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle,h)}(R,I)$. For simplicity, let us continue to assume that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively. One of the assumptions for having an object in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle,h)}(R,I)$ is that there exists a totally isotropic embedding $\mathrm{Hom}_{\mathbb{R}}(X\underset{\mathbb{Z}}{\otimes}\mathbb{R},\mathbb{R}(1))\hookrightarrow(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})$ (see Definition 5.1.2.6). In this case, there is an induced filtration $\mathtt{Z}_{\mathbb{R}}=\{\mathtt{Z}_{-i,\mathbb{R}}\}_i$ on $L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ determining a unique isomorphism class of the induced symplectic module $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}},\langle\,\cdot\,,\,\cdot\,\rangle_{11,\mathbb{R}})$ over $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ (see Proposition 5.1.2.2).

Motivated by Lemma 5.2.2.2 and its proof,

Definition 5.2.7.1. *We say that a symplectic admissible filtration $\mathtt{Z}$ on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$ is **fully symplectic with respect to** $(L,\langle\,\cdot\,,\,\cdot\,\rangle)$ if there is a symplectic admissible filtration $\mathtt{Z}_{\mathbb{A}^\square}=\{\mathtt{Z}_{-i,\mathbb{A}^\square}\}_i$ on $L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^\square$ that **extends** $\mathtt{Z}$ in the sense that $\mathtt{Z}_{-i,\mathbb{A}^\square}\cap(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square)=\mathtt{Z}_{-i}$ in $L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^\square$ for all i.*

Remark 5.2.7.2. Implicit in Definition 5.2.7.1 is that $\mathtt{Z}_{-i,\mathbb{A}^\Box}$ is *integrable* for every i. In this case, there exists (noncanonically) a PEL-type $\mathcal{O}$-lattice $(L^{\mathtt{Z}}, \langle\cdot,\cdot\rangle^{\mathtt{Z}}, h^{\mathtt{Z}})$ such that there exists an isomorphism $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{A}^\Box}, \langle\cdot,\cdot\rangle_{11}, h_{-1}) \xrightarrow{\sim} (L^{\mathtt{Z}} \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\Box, \langle\cdot,\cdot\rangle^{\mathtt{Z}}, h^{\mathtt{Z}})$ (over $\mathbb{A}^\Box$). By modifying $(L^{\mathtt{Z}}, \langle\cdot,\cdot\rangle^{\mathtt{Z}})$ if necessary, we may assume that there exist isomorphisms $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\cdot,\cdot\rangle_{11}) \xrightarrow{\sim} (L^{\mathtt{Z}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle^{\mathtt{Z}})$ (over $\hat{\mathbb{Z}}^\Box$) and $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}}, \langle\cdot,\cdot\rangle_{11,\mathbb{R}}, h_{-1}) \xrightarrow{\sim} (L^{\mathtt{Z}} \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\cdot,\cdot\rangle^{\mathtt{Z}}, h^{\mathtt{Z}})$ (over $\mathbb{R}$), and that $[(L^{\mathtt{Z}})^\# : L^{\mathtt{Z}}]$ contains no prime factors other than those of $[L^\# : L]$.

Definition 5.2.7.3. *A symplectic-liftable admissible filtration $\mathtt{Z}_n$ on L/nL is called* ***fully symplectic-liftable*** *with respect to $(L, \langle\cdot,\cdot\rangle)$ if it is the reduction modulo n of some admissible filtration $\mathtt{Z}$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$ that is fully symplectic with respect to $(L, \langle\cdot,\cdot\rangle)$ as in Definition 5.2.7.1.*

Remark 5.2.7.4. As explained in Remark 5.2.2.8, even when $n = 1$, in which case the whole space L/nL is trivial, we shall still distinguish the filtrations by their equipped $\mathcal{O}$-multiranks.

Lemma 5.2.7.5. *Let $\mathtt{Z}_n$ be an admissible filtration on L/nL that is fully symplectic-liftable with respect to $(L, \langle\cdot,\cdot\rangle)$. Let $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\cdot,\cdot\rangle_{11})$ be induced by some fully symplectic lifting $\mathtt{Z}$ of $\mathtt{Z}_n$, and let $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}}, \langle\cdot,\cdot\rangle_{11,\mathbb{R}}, h_{-1})$ be determined as in Proposition 5.1.2.2 (which has the same reflex field F_0 as $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\cdot,\cdot\rangle, h)$ does). Then there is a (noncanonical) PEL-type $\mathcal{O}$-lattice $(L^{\mathtt{Z}_n}, \langle\cdot,\cdot\rangle^{\mathtt{Z}_n}, h^{\mathtt{Z}_n})$ satisfying Condition 1.4.3.10 such that*

1. *$[(L^{\mathtt{Z}_n})^\# : L^{\mathtt{Z}_n}]$ is prime-to-$\Box$;*
2. *there exist (noncanonical) $\mathcal{O}$-equivariant isomorphisms $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\cdot,\cdot\rangle_{11}) \xrightarrow{\sim} (L^{\mathtt{Z}_n} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle^{\mathtt{Z}_n})$ (over $\hat{\mathbb{Z}}^\Box$) and $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}}, \langle\cdot,\cdot\rangle_{11,\mathbb{R}}, h_{-1}) \xrightarrow{\sim} (L^{\mathtt{Z}_n} \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\cdot,\cdot\rangle^{\mathtt{Z}_n}, h^{\mathtt{Z}_n})$ (over $\mathbb{R}$);*
3. *the moduli problem $\mathsf{M}^{\mathtt{Z}_n}_n$ (over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$) defined by the noncanonical $(L^{\mathtt{Z}_n}, \langle\cdot,\cdot\rangle^{\mathtt{Z}_n}, h^{\mathtt{Z}_n})$, as in Definition 1.4.1.2, is canonical in the sense that it depends (up to isomorphism) only on $\mathtt{Z}_n$, but not on the choice of $(L^{\mathtt{Z}_n}, \langle\cdot,\cdot\rangle^{\mathtt{Z}_n}, h^{\mathtt{Z}_n})$.*

Proof. The existence of $(L^{\mathtt{Z}_n}, \langle\cdot,\cdot\rangle^{\mathtt{Z}_n}, h^{\mathtt{Z}_n})$ with the given properties is explained in Remark 5.2.7.2. As pointed out in Remark 1.4.3.14, the moduli problem $\mathsf{M}^{\mathtt{Z}_n}_n$ is smooth and has at least one complex point. $\Box$

Remark 5.2.7.6. To avoid unnecessarily introducing more noncanonical data, we shall sometimes suppress the choice of $(L^{\mathtt{Z}_n}, \langle\cdot,\cdot\rangle^{\mathtt{Z}_n}, h^{\mathtt{Z}_n})$, and say that $\mathsf{M}^{\mathtt{Z}_n}_n$ is defined by $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\cdot,\cdot\rangle_{11})$ and $(\mathrm{Gr}^{\mathtt{Z}}_{-1,\mathbb{R}}, \langle\cdot,\cdot\rangle_{11,\mathbb{R}}, h_{-1})$.

Let us begin with a symplectic-liftable admissible filtration $\mathtt{Z}_n = \{\mathtt{Z}_{-i,n}\}_i$ on L/nL that is fully symplectic with respect to $(L, \langle\cdot,\cdot\rangle)$. By Proposition 5.2.2.23, this is the most basic information we have about a level-n structure. Our goal is to describe the additional data for producing a level-n structure of $(G_{\tilde{\eta}}, \lambda_{\tilde{\eta}}, i_{\tilde{\eta}})$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle)$ over $\tilde{\eta}$.

In particular, being symplectic-liftable, the admissible filtration $\mathtt{Z}_n$ is the reduction modulo n of some symplectic admissible filtration $\mathtt{Z} = \{\mathtt{Z}_{-i}\}_i$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$. The pairing $\langle\cdot,\cdot\rangle$ induces the pairings

$$\langle\cdot,\cdot\rangle_{20} : \mathrm{Gr}^{\mathtt{Z}}_{-2} \times \mathrm{Gr}^{\mathtt{Z}}_{0} \to \hat{\mathbb{Z}}^\Box(1)$$

and

$$\langle\cdot,\cdot\rangle_{11} : \mathrm{Gr}^{\mathtt{Z}}_{-1} \times \mathrm{Gr}^{\mathtt{Z}}_{-1} \to \hat{\mathbb{Z}}^\Box(1),$$

both satisfying $\langle bx, y\rangle_{ij} = \langle x, b^\star y\rangle_{ij}$ for all $x \in \mathrm{Gr}^{\mathtt{Z}}_{-i}$, $y \in \mathrm{Gr}^{\mathtt{Z}}_{-j}$, and $b \in \mathcal{O}$.

For simplicity, let us assume for the moment that the étale sheaves $\underline{X}$ and $\underline{Y}$ in the datum $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ are constant with respective values X and Y.

The data X, Y, and $\phi : Y \hookrightarrow X$ define a pairing

$$\langle\cdot,\cdot\rangle_\phi : \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}(1)) \times Y \to \mathbb{Z}(1)$$

by sending (x, y) to $x(\phi(y))$ for all $x \in \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}(1))$ and $y \in Y$. The $\mathcal{O}$-module structure $\mathcal{O} \to \mathrm{End}_{\mathbb{Z}}(X)$ induces by transposition a right $\mathcal{O}^{\mathrm{op}}$-module structure $\mathcal{O}^{\mathrm{op}} \to \mathrm{End}_{\mathbb{Z}}(\mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}(1)))$, and hence an $\mathcal{O}$-module structure by precomposition with the natural anti-isomorphism $\mathcal{O} \to \mathcal{O}^{\mathrm{op}} : b \mapsto b^\star$. The $\mathcal{O}$-equivariance of ϕ implies that

$$\langle bx, y\rangle_\phi = (bx)(\phi(y)) = x(b^\star \phi(y)) = \langle x, b^\star y\rangle_\phi$$

for all $x \in \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z}(1))$, $y \in Y$, and $b \in \mathcal{O}$. By extension of scalars, the pairing $\langle\cdot,\cdot\rangle_\phi$ induces naturally a pairing

$$\langle\cdot,\cdot\rangle_\phi : \mathrm{Hom}_{\mathbb{Z}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \hat{\mathbb{Z}}^\Box(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \to \hat{\mathbb{Z}}^\Box(1).$$

Then it makes sense to consider pairs $(\varphi_{-2}, \varphi_0)$ of $\mathcal{O}$-equivariant isomorphisms

$$\varphi_{-2} : \mathrm{Gr}^{\mathtt{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \hat{\mathbb{Z}}^\Box(1))$$

and

$$\varphi_0 : \mathrm{Gr}^{\mathtt{Z}}_0 \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$$

such that

$$\langle x, y\rangle_\phi = \langle \varphi_{-2}^{-1}(x), \varphi_0^{-1}(y)\rangle_{20} \tag{5.2.7.7}$$

for all $x \in \mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \hat{\mathbb{Z}}^\Box(1))$ and $y \in Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$. (In order to make later constructions compatible, the sign convention for $\langle\cdot,\cdot\rangle_\phi$ is chosen to be analogous to the one for e^ϕ in Proposition 5.2.2.1.)

On the other hand, $\mathrm{Gr}^{\mathtt{Z}}_{-1}$ is paired with itself under $\langle\cdot,\cdot\rangle_{11}$, and $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\cdot,\cdot\rangle_{11})$ is symplectic isomorphic to $(L^{\mathtt{Z}} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle^{\mathtt{Z}})$ for some PEL-type $\mathcal{O}$-lattice $(L^{\mathtt{Z}}, \langle\cdot,\cdot\rangle^{\mathtt{Z}}, h^{\mathtt{Z}})$ by Lemma 5.2.7.5. Therefore it makes sense to consider level-n structures $\varphi_{-1,n} : \mathrm{Gr}^{\mathtt{Z}}_{-1,n} \xrightarrow{\sim} A[n]_{\tilde{\eta}}$ of type $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\cdot,\cdot\rangle_{11})$ as in Definition 1.3.6.2.

Suppose that we have chosen a liftable splitting $\delta_n : \mathrm{Gr}^{\mathtt{Z}}_n \xrightarrow{\sim} L/nL$, which can be lifted to some $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$. Then, as in Section 5.2.2, the pairing $\langle\cdot,\cdot\rangle$ can be expressed in matrix form as some

$$\begin{pmatrix} & & \langle\cdot,\cdot\rangle_{20} \\ & \langle\cdot,\cdot\rangle_{11} & \langle\cdot,\cdot\rangle_{10} \\ \langle\cdot,\cdot\rangle_{02} & \langle\cdot,\cdot\rangle_{01} & \langle\cdot,\cdot\rangle_{00} \end{pmatrix},$$

extending the pairings $\langle\cdot,\cdot\rangle_{20}$ and $\langle\cdot,\cdot\rangle_{11}$ above between the graded pieces. The level-n structure $\varphi_{-1,n} : \mathrm{Gr}^{\mathtt{Z}}_{-1,n} \xrightarrow{\sim} A[n]_{\tilde{\eta}}$ can be lifted noncanonically to some symplectic isomorphism $\varphi_{-1} : \mathrm{Gr}^{\mathtt{Z}}_{-1} \xrightarrow{\sim} \mathrm{T}^\Box A_{\tilde{\eta}}$, which we can take as $\hat{f}_{-1} : \mathrm{Gr}^{\mathtt{Z}}_{-1} \xrightarrow{\sim} \mathrm{Gr}^{\mathtt{W}}_{-1}$. The symplectic isomorphism φ_{-1} gives, in particular, an isomorphism $\nu(\varphi_{-1}) : \hat{\mathbb{Z}}^\Box(1) \xrightarrow{\sim} \mathrm{T}^\Box \mathbf{G}_{\mathrm{m},\tilde{\eta}}$. Suppose we have a pair $(\varphi_{-2}, \varphi_0)$ as above satisfying (5.2.7.7).

Then we can define isomorphisms

$$\begin{aligned}\hat{f}_{-2} : \mathtt{Z}_{-2} &\overset{\varphi_{-2}}{\overset{\sim}{\to}} \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1)) \\ &\overset{\nu(\varphi_{-1})}{\overset{\sim}{\to}} \underline{\mathrm{Hom}}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \mathbf{G}_{\mathrm{m},\bar\eta}) \overset{\text{can.}}{\overset{\sim}{\to}} \mathrm{T}^\square T_{\bar\eta} \cong \mathrm{Gr}^{\mathtt{W}}_{-2},\end{aligned}$$

and $\hat{f}_0 : \mathrm{Gr}^{\mathtt{Z}}_0 \overset{\varphi_0}{\overset{\sim}{\to}} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square \cong \mathrm{Gr}^{\mathtt{W}}_0$, and a graded isomorphism $\hat{f} := \underset{i}{\oplus} \hat{f}_i$. As soon as $\hat{f}$ is symplectic (as in Definition 5.2.2.13) with respect to some suitable choices of the isomorphism $\nu(\hat{f}) : \hat{\mathbb{Z}}^\square(1) \overset{\sim}{\to} \mathrm{T}^\square \mathbf{G}_{\mathrm{m},\bar\eta}$ and of the splitting $\hat{\varsigma} : \mathrm{Gr}^{\mathtt{W}} \overset{\sim}{\to} \mathrm{T}^\square G_{\bar\eta}$, we will obtain a symplectic isomorphism $\hat\alpha$ such that $\mathrm{Gr}(\hat\alpha) = \hat{f}$ by applying Proposition 5.2.2.22.

To find the condition for $\hat{f}$ to be symplectic, or equivalently the condition for $\hat\varsigma$ to make $\hat{f}$ symplectic, let us assume that a liftable splitting $\varsigma_n : \mathrm{Gr}^{\mathtt{W}}_n \overset{\sim}{\to} G[n]_{\bar\eta}$ is given in terms of a liftable triple $(c_n, c_n^\vee, \tau_n)$ over $\tilde\eta$, which can be lifted to a splitting $\hat\varsigma : \mathrm{Gr}^{\mathtt{W}} \overset{\sim}{\to} \mathrm{T}^\square G_{\bar\eta}$ given in terms of a triple $(\hat{c}, \hat{c}^\vee, \hat\tau)$. Then we can write the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}$ on $\mathrm{T}^\square G_{\bar\eta}$ in matrix form as

$$\begin{pmatrix} & & \mathrm{e}_{20} \\ & \mathrm{e}_{11} & \mathrm{e}_{10} \\ \mathrm{e}_{02} & \mathrm{e}_{01} & \mathrm{e}_{00} \end{pmatrix}.$$

By Lemma 5.2.2.14, the condition for $\hat{f}$ to be symplectic is the condition that $\hat{f}^*(\mathrm{e}_{ij}) = \nu(\hat{f}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{ij}$ for all i and j for the isomorphism $\nu(\hat{f})$ accompanying $\hat{f}$ (which we have not chosen yet). By the construction of $\hat{f} = \underset{i}{\oplus} \hat{f}_{-i}$, and by Proposition 5.2.2.1, we know that if we take $\nu(\hat{f}) = \nu(\varphi_{-1})$, then it is automatic that $\hat{f}^*(\mathrm{e}_{20}) = \nu(\hat{f}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{20}$ and $\hat{f}^*(\mathrm{e}_{11}) = \nu(\hat{f}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{11}$. This forces the choice of $\nu(\hat{f})$ if $\hat{f}$ is ever going to be symplectic. Therefore we see that the condition is that e_{10} and e_{00} must satisfy $\hat{f}^*(\mathrm{e}_{10}) = \nu(\varphi_{-1}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{10}$ and $\hat{f}^*(\mathrm{e}_{00}) = \nu(\varphi_{-1}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{00}$.

By Corollaries 5.2.3.15 and 5.2.3.5, we know that e_{10} and e_{00} must agree with the pairings $\mathtt{d}_{10}$ and $\mathtt{d}_{00}$, respectively, defined using the triple $(\hat{c}, \hat{c}^\vee, \hat\tau)$ corresponding to the splitting $\hat\varsigma : \mathrm{Gr}^{\mathtt{W}} \overset{\sim}{\to} \mathrm{T}^\square G_{\bar\eta}$ that we have not specified yet. Therefore we must require the condition that $(c_n, c_n^\vee, \tau_n)$ is liftable to some triple $(\hat{c}, \hat{c}^\vee, \hat\tau)$ that allows the existence of a splitting $\hat\varsigma$ satisfying the condition we have just found.

Assume that this condition is achieved. Then the graded isomorphism $\hat{f} : \mathrm{Gr}^{\mathtt{Z}} \overset{\sim}{\to} \mathrm{Gr}^{\mathtt{W}}$ defined above is symplectic by Lemma 5.2.2.14. By Proposition 5.2.2.22, the symplectic triple $(\hat\delta, \hat\varsigma, \hat{f})$ defines a symplectic isomorphism $\hat\alpha : L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square \overset{\sim}{\to} \mathrm{T}^\square G_{\bar\eta}$ by $\hat\alpha := \hat\varsigma \circ \hat{f} \circ \hat\delta^{-1}$. Then $\hat\alpha$ necessarily respects the filtrations, and $\mathrm{Gr}(\hat\alpha) = \hat{f}$ necessarily induces the isomorphisms we have specified on the graded pieces. By reduction modulo n, we obtain a level-n structure $\alpha_n : L/nL \overset{\sim}{\to} G[n]_{\bar\eta}$, as desired. This α_n depends only on the reduction modulo n of the above choices $(\mathtt{Z}, \varphi_{-2}, \varphi_{-1}, \varphi_0, \hat\delta, \hat{c}, \hat{c}^\vee, \hat\tau)$, which we denote by a tuple $(\mathtt{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$.

This gives a recipe for producing level-n structures from tuples of the form above, which exhausts all possible level-n structures defined over $\tilde\eta$ by Proposition 5.2.2.22. In order to state the result in general, we shall assume from now on that étale sheaves such as $\underline{X}$ and $\underline{Y}$ are not necessarily constant. Since the statements about Weil pairings can always be verified by passing to étale localizations, our arguments remain valid in the more general setting by étale descent.

Definition 5.2.7.8. *With the setting as in Section 5.2.1, suppose we are given a tuple $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}}, (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h)}(R, I)$. A* ***pre-level-$n$ structure datum of type*** $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \langle\,\cdot\,,\,\cdot\,\rangle)$ *over $\tilde\eta$ is a tuple*

$$\alpha_n^\natural := (\mathtt{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$$

consisting of the following data:

1. *$\mathtt{Z}_n$ is an admissible filtration on L/nL that is fully symplectic-liftable with respect to $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ (see Definition 5.2.7.3). The admissible filtration $\mathtt{Z}_n$, being, in particular, symplectic-liftable, is the reduction modulo n of some symplectic admissible filtration $\mathtt{Z}$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$. This determines the pairings $\langle\,\cdot\,,\,\cdot\,\rangle_{20} : \mathrm{Gr}^{\mathtt{Z}}_{-2} \times \mathrm{Gr}^{\mathtt{Z}}_0 \to \hat{\mathbb{Z}}^\square(1)$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{11} : \mathrm{Gr}^{\mathtt{Z}}_{-1} \times \mathrm{Gr}^{\mathtt{Z}}_{-1} \to \hat{\mathbb{Z}}^\square(1)$, whose reduction modulo n are pairings $\langle\,\cdot\,,\,\cdot\,\rangle_{20,n}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{11,n}$ depending only on $\mathtt{Z}_n$ and not on the choice of $\mathtt{Z}$.*

2. *$\varphi_{-1,n} : \mathrm{Gr}^{\mathtt{Z}}_{-1,n} \overset{\sim}{\to} A[n]_{\tilde\eta}$ is a principal level-n structure of $(A_{\tilde\eta}, \lambda_{A,\tilde\eta}, i_{A,\tilde\eta})$ of type $(\mathrm{Gr}^{\mathtt{Z}}_{-1}, \langle\,\cdot\,,\,\cdot\,\rangle_{11})$ over $\tilde\eta$ (see Lemma 5.2.7.5 and Remark 5.2.7.6). By definition (see Definition 1.3.6.2), $\varphi_{-1,n}$ comes together with an isomorphism $\nu(\varphi_{-1,n}) : (\mathbb{Z}/n\mathbb{Z})(1) \overset{\sim}{\to} \boldsymbol{\mu}_{n,\tilde\eta}$, such that $\varphi_{-1,n}$ and $\nu(\varphi_{-1,n})$ are the reductions modulo n of some isomorphisms $\varphi_{-1} : \mathrm{Gr}^{\mathtt{Z}}_{-1} \overset{\sim}{\to} \mathrm{T}^\square A_{\bar\eta}$ and $\nu(\varphi_{-1}) : \hat{\mathbb{Z}}^\square(1) \overset{\sim}{\to} \mathrm{T}^\square \mathbf{G}_{\mathrm{m},\bar\eta}$ forming a symplectic isomorphism $\varphi_{-1} : \mathrm{Gr}^{\mathtt{Z}}_{-1} \overset{\sim}{\to} \mathrm{T}^\square A_{\bar\eta}$ in the sense that they match the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{11}$ on $\mathrm{Gr}^{\mathtt{Z}}_{-1}$ with the λ_A-Weil pairing on $\mathrm{T}^\square A_{\bar\eta}$.*

3. *$\varphi_{-2,n} : \mathrm{Gr}^{\mathtt{Z}}_{-2,n} \overset{\sim}{\to} \underline{\mathrm{Hom}}_{\tilde\eta}((\underline{X}/n\underline{X})_{\tilde\eta}, (\mathbb{Z}/n\mathbb{Z})(1))$ and $\varphi_{0,n} : \mathrm{Gr}^{\mathtt{Z}}_{0,n} \overset{\sim}{\to} (\underline{Y}/n\underline{Y})_{\tilde\eta}$ are isomorphisms that are liftable to some $\varphi_{-2} : \mathrm{Gr}^{\mathtt{Z}}_{-2} \overset{\sim}{\to} \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1))$ and $\varphi_0 : \mathrm{Gr}^{\mathtt{Z}}_0 \overset{\sim}{\to} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ over $\bar\eta$, such that the pairing e^ϕ is pulled back via $(\nu(\varphi_{-1}) \circ \varphi_{-2,n}) \times \varphi_{0,n}$ to the pairing $\nu(\varphi_{-1}) \circ \langle\,\cdot\,,\,\cdot\,\rangle_{20}$, where $\mathrm{e}^\phi : \mathrm{T}^\square T_{\bar\eta} \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square) \to \mathrm{T}^\square \mathbf{G}_{\mathrm{m},\bar\eta}$ is the canonical pairing defined as in Proposition 5.2.2.1.*

4. *$\delta_n : \mathrm{Gr}^{\mathtt{Z}}_n \overset{\sim}{\to} L/nL$ is a liftable splitting as in Definition 5.2.2.10, which is the reduction modulo n of some splitting $\hat\delta : \mathrm{Gr}^{\mathtt{Z}} \overset{\sim}{\to} L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$. This $\hat\delta$ determines the pairings $\langle\,\cdot\,,\,\cdot\,\rangle_{10} : \mathrm{Gr}^{\mathtt{Z}}_{-1} \times \mathrm{Gr}^{\mathtt{Z}}_0 \to \hat{\mathbb{Z}}^\square(1)$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{00} : \mathrm{Gr}^{\mathtt{Z}}_0 \times \mathrm{Gr}^{\mathtt{Z}}_0 \to \hat{\mathbb{Z}}^\square(1)$, whose reductions modulo each integer m such that $n|m$ and $\square \nmid m$ define pairings $\langle\,\cdot\,,\,\cdot\,\rangle_{10,m}$ and $\langle\,\cdot\,,\,\cdot\,\rangle_{00,m}$, depending only on $\mathtt{Z}_m$ and not on the full $\mathtt{Z}$.*

5. *The homomorphism $c_n : \frac{1}{n}\underline{Y}_{\tilde\eta} \to A_{\tilde\eta}$, (resp. $c_n^\vee : \frac{1}{n}\underline{X}_{\tilde\eta} \to A^\vee_{\tilde\eta}$, resp. $\tau_n : \mathbf{1}_{\frac{1}{n}\underline{Y} \underset{S}{\times} \underline{X}, \tilde\eta} \overset{\sim}{\to} (c_n^\vee, c_{\tilde\eta})^* \mathcal{P}^{\otimes -1}_{A,\tilde\eta}$) is a lifting of c (resp. $c^\vee$, resp. τ) over $\tilde\eta$, such that the triple $(c_n, c_n^\vee, \tau_n)$ is liftable to some compatible system of liftings $(\hat{c}, \hat{c}^\vee, \hat\tau) = \{(c_m, c_m^\vee, \tau_m)\}_{n|m, \square\nmid m}$ (as in Definition 5.2.3.4), which determines two compatible systems of pairings $\{\mathtt{d}_{10,m} : A[m]_{\bar\eta} \times (\frac{1}{m}Y/Y) \to \boldsymbol{\mu}_{m,\bar\eta}\}_{n|m,\square\nmid m}$ and $\{\mathtt{d}_{00,m} : (\frac{1}{m}Y/Y) \times (\frac{1}{m}Y/Y) \to \boldsymbol{\mu}_{m,\bar\eta}\}_{n|m,\square\nmid m}$ (as in Lemma 5.2.3.12), and hence two pairings $\mathtt{d}_{10} : \mathrm{T}^\square A_{\bar\eta} \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square) \to \mathrm{T}^\square \mathbf{G}_{\mathrm{m},\bar\eta}$ and*

$\mathtt{d}_{00} : (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) \to \mathbf{G}_{\mathrm{m},\tilde\eta}$ *(as in Corollary 5.2.3.13), by setting*

$$\mathtt{d}_{10,m}(a, \tfrac{1}{m}y) := \mathrm{e}_{A[m]}(a, (\lambda_A c_m^\vee - c_m\phi_m)(\tfrac{1}{m}y)) \in \boldsymbol{\mu}_m(\tilde\eta)$$

for each $a \in A[m]_{\tilde\eta}$ *and* $\frac{1}{m}y \in \frac{1}{m}Y$*, and by setting*

$$\mathtt{d}_{00,m}(\tfrac{1}{m}y, \tfrac{1}{m}y') := \tau_m(\tfrac{1}{m}y, \phi(y'))\tau_m(\tfrac{1}{m}y', \phi(y))^{-1} \in \boldsymbol{\mu}_m(\tilde\eta)$$

for each $\frac{1}{m}y, \frac{1}{m}y' \in \frac{1}{m}Y$.

We say that the pre-level-n structure datum $\alpha_n^\natural$ is ***symplectic-liftable****, and call it a* ***level-n structure datum of type*** $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle)$ *over $\tilde\eta$, if the following condition is satisfied: There exists some lifting $\hat\alpha^\natural := (\mathtt{Z}, \varphi_{-2}, \varphi_{-1}, \varphi_0, \hat\delta, \hat c, \hat c^\vee, \hat\tau)$ of $\alpha_n^\natural$ as above, which is* ***symplectic*** *in the sense that*

$$(\varphi_{-1} \times \varphi_0)^*(\mathtt{d}_{10}) = \nu(\varphi_{-1}) \circ \langle\cdot,\cdot\rangle_{10}$$

and

$$(\varphi_0 \times \varphi_0)^*(\mathtt{d}_{00}) = \nu(\varphi_{-1}) \circ \langle\cdot,\cdot\rangle_{00}$$

(see Lemma 5.2.2.14).

Proposition 5.2.7.9. *With the setting as in Section 5.2.1, suppose we are given a tuple $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\cdot,\cdot\rangle, h)}(R, I)$ corresponding to a triple (G, λ, i) in $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\cdot,\cdot\rangle, h)}(R, I)$ via Theorem 5.1.2.7, and a level-n structure datum $\alpha_n^\natural$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle)$ defined over $\tilde\eta$ as in Definition 5.2.7.8 (without the assumption that $\underline{X}$ and $\underline{Y}$ are constant). Then the datum $\alpha_n^\natural$ gives, in particular, a splitting $\varsigma_n : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} G[n]_{\tilde\eta}$ of the filtration*

$$0 \subset \mathtt{W}_{-2,n} = T[n]_{\tilde\eta} \subset \mathtt{W}_{-1,n} = G^\natural[n]_{\tilde\eta} \subset \mathtt{W}_{0,n} = G[n]_{\tilde\eta},$$

and a graded symplectic isomorphism $f_n : \mathrm{Gr}_n^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}_n^{\mathtt{W}}$ defined over $\tilde\eta$, such that $\alpha_n = \varsigma_n \circ f_n \circ \delta_n^{-1} : L/nL \xrightarrow{\sim} G[n]_{\tilde\eta}$ defines a level-n structure of $(G_{\tilde\eta}, \lambda_{\tilde\eta}, i_{\tilde\eta})$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle)$ as in Definition 1.3.6.2. Moreover, every level-n structure $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde\eta}$ of $(G_{\tilde\eta}, \lambda_{\tilde\eta}, i_{\tilde\eta})$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box, \langle\cdot,\cdot\rangle)$ arises in this way from some $\alpha_n^\natural$ (defined over $\tilde\eta$).

However, this association is not one to one. In general, there can be different level-n structure data $\alpha_n^\natural$ and ${\alpha_n^\natural}'$ that produce the same level-n structure α_n. Therefore we would like to introduce equivalences among the level-n structure data, such that the equivalence classes of them correspond bijectively to the level-n structures.

If we start with a level-n structure $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde\eta}$, then we see that the filtration $\mathtt{Z}_n$ is necessarily determined by $\mathtt{W}_n$ under α_n, and moreover $\mathrm{Gr}_{-2,n}(\alpha_n) = \nu(\varphi_{-1,n}) \circ \varphi_{-2,n}$, $\mathrm{Gr}_{-1,n}(\alpha_n) = \varphi_{-1,n}$, and $\mathrm{Gr}_{0,n}(\alpha_n) = \varphi_{0,n}$ are all uniquely determined by $\mathrm{Gr}_n(\alpha_n)$. Hence the only possible difference between two objects giving the same level-n structure is in the choices of $(\delta_n, c_n, c_n^\vee, \tau_n)$. Let us suppose that we have a different tuple $(\delta_n', c_n', (c_n^\vee)', \tau_n')$ giving the same level structure α_n.

On the one hand, as elaborated on in Section 5.2.3, the liftable triple $(c_n, c_n^\vee, \tau_n)$ corresponds to a liftable splitting $\varsigma_n : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} G[n]_{\tilde\eta}$, and the different choice $(c_n', (c_n^\vee)', \tau_n')$ corresponds to a different splitting $\varsigma_n' : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} G[n]_{\tilde\eta}$ that is related to ς_n by a liftable change of basis $\mathtt{w}_n : \mathrm{Gr}_n^{\mathtt{W}} \xrightarrow{\sim} \mathrm{Gr}_n^{\mathtt{W}}$ as in Definition 5.2.2.20, in the sense that $\varsigma_n' = \varsigma_n \circ \mathtt{w}_n$. Here the matrix entries are homomorphisms $\mathtt{w}_{21,n} : A[n]_{\tilde\eta} \to T[n]_{\tilde\eta}$, $\mathtt{w}_{10,n} : (\frac{1}{n}\underline{Y}/\underline{Y})_{\tilde\eta} \to A[n]_{\tilde\eta}$, and $\mathtt{w}_{20,n} : (\frac{1}{n}\underline{Y}/\underline{Y})_{\tilde\eta} \to T[n]_{\tilde\eta}$, which correspond to a homomorphism $d_n : \frac{1}{n}\underline{X} \to A^\vee[n]_{\tilde\eta}$, a homomorphism $d_n^\vee : \frac{1}{n}\underline{Y} \to A[n]_{\tilde\eta}$, and a pairing $e_n : \frac{1}{n}\underline{Y}_{\tilde\eta} \times \underline{X}_{\tilde\eta} \to \boldsymbol{\mu}_{n,\tilde\eta}$, respectively. As we saw in Section 5.2.5, the triples $(d_n, d_n^\vee, e_n)$ and $(c_n', (c_n^\vee)', \tau_n')$ can be related by $c_n' = c_n + d_n$, $(c_n^\vee)' = c_n^\vee + d_n^\vee$, and more elaborately (over an étale covering over which both $\underline{X}$ and $\underline{Y}$ become constant) $\tau_n'(\frac{1}{n}y, \chi) = \tau_n(\frac{1}{n}y, \chi) r(d_n^\vee(\frac{1}{n}y), c_n(\frac{1}{n}\chi))\chi(e_n(\frac{1}{n}y))$ for each $\frac{1}{n}y \in \frac{1}{n}Y$ and $\chi \in X$ as in (5.2.5.1) (see Section 5.2.5 for details).

Definition 5.2.7.10. *We say in this case that $(d_n, d_n^\vee, e_n)$* ***translates*** *$(c_n, c_n^\vee, \tau_n)$ to $(c_n', (c_n^\vee)', \tau_n')$.*

Hence we have a dictionary between the change of basis $\mathtt{w}_n$ and the translation by the triple $(d_n, d_n^\vee, e_n)$, both giving the difference between ς_n and ς_n'.

On the other hand, the two different choices of liftable splittings $\delta_n, \delta_n' : \mathrm{Gr}_n^{\mathtt{Z}} \xrightarrow{\sim} L/nL$ are related by a liftable change of basis $\mathtt{z}_n : \mathrm{Gr}_n^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}_n^{\mathtt{Z}}$ as in Definition 5.2.2.19, in the sense that $\delta_n' = \delta_n \circ \mathtt{z}_n$.

By Proposition 5.2.2.23, we know that level-n structures are in bijection with equivalence classes of symplectic-liftable triples as in Definition 5.2.2.21. Therefore, the key point of the equivalence is the commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{Gr}_n^{\mathtt{Z}} = \mathrm{Gr}_{-2,n}^{\mathtt{Z}} \oplus \mathrm{Gr}_{-1,n}^{\mathtt{Z}} \oplus \mathrm{Gr}_{0,n}^{\mathtt{Z}} & \xrightarrow[\sim]{\mathtt{z}_n} & \mathrm{Gr}_n^{\mathtt{Z}} = \mathrm{Gr}_{-2,n}^{\mathtt{Z}} \oplus \mathrm{Gr}_{-1,n}^{\mathtt{Z}} \oplus \mathrm{Gr}_{0,n}^{\mathtt{Z}} \\
{\scriptstyle \mathrm{Gr}_n(\alpha_n)} \downarrow \wr & & \wr \downarrow {\scriptstyle \mathrm{Gr}_n(\alpha_n)} \\
\mathrm{Gr}_n^{\mathtt{W}} \cong T[n]_{\tilde\eta} \oplus A[n]_{\tilde\eta} \oplus (\frac{1}{n}\underline{Y}/\underline{Y})_{\tilde\eta} & \xrightarrow[\mathtt{w}_n]{\sim} & \mathrm{Gr}_n^{\mathtt{W}} \cong T[n]_{\tilde\eta} \oplus A[n]_{\tilde\eta} \oplus (\frac{1}{n}\underline{Y}/\underline{Y})_{\tilde\eta}
\end{array}$$

in which $\mathrm{Gr}_n(\alpha_n) = (\nu(\varphi_{-1,n}) \circ \varphi_{-2,n}) \oplus \varphi_{-1,n} \oplus \varphi_{0,n}$, which is equivalent to the compatibilities $(\nu(\varphi_{-1,n}) \circ \varphi_{-2,n}) \circ \mathtt{z}_{21} = \mathtt{w}_{21} \circ \varphi_{-1,n}$, $\varphi_{-1,n} \circ \mathtt{z}_{10} = \mathtt{w}_{10} \circ \varphi_{0,n}$, and $(\nu(\varphi_{-1,n}) \circ \varphi_{-2,n}) \circ \mathtt{z}_{20} = \mathtt{w}_{20} \circ \varphi_{0,n}$.

Let us formulate this observation as follows:

Definition 5.2.7.11. *With the setting as in Section 5.2.1, suppose we are given a tuple $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\cdot,\cdot\rangle, h)}(R, I)$. Two level-$n$ structure data*

$$\alpha_n^\natural = (\mathtt{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$$

and

$$(\alpha_n^\natural)' = (\mathtt{Z}_n', \varphi_{-2,n}', \varphi_{-1,n}', \varphi_{0,n}', \delta_n', c_n', (c_n^\vee)', \tau_n')$$

of type M_n defined over $\tilde\eta$ (as in Definition 5.2.7.8) are called ***equivalent*** *if the following conditions are satisfied:*

1. *The following identities hold: $\mathtt{Z}_n = \mathtt{Z}_n'$, $\varphi_{-2,n} = \varphi_{-2,n}'$, $\varphi_{-1,n} = \varphi_{-1,n}'$, and $\varphi_{0,n} = \varphi_{0,n}'$.*

2. *There is a liftable change of basis $\mathtt{z}_n : \mathrm{Gr}_n^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Gr}_n^{\mathtt{Z}}$, given in matrix form by $\mathtt{z}_n = \begin{pmatrix} 1 & \mathtt{z}_{21,n} & \mathtt{z}_{20,n} \\ & 1 & \mathtt{z}_{10,n} \\ & & 1 \end{pmatrix}$, such that $\delta_n' = \delta_n \circ \mathtt{z}_n$.*

3. *If we consider the homomorphisms $d_n : \frac{1}{n}\underline{X} \to A^\vee[n]_{\tilde\eta}$, $d_n^\vee : \frac{1}{n}\underline{Y} \to A[n]_{\tilde\eta}$, and $e_n : \frac{1}{n}\underline{Y}_{\tilde\eta} \to T[n]_{\tilde\eta}$ defined respectively by the relations*
$$\mathrm{e}_{A[n]}(a, d_n(\tfrac{1}{n}\chi)) = \chi(\nu(\varphi_{-1,n}) \circ \varphi_{-2,n} \circ \mathtt{z}_{21,n} \circ \varphi_{-1,n}^{-1}(a)),$$
$$d_n^\vee(\tfrac{1}{n}y) = \varphi_{-1,n} \circ \mathtt{z}_{10,n} \circ \varphi_{0,n}^{-1}(\tfrac{1}{n}y),$$
$$e_n(\tfrac{1}{n}y) = \nu(\varphi_{-1,n}) \circ \varphi_{-2,n} \circ \mathtt{z}_{20,n} \circ \varphi_{0,n}^{-1}(\tfrac{1}{n}y),$$

for each $a \in A[n]_{\tilde{\eta}}$, $\chi \in \underline{X}_{\tilde{\eta}}$, and $\frac{1}{n}y \in \frac{1}{n}\underline{Y}_{\tilde{\eta}}$, then $(d_n, d_n^\vee, e_n)$ translates $(c_n, c_n^\vee, \tau_n)$ to $(c_n', (c_n^\vee)', \tau_n')$ in the sense of Definition 5.2.7.10. (Here we identify $\varphi_{0,n} : \mathrm{Gr}_{0,n}^{\mathtt{Z}} \xrightarrow{\sim} Y/nY$ also as an isomorphism $\mathrm{Gr}_{0,n}^{\mathtt{Z}} \xrightarrow{\sim} \frac{1}{n}Y/Y$ by the canonical isomorphism $Y/nY \xrightarrow{\sim} \frac{1}{n}Y/Y$.)

Definition 5.2.7.12. *With the setting as in Section 5.2.1, the category $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}}(R, I)$ has objects of the form $(G, \lambda, i, \alpha_n)$ (over $S = \mathrm{Spec}(R)$), where*

1. *(G, λ, i) defines an object in $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle,h)}(R, I)$;*
2. *$\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ is a level-n structure of $(G_{\tilde{\eta}}, \lambda_{\tilde{\eta}}, i_{\tilde{\eta}})$ of type $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square, \langle\,\cdot\,,\,\cdot\,\rangle)$ (see Definition 1.3.6.2).*

For simplicity, when $\tilde{\eta} = \eta$, we shall denote $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_n,\eta}(R, I)$ by $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_n}(R, I)$.

Definition 5.2.7.13. *With the setting as in Section 5.2.1, the category $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}}(R, I)$ has objects of the form $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha_n^\natural])$, where*

1. *$(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ is an object in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle,h)}(R, I)$;*
2. *$[\alpha_n^\natural]$ is an equivalence class of level-n structure data $\alpha_n^\natural$ of type $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square, \langle\,\cdot\,,\,\cdot\,\rangle)$ defined over $\tilde{\eta}$ (see Definitions 5.2.7.8 and 5.2.7.11).*

For simplicity, when $\tilde{\eta} = \eta$, we shall denote $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}}(R, I)$ by $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}(R, I)$.

We can now replace Proposition 5.2.7.9 with the following theorem, which is our main result of Section 5.2:

Theorem 5.2.7.14. *There is an equivalence of categories*

$$M_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}}(R, I) : \mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}}(R, I) \to \mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}}(R, I) :$$
$$(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha_n^\natural]) \mapsto (G, \lambda, i, \alpha_n).$$

For simplicity, when $\tilde{\eta} = \eta$, we shall denote $M_{\mathrm{PEL},\mathsf{M}_n,\tilde{\eta}}(R, I)$ by $M_{\mathrm{PEL},\mathsf{M}_n}(R, I)$.

5.3 Data for General PEL-Structures

Let us continue with the same setting as in Section 5.2 in this section.

5.3.1 Formation of Étale Orbits

With the setting as in Section 5.2.1, let (G, λ, i) be an object in $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle,h)}(R, I)$ associated with a tuple $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle,h)}(R, I)$ by the equivalence $M_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle,h)}(R, I)$ in Theorem 5.1.2.7.

Let $\mathcal{H}$ be an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^\square)$, and let $n \geq 1$ be an integer such that $\square \nmid n$ and $\mathcal{U}^\square(n) \subset \mathcal{H}$. Let $H_n := \mathcal{H}/\mathcal{U}^\square(n)$ (see Remark 1.2.1.9 for the definition of $\mathcal{U}^\square(n)$). By definition, an integral level-$\mathcal{H}$ structure $\alpha_\mathcal{H}$ of $(G_\eta, \lambda_\eta, i_\eta)$ is given by an H_n-orbit α_{H_n} of étale-locally-defined level-n structures. In other words, there exists an étale morphism $\tilde{\eta} \to \eta$ (which we may assume to be defined by a field extension as in Section 5.2.1) such that the pullback of H_n to $\tilde{\eta}$ is the disjoint union of all elements in the H_n-orbit of some level-n structure $\alpha_n : L/nL \xrightarrow{\sim} G[n]_{\tilde{\eta}}$ defined over $\tilde{\eta}$. According to Theorem 5.2.7.14, α_n is associated with an equivalence class $[\alpha_n^\natural]$ of level-n structure data $\alpha_n^\natural = (\mathtt{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$. Then the H_n-orbit of α_n is naturally associated with the H_n-orbit of $\alpha_n^\natural$, if we can explain how the action of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ on the set of α_n's (defined by $\alpha_n \mapsto \alpha_n \circ g_n$) is translated into an action of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ on the set of $\alpha_n^\natural$'s. Even better, we would like to work out a direct way to define an action of $\mathrm{G}(\hat{\mathbb{Z}}^\square)$ on the set of $\hat{\alpha}^\natural$'s, which induces the action of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ on the set of $\alpha_n^\natural$'s by reduction modulo n. This is the goal of this section.

Let us fix a choice of α_n that corresponds to $\alpha_n^\natural$ as above. Let $\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square \xrightarrow{\sim} \mathrm{T}^\square G_{\tilde{\eta}}$ be any symplectic isomorphism lifting α_n. Let $\hat{\alpha}^\natural = (\mathtt{Z}, \varphi_{-2}, \varphi_{-1}, \varphi_0, \hat{\delta}, \hat{c}, \hat{c}^\vee, \hat{\tau})$ be any element lifting $\alpha_n^\natural$ (as a reduction modulo n). By construction of $\alpha_n^\natural$, we may arrange that $\mathtt{Z}$ is the pullback of $\mathtt{W}$ under $\hat{\alpha}$, and that φ_{-2}, φ_{-1}, and φ_0 are determined uniquely by $\mathrm{Gr}(\hat{\alpha})$. There is a freedom in making the choice of the splitting $\hat{\delta} : \mathrm{Gr}^{\mathtt{Z}} \xrightarrow{\sim} L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$, but then the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ and $\hat{\alpha}$ force a unique choice of $(\hat{c}, \hat{c}^\vee, \hat{\tau})$ for each particular choice of δ.

Let g_n be any element of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$, and let g be any element of $\mathrm{G}(\hat{\mathbb{Z}}^\square)$ lifting g_n. If we replace $\hat{\alpha}$ with $\hat{\alpha} \circ g$, then the relation $\mathtt{W} = \hat{\alpha}(\mathtt{Z})$ is replaced with $\mathtt{W} = (\hat{\alpha} \circ g)(g^{-1}(\mathtt{Z}))$. Hence we see that we should replace $\mathtt{Z}$ with $\mathtt{Z}' := g^{-1}(\mathtt{Z})$, which is related to $\mathtt{Z}$ by the induced isomorphisms $g : \mathtt{Z}' = g^{-1}(\mathtt{Z}_{-i}) \xrightarrow{\sim} \mathtt{Z}_{-i}$ for $i = 0, 1, 2$. This determines the isomorphisms $\mathrm{Gr}_{-i}(g) : \mathrm{Gr}_{-i}^{\mathtt{Z}'} \xrightarrow{\sim} \mathrm{Gr}_{-i}^{\mathtt{Z}}$ on the graded pieces, and suggests the following accordingly:

Construction 5.3.1.1.
1. We shall replace $\varphi_0 : \mathrm{Gr}_0^{\mathtt{Z}} \xrightarrow{\sim} Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square$ with $\varphi_0' := \varphi_0 \circ \mathrm{Gr}_0(g)$.
2. We shall replace $\varphi_{-1} : \mathrm{Gr}_{-1}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{T}^\square A_{\tilde{\eta}}$ with $\varphi_{-1}' := \varphi_{-1} \circ \mathrm{Gr}_{-1}(g)$, and accordingly $\nu(\varphi_{-1}) : \hat{\mathbb{Z}}^\square(1) \xrightarrow{\sim} \mathrm{T}^\square \mathbf{G}_{\mathrm{m},\tilde{\eta}}$ with $\nu(\varphi_{-1}') = \nu(\varphi_{-1}) \circ \nu(g)$.
3. We shall replace $\varphi_{-2} : \mathrm{Gr}_{-2}^{\mathtt{Z}} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1))$ with $\varphi_{-2}' := \nu(g)^{-1} \circ \varphi_{-2} \circ \mathrm{Gr}_{-2}(g)$. This is because (when $X \neq 0$ and hence $L \neq 0$) the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Gr}_{-2}^{\mathtt{Z}} \times \mathrm{Gr}_0^{\mathtt{Z}} \arrow[r, "\mathrm{Gr}_{-2}(\hat{\alpha}) \times \mathrm{Gr}_0(\hat{\alpha})"] & \mathrm{T}^\square T_{\tilde{\eta}} \times (Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square) \arrow[r, "e^\phi"] \arrow[d, equal] & \mathrm{T}^\square \mathbf{G}_{\mathrm{m},\tilde{\eta}} \arrow[d, equal] \\
\mathrm{Gr}_{-2}^{\mathtt{Z}'} \times \mathrm{Gr}_0^{\mathtt{Z}'} \arrow[u, "\wr", "\mathrm{Gr}_{-2}(g) \times \mathrm{Gr}_0(g)"'] \arrow[r, "\mathrm{Gr}_{-2}(\hat{\alpha}\circ g) \times \mathrm{Gr}_0(\hat{\alpha}\circ g)"] & \mathrm{T}^\square T_{\tilde{\eta}} \times (Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square) \arrow[r, "e^\phi"] & \mathrm{T}^\square \mathbf{G}_{\mathrm{m},\tilde{\eta}}
\end{tikzcd}$$

forces the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Gr}_{-2}^{\mathtt{Z}} \times \mathrm{Gr}_0^{\mathtt{Z}} \arrow[r, "\varphi_{-2} \times \varphi_0"] & \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1)) \times (Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square) \arrow[r, "\mathrm{can.}"] & \hat{\mathbb{Z}}^\square(1) \\
\mathrm{Gr}_{-2}^{\mathtt{Z}'} \times \mathrm{Gr}_0^{\mathtt{Z}'} \arrow[u, "\wr", "\mathrm{Gr}_{-2}(g) \times \mathrm{Gr}_0(g)"'] \arrow[r, "\varphi_{-2}' \times \varphi_0'"] & \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1)) \times (Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square) \arrow[u, "\wr", "\nu(g) \times \mathrm{Id}"'] \arrow[r, "\mathrm{can.}"] & \hat{\mathbb{Z}}^\square(1) \arrow[u, "\wr", "\nu(g)"']
\end{tikzcd}$$

with the canonical pairing
$$\mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box,\hat{\mathbb{Z}}^\Box(1))\times(Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\to\hat{\mathbb{Z}}^\Box(1)$$
given by the composition of $\mathrm{Id}\times\phi$ with the canonical pairing $\mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box,\hat{\mathbb{Z}}^\Box(1))\times(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\to\hat{\mathbb{Z}}^\Box(1)$.

4. Then $\hat{f}_{-2}:=\nu(\varphi_{-1})\circ\varphi_{-2}$, $\hat{f}_{-1}:=\varphi_{-1}$, and $\hat{f}_0:=\varphi_0$ are replaced with $\hat{f}'_{-2}:=\nu(f'_{-2})\circ\varphi'_{-2}=\nu(f_{-2})\circ\nu(g)\circ\varphi_{-2}\circ\mathrm{Gr}_{-2}(g)=\hat{f}_{-2}\circ(\nu(g)\circ\mathrm{Gr}_{-2}(g))$, with $\hat{f}'_{-1}=\hat{f}_{-1}\circ\mathrm{Gr}_{-1}(g)$, and with $\hat{f}'_0=\hat{f}_0\circ\mathrm{Gr}_0(g)$, respectively. This replaces the symplectic isomorphism $\hat{f}=\hat{f}_{-2}\oplus\hat{f}_{-1}\oplus\hat{f}_0:\mathrm{Gr}^{\mathtt{Z}}\xrightarrow{\sim}\mathrm{Gr}^{\mathtt{W}}$ with the symplectic isomorphism $\hat{f}'=\hat{f}'_{-2}\oplus\hat{f}'_{-1}\oplus\hat{f}'_0:\mathrm{Gr}^{\mathtt{Z}'}\xrightarrow{\sim}\mathrm{Gr}^{\mathtt{W}}$, so that $\hat{f}'=\hat{f}\circ\mathrm{Gr}(g)$ and accordingly $\nu(\hat{f}')=\nu(\hat{f})\circ\nu(g)$.

5. Let $\hat{\varsigma}:\mathtt{W}\xrightarrow{\sim}\mathrm{T}^\Box G_{\bar{\eta}}$ be the splitting determined by $(\hat{c},\hat{c}^\vee,\hat{\tau})$. Then the relation $\hat{\alpha}=\varsigma\circ\hat{f}\circ\hat{\delta}^{-1}$ can be rewritten as $\hat{\alpha}\circ g=\hat{\varsigma}\circ\hat{f}'\circ(g^{-1}\circ\hat{\delta}\circ\mathrm{Gr}(g))^{-1}$. This shows that, if we take $\hat{\delta}':=g^{-1}\circ\hat{\delta}\circ\mathrm{Gr}(g)$ as one of the possible ways to modify it (as it is not canonical), then we may set $\hat{\varsigma}':=\hat{\varsigma}$ and retain the relation $\hat{\alpha}\circ g=\hat{\varsigma}'\circ\hat{f}'\circ(\hat{\delta}')^{-1}$. (Note that $\nu(\hat{\delta})$ and $\nu(\hat{\delta}')$ are both the identity because $\nu(g^{-1})$ and $\nu(\mathrm{Gr}(g))$ *cancel* each other.)

Summarizing,

Proposition 5.3.1.2. *There is a natural action of* $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ *on tuples of the form* $\hat{\alpha}^\natural=(\mathtt{Z},\varphi_{-2},\varphi_{-1},\varphi_0,\hat{\delta},\hat{c},\hat{c}^\vee,\hat{\tau})$, *defined for each* $g\in\mathrm{G}(\hat{\mathbb{Z}}^\Box)$, *by sending* $\hat{\alpha}^\natural$ *as above to*
$$(\mathtt{Z}',\varphi'_{-2},\varphi'_{-1},\varphi'_0,\hat{\delta}',\hat{c},\hat{c}^\vee,\hat{\tau}),$$
where

1. $\mathtt{Z}':=g^{-1}(\mathtt{Z})$;
2. $\varphi'_{-2}:=\nu(g)\circ\varphi_{-2}\circ\mathrm{Gr}_{-2}(g)$;
3. $\varphi'_{-1}:=\varphi_{-1}\circ\mathrm{Gr}_{-1}(g)$ *and accordingly* $\nu(\varphi'_{-1}):=\nu(\varphi_{-1})\circ\nu(g)^{-1}$;
4. $\varphi'_0:=\varphi_0\circ\mathrm{Gr}_0(g)$;
5. $\hat{\delta}':=g^{-1}\circ\hat{\delta}\circ\mathrm{Gr}(g)$;
6. $(\hat{c},\hat{c}^\vee,\hat{\tau})$ *is unchanged.*

Proposition 5.3.1.3. *By taking reduction modulo n of the action defined in Proposition 5.3.1.2, we obtain an action of* $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ *on the level-n structure data of the form* $\alpha_n^\natural=(\mathtt{Z}_n,\varphi_{-2,n},\varphi_{-1,n},\varphi_{0,n},\delta_n,c_n,c_n^\vee,\tau_n)$, *defined for each* $g_n\in\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ *by sending* $\alpha_n^\natural$ *as above to*
$$(\mathtt{Z}'_n,\varphi'_{-2,n},\varphi'_{-1,n},\varphi'_{0,n},\delta'_n,c_n,c_n^\vee,\tau_n),$$
where

1. $\mathtt{Z}'_n:=g_n^{-1}(\mathtt{Z}_n)$;
2. $\varphi'_{-2,n}:=\nu(g_n)\circ\varphi_{-2,n}\circ\mathrm{Gr}_{-2}(g_n)$;
3. $\varphi'_{-1,n}:=\varphi_{-1,n}\circ\mathrm{Gr}_{-1,n}(g_n)$ *and accordingly* $\nu(\varphi'_{-1,n}):=\nu(\varphi_{-1,n})\circ\nu(g_n)^{-1}$;
4. $\varphi'_{0,n}:=\varphi_{0,n}\circ\mathrm{Gr}_{0,n}(g_n)$;
5. $\delta'_n:=g_n^{-1}\circ\delta_n\circ\mathrm{Gr}_n(g_n)$;
6. $(c_n,c_n^\vee,\tau_n)$ *is unchanged.*

This action respects the equivalence relations among level-n structure data, and hence induces an action on the equivalence classes $[\alpha_n^\natural]$ *as well.*

Proof. Everything is clear except the last statement. The last statement is essentially a tautology, because the equivalence classes of level-n structure data correspond bijectively to level-n structures, and the action is defined by the action on level-n structures. ☐

Now let H_n be a subgroup of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ as above, which is $H_n=\mathcal{H}/\mathcal{U}^\Box(n)$ for some open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$. We would like to find the correct formulation of an H_n-orbit of $[\alpha_n^\natural]$ over $\tilde{\eta}$, so that we can *descend* the orbit to some similar object over η.

Definition 5.3.1.4. *Let* $\mathtt{Z}_n$ *be a fully symplectic-liftable filtration on* L/nL. *Then we define the following subgroups or quotients of subgroups of* $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$:
$$\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}:=\{g_n\in\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z}):g_n^{-1}(\mathtt{Z}_n)=\mathtt{Z}_n\},$$
$$\mathrm{Z}^{\mathrm{ess}}_{\mathtt{Z}_n}:=\{g_n\in\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}:\mathrm{Gr}_{-1,n}(g_n)=\mathrm{Id}_{\mathrm{Gr}^{\mathtt{Z}}_{-1,n}}\text{ and }\nu(g_n)=1\},$$
$$\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}:=\{g_n\in\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}:\mathrm{Gr}_n(g_n)=\mathrm{Id}_{\mathrm{Gr}^{\mathtt{Z}}_n}\text{ and }\nu(g_n)=1\},$$
$$\mathrm{G}^{\mathrm{ess}}_{h,\mathtt{Z}_n}:=\left\{\begin{array}{c}(g_{-1,n},r_n)\in\mathrm{GL}_{\mathcal{O}}(\mathrm{Gr}^{\mathtt{Z}}_{-1,n})\times\mathbf{G}_{\mathrm{m}}(\mathbb{Z}/n\mathbb{Z}):\\ \exists g_n\in\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}\text{ s.t. }\mathrm{Gr}_{-1,n}(g_n)=g_{-1,n}\text{ and }\nu(g_n)=r_n\end{array}\right\},$$
$$\mathrm{G}^{\mathrm{ess}}_{l,\mathtt{Z}_n}:=\left\{\begin{array}{c}(g_{-2,n},g_{0,n})\in\mathrm{GL}_{\mathcal{O}}(\mathrm{Gr}^{\mathtt{Z}}_{-2,n})\times\mathrm{GL}_{\mathcal{O}}(\mathrm{Gr}^{\mathtt{Z}}_{0,n}):\\ \exists g_n\in\mathrm{Z}^{\mathrm{ess}}_{\mathtt{Z}_n}\text{ s.t. }\mathrm{Gr}_{-2,n}(g_n)=g_{-2,n}\text{ and }\mathrm{Gr}_{0,n}(g_n)=g_{0,n}\end{array}\right\},$$
$$\mathrm{U}^{\mathrm{ess}}_{2,\mathtt{Z}_n}:=\left\{\begin{array}{c}g_{20,n}\in\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathtt{Z}}_{0,n},\mathrm{Gr}^{\mathtt{Z}}_{-2,n}):\\ \exists g_n\in\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}\text{ s.t. }\delta_n^{-1}\circ g_n\circ\delta_n=\begin{pmatrix}1&&g_{20,n}\\&1&\\&&1\end{pmatrix}\end{array}\right\},$$
$$\mathrm{U}^{\mathrm{ess}}_{1,\mathtt{Z}_n}:=\left\{\begin{array}{c}(g_{21,n},g_{10,n})\in\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathtt{Z}}_{-1,n},\mathrm{Gr}^{\mathtt{Z}}_{-2,n})\times\mathrm{Hom}_{\mathcal{O}}(\mathrm{Gr}^{\mathtt{Z}}_{0,n},\mathrm{Gr}^{\mathtt{Z}}_{-1,n}):\\ \exists g_n\in\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}\text{ s.t. }\delta_n^{-1}\circ g_n\circ\delta_n=\begin{pmatrix}1&g_{21,n}&g_{20,n}\\&1&g_{10,n}\\&&1\end{pmatrix}\text{ for some }g_{20,n}\end{array}\right\}.$$

Remark 5.3.1.5. Since $\nu(\mathrm{Gr}_{-1,n}(g_n))=\nu(g_n)$ by definition, the condition $\nu(g_n)=1$ in the definition of $\mathrm{Z}^{\mathrm{ess}}_{\mathtt{Z}_n}$ is redundant if we interpret $\mathrm{Gr}_{-1,n}(g_n)=\mathrm{Id}_{\mathrm{Gr}^{\mathtt{Z}}_{-1,n}}$ as an identity of symplectic isomorphisms (see Definition 1.1.4.8).

Lemma 5.3.1.6. *By definition, there are natural inclusions*
$$\mathrm{U}^{\mathrm{ess}}_{2,\mathtt{Z}_n}\subset\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}\subset\mathrm{Z}^{\mathrm{ess}}_{\mathtt{Z}_n}\subset\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}\subset\mathrm{G}^{\mathrm{ess}}_{\mathtt{Z}_n},\tag{5.3.1.7}$$
and natural exact sequences
$$1\to\mathrm{Z}^{\mathrm{ess}}_{\mathtt{Z}_n}\to\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}\to\mathrm{G}^{\mathrm{ess}}_{h,\mathtt{Z}_n}\to1,\tag{5.3.1.8}$$
$$1\to\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}\to\mathrm{Z}^{\mathrm{ess}}_{\mathtt{Z}_n}\to\mathrm{G}^{\mathrm{ess}}_{l,\mathtt{Z}_n}\to1,\tag{5.3.1.9}$$
$$1\to\mathrm{U}^{\mathrm{ess}}_{2,\mathtt{Z}_n}\to\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}\to\mathrm{U}^{\mathrm{ess}}_{1,\mathtt{Z}_n}\to1.\tag{5.3.1.10}$$

Definition 5.3.1.11. *Let* H_n *be a subgroup of* $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ *as above. For each of the subgroups* $\divideontimes$ *in (5.3.1.7), we define* $H_{n,\divideontimes}:=H_n\cap\divideontimes$. *For each of the quotients of two groups* $\divideontimes=\divideontimes_1/\divideontimes_2$ *in (5.3.1.7), (5.3.1.8), (5.3.1.9), or (5.3.1.10), we define*

$H_{n,\divideontimes} := H_{n,\divideontimes_1}/H_{n,\divideontimes_2}$. *Thus we have defined the groups* $H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}}$, $H_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathtt{Z}_n}}$, $H_{n,\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}}$, $H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathtt{Z}_n}}$, $H_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathtt{Z}_n}}$, $H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathtt{Z}_n}}$, *and* $H_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathtt{Z}_n}}$, *so that we have the natural inclusions*

$$H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathtt{Z}_n}} \subset H_{n,\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}} \subset H_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathtt{Z}_n}} \subset H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}} \subset H_n$$

and natural exact sequences

$$1 \to H_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathtt{Z}_n}} \to H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}} \to H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathtt{Z}_n}} \to 1,$$
$$1 \to H_{n,\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}} \to H_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathtt{Z}_n}} \to H_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathtt{Z}_n}} \to 1,$$
$$1 \to H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathtt{Z}_n}} \to H_{n,\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}} \to H_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathtt{Z}_n}} \to 1.$$

Let us return to the question of the description of H_n-orbits of $[\alpha_n^\natural]$. Note that the filtration $\mathtt{Z}_n$ in each equivalence class $[\alpha_n^\natural]$ is independent of the representative $\alpha_n^\natural$ we take for $[\alpha_n^\natural]$.

Given any particular fully symplectic-liftable filtration $\mathtt{Z}_n$ on L/nL, the elements in its H_n-orbit can be parameterized by the left cosets $H_{n,\mathrm{P}_{\mathtt{Z}_n}}\backslash H_n$. Then an étale-locally-defined H_n-orbit of $\mathtt{Z}_n$ is a scheme $\mathtt{Z}_{H_n}$ finite étale over η that is isomorphic to the constant scheme $H_{n,\mathrm{P}_{\mathtt{Z}_n}}\backslash H_n$ over some finite étale scheme over η.

Let us work over $\mathtt{Z}_{H_n}$. Suppose $\tilde{\eta}$ is a point of $\mathtt{Z}_{H_n}$, which is finite étale over η, over which we have a representative $\mathtt{Z}_n$ in the orbit $\mathtt{Z}_{H_n}$. Let us investigate the $H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}}$-orbits of the remaining objects. The next natural object to study is the $H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}}$-orbit of principal level-n structures $\varphi_{-1,n} : \mathrm{Gr}^{\mathtt{Z}}_{-1,n} \xrightarrow{\sim} A[n]_{\tilde{\eta}}$ of $(A_{\tilde{\eta}}, \lambda_{A,\tilde{\eta}}, i_{A,\tilde{\eta}})$ of type $(\mathrm{Gr}^{\mathtt{Z}}_{-1,n}, \langle\,\cdot\,,\,\cdot\,\rangle_{11})$ (see Lemma 5.2.7.5 and Remark 5.2.7.6). Since the action of $g_n \in H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}}$ is realized as $\mathrm{Gr}^{\mathtt{Z}}_{-1,n}(g_n)$ for each $g_n \in \mathrm{P}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$, an $H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathtt{Z}_n}}$-orbit of $\varphi_{-1,n}$ is also an $H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathtt{Z}_n}}$-orbit. By patching over points of $\mathtt{Z}_{H_n}$ over finite étale extensions of η, we obtain an étale sheaf $\mathrm{Gr}^{\mathtt{Z}_{H_n}}_{-1,n}$ over $\mathtt{Z}_{H_n}$, whose fibers over each $\tilde{\eta}$ as above is the middle graded piece $\mathrm{Gr}^{\mathtt{Z}}_{-1,n}$ of the filtration $\mathtt{Z}_n$ at that point, and we obtain an étale subscheme φ_{-1,H_n} of

$$\underline{\mathrm{Isom}}_{\mathtt{Z}_{H_n}}(\mathrm{Gr}^{\mathtt{Z}_{H_n}}_{-1,n}, A[n]_{\mathtt{Z}_{H_n}}) \underset{\mathtt{Z}_{H_n}}{\times} \underline{\mathrm{Isom}}_{\mathtt{Z}_{H_n}}(((\mathbb{Z}/n\mathbb{Z})(1))_{\mathtt{Z}_{H_n}}, \boldsymbol{\mu}_{n,\mathtt{Z}_{H_n}}),$$

whose fiber over each $\tilde{\eta}$ as above is an $H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathtt{Z}_n}}$-orbit of étale-locally-defined level-n structures of $(A_{\tilde{\eta}}, \lambda_{A,\tilde{\eta}}, i_{A,\tilde{\eta}})$ of type $(\mathrm{Gr}^{\mathtt{Z}}_{-1,n}, \langle\,\cdot\,,\,\cdot\,\rangle_{11})$. Then $\varphi_{-1,H_n} \twoheadrightarrow \eta$ is finite étale.

Over φ_{-1,H_n}, we have étale locally over each point a choice of some $(\mathtt{Z}_n, \varphi_{-1,n})$ in its H_n-orbit. The next natural object to study is the $H_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathtt{Z}_n}}$-orbit of $(\varphi_{-2,n}, \varphi_{0,n})$. The natural group to consider is the subgroup $H_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathtt{Z}_n}}$ of $\mathrm{G}^{\mathrm{ess}}_{l,\mathtt{Z}_n} \cong \mathrm{Z}^{\mathrm{ess}}_{\mathtt{Z}_n}/\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}$. By abuse of notation, let us denote the $H_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathtt{Z}_n}}$-orbit of a particular pair $(\varphi_{-2,n}, \varphi_{0,n})$ by $(\varphi_{-2,H_n}, \varphi_{0,H_n})$. Although the notation of using a pair is misleading, we shall interpret $(\varphi_{-2,H_n}, \varphi_{0,H_n})$ as an étale subscheme of the pullback of

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\mathrm{Gr}^{\mathtt{Z}}_{-2,n}, \underline{\mathrm{Hom}}_{\eta}((\underline{X}/n\underline{X})_\eta, ((\mathbb{Z}/n\mathbb{Z})(1))_\eta)) \underset{\eta}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\mathrm{Gr}^{\mathtt{Z}}_{0,n}, (\underline{Y}/n\underline{Y})_\eta)$$

to φ_{-1,H_n}. Then $(\varphi_{-2,H_n}, \varphi_{0,H_n}) \twoheadrightarrow \eta$ is finite étale.

Over $(\varphi_{-2,H_n}, \varphi_{0,H_n})$, we have étale locally over each point a choice of some $(\mathtt{Z}_{H_n}, \varphi_{-2,H_n}, \varphi_{-1,H_n}, \varphi_{0,H_n})$ in its H_n-orbit. Let us study the $H_{n,\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}}$-orbit of the remaining objects $(\delta_n, c_n, c_n^\vee, \tau_n)$ *up to equivalence.*

If we use the action we have defined so far on the representatives, then the action is realized on δ_n, but the objects $(c_n, c_n^\vee, \tau_n)$ over $\tilde{\eta}$ are never changed under the actions. This is not pertinent for our purpose because we want to have descended forms of them over η. On the other hand, if we allow ourselves to take equivalent objects, then we may modify the action of $\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}$ on the representatives as follows: The elements $g_n \in \mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}$ satisfy $\mathrm{Gr}_n(g_n) = \mathrm{Id}_{\mathrm{Gr}^{\mathtt{Z}}_n}$ by definition. Hence the action of g_n sends $(\delta_n, c_n, c_n^\vee, \tau_n)$ to $(g_n^{-1} \circ \delta_n, c_n, c_n^\vee, \tau_n)$. Suppose $\delta_n^{-1} \circ g_n \circ \delta_n = \begin{pmatrix} 1 & g_{21,n} & g_{20,n} \\ & 1 & g_{10,n} \\ & & 1 \end{pmatrix}$. Let us write $g_n^{-1} \circ \delta_n = \delta_n \circ (\delta_n^{-1} \circ g_n^{-1} \circ \delta_n)$, where $\mathtt{z}_n := \delta_n^{-1} \circ g_n^{-1} \circ \delta_n$ is now viewed as a change of basis. Then (as in Proposition 5.2.7.11) we have morphisms d_n, $d_n^\vee$, and e_n determined by $g_{21,n}$, $g_{10,n}$, and $g_{20,n}$, respectively, so that $(g_n^{-1} \circ \delta_n, c_n, c_n^\vee, \tau_n)$ is equivalent to $(\delta_n, c_n', (c_n^\vee)', \tau_n')$, where $(c_n', (c_n^\vee)', \tau_n')$ is the translation of $(c_n, c_n^\vee, \tau_n)$ by the $(d_n, d_n^\vee, e_n)$ (as in Definition 5.2.7.10).

By the explicit formulas, the modified action of $\mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}$ on $(c_n, c_n^\vee)$ factors through $\mathrm{U}^{\mathrm{ess}}_{1,\mathtt{Z}_n} = \mathrm{U}^{\mathrm{ess}}_{\mathtt{Z}_n}/\mathrm{U}^{\mathrm{ess}}_{2,\mathtt{Z}_n}$. Hence it makes sense to form $H_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathtt{Z}_n}}$-orbits of $(c_n, c_n^\vee)$ and denote it by $(c_{H_n}, c_{H_n}^\vee)$. Again, although the notation of using a pair is misleading, we shall interpret $(c_{H_n}, c_{H_n}^\vee)$ as a subscheme of the pullback of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}\underline{X}_\eta, A_\eta^\vee) \times \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}\underline{Y}_\eta, A_\eta)$ to $(\varphi_{-2,H_n}, \varphi_{0,H_n})$. (In general, such an ambient scheme is larger than the parameter space of all pairs that we allow. A more precise construction will be given in Section 6.2.3.) Then $(c_{H_n}, c_{H_n}^\vee) \twoheadrightarrow \eta$ is finite étale.

Over $(c_{H_n}, c_{H_n}^\vee)$, we have étale locally over each point a choice of some representative $(c_{H_n}, c_{H_n}^\vee)$ in its $H_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathtt{Z}_n}}$-orbit. Then it only remains to understand the action of $H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathtt{Z}_n}}$ on τ_n. Let us denote by τ_{n,H_n} the $H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathtt{Z}_n}}$-orbit of τ_n, and denote by ι_{n,H_n} the $H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathtt{Z}_n}}$-orbit of ι_n. Similarly to the case of $(c_{H_n}, c_{H_n}^\vee)$ above, we shall identify τ_{n,H_n} with ι_{n,H_n} as schemes over η, and interpret ι_{n,H_n} as a subscheme of the pullback of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}\underline{Y}_\eta, G_\eta^\natural)$ to $(c_{H_n}, c_{H_n}^\vee)$, although this is not the precise parameter space we wanted. Then $\tau_{n,H_n} \twoheadrightarrow \eta$ is finite étale.

Finally, over τ_{n,H_n}, we have étale locally over each point a choice of all the data, including the unique choice of δ_n, which we shall denote by δ_{H_n}. If we replace δ_n with $\delta_n \circ \mathtt{z}_n$ before we construct $(c_{H_n}, c_{H_n}^\vee, \tau_{H_n})$, then we shall replace accordingly the whole orbit $(c_{H_n}, c_{H_n}^\vee, \tau_{H_n})$ with another triple $(c_{H_n}', (c_{H_n}^\vee)', \tau_{H_n}')$ (which is another scheme finite étale over η). (Therefore the notion of equivalences carries over naturally to the context of H_n-orbits.)

To facilitate the language, we shall often ignore the fact that the above objects are schemes finite étale over η, and denote them by tuples as if they were usual objects:

Definition 5.3.1.12. *With the setting as in Section 5.2.1, suppose we are given a tuple* $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ *in* $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\,\cdot\,,\,\cdot\,\rangle,h)}(R, I)$. *Let* $n \geq 1$ *be an integer such that* $\Box \nmid n$. *Let* H_n *be a subgroup of* $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$. *By an* ***$H_n$-orbit of étale-locally-defined level-n structure data****, we mean a scheme*

$$\alpha^\natural_{H_n} = (\mathtt{Z}_{H_n}, \varphi_{-2,H_n}, \varphi_{-1,H_n}, \varphi_{0,H_n}, \delta_{H_n}, c_{H_n}, c^\vee_{H_n}, \tau_{H_n})$$

(or rather just τ_{H_n}*) finite étale over* η*, which is étale locally (over* η*) the disjoint union of all elements in some* H_n*-orbit of level-n structure data (as in Definition 5.2.7.8). We use the same terminology,* ***H_n-orbit of étale-locally-defined****, for each of the entries in* $\alpha^\natural_{H_n}$.

Definition 5.3.1.13. *With the setting as in Definition 5.3.1.12, we say that two* H_n*-orbits* $\alpha^\natural_{H_n}$ *and* $(\alpha^\natural_{H_n})'$ *are equivalent if there is a finite étale extension* $\tilde{\eta} \to \eta$ *over which* $\alpha^\natural_{H_n}$ *contains some level-n structure datum that is equivalent (as in Definition 5.2.7.11) to some level-n structure datum in* $(\alpha^\natural_{H_n})'$.

Definition 5.3.1.14. *With the setting as in Section 5.2.1, suppose we are given a tuple* $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ *in* $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle,h)}(R, I)$. *Let* $\mathcal{H}$ *be any open compact subgroup of* $\mathrm{G}(\hat{\mathbb{Z}}^\square)$. *For each integer* $n \geq 1$ *such that* $\square \nmid n$ *and* $\mathcal{U}^\square(n) \subset \mathcal{H}$, *set* $H_n := \mathcal{H}/\mathcal{U}^\square(n)$ *as usual. Then a* ***level-$\mathcal{H}$ structure datum of type*** $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square, \langle\cdot,\cdot\rangle)$ *over* η *is a collection* $\alpha^\natural_\mathcal{H} = \{\alpha^\natural_{H_n}\}_n$ *indexed by integers* $n \geq 1$ *such that* $\square \nmid n$ *and* $\mathcal{U}^\square(n) \subset \mathcal{H}$, *with elements* $\alpha^\natural_{H_n}$ *described as follows:*

1. *For each index* n, *the element* $\alpha^\natural_{H_n}$ *is an* H_n*-orbit of étale-locally-defined level-*n *structure data as in Definition 5.3.1.12.*
2. *For all indices* n *and* m *such that* $n|m$, *the* H_n*-orbit* $\alpha^\natural_{H_n}$ *is determined by the* H_m*-orbit* $\alpha^\natural_{H_m}$ *by reduction modulo* n.

It is customary to denote $\alpha^\natural_\mathcal{H}$ *by a tuple*

$$\alpha^\natural_\mathcal{H} = (\mathsf{Z}_\mathcal{H}, \varphi_{-2,\mathcal{H}}, \varphi_{-1,\mathcal{H}}, \varphi_{0,\mathcal{H}}, \delta_\mathcal{H}, c_\mathcal{H}, c^\vee_\mathcal{H}, \tau_\mathcal{H}),$$

each subtuple or entry being a collection indexed by n *as* $\alpha^\natural_\mathcal{H}$ *is, and to denote by* $\iota_\mathcal{H}$ *the collection corresponding to* $\tau_\mathcal{H}$.

Convention 5.3.1.15. *To facilitate the language, we shall call* $\alpha^\natural_\mathcal{H}$ *an* $\mathcal{H}$*-orbit, with similar usages applied to other objects with subscript* $\mathcal{H}$. *If we have two open compact subgroups* $\mathcal{H}' \subset \mathcal{H}$, *and if we have an object* $\alpha^\natural_{\mathcal{H}'}$ *at level* $\mathcal{H}'$, *then there is a natural meaning of the object* $\alpha^\natural_\mathcal{H}$ *at level* $\mathcal{H}$ *determined by* $\alpha^\natural_{\mathcal{H}'}$. *We say in this case that* $\alpha^\natural_\mathcal{H}$ *is the* $\mathcal{H}$*-orbit of* $\alpha^\natural_{\mathcal{H}'}$.

Definition 5.3.1.16. *With the setting as in Definition 5.3.1.14, we say that two level-*$\mathcal{H}$ *structure data* $\alpha^\natural_\mathcal{H} = \{\alpha^\natural_{H_n}\}_n$ *and* $(\alpha^\natural_\mathcal{H})' = \{(\alpha^\natural_{H_n})'\}_n$ *are equivalent if there is an index* n *such that* $\alpha^\natural_{H_n}$ *and* $(\alpha^\natural_{H_n})'$ *are equivalent as in Definition 5.3.1.13.*

Definition 5.3.1.17. *With the setting as in Section 5.2.1, the category* $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_\mathcal{H}}(R, I)$ *has objects of the form* $(G, \lambda, i, \alpha_\mathcal{H})$ *(over* S*), where*

1. (G, λ, i) *defines an object in* $\mathrm{DEG}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle,h)}(R, I)$;
2. $\alpha_\mathcal{H}$ *is a level-*$\mathcal{H}$ *structure of* $(G_\eta, \lambda_\eta, i_\eta)$ *of type* $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square, \langle\cdot,\cdot\rangle)$ *as in Definition 1.3.7.6.*

Definition 5.3.1.18. *With the setting as in Section 5.2.1, the category* $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_\mathcal{H}}(R, I)$ *has objects of the form* $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha^\natural_\mathcal{H}])$, *where*

1. $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau)$ *is an object in* $\mathrm{DD}_{\mathrm{PE}_{\mathrm{Lie}},(L\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle,h)}(R, I)$;
2. $[\alpha^\natural_\mathcal{H}]$ *is an equivalence class of level-*$\mathcal{H}$ *structure data* $\alpha^\natural_\mathcal{H}$ *of type* $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square, \langle\cdot,\cdot\rangle)$ *defined over* η *(see Definitions 5.3.1.14 and 5.3.1.16).*

Then we have the following consequence of Theorem 5.2.7.14:

Theorem 5.3.1.19. *There is an equivalence of categories*

$$\mathrm{M}_{\mathrm{PEL},\mathsf{M}_\mathcal{H}}(R, I) : \mathrm{DD}_{\mathrm{PEL},\mathsf{M}_\mathcal{H}}(R, I) \to \mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_\mathcal{H}}(R, I) :$$
$$(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha^\natural_\mathcal{H}]) \mapsto (G, \lambda, i, \alpha_\mathcal{H}).$$

5.3.2 Degenerating Families

For ease of exposition later, let us make the following definitions:

Definition 5.3.2.1. *Let* S *be a normal locally noetherian algebraic stack. A tuple* $(G, \lambda, i, \alpha_\mathcal{H})$ *over* S *is called a* ***degenerating family of type*** $\mathsf{M}_\mathcal{H}$, *or simply a* ***degenerating family*** *when the context is clear, if there exists a dense subalgebraic stack* S_1 *of* S, *such that* S_1 *is defined over* $\mathrm{Spec}(\mathcal{O}_{F_0,(\square)})$, *and such that we have the following:*

1. *By viewing group schemes as relative objects,* G *is a semi-abelian scheme over* S *whose restriction* G_{S_1} *to* S_1 *is an abelian scheme. In this case, the dual semi-abelian scheme* $G^\vee$ *(as in Theorem 3.4.3.2) exists (up to unique isomorphism), whose restriction* $G^\vee_{S_1}$ *to* S_1 *is the dual abelian scheme of* G_{S_1} *(see Proposition 3.3.1.5).*
2. $\lambda : G \xrightarrow{\sim} G^\vee$ *is a homomorphism that induces by restriction a prime-to-*$\square$ *polarization* λ_{S_1} *of* G_{S_1}.
3. $i : \mathcal{O} \to \mathrm{End}_S(G)$ *is a homomorphism that defines by restriction an* $\mathcal{O}$*-structure* $i_{S_1} : \mathcal{O} \to \mathrm{End}_{S_1}(G_{S_1})$ *of* (G_{S_1}, λ_{S_1}).
4. $\underline{\mathrm{Lie}}_{G_{S_1}/S_1}$ *with its* $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\square)}$*-module structure given naturally by* i_{S_1} *satisfies the determinantal condition in Definition 1.3.4.1 given by* $(L\underset{\mathbb{Z}}{\otimes}\mathbb{R}, \langle\cdot,\cdot\rangle, h)$.
5. $\alpha_\mathcal{H}$ *is a level-*$\mathcal{H}$ *structure for* $(G_{S_1}, \lambda_{S_1}, i_{S_1})$ *of type* $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square, \langle\cdot,\cdot\rangle)$ *as in Definition 1.3.7.6, defined over* S_1.

In other words, we require $(G_{S_1}, \lambda_{S_1}, i_{S_1}, \alpha_\mathcal{H}) \to S_1$ *to define a tuple parameterized by the moduli problem* $\mathsf{M}_\mathcal{H}$.

Remark 5.3.2.2. Conditions 2, 3, 4, and 5 are closed conditions for structures on *abelian schemes* defined over $\mathrm{Spec}(\mathcal{O}_{F_0,(\square)})$. Hence the rather weak condition for S_1 in Definition 5.3.2.1 is justified because S_1 can always be replaced with the largest subalgebraic stack of S over $\mathrm{Spec}(\mathcal{O}_{F_0,(\square)})$ (which is open dense in S) such that G_{S_1} is an abelian scheme. (Conditions 2 and 3 are closed by Lemma 4.2.1.6, Corollary 1.3.1.12, and Proposition 3.3.1.5. Condition 4 is closed because it is defined by an equality of polynomial functions. Condition 5 is closed because $\alpha_\mathcal{H}$ is defined by isomorphisms between finite étale group schemes.)

Definition 5.3.2.3. *With the setting as in Definition 5.3.2.1, suppose* $n \geq 1$ *is an integer prime-to-*$\square$. *Then a tuple* $(G, \lambda, i, \alpha_n)$ *over* S *is called a* ***degenerating family of type*** M_n *if it satisfies the same definitions as in Definition 5.3.2.1 except that 5 in Definition 5.3.2.1 is replaced with*

5′. α_n *is a principal level-*n *structure for* $(G_{S_1}, \lambda_{S_1}, i_{S_1})$ *of type* $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\square, \langle\cdot,\cdot\rangle)$ *as in Definition 1.3.6.2, defined over* S_1.

Definition 5.3.2.4. *We define a tuple* (G, λ, i) *over* S *to be a* ***degenerating family of type*** $\mathsf{M}_\mathcal{H}$ *(resp.* M_n*)* ***without level structures***, *or still simply a* ***degenerating family*** *when the context is clear, if* G, λ, *and* i *satisfy 1, 2, 3, and 4 as in Definition 5.3.2.1, without a level structure described by 5 (resp. 5′) as in the definition of* $\mathsf{M}_\mathcal{H}$ *(resp.* M_n*).*

5.3.3 Criterion for Properness

Here is an interesting consequence of Theorem 5.3.1.19:

Theorem 5.3.3.1. *Let M' be an algebraic stack separated and of finite-type over an arbitrary locally noetherian algebraic stack S'. (In particular, we allow any residue characteristic to appear in S'.) Suppose there is an open dense subalgebraic stack S'_1 of S' such that the following conditions are satisfied:*

1. *$\mathsf{M}' \underset{\mathsf{S}'}{\times} \mathsf{S}'_1$ is open dense in M'.*
2. *There exists a morphism $\mathsf{S}'_1 \to \mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ and a morphism $\mathsf{M}' \underset{\mathsf{S}'}{\times} \mathsf{S}'_1 \xrightarrow{f} \mathsf{M}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{S}'_1$ such that, for each complete discrete valuation ring V with algebraically closed residue field, a morphism $\xi_1 : \mathrm{Spec}(\mathrm{Frac}(V)) \to \mathsf{M}' \underset{\mathsf{S}'}{\times} \mathsf{S}'_1$ defining a tuple $(G, \lambda, i, \alpha_{\mathcal{H}})$ by composition with f extends to a morphism $\xi : \mathrm{Spec}(V) \to \mathsf{M}'$ whenever the abelian scheme G extends to an abelian scheme over $\mathrm{Spec}(V)$.*
3. *Every admissible filtration Z on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ that is fully symplectic with respect to $(L, \langle \cdot, \cdot \rangle)$ has $\mathcal{O}$-multirank zero (see Definitions 5.2.7.1 and 5.2.2.6).*

Then $\mathsf{M}' \to \mathsf{S}'$ is proper.

Proof. To show that $\mathsf{M}' \to \mathsf{S}'$ is proper, we need to verify the valuative criterion for it. By 1 (and by Remark A.7.2.13), it suffices to show that, for each $\mathrm{Spec}(V) \to \mathsf{S}'$ where V is a complete discrete valuation ring with algebraically closed residue field, each morphism $\xi_1 : \mathrm{Spec}(\mathrm{Frac}(V)) \to \mathsf{M}' \underset{\mathsf{S}'}{\times} \mathsf{S}'_1$ extends to a morphism $\xi : \mathrm{Spec}(V) \to \mathsf{M}'$. By composition with the morphism f in 2, each morphism $\xi_1 : \mathrm{Spec}(\mathrm{Frac}(V)) \to \mathsf{M}' \underset{\mathsf{S}'}{\times} \mathsf{S}'_1$ induces a morphism $f \circ \xi_1 : \mathrm{Spec}(\mathrm{Frac}(V)) \to \mathsf{M}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{S}'_1$ defining an object $(G, \lambda, i, \alpha_{\mathcal{H}})$ of $\mathsf{M}_{\mathcal{H}}$ over $\mathrm{Spec}(\mathrm{Frac}(V))$. By Theorem 3.3.2.4, we know that G extends to a semi-abelian scheme over $\mathrm{Spec}(V)$. By Proposition 3.3.1.5, both λ and i extends uniquely over $\mathrm{Spec}(V)$. Note that $\mathrm{Spec}(\mathrm{Frac}(V))$ is defined over S_0 via the morphism $\mathsf{S}'_1 \to \mathsf{S}_0$ in 2. Therefore it makes sense to say that we have a degenerating family of type $\mathsf{M}_{\mathcal{H}}$ over $\mathrm{Spec}(V)$ (see Definition 5.3.2.1) extending $(G, \lambda, i, \alpha_{\mathcal{H}})$. Since every torus over the algebraically closed residue field of V is trivial, this degenerating family satisfies the isotriviality condition in Definition 4.2.1.1 and defines an object of $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(V)$. By Theorem 5.3.1.19, this corresponds to an object of $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(V)$. In particular, we obtain an $\mathcal{H}$-orbit of admissible filtrations Z on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ that are fully symplectic with respect to $(L, \langle \cdot, \cdot \rangle)$. By 3, the $\mathcal{O}$-multirank of every such Z is zero. Therefore, the torus part of the semi-abelian scheme extending G must be trivial, which means G extends to an abelian scheme over $\mathrm{Spec}(V)$ (see Remark 3.3.1.3 and Lemma 3.3.1.4). Then the valuative criterion is verified by 2, as desired. $\Box$

Remark 5.3.3.2. Condition 2 in Theorem 5.3.3.1 is satisfied when $\mathsf{M}' \to \mathsf{S}'$ is $\mathsf{M}_{\mathcal{H}} \to \mathsf{S}_0$ (see Remark 5.3.2.2; cf. the proof of properness in [76, end of §5]). Condition 3 in Theorem 5.3.3.1 is satisfied, for example, when B is simple and the signatures $(p_\tau, q_\tau)_\tau$ of $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h)$ (see Definition 1.2.5.2) satisfy $\min(p_\tau, q_\tau) = 0$ for at least one τ. (In this case, instead of applying Theorem 5.3.1.19 in the proof of Theorem 5.3.1.19, we may apply the much simpler Theorem 5.1.2.7.) More generally, condition 3 is satisfied if the group $\mathrm{G}(\mathbb{Q}_v)$ contains no unipotent elements for some (finite or infinite) place v of $\mathbb{Q}$ prime-to-$\Box$.

Remark 5.3.3.3. Theorem 5.3.3.1 is applicable, for example, to proving the properness of moduli problems defining integral models of Shimura varieties with reasonable *bad reductions* (satisfying conditions 1 and 2).

5.4 Notion of Cusp Labels

5.4.1 Principal Cusp Labels

Definition 5.4.1.1. *With the setting as in Section 5.2.1, the category $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}(R, I)$ has objects of the form*

$$(\mathsf{Z}_n, (\underline{X}, \underline{Y}, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n)),$$

where

1. *$\underline{X}$ (resp. $\underline{Y}$) is constant with value X (resp. Y);*
2. *$c_n|_X : X \to A_\eta^\vee$ (resp. $c_n^\vee|_Y : Y \to A_\eta$) extends to an $\mathcal{O}$-equivariant homomorphism $c : X \to A^\vee$ (resp. $c^\vee : Y \to A$) over S (which is unique by Corollary 5.2.3.11);*
3. *if we set $\tau := \tau_n|_{\mathbf{1}_{Y \times X,\eta}}$ and $\alpha_n^\natural := (\mathsf{Z}_n, \varphi_{-2,n}, \varphi_{-1,n}, \varphi_{0,n}, \delta_n, c_n, c_n^\vee, \tau_n)$, then $\alpha_n^\natural$ is a level-n structure datum as in Definition 5.2.7.8, and $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha_n^\natural])$ is an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}(R, I)$ as in Definition 5.2.7.13.*

Remark 5.4.1.2. There is a natural functor $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}(R, I) \to \mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}(R, I)$ defined by assigning the class $[\alpha_n^\natural]$ to the representative $\alpha_n^\natural$.

Now let us introduce the idea of cusp labels. We would like to define our cusp labels as equivalence classes of the tuples $(\mathsf{Z}_n, (\underline{X}, \underline{Y}, \phi, \varphi_{-2,n}, \varphi_{0,n}), \delta_n)$, because it is the part of the degeneration datum that is discrete in nature. Later our construction will produce for each cusp label a formal scheme over which we have tautological data of $(A, \lambda_A, i_A, \varphi_{-1,n})$ and $(c_n, c_n^\vee, \tau_n)$. These formal schemes should be interpreted as our boundary components, because their suitable algebraic approximations will be glued to the moduli problem M_n in the étale topology (and then form the desired compactification). Since the gluing process will be carried out in the étale topology, there is no loss of generality to assume (in the definition of cusp labels) that $\underline{X}$ and $\underline{Y}$ are constant with values X and Y, respectively.

Definition 5.4.1.3. *Given a fully symplectic admissible filtration Z on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ with respect to $(L, \langle \cdot, \cdot \rangle)$ as in Definition 5.2.7.1, a* ***torus argument*** *Φ for Z is a tuple $\Phi := (X, Y, \phi, \varphi_{-2}, \varphi_0)$, where*

1. *X and Y are $\mathcal{O}$-lattices of the same $\mathcal{O}$-multirank (see Definition 1.2.1.21), and $\phi : Y \hookrightarrow X$ is an $\mathcal{O}$-equivariant embedding;*
2. *$\varphi_{-2} : \mathrm{Gr}^{\mathsf{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^{\Box}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \hat{\mathbb{Z}}^{\Box}(1))$ and $\varphi_0 : \mathrm{Gr}^{\mathsf{Z}}_0 \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ are isomorphisms such that the pairing $\langle \cdot, \cdot \rangle_{20} : \mathrm{Gr}^{\mathsf{Z}}_{-2} \times \mathrm{Gr}^{\mathsf{Z}}_0 \to \hat{\mathbb{Z}}^{\Box}(1)$ defined by Z is the*

pullback of the pairing

$$\langle\cdot,\cdot\rangle_\phi : \mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box,\hat{\mathbb{Z}}^\Box(1))\times(Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\to\hat{\mathbb{Z}}^\Box(1)$$

defined by the composition

$$\mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box,\hat{\mathbb{Z}}^\Box(1))\times(Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$$
$$\overset{\mathrm{Id}\times\phi}{\to}\mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box,\hat{\mathbb{Z}}^\Box(1))\times(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\to\hat{\mathbb{Z}}^\Box(1),$$

with the sign convention that $\langle\cdot,\cdot\rangle_\phi(x,y)=x(\phi(y))=(\phi(y))(x)$ *for all* $x\in\mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box,\hat{\mathbb{Z}}^\Box(1))$ *and* $y\in Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$.

Definition 5.4.1.4. *Given a fully symplectic-liftable admissible filtration* Z_n *on* L/nL *with respect to* $(L,\langle\cdot,\cdot\rangle)$ *as in Definition 5.2.7.3, a* ***torus argument*** Φ_n ***at level*** n *for* Z_n *is a tuple* $\Phi_n:=(X,Y,\phi,\varphi_{-2,n},\varphi_{0,n})$*, where*

1. X *and* Y *are* $\mathcal{O}$*-lattices of the same* $\mathcal{O}$*-multirank (see Definition 1.2.1.21), and* $\phi:Y\hookrightarrow X$ *is an* $\mathcal{O}$*-equivariant embedding;*
2. $\varphi_{-2,n}:\mathrm{Gr}^{\mathsf{Z}}_{-2,n}\overset{\sim}{\to}\mathrm{Hom}(X/nX,(\mathbb{Z}/n\mathbb{Z})(1))$ *and* $\varphi_{0,n}:\mathrm{Gr}^{\mathsf{Z}}_{0,n}\overset{\sim}{\to}Y/nY$ *are isomorphisms that are reductions modulo* n *of some isomorphisms* $\varphi_{-2}:\mathrm{Gr}^{\mathsf{Z}}_{-2}\overset{\sim}{\to}\mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box,\hat{\mathbb{Z}}^\Box(1))$ *and* $\varphi_0:\mathrm{Gr}^{\mathsf{Z}}_0\overset{\sim}{\to}(Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$*, respectively, such that* $\Phi=(X,Y,\phi,\varphi_{-2},\varphi_0)$ *form a torus argument as in Definition 5.4.1.3.*

We say in this case that Φ_n *is the reduction modulo* n *of* Φ.

Definition 5.4.1.5. *Let* $\Phi_n=(X,Y,\phi,\varphi_{-2,n},\varphi_{0,n})$ *and* $\Phi'_n=(X',Y',\phi',\varphi'_{-2,n},\varphi'_{0,n})$ *be two torus arguments at level* n*. We say that* Φ_n *and* Φ'_n *are* ***equivalent*** *if there exists a pair of isomorphisms* $(\gamma_X:X'\overset{\sim}{\to}X,\gamma_Y:Y\overset{\sim}{\to}Y')$ *such that* $\phi=\gamma_X\phi'\gamma_Y$*,* $\varphi'_{-2,n}={}^t\gamma_X\varphi_{-2,n}$*, and* $\varphi'_{0,n}=\gamma_Y\varphi_{0,n}$*. In this case, we say that* Φ_n *and* Φ'_n *are equivalent under the pair of isomorphisms* (γ_X,γ_Y)*, which we denote by* $(\gamma_X,\gamma_Y):\Phi_n\overset{\sim}{\to}\Phi'_n$.

Definition 5.4.1.6. *The group functor* $\mathrm{GL}_\phi=\mathrm{GL}_{X,Y,\phi}$ *over* $\mathrm{Spec}(\mathbb{Z})$ *is defined by assigning to each* $\mathbb{Z}$*-algebra* R *the group*

$$\mathrm{GL}_\phi(R):=\{(g_X,g_Y)\in\mathrm{GL}_{\mathcal{O}}(X)\times\mathrm{GL}_{\mathcal{O}}(Y):\phi=g_X\phi g_Y\}.$$

Definition 5.4.1.7. *The group* $\Gamma_\phi=\Gamma_{X,Y,\phi}$ *is the group* $\mathrm{GL}_\phi(\mathbb{Z})$*, or equivalently the group of pairs of isomorphisms* $(\gamma_X:X\overset{\sim}{\to}X,\gamma_Y:Y\overset{\sim}{\to}Y)$ *in* $\mathrm{GL}_{\mathcal{O}}(Y)\times\mathrm{GL}_{\mathcal{O}}(X)$ *such that* $\phi=\gamma_X\phi\gamma_Y$.

By functoriality, we have a natural homomorphism

$$\Gamma_\phi=\mathrm{GL}_\phi(\mathbb{Z})\to\mathrm{GL}_\phi(R)\hookrightarrow\mathrm{GL}_{\mathcal{O}}(X\underset{\mathbb{Z}}{\otimes}R)\times\mathrm{GL}_{\mathcal{O}}(Y\underset{\mathbb{Z}}{\otimes}R)$$

for each $\mathbb{Z}$-algebra R. (Here Γ_ϕ is understood as a group independent of R.)

We have the following simple observation:

Lemma 5.4.1.8. *For each triple* (X,Y,ϕ)*, if two torus arguments* $\Phi_n=(X,Y,\phi,\varphi_{-2,n},\varphi_{0,n})$ *and* $\Phi'_n=(X,Y,\phi,\varphi'_{-2,n},\varphi'_{0,n})$ *at level* n *are equivalent under some* (γ_X,γ_Y)*, then necessarily* $(\gamma_X,\gamma_Y)\in\Gamma_\phi$.

Definition 5.4.1.9. *A* ***(principal) cusp label at level*** n *for a PEL-type* $\mathcal{O}$*-lattice* $(L,\langle\cdot,\cdot\rangle,h)$*, or a cusp label of the moduli problem* M_n*, is an equivalence class* $[(\mathsf{Z}_n,\Phi_n,\delta_n)]$ *of triples* $(\mathsf{Z}_n,\Phi_n,\delta_n)$*, where*

1. Z_n *is an admissible filtration on* L/nL *that is fully symplectic-liftable in the sense of Definition 5.2.7.3;*
2. Φ_n *is a torus argument at level* n *for* Z_n*;*
3. $\delta_n:\mathrm{Gr}^{\mathsf{Z}}_n\overset{\sim}{\to}L/nL$ *is a liftable splitting.*

Two triples $(\mathsf{Z}_n,\Phi_n,\delta_n)$ *and* $(\mathsf{Z}'_n,\Phi'_n,\delta'_n)$ *are equivalent if* Z_n *and* Z'_n *are identical, and if* Φ_n *and* Φ'_n *are equivalent as in Definition 5.4.1.5.*

Convention 5.4.1.10. *For simplicity, we shall often suppress* Z_n *from the notation* $(\mathsf{Z}_n,\Phi_n,\delta_n)$*, with the understanding that the data* Φ_n *and* δ_n *require an implicit choice of* Z_n*. Even though we suppress* Z_n *in the representatives of cusp labels, we shall still retain the notation* $(L^{\mathsf{Z}_n},\langle\cdot,\cdot\rangle^{\mathsf{Z}_n},h^{\mathsf{Z}_n})$ *and* $\mathsf{M}_n^{\mathsf{Z}_n}$ *as in Lemma 5.2.7.5.*

Lemma 5.4.1.11. *Suppose we are given a representative* $(\mathsf{Z}_n,\Phi_n,\delta_n)$ *of a cusp label at level* n*, with* $\Phi_n=(X,Y,\phi,\varphi_{-2,n},\varphi_{0,n})$*. Then, for each object* $(A,\lambda_A,i_A,\underline{X},\underline{Y},\phi,c,c^\vee,\tau,[\alpha_n^\natural])$ *in* $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}(R,I)$ *such that* $\underline{X}$ *and* $\underline{Y}$ *are constant with values* X *and* Y*, respectively, there is a unique object*

$$(\mathsf{Z}_n,(X,Y,\phi,\varphi_{-2,n},\varphi_{0,n}),(A,\lambda_A,i_A,\varphi_{-1,n}),\delta_n,(c_n,c_n^\vee,\tau_n))$$

in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}(R,I)$ *such that* $\alpha_n^\natural=(\mathsf{Z}_n,\varphi_{-2,n},\varphi_{-1,n},\varphi_{0,n},\delta_n,c_n,c_n^\vee,\tau_n)$ *is a representative of* $[\alpha_n^\natural]$.

Definition 5.4.1.12. *The* ***$\mathcal{O}$-multirank*** *of a cusp label at level* n *represented by some* $(\mathsf{Z}_n,\Phi_n,\delta_n)$ *is the* $\mathcal{O}$*-multirank of* Z_n *(see Definitions 5.2.2.6, 5.2.2.9, and Remark 5.2.2.8).*

Lemma 5.4.1.13. *Suppose we have a cusp label at level* n *represented by some* $(\mathsf{Z}_n,\Phi_n,\delta_n)$*, with* $\Phi_n=(X,Y,\phi,\varphi_{-2,n},\varphi_{0,n})$*. For each admissible surjection* $s_X:X\twoheadrightarrow X'$ *of* $\mathcal{O}$*-lattices (see Definition 1.2.6.7), there is a canonically determined cusp label at level* n *which can be represented by some* $(\mathsf{Z}'_n,\Phi'_n,\delta'_n)$ *with* $\Phi'_n=(X',Y',\phi',\varphi_{-2,n},\varphi_{0,n})$ *described as follows:*

1. X' *is the same* X' *we have in the surjection* s_X.
2. *An admissible surjection* $s_Y:Y\twoheadrightarrow Y'$ *of* $\mathcal{O}$*-lattices for some* Y' *determined by setting* $\ker(s_Y)=\phi^{-1}(\ker(s_X))$.
3. *The definitions of* s_Y *and* Y' *induce an embedding* $\phi':Y'\to X'$ *such that* $s_X\phi=\phi's_Y$.
4. *The admissible surjection* $s_X:X\twoheadrightarrow X'$ *defines an admissible embedding*
$$s_X^*:\mathrm{Hom}(X'/nX',(\mathbb{Z}/n\mathbb{Z})(1))\hookrightarrow\mathrm{Hom}(X/nX,(\mathbb{Z}/n\mathbb{Z})(1)).$$
The image of s_X^* *is mapped to an admissible submodule* $\mathsf{Z}'_{-2,n}$ *of* $\mathsf{Z}_{-2,n}$*, and defines an isomorphism* $\varphi'_{-2}:\mathrm{Gr}^{\mathsf{Z}'}_{-2,n}=\mathsf{Z}'_{-2,n}\overset{\sim}{\to}\mathrm{Hom}(X'/nX',(\mathbb{Z}/n\mathbb{Z})(1))$.
The composition of the admissible surjection $\mathsf{Z}_{0,n}\twoheadrightarrow\mathsf{Z}_{0,n}/\mathsf{Z}_{-1,n}=\mathrm{Gr}^{\mathsf{Z}}_{0,n}$ *with* $\varphi_{0,n}:\mathrm{Gr}^{\mathsf{Z}}_{0,n}\overset{\sim}{\to}Y/nY$ *defines a surjection* $\mathsf{Z}_{0,n}\twoheadrightarrow Y/nY$*, and hence the admissible surjection* $s_Y:Y\twoheadrightarrow Y'$ *defines an admissible surjection* $\mathsf{Z}_{0,n}\twoheadrightarrow Y'/nY'$*, whose kernel defines an admissible submodule* $\mathsf{Z}'_{-1,n}$ *of* $\mathsf{Z}_{0,n}=L/nL$*. This defines an isomorphism* $\varphi'_{0,n}:\mathrm{Gr}^{\mathsf{Z}'}_{0,n}\overset{\sim}{\to}Y'/nY'$.
These two submodules $\mathsf{Z}'_{-2,n}$ *and* $\mathsf{Z}'_{-1,n}$ *of* L/nL *define a fully symplectic-liftable admissible filtration* $\mathsf{Z}'_n=\{\mathsf{Z}'_{-i,n}\}_i$ *on* L/nL.

5. $\delta'_n : \mathrm{Gr}^{\mathsf{Z}}_n \xrightarrow{\sim} L/nL$ *is just any liftable splitting.*

This construction determines a unique pair (Z'_n, Φ'_n)*, and hence a unique cusp label at level* n*.*

Proof. The upshot is to show that the tuple $(\mathsf{Z}'_n, \Phi'_n, \delta'_n)$ we arrive at, with $\Phi'_n = (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n})$, is a representative of a cusp label. In other words, the filtration Z'_n has to be fully symplectic-liftable and admissible, and the torus argument Φ'_n at level n has to be a reduction modulo n of some torus argument Φ' for some fully symplectic lifting Z of Z_n.

Let us fix some symplectic lifting Z of Z_n, and fix some liftings $\varphi_{-2} : \mathrm{Gr}^{\mathsf{Z}}_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1))$ and $\varphi_0 : \mathrm{Gr}^{\mathsf{Z}}_0 \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ of $\varphi_{-2,n} : \mathrm{Gr}^{\mathsf{Z}}_{-2,n} \xrightarrow{\sim} \mathrm{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1))$ and $\varphi_{0,n} : \mathrm{Gr}^{\mathsf{Z}}_{0,n} \xrightarrow{\sim} Y/nY$, respectively.

The admissible surjection $s_X : X \twoheadrightarrow X'$ defines an admissible embedding
$$s_X^* : \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1)) \hookrightarrow \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1)).$$
By the choice of φ_{-2} above, this defines an admissible submodule Z'_{-2} of Z_{-2}, whose reduction modulo n is the admissible submodule $\mathsf{Z}'_{-2,n}$ of $\mathsf{Z}_{-2,n}$. This defines an isomorphism $\varphi'_{-2} : \mathrm{Gr}^{\mathsf{Z}'}_{-2} = \mathsf{Z}'_{-2} \xrightarrow{\sim} \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1))$ lifting the isomorphism $\varphi'_{-2,n} : \mathrm{Gr}^{\mathsf{Z}'}_{-2,n} = \mathsf{Z}'_{-2,n} \xrightarrow{\sim} \mathrm{Hom}(X'/nX', (\mathbb{Z}/n\mathbb{Z})(1))$.

On the other hand, the composition of the admissible surjection $\mathsf{Z}_0 \twoheadrightarrow \mathsf{Z}_0/\mathsf{Z}_{-1} = \mathrm{Gr}^{\mathsf{Z}}_0$ with the isomorphism $\varphi_0^{-1} : \mathrm{Gr}^{\mathsf{Z}}_0 \xrightarrow{\sim} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ defines an admissible surjection $\mathsf{Z}_0 \twoheadrightarrow Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$, and hence the admissible surjection $s_Y : Y \twoheadrightarrow Y'$ defines an admissible surjection $\mathsf{Z}_0 \twoheadrightarrow Y' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$. The kernel of $\mathsf{Z}_0 \twoheadrightarrow Y' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ defines an admissible submodule Z'_{-1} of $\mathsf{Z}_0 = L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$, whose reduction modulo n is the admissible submodule $\mathsf{Z}_{-1,n}$ of $\mathsf{Z}_{0,n} = L/nL$. This defines an isomorphism $\varphi'_0 : \mathrm{Gr}^{\mathsf{Z}'}_0 \xrightarrow{\sim} Y' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ lifting the isomorphism $\varphi'_{0,n} : \mathrm{Gr}^{\mathsf{Z}'}_{0,n} \xrightarrow{\sim} Y'/nY'$.

The filtration $\mathsf{Z}' = \{\mathsf{Z}'_{-i}\}_i$ thus defined is admissible by construction, and it is symplectic because of the definition of ϕ and ϕ'. Indeed, we can define Z'_{-1} as the annihilator of Z'_{-2}. Also, it is fully symplectic because Z'_{-2} is a submodule of Z_{-2}. Hence the filtration $\mathsf{Z}'_n = \{\mathsf{Z}'_{-i,n}\}_i$ is fully symplectic-liftable and admissible, as desired. □

The realization of Y' as an admissible quotient of Y is unique in the construction of Lemma 5.4.1.13. However, once we fix a particular realization of Y' as a quotient of Y, by possibly a different quotient homomorphism, the remaining construction in the proof of Lemma 5.4.1.13 carries over without any necessary modification. Hence, for comparing two objects in general, it will be more convenient if we can allow a twist of the identification of Y' by an isomorphism, as long as the kernel of $s_Y : Y \twoheadrightarrow Y'$ remains the same.

Definition 5.4.1.14. *A **surjection*** $(\mathsf{Z}_n, \Phi_n, \delta_n) \twoheadrightarrow (\mathsf{Z}'_n, \Phi'_n, \delta'_n)$ *between representatives of cusp labels at level* n*, where* $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ *and* $\Phi'_n = (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n})$*, is a pair (of surjections)* $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y')$ *such that*

1. *both* s_X *and* s_Y *are admissible surjections, and they are compatible with* ϕ *and* ϕ' *in the sense that* $s_X\phi = \phi' s_Y$*;*
2. $\mathsf{Z}'_{-2,n}$ *is an admissible submodule of* $\mathsf{Z}_{-2,n}$*, and the natural embedding* $\mathrm{Gr}^{\mathsf{Z}'}_{-2,n} \hookrightarrow \mathrm{Gr}^{\mathsf{Z}}_{-2,n}$ *satisfies* $\varphi_{-2,n} \circ (\mathrm{Gr}^{\mathsf{Z}'}_{-2,n} \hookrightarrow \mathrm{Gr}^{\mathsf{Z}}_{-2,n}) = s_X^* \circ \varphi'_{-2,n}$*;*
3. $\mathsf{Z}_{-1,n}$ *is an admissible submodule of* $\mathsf{Z}'_{-1,n}$*, and the natural surjection* $\mathrm{Gr}^{\mathsf{Z}}_{0,n} \twoheadrightarrow \mathrm{Gr}^{\mathsf{Z}'}_{0,n}$ *satisfies* $s_Y \circ \varphi_{0,n} = \varphi'_{0,n} \circ (\mathrm{Gr}^{\mathsf{Z}}_{0,n} \twoheadrightarrow \mathrm{Gr}^{\mathsf{Z}'}_{0,n})$.

(In other words, Z'_n *and* $(\varphi'_{-2,n}, \varphi'_{0,n})$ *are assigned to* Z_n *and* $(\varphi_{-2,n}, \varphi_{0,n})$ *respectively under* (s_X, s_Y) *as in Lemma 5.4.1.13.) In this case, we write* $(s_X, s_Y) : (\mathsf{Z}_n, \Phi_n, \delta_n) \twoheadrightarrow (\mathsf{Z}'_n, \Phi'_n, \delta'_n)$.

Lemma 5.4.1.15. *Let* (Φ_n, δ_n), (Φ'_n, δ'_n), (Φ''_n, δ''_n)*, and* (Φ'''_n, δ'''_n) *be representatives of cusp labels at level n (see Convention 5.4.1.10), with* $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$, $\Phi'_n = (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n})$*, etc. Suppose* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ *defines an isomorphism* $(\Phi_n, \delta_n) \xrightarrow{\sim} (\Phi'_n, \delta'_n)$, $(s_{X'} : X' \twoheadrightarrow X'', s_{Y'} : Y' \twoheadrightarrow Y'')$ *defines a surjection* $(\Phi'_n, \delta'_n) \twoheadrightarrow (\Phi''_n, \delta''_n)$*, and* $(\gamma_{X''} : X''' \xrightarrow{\sim} X'', \gamma_{Y''} : Y'' \xrightarrow{\sim} Y''')$ *defines an isomorphism* $(\Phi''_n, \delta''_n) \xrightarrow{\sim} (\Phi'''_n, \delta'''_n)$*. Then* $(\gamma_{X''}^{-1} s_{X'} \gamma_X^{-1} : X \twoheadrightarrow X''', \gamma_{Y''} s_{Y'} \gamma_Y : Y \twoheadrightarrow Y''')$ *defines a surjection* $(\Phi_n, \delta_n) \twoheadrightarrow (\Phi'''_n, \delta'''_n)$.

This justifies the following:

Definition 5.4.1.16. *We say that there is a **surjection** from a cusp label at level n represented by some* $(\mathsf{Z}_n, \Phi_n, \delta_n)$ *to another cusp label at level n represented by some* $(\mathsf{Z}'_n, \Phi'_n, \delta'_n)$ *if there is a surjection* (s_X, s_Y) *from* $(\mathsf{Z}_n, \Phi_n, \delta_n)$ *to* $(\mathsf{Z}'_n, \Phi'_n, \delta'_n)$.

5.4.2 General Cusp Labels

In this section, we give analogues of the definitions in Section 5.4.1 whenever it is appropriate in the context.

Suppose we are given a collection of orbits $\mathsf{Z}_{\mathcal{H}} = \{\mathsf{Z}_{H_n}\}_n$ as in Definition 5.3.1.14. For each integer $n \geq 1$ such that $\square \nmid n$ and $\mathcal{U}^\square(n) \subset \mathcal{H}$, we have an H_n-orbit Z_{H_n} of fully symplectic-liftable admissible filtrations on L/nL, which we shall *assume* to be a constant scheme over the base scheme. (This is harmless for our purpose.) We say in this case that $\mathsf{Z}_{\mathcal{H}}$ and each of Z_{H_n} are *split*. Over each point, namely, for each choice of Z_n in this orbit, it determines a subgroup $\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}$. Each element $g_n \in \mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}$ acts on the torus arguments at level n by sending $(X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ to $(X, Y, \phi, \varphi'_{-2,n}, \varphi'_{0,n})$ with $\varphi'_{-2,n} := \nu(g_n)^{-1} \circ \varphi_{-2,n} \circ \mathrm{Gr}_{-2,n}(g_n)$ and $\varphi'_{0,n} := \varphi_{0,n} \circ \mathrm{Gr}_{0,n}(g_n)$. Then it makes sense to talk about the $H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}}$-orbits of torus arguments over Z_n, and hence about H_n-orbits Φ_{H_n} of étale-locally-defined torus arguments over Z_{H_n}.

We would like to focus on the situation when Φ_{H_n} is also *split* in the sense that it is formed by orbits that are already defined over the same base (without having to make étale localizations):

Definition 5.4.2.1. *Given a collection of split orbits* $\mathsf{Z}_{\mathcal{H}} = \{\mathsf{Z}_{H_n}\}_n$ *as in Definition 5.3.1.14, a **torus argument*** $\Phi_{\mathcal{H}}$ ***at level*** $\mathcal{H}$ *for* $\mathsf{Z}_{\mathcal{H}}$ *is a tuple* $\Phi_{\mathcal{H}} := (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ *which is a collection* $\Phi_{\mathcal{H}} = \{\Phi_{H_n}\}_n$ *of* H_n*-orbits of torus arguments at levels* n *with elements* Φ_{H_n} *described as follows:*

1. *For each index n, there is an element Z_n in Z_{H_n} and a torus argument $(X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ at level n for Z_n, such that Φ_{H_n} is the H_n-orbit of $(X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$.*
2. *For all indices n and m such that $n|m$, the H_n-orbit Φ_{H_n} is determined by the H_m-orbit Φ_{H_m} by reduction modulo n.*

By abuse of notation, we shall write $\Phi_{H_n} = (X, Y, \phi, \varphi_{-2,H_n}, \varphi_{0,H_n})$ and $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$.

Definition 5.4.2.2. *Two torus arguments $\Phi_{\mathcal{H}} = \{\Phi_{H_n}\}_n$ and $\Phi'_{\mathcal{H}} = \{\Phi'_{H_n}\}_n$ at level $\mathcal{H}$ are **equivalent** if and only if there exists a pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ and some index n such that $\Phi_{\mathcal{H}}$ contains some torus argument at level n that is equivalent under (γ_X, γ_Y) (as in Definition 5.4.1.5) to some torus argument at level n in $\Phi'_{\mathcal{H}}$. In this case, we say that $\Phi_{\mathcal{H}}$ and $\Phi'_{\mathcal{H}}$ are equivalent under the isomorphism (γ_X, γ_Y), which we denote by $(\gamma_X, \gamma_Y) : \Phi_{\mathcal{H}} \xrightarrow{\sim} \Phi'_{\mathcal{H}}$.*

The following is a trivial reformulation of Lemma 5.4.1.8:

Lemma 5.4.2.3. *If two torus arguments $\Phi_{\mathcal{H}}$ and $\Phi'_{\mathcal{H}}$ at level n are equivalent under some (γ_X, γ_Y), then necessarily $(\gamma_X, \gamma_Y) \in \Gamma_\phi$.*

On the other hand, over each point of Z_{H_n}, namely, for each choice of Z_n in this orbit, we can make sense of a liftable splitting δ_n over it, and hence we can talk about H_n-orbits of $(\mathsf{Z}_{H_n}, \delta_{H_n})$ with the understanding that δ_{H_n} is defined pointwise over Z_{H_n}. We say in this case that δ_{H_n} is a splitting of Z_{H_n}. Hence it also makes sense to talk about a splitting $\delta_{\mathcal{H}}$ of $\mathsf{Z}_{\mathcal{H}}$.

Definition 5.4.2.4. *A **cusp label at level** $\mathcal{H}$ for a PEL-type $\mathcal{O}$-lattice $(L, \langle \cdot, \cdot \rangle, h)$, or a cusp label of the moduli problem $\mathsf{M}_{\mathcal{H}}$, is an equivalence class of triples $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$, where*

1. *$\mathsf{Z}_{\mathcal{H}} = \{\mathsf{Z}_{H_n}\}_n$ is a collection of orbits of admissible filtration on L/nL that are fully symplectic-liftable in the sense of Definition 5.2.7.3;*
2. *$\Phi_{\mathcal{H}}$ is a torus argument at level $\mathcal{H}$ for $\mathsf{Z}_{\mathcal{H}}$;*
3. *$\delta_{\mathcal{H}}$ is a liftable splitting of $\mathsf{Z}_{\mathcal{H}}$.*

Two triples $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent if $\mathsf{Z}_{\mathcal{H}}$ and $\mathsf{Z}'_{\mathcal{H}}$ are identical, and if $\Phi_{\mathcal{H}}$ and $\Phi'_{\mathcal{H}}$ are equivalent as in Definition 5.4.2.2. If $\Phi_{\mathcal{H}}$ and $\Phi'_{\mathcal{H}}$ are equivalent under some (γ_X, γ_Y) as in Definition 5.4.2.2, then we say that $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent under (γ_X, γ_Y).

Convention 5.4.2.5. *As in Convention 5.4.1.10, to simplify the notation, we shall often suppress $\mathsf{Z}_{\mathcal{H}}$ from the notation $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$, with the understanding that the data $\Phi_{\mathcal{H}}$ and $\delta_{\mathcal{H}}$ require an implicit choice of $\mathsf{Z}_{\mathcal{H}}$.*

We shall nevertheless define the notation $(L^{\mathsf{Z}_{\mathcal{H}}}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_{\mathcal{H}}}, h^{\mathsf{Z}_{\mathcal{H}}})$ and $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ even when we suppress $\mathsf{Z}_{\mathcal{H}}$ from the notation in the representatives of cusp labels (see Convention 5.4.1.10 and Lemma 5.2.7.5):

Definition 5.4.2.6. *The PEL-type $\mathcal{O}$-lattice $(L^{\mathsf{Z}_{\mathcal{H}}}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_{\mathcal{H}}}, h^{\mathsf{Z}_{\mathcal{H}}})$ is a fixed (noncanonical) choice of any of the PEL-type $\mathcal{O}$-lattice $(L^{\mathsf{Z}_n}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ in Lemma 5.2.7.5 for any element Z_n in any Z_{H_n} (in $\mathsf{Z}_{\mathcal{H}} = \{\mathsf{Z}_{H_n}\}_n$). It determines some group functor $\mathrm{G}_{(L^{\mathsf{Z}_n}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})}$, such that there is an identification $\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n} \cong \mathrm{G}^{\mathrm{ess}}_{(L^{\Phi_n}, \langle \cdot, \cdot \rangle^{\Phi_n}, h^{\mathsf{Z}_n})}(\mathbb{Z}/n\mathbb{Z})$. Let $\mathcal{H}_h$ be the preimage of $H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$ under the surjection $\mathrm{G}_{(L^{\mathsf{Z}_n}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})}(\hat{\mathbb{Z}}^\square) \twoheadrightarrow \mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}$. Then we define $\mathsf{M}_{\mathcal{H}_h}$ to be the moduli problem defined by $(L^{\mathsf{Z}_n}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ with level-$\mathcal{H}_h$ structures as in Lemma 5.2.7.5. (The isomorphism class of $\mathsf{M}_{\mathcal{H}_h}$ is well defined and independent of the choice of $(L^{\mathsf{Z}_{\mathcal{H}}}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_{\mathcal{H}}}, h^{\mathsf{Z}_{\mathcal{H}}}) = (L^{\mathsf{Z}_n}, \langle \cdot, \cdot \rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$.) We define $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$ to be the quotient of $\coprod \mathsf{M}_n^{\mathsf{Z}_n}$ by H_n, where the disjoint union is over representatives $(\mathsf{Z}_n, \Phi_n, \delta_n)$ (with the same (X, Y, ϕ)) in $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$, which is finite étale over $\mathsf{M}_{\mathcal{H}_h}$ by construction. (The isomorphism class of $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$ is independent of the choice of n and the representatives $(\mathsf{Z}_n, \Phi_n, \delta_n)$ we use. We have $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H}_h}$ when, for some (and hence every) choice of a representative $(\mathsf{Z}_n, \Phi_n, \delta_n)$ in $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$, the canonical image of $H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}}$ in $\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n} \times \mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n}$ is the direct product $H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}} \times H_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n}}$.) We then (abusively) define $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ to be the quotient of $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$ by the subgroup of Γ_ϕ stabilizing $\Phi_{\mathcal{H}}$ (whose action factors through a finite quotient group), which depends only on the cusp label $[(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$, but not on the choice of the representative $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$. By construction, we have finite étale morphisms $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}_h}$ (which can be identified with $\mathsf{M}_{\mathcal{H}'_h} \to \mathsf{M}_{\mathcal{H}''_h} \to \mathsf{M}_{\mathcal{H}_h}$ for some canonically determined open compact subgroups $\mathcal{H}'_h \subset \mathcal{H}''_h \subset \mathcal{H}_h$).*

Definition 5.4.2.7. *The **$\mathcal{O}$-multirank** of a cusp label at level $\mathcal{H}$ represented by some $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ is the $\mathcal{O}$-multirank (see Definition 5.4.1.12) of any Z_n in any Z_{H_n} in $\mathsf{Z}_{\mathcal{H}}$.*

Definition 5.4.2.8. *With the setting as in Section 5.2.1, the category $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R, I)$ has objects of the form*

$$(\mathsf{Z}_{\mathcal{H}}, (\underline{X}, \underline{Y}, \phi, \varphi^{\sim}_{-2,\mathcal{H}}, \varphi^{\sim}_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c^{\vee}_{\mathcal{H}}, \tau_{\mathcal{H}}))$$

such that

1. *$\underline{X}$ (resp. $\underline{Y}$) is constant with value X (resp. Y);*
2. *$(\varphi^{\sim}_{-2,\mathcal{H}}, \varphi^{\sim}_{0,\mathcal{H}})$ is a subscheme of $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}) \underset{\mathsf{Z}_{\mathcal{H}}}{\times} \varphi_{-1,\mathcal{H}}$, where $(X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ is a torus argument at level $\mathcal{H}$ as in Definition 5.4.2.1 above, and where $(\varphi^{\sim}_{-2,\mathcal{H}}, \varphi^{\sim}_{0,\mathcal{H}})$ is an étale-locally-defined H_n-orbit which surjects under the two projections to the orbits defining $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ and $\varphi_{-1,\mathcal{H}}$; in this case we say that $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ is induced by $(\varphi^{\sim}_{-2,\mathcal{H}}, \varphi^{\sim}_{0,\mathcal{H}})$ (then, by the universal property of $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$ because of its very construction, the torus part $(\mathsf{Z}_{\mathcal{H}}, \Phi^{\sim}_{\mathcal{H}} = (X, Y, \phi, \varphi^{\sim}_{-2,\mathcal{H}}, \varphi^{\sim}_{0,\mathcal{H}}), \delta_{\mathcal{H}})$ and abelian part $(A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}})$ canonically define a morphism $S = \mathrm{Spec}(R) \to \mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$);*
3. *$c_{\mathcal{H}}|_X : X \to A^{\vee}_\eta$ (resp. $c^{\vee}_{\mathcal{H}}|_Y : Y \to A_\eta$; these restrictions are defined by taking the common induced element in the étale-locally-defined orbit and performing descent) extends to an $\mathcal{O}$-equivariant homomorphism $c : X \to A^{\vee}$ (resp. $c^{\vee} : Y \to A$) over S (which is unique by Corollary 5.2.3.11);*
4. *if we set $\tau := \tau_{\mathcal{H}}|_{\mathbf{1}_{Y \times X, \eta}}$ (defined by descent as in the case of $c_{\mathcal{H}}|_X$ and $c^{\vee}_{\mathcal{H}}|_Y$) and $\alpha^\natural_{\mathcal{H}} := (\mathsf{Z}_{\mathcal{H}}, \varphi_{-2,\mathcal{H}}, \varphi_{-1,\mathcal{H}}, \varphi_{0,\mathcal{H}}, \delta_{\mathcal{H}}, c_{\mathcal{H}}, c^{\vee}_{\mathcal{H}}, \tau_{\mathcal{H}})$, then $\alpha^\natural_{\mathcal{H}}$ is a level-$\mathcal{H}$ structure datum as in Definition 5.3.1.14, and $(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^{\vee}, \tau, [\alpha^\natural_{\mathcal{H}}])$ is an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R, I)$ as in Definition 5.3.1.18.*

Remark 5.4.2.9. There is a natural functor $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R,I) \to \mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R,I)$ defined by assigning the class $[\alpha^{\natural}_{\mathcal{H}}]$ to the representative $\alpha^{\natural}_{\mathcal{H}}$.

Lemma 5.4.2.10. *Suppose we are given a representative $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of a cusp label at level $\mathcal{H}$, with $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$. Then for each object*

$$(A, \lambda_A, i_A, \underline{X}, \underline{Y}, \phi, c, c^\vee, \tau, [\alpha^{\natural}_{\mathcal{H}}])$$

in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R,I)$ with split $\mathsf{Z}_{\mathcal{H}}$ and constant $\underline{X}$ and $\underline{Y}$ in any representative of $[\alpha^{\natural}_{\mathcal{H}}]$, there is (up to isomorphisms inducing automorphisms of $(X, Y, \lambda, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$) a unique object

$$(\mathsf{Z}_{\mathcal{H}}, (X, Y, \phi, \varphi^{\sim}_{-2,\mathcal{H}}, \varphi^{\sim}_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c^\vee_{\mathcal{H}}, \tau_{\mathcal{H}}))$$

in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R,I)$ (where $(\varphi^{\sim}_{-2,\mathcal{H}}, \varphi^{\sim}_{0,\mathcal{H}})$ induces $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ as in Definition 5.4.2.8) such that $\alpha^{\natural}_{\mathcal{H}} = (\mathsf{Z}_{\mathcal{H}}, \varphi^{\sim}_{-2,\mathcal{H}}, \varphi_{-1,\mathcal{H}}, \varphi^{\sim}_{0,\mathcal{H}}, \delta_{\mathcal{H}}, c_{\mathcal{H}}, c^\vee_{\mathcal{H}}, \tau_{\mathcal{H}})$ is a representative of $[\alpha^{\natural}_{\mathcal{H}}]$.

Lemma 5.4.2.11. *Suppose we have a cusp label at level $\mathcal{H}$ represented by some $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$, with $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$. For each admissible surjection $s_X : X \twoheadrightarrow X'$ of $\mathcal{O}$-lattices (see Definition 1.2.6.7), there is canonically determined cusp label at level $\mathcal{H}$ which can be represented by some $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ with $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ described as follows:*

1. *X' is the same X' we have in the surjection s_X.*
2. *An admissible surjection $s_Y : Y \twoheadrightarrow Y'$ of $\mathcal{O}$-lattices for some Y' determined by setting $\ker(s_Y) = \phi^{-1}(\ker(s_X))$.*
3. *The definitions of s_Y and Y' induce an embedding $\phi' : Y' \to X'$ such that $s_X \phi = \phi' s_Y$.*
4. *Let us write $\mathsf{Z}_{\mathcal{H}} = \{\mathsf{Z}_{H_n}\}_n$, with indices given by integers $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$. For each index n, the recipe in Lemma 5.4.1.13 determines an assignment of Z'_n and $(\varphi'_{-2,n}, \varphi'_{0,n})$ to Z_n and $(\varphi_{-2,n}, \varphi_{0,n})$ respectively under (s_X, s_Y), which is compatible with the process of taking orbits. Hence we have an induced assignment of Z'_{H_n} and $(\varphi'_{-2,H_n}, \varphi'_{0,H_n})$ to Z_{H_n} and $(\varphi_{-2,H_n}, \varphi_{0,H_n})$ respectively, and hence an induced assignment of $\mathsf{Z}'_{\mathcal{H}}$ and $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ to $\mathsf{Z}_{\mathcal{H}}$ and $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ respectively under (s_X, s_Y).*
5. *$\delta'_{\mathcal{H}}$ is just any liftable splitting of $\mathsf{Z}'_{\mathcal{H}}$*

This construction determines a unique pair $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}})$, and hence a unique cusp label at level $\mathcal{H}$.

Definition 5.4.2.12. *A **surjection** $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ between representatives of cusp labels at level $\mathcal{H}$, where $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ and $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$, is a pair (of surjections) $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y')$ such that*

1. *both s_X and s_Y are admissible surjections, and they are compatible with ϕ and ϕ' in the sense that $s_X \phi = \phi' s_Y$;*
2. *$\mathsf{Z}'_{\mathcal{H}}$ and $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ are assigned to $\mathsf{Z}_{\mathcal{H}}$ and $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ respectively under (s_X, s_Y) as in Lemma 5.4.2.11. (We do not need to know if s_Y is the canonically determined surjection as in Lemma 5.4.2.11 for the construction there.)*

In this case, we write $(s_X, s_Y) : (\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$.

Alternatively, we can view such a surjection as a collection of orbits of surjections described in Definition 5.4.1.14, in its natural sense.

Then a trivial analogue of Lemma 5.4.1.15 justifies the following:

Definition 5.4.2.13. *We say that there is a **surjection** from a cusp label at level $\mathcal{H}$ represented by some $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ to a cusp label at level $\mathcal{H}$ represented by some $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ if there is a surjection (s_X, s_Y) from $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ to $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$.*

5.4.3 Hecke Actions on Cusp Labels

With the setting as at the beginning of Section 5.4.1, suppose we have an element $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$, and suppose we have two open compact subgroups $\mathcal{H}'$ and $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ such that $g^{-1}\mathcal{H}'g \subset \mathcal{H}$.

Construction 5.4.3.1. Suppose we have a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}'})$ of type $\mathsf{M}_{\mathcal{H}'}$ over S. By definition, this means its restriction $(G_\eta, \lambda_\eta, i_\eta, \alpha_{\mathcal{H}'})$ to the generic point η of S defines an object parameterized by $\mathsf{M}_{\mathcal{H}'}$. Let $\bar{\eta}$ be a geometric point over η. By Proposition 1.4.3.4, the object defined by $(G_\eta, \lambda_\eta, i_\eta, \alpha_{\mathcal{H}'})$ corresponds to an object defined by $(G_\eta, \lambda_\eta, i_\eta, [\hat{\alpha}]_{\mathcal{H}'})$. We may interpret $[\hat{\alpha}]_{\mathcal{H}'}$ as based at $\bar{\eta}$ (see Definitions 1.3.8.2 and 1.3.8.7, and Convention 1.3.8.8). For each representative $\hat{\alpha}$ of the $\mathcal{H}'$-orbit $[\hat{\alpha}]_{\mathcal{H}'}$, we may consider the composition $\hat{\alpha} \circ g$. If we take a different representative $\hat{\alpha}'$, which is by definition $\hat{\alpha} \circ u$ for some $u \in \mathcal{H}'$, then $\hat{\alpha}' \circ g = \hat{\alpha} \circ g \circ (g^{-1}ug)$. Since $g^{-1}\mathcal{H}'g \subset \mathcal{H}$, we obtain a well-defined $\mathcal{H}$-orbit $[\hat{\alpha} \circ g]_{\mathcal{H}}$. This defines an object $(G_\eta, \lambda_\eta, i_\eta, [\hat{\alpha} \circ g]_{\mathcal{H}})$ in $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$. (This is consistent with the *action* of g on the tower $\mathsf{M}^{\Box}(\eta) = \varprojlim_{\mathcal{H}, \mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box})} \mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}(\eta)$ mentioned in Remark 1.4.3.11.)

Construction 5.4.3.2. For the purpose of studying degenerations, it is desirable that we translate the object $(G_\eta, \lambda_\eta, i_\eta, [\hat{\alpha} \circ g]_{\mathcal{H}})$ of $\mathsf{M}^{\mathrm{rat}}_{\mathcal{H}}$ (up to $\mathbb{Z}^\times_{(\Box)}$-isogeny) back to some object $(G'_\eta, \lambda'_\eta, i'_\eta, \alpha'_{\mathcal{H}})$ of $\mathsf{M}_{\mathcal{H}}$ (up to isomorphism). As in the proof of Proposition 1.4.3.4, this is achieved by a $\mathbb{Z}^\times_{(\Box)}$-isogeny $f_\eta : G_\eta \to G'_\eta$ defined as follows:

Take any representative $\hat{\alpha} : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} G_{\bar{\eta}}$ of $[\hat{\alpha}]_{\mathcal{H}}$, which by construction sends $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to $\mathrm{T}^{\Box} G_{\bar{\eta}}$. Then $\hat{\alpha} \circ g : L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^{\Box} G_{\bar{\eta}}$ sends $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ to the $\mathcal{O}$-invariant open compact subgroup $\hat{\alpha}(g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}))$ of $\mathrm{V}^{\Box} G_{\bar{\eta}}$. Since the $\mathcal{H}'$-orbit of $\hat{\alpha}$ is $\pi_1(\eta, \bar{\eta})$-invariant, and since $g^{-1}\mathcal{H}'g \subset \mathcal{H}$, the $\mathcal{H}$-orbit of $\hat{\alpha} \circ g$ is also $\pi_1(\eta, \bar{\eta})$-invariant. Since $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, this shows that $\hat{\alpha}(g(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}))$ is $\pi_1(\eta, \bar{\eta})$-invariant, which by Corollary 1.3.5.4 corresponds to some $\mathbb{Z}^\times_{(\Box)}$-isogeny $f_\eta : G_\eta \to G'_\eta$ to an abelian scheme G'_η.

The inclusion $\hat{\alpha}(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \hookrightarrow \hat{\alpha}(L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ in $\mathrm{V}^{\Box} G_{\bar{\eta}}$ corresponds by Corollary 1.3.5.4 to the class of the polarization $\lambda_\eta : G_\eta \to G^\vee_\eta$. Among $\mathcal{O}$-invariant open compact subgroups in $\mathrm{V}^{\Box} G_{\bar{\eta}}$ isomorphic to $\hat{\alpha}(L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$, it is the condition that $\hat{\alpha}(L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ is dual to $\hat{\alpha}(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ under the λ-Weil pairing e^{λ}, or rather the condition that $L^{\#} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ is dual to $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ under the pairing $\langle \cdot, \cdot \rangle$, that characterizes this class of $\mathbb{Z}^\times_{(\Box)}$-isogeny. (The class of a $\mathbb{Z}^\times_{(\Box)}$-isogeny from G_η to an abelian scheme is only defined up to isomorphism on the target.)

Since $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$ satisfies $\langle gx, gy\rangle = \nu(g)\langle x,y\rangle$ for all $x,y \in L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}$, we see that the perfect duality $\langle\cdot,\cdot\rangle : (L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\times(L^\#\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box) \to \hat{\mathbb{Z}}^\Box(1)$ is carried to the perfect duality $\langle\cdot,\cdot\rangle : g(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\times g(L^\#\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box) \to \nu(g)\hat{\mathbb{Z}}^\Box(1)$. In particular, the dual of $g(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$ under $\langle\cdot,\cdot\rangle$ is $\nu(g)^{-1}g(L^\#\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$. Under $\hat{\alpha}$, this translates to the statement that the dual of the $\mathbb{Z}^\times_{(\Box)}$-isogeny f_η is the $\mathbb{Z}^\times_{(\Box)}$-isogeny $f^\vee_\eta : (G'_\eta)^\vee \to G^\vee_\eta$ with source and target defined respectively by the open compact subgroups $\nu(g)^{-1}g(L^\#\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$ and $L^\#\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$.

The composition $(f^\vee_\eta)^{-1}\circ\lambda_\eta\circ f^{-1}_\eta : G'_\eta \to (G'_\eta)^\vee$ has source and target defined by $g(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$ and $\nu(g)^{-1}g(L^\#\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$, respectively, and we know by Corollary 1.3.2.21 that this $\mathbb{Z}^\times_{(\Box)}$-isogeny is a $\mathbb{Z}^\times_{(\Box)}$-polarization. Using the approximation $\mathbb{A}^{\infty,\Box,\times} = \mathbb{Z}^\times_{(\Box),>0}\cdot\hat{\mathbb{Z}}^{\Box,\times}$, there is a unique element $r \in \mathbb{Z}^\times_{(\Box),>0}$ such that $\nu(g) = ru$ for some $u \in \hat{\mathbb{Z}}^{\Box,\times}$. Set $\lambda'_\eta := r^{-1}(f^\vee_\eta)^{-1}\circ\lambda_\eta\circ f^{-1}_\eta$. Then the source and target of the class of this $\mathbb{Z}^\times_{(\Box)}$-isogeny correspond to the open compact subgroups $g(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$ and $g(L^\#\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box) = r\nu(g)^{-1}g(L^\#\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$, respectively. This defines a polarization $\lambda' : G'_\eta \to (G'_\eta)^\vee$, which satisfies $f^\vee_\eta\circ\lambda'_\eta\circ f_\eta = r\lambda_\eta$.

Since $g(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)$ is invariant under $\mathcal{O}$, we see that $i_\eta : \mathcal{O} \to \mathrm{End}_\eta(G_\eta)$ induces an $\mathcal{O}$-endomorphism structure $i'_\eta : \mathcal{O} \to \mathrm{End}_\eta(G'_\eta)$ (with image in $\mathrm{End}_\eta(G'_\eta)$ instead of $\mathrm{End}_\eta(G'_\eta)\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$).

Finally, the symplectic isomorphism $\hat{\alpha}' := \mathrm{V}^\Box(f_\eta)\circ\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^\Box\, G'_{\bar{\eta}}$ sends $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$ to $\mathrm{T}^\Box\, G'_{\bar{\eta}}$. Since $g^{-1}\mathcal{H}'g \subset \mathcal{H}$, the $\mathcal{H}$-orbit of $\hat{\alpha}'$ is independent of the choice of $\hat{\alpha}$. Hence it is necessarily defined over η, and defines a rational level-$\mathcal{H}$ structure $[\hat{\alpha}']_\mathcal{H}$.

Thus we have constructed an explicit $\mathbb{Z}^\times_{(\Box)}$-isogeny $f_\eta : G_\eta \to G'_\eta$ that defines the equivalence $(G_\eta,\lambda_\eta,i_\eta,[\hat{\alpha}\circ g]_\mathcal{H}) \sim_{\mathbb{Z}^\times_{(\Box)}\text{-isog.}} (G'_\eta,\lambda'_\eta,i'_\eta,[\hat{\alpha}']_\mathcal{H})$. By Lemma 1.3.8.5, $(G'_\eta,\lambda'_\eta,i'_\eta,[\hat{\alpha}']_\mathcal{H})$ comes from a unique $(G'_\eta,\lambda'_\eta,i'_\eta,\alpha'_\mathcal{H})$ under Construction 1.3.8.4. (This finishes Construction 5.4.3.2.)

For the rest of this subsection, let us assume for simplicity that the target G'_η of the $\mathbb{Z}^\times_{(\Box)}$-isogeny $G_\eta \to G'_\eta$ extends (uniquely) to a semi-abelian scheme G' over S. This is true, for example, if we assume that R is a complete discrete valuation ring with maximal ideal I and with algebraically closed residue field k, in which case the semistable reduction theorem (see Theorem 3.3.2.4) applies. (Such an assumption is harmless for our purpose of determining Hecke actions on cusp labels.)

Construction 5.4.3.3. Let us take any integer $N \geq 1$ prime-to-$\Box$ such that Nf_η is a prime-to-$\Box$ isogeny. Then, by Proposition 3.3.1.5, the isogeny $Nf_\eta : G_\eta \twoheadrightarrow G'_\eta$ extends uniquely to an isogeny $Nf : G \twoheadrightarrow G'$ between semi-abelian schemes. Let $K := \ker(Nf)$, which is (quasi-finite and) flat by Lemma 1.3.1.11. (By Lemma 3.4.3.1, we could have defined G' as the quotient of G by the schematic closure of $K_\eta := \ker(Nf_\eta)$ if we know for a different reason that the schematic closure of K_η in G is quasi-finite and flat.) By symbolically taking f to be $N^{-1}\circ(Nf)$, we obtain a "$\mathbb{Z}^\times_{(\Box)}$-isogeny" $f : G \to G'$. This "$\mathbb{Z}^\times_{(\Box)}$-isogeny" is independent of the N we have chosen in the sense that, if we have chosen any other N' with $N|N'$, then necessarily $(N'/N)\circ(Nf) = N'f$.

Let us investigate the effect of the action of g on the degeneration data. Suppose that $(G,\lambda,i,\alpha_{\mathcal{H}'})$ (resp. $(G',\lambda',i',\alpha'_\mathcal{H})$) is an object of $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}'}}(R,I)$ (resp. $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_\mathcal{H}}(R,I)$) over $S = \mathrm{Spec}(R)$, with associated degeneration data $(A,\lambda_A,i_A,\underline{X},\underline{Y},\phi,c,c^\vee,\tau,[\alpha^\natural_{\mathcal{H}'}])$ in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}'}}(R,I)$ (resp. $(A',\lambda'_A,i'_A,\underline{X}',\underline{Y}',\phi',c',(c^\vee)',\tau',[(\alpha'_\mathcal{H})^\natural])$ in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_\mathcal{H}}(R,I)$). For our purpose, we would like to assume that $\underline{X}$, $\underline{Y}$, $\underline{X}'$, and $\underline{Y}'$ are constant with respective values X, Y, X', and Y'. Suppose that

$$\alpha^\natural_{\mathcal{H}'} = (\mathsf{Z}_{\mathcal{H}'},\varphi_{-2,\mathcal{H}'},\varphi_{-1,\mathcal{H}'},\varphi_{0,\mathcal{H}'},\delta_{\mathcal{H}'},c_{\mathcal{H}'},c^\vee_{\mathcal{H}'},\tau_{\mathcal{H}'})$$

is a representative of $[\alpha^\natural_{\mathcal{H}'}]$, and that

$$(\alpha'_\mathcal{H})^\natural = (\mathsf{Z}'_\mathcal{H},\varphi'_{-2,\mathcal{H}},\varphi'_{-1,\mathcal{H}},\varphi'_{0,\mathcal{H}},\delta'_\mathcal{H},c'_\mathcal{H},(c^\vee_\mathcal{H})',\tau'_\mathcal{H})$$

is a representative of $[(\alpha'_\mathcal{H})^\natural]$. Suppose moreover that $\mathsf{Z}_{\mathcal{H}'}$ and $\Phi_{\mathcal{H}'} = (X,Y,\phi,\varphi_{-2,\mathcal{H}'},\varphi_{0,\mathcal{H}'})$ (resp. $\mathsf{Z}'_\mathcal{H}$ and $\Phi'_\mathcal{H} = (X',Y',\phi',\varphi'_{-2,\mathcal{H}},\varphi'_{0,\mathcal{H}})$) are *split* in the sense that the latter defines a torus argument at level $\mathcal{H}'$ (resp. $\mathcal{H}$) (see Definition 5.4.2.1). Then we obtain two cusp labels represented by triples $(\mathsf{Z}_{\mathcal{H}'},\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})$ and $(\mathsf{Z}'_\mathcal{H},\Phi'_\mathcal{H},\delta'_\mathcal{H})$, respectively. Our goal is to find the relation between the two cusp labels.

Let $n,m \geq 1$ be integers such that $\mathcal{U}^\Box(m) \subset \mathcal{H}'$, $\mathcal{U}^\Box(n) \subset \mathcal{H}$, and $g^{-1}\mathcal{U}^\Box(m)g \subset \mathcal{U}^\Box(n)$. Let $H_n := \mathcal{H}/\mathcal{U}^\Box(n)$ and let $H'_m := \mathcal{H}'/\mathcal{U}^\Box(m)$. Let

$$\alpha^\natural_{H'_m} = (\mathsf{Z}_{H'_m},\varphi_{-2,H'_m},\varphi_{-1,H'_m},\varphi_{0,H'_m},\delta_{H'_m},c_{H'_m},c^\vee_{H'_m},\tau_{H'_m})$$

be the H'_m-orbit in $\alpha^\natural_{\mathcal{H}'}$, and let

$$(\alpha'_{H_n})^\natural = (\mathsf{Z}'_{H_n},\varphi'_{-2,H_n},\varphi'_{-1,H_n},\varphi'_{0,H_n},\delta'_{H_n},c'_{H_n},(c^\vee_{H_n})',\tau'_{H_n})$$

be the H_n-orbit in $\alpha^\natural_\mathcal{H}$.

First let us determine the relation between $\mathsf{Z}_{\mathcal{H}'}$ and $\mathsf{Z}'_\mathcal{H}$, or equivalently the relation between $\mathsf{Z}_{H'_m}$ and Z_{H_n}. Since this question is related only to the group schemes $G[m]$ and $G'[n]$ that are independent of the choice of N in Nf, we may take any N and consider the isogeny $Nf : G \twoheadrightarrow G'$ with kernel K as in Construction 5.4.3.3 above. Let us extend the filtration W on $\mathrm{T}^\Box\, G_{\bar{\eta}}$ to a filtration $\mathsf{W}_{\mathbb{A}^{\infty,\Box}}$ on $\mathrm{V}^\Box\, G_{\bar{\eta}}$, given explicitly by

$$0 \subset \mathsf{W}_{-2,\mathbb{A}^{\infty,\Box}} = \mathrm{V}^\Box\, T_{\bar{\eta}} \subset \mathsf{W}_{-1,\mathbb{A}^{\infty,\Box}} = \mathrm{V}^\Box\, G^\natural_{\bar{\eta}} \subset \mathsf{W}_{0,\mathbb{A}^{\infty,\Box}} = \mathrm{V}^\Box\, G_{\bar{\eta}}.$$

Similarly, we have a filtration W' on $\mathrm{T}^\Box\, G'_{\bar{\eta}}$, which extends to a filtration $\mathsf{W}'_{\mathbb{A}^{\infty,\Box}}$ on $\mathrm{V}^\Box\, G'_{\bar{\eta}}$. As in Section 3.4.1, the finite group scheme K_η has a natural filtration

$$0 \subset K^\mu_\eta \subset K^\mathrm{f}_\eta \subset K_\eta.$$

Since $Nf_\eta : G_\eta \to G'_\eta$ is the quotient of G_η by K_η, and since the filtrations are compatible with this quotient, the isomorphism $\mathrm{V}^\Box(f_\eta) : \mathrm{V}^\Box\, G_{\bar{\eta}} \xrightarrow{\sim} \mathrm{V}^\Box\, G'_{\bar{\eta}}$ identifies the two filtrations W and W'.

Let us take any symplectic lifting $\hat{\alpha}$ of the level-$\mathcal{H}'$ structure $\alpha_{\mathcal{H}'}$, namely, any representative of the $\mathcal{H}'$-orbit $[\hat{\alpha}]_{\mathcal{H}'}$ associated with $\alpha_{\mathcal{H}'}$ which sends $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$ to $\mathrm{T}^\Box\, G_{\bar{\eta}}$, and consider the induced isomorphism $\hat{\alpha} : L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box} \xrightarrow{\sim} \mathrm{V}^\Box\, G_{\bar{\eta}}$. Then the pullback of $\mathtt{W}$ by $\hat{\alpha}$ is a filtration $\mathtt{Z}$ on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$ whose reduction modulo m is a filtration $\mathtt{Z}_m$ in the H'_m-orbit $\mathtt{Z}_{H'_m}$, and the pullback of $\mathtt{W}_{\mathbb{A}^{\infty,\Box}}$ by $\hat{\alpha}$ is the natural filtration $\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}$ extending $\mathtt{Z}$. In other words, we can interpret $\mathtt{Z}_{\mathcal{H}'}$ as the $\mathcal{H}'$-orbit of $\mathtt{Z}$. By construction of f_η, we have $\hat{\alpha}(g(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)) = \mathrm{V}^\Box(f_\eta)^{-1}(\mathrm{T}^\Box G'_{\bar{\eta}})$, or equivalently $\hat{\alpha}' = \mathrm{V}^\Box(f_\eta)\circ\hat{\alpha}\circ g$. Therefore, the pullback of the filtration $\mathtt{W}'_{\mathbb{A}^{\infty,\Box}} = \mathrm{V}^\Box(f_\eta)(\mathtt{W}_{\mathbb{A}^{\infty,\Box}})$ by $\hat{\alpha}'$ is $\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}} := (\hat{\alpha}')^{-1}(\mathtt{W}'_{\mathbb{A}^{\infty,\Box}}) = g^{-1}(\hat{\alpha}^{-1}(\mathtt{W}_{\mathbb{A}^{\infty,\Box}})) = g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$. Equivalently, this means $\mathtt{Z}' = g^{-1}(g(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\cap\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}) = (L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\cap g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$. By taking reduction modulo n, we obtain a filtration $\mathtt{Z}'_n$ on L/nL in the H_n-orbit $\mathtt{Z}_{H_n}$. If we replace the lifting $\hat{\alpha}$ of $\alpha_{\mathcal{H}'}$ with $\hat{\alpha}\circ u$ for some $u\in\mathcal{H}'$, then the filtration $\mathtt{Z}_{\mathbb{A}^{\infty,\Box}} = \hat{\alpha}^{-1}(\mathtt{W}_{\mathbb{A}^{\infty,\Box}})$ is replaced with $u^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$, and hence the filtration $\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}} = g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$ is replaced with $g^{-1}(u^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})) = (g^{-1}u^{-1}g)g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}) = (g^{-1}u^{-1}g)(\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}})$. Accordingly, the filtration $\mathtt{Z}' = (L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\cap g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$ of $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$ is replaced with $(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\cap(g^{-1}u^{-1}g)(g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}))$. Since $g^{-1}\mathcal{H}'g\subset\mathcal{H}$ and $g^{-1}\mathcal{U}^\Box(m)g\subset\mathcal{U}^\Box(n)$ by assumption, we see that its $\mathcal{H}$-orbit remains the same, and that the reduction modulo n of $\mathtt{Z}'$ remains in the same H_n-orbit. Hence the assignment of $\mathtt{Z}'_{H_n}$ to $\mathtt{Z}_{H'_m}$ as described above is well defined and does not depend on the choice of $\hat{\alpha}$. (Reformulating what we have said, if we interpret alternatively $\mathtt{Z}_{\mathcal{H}'}$ (resp. $\mathtt{Z}_{\mathcal{H}}$) as an $\mathcal{H}'$-orbit (resp. $\mathcal{H}$-orbit) of some fully symplectic filtration $\mathtt{Z}$ (resp. $\mathtt{Z}'$) on $L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$, and if we set $\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}$ (resp. $\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}{}'$) to be the fully symplectic filtrations on $L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}$ induced by $\mathtt{Z}$ (resp. $\mathtt{Z}'$), then the $\mathcal{H}'$-orbit of $g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$ is contained in the $\mathcal{H}$-orbit of $\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}}$.)

Next let us investigate the relation between $\Phi_{\mathcal{H}'}$ and $\Phi'_{\mathcal{H}}$, or equivalently the relation between $\Phi_{H'_m}$ and Φ'_{H_n}. Consider the exact sequence $0\to\mathrm{T}^\Box G_{\bar{\eta}}\to\mathrm{V}^\Box G_{\bar{\eta}}\to(G_{\bar{\eta}})^\Box_{\mathrm{tors}}\to 0$, which was first introduced in Section 1.3.5. Then the above-chosen isomorphism $\hat{\alpha}: L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box\xrightarrow{\sim}\mathrm{T}^\Box G_{\bar{\eta}}$ and its natural extension $\hat{\alpha}: L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}\xrightarrow{\sim}\mathrm{V}^\Box G_{\bar{\eta}}$ defines an isomorphism $(L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})/(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\xrightarrow{\sim}(G_{\bar{\eta}})^\Box_{\mathrm{tors}}$. Let us still denote this isomorphism by $\hat{\alpha}$. Then, by abusing the difference between a finite étale group scheme over $\bar{\eta}$ and its group of closed points, we may identify $K_{\bar{\eta}} = \hat{\alpha}((N^{-1}g(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box))/(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box))$, with filtration given by $K^{\mathrm{f}}_{\bar{\eta}} = \hat{\alpha}((N^{-1}g(\mathtt{Z}'_{-1}))/\mathtt{Z}_{-1})$ and $K^\mu_{\bar{\eta}} = \hat{\alpha}((N^{-1}g(\mathtt{Z}'_{-2}))/\mathtt{Z}_{-2})$.

The group scheme K^μ_η extends uniquely to a finite flat subgroup scheme K^μ of G over S, which is isomorphic to a finite flat subgroup scheme $K^\flat$ of T. The group scheme $K^\flat$ is the kernel of the isogeny $Nf_T: T\twoheadrightarrow T'$ induced by Nf, and therefore is dual to the cokernel of the inclusion $Nf_X: X'\hookrightarrow X$ of character groups. In particular, we may identify X' as an $\mathcal{O}$-lattice in $X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$, and define an isomorphism $f_X: X'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}\xrightarrow{\sim}X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$ by $f_X = N^{-1}\circ Nf_X$. This is dual to the "$\mathbb{Z}^\times_{(\Box)}$-isogeny" $f_T: T\to T'$.

On the other hand, the group scheme K^{f}_η extends uniquely to a finite flat subgroup scheme K^{f} of G over S, which is isomorphic to a finite flat subgroup scheme $K^\natural$ of $G^\natural$. The group scheme $K^\natural$ is the kernel of the isogeny $Nf^\natural: G^\natural\twoheadrightarrow(G')^\natural$ induced by Nf. The quotient $K/K^\natural$ of K by $K^\natural$ is an étale group scheme (which is not finite in general) over S, and its restriction $(K/K^\natural)_\eta$ to η is isomorphic to the cokernel of the inclusion $Nf_Y: Y\hookrightarrow Y'$ of character groups. Then we can identify Y' as an $\mathcal{O}$-lattice in $Y\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$, and define an isomorphism $f_Y: Y\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}\xrightarrow{\sim}Y'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\Box)}$ by $f_Y = N^{-1}\circ Nf_Y$. We can also interpret this f_Y as the dual of some "$\mathbb{Z}^\times_{(\Box)}$-isogeny" $(T^\vee)'\to T^\vee$, as follows: By Theorem 3.4.3.2, $G^\vee_\eta$ and $(G'_\eta)^\vee$ extend to semi-abelian schemes $G^\vee$ and $(G')^\vee$, respectively. By Proposition 3.3.1.5, the dual $(Nf_\eta)^\vee: (G'_\eta)^\vee\twoheadrightarrow G^\vee_\eta$ of the isogeny $Nf_\eta: G_\eta\twoheadrightarrow G'_\eta$ extends uniquely to an isogeny $(G')^\vee\twoheadrightarrow G^\vee$, which we denote by $(Nf)^\vee$. Then (as in Construction 5.4.3.3) we symbolically take $f^\vee: (G')^\vee\to G^\vee$ to be $N^{-1}\circ(Nf)^\vee$, which is the "dual $\mathbb{Z}^\times_{(\Box)}$-isogeny" $f^\vee: (G')^\vee\to G^\vee$ that extends $f^\vee_\eta: (G'_\eta)^\vee\to G^\vee_\eta$. Then the restriction of $f^\vee$ to the torus part $(T^\vee)'$ of G' gives the above $(T^\vee)'\to T^\vee$.

Under the isomorphism $\mathrm{V}^\Box(f_\eta):\mathrm{V}^\Box G_{\bar{\eta}}\xrightarrow{\sim}\mathrm{V}^\Box G'_{\bar{\eta}}$, and under the two isomorphisms $\mathrm{V}^\Box(f_{T,\eta}):\mathrm{V}^\Box T_{\bar{\eta}}\xrightarrow{\sim}\mathrm{V}^\Box T'_{\bar{\eta}}$ and $f_Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}: Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}\xrightarrow{\sim}Y'\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}$, the canonical pairing $\mathrm{e}^\phi:\mathrm{V}^\Box T_{\bar{\eta}}\times(Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})\to\mathrm{V}^\Box\mathbf{G}_{\mathrm{m},\bar{\eta}}$ induced by the λ_η-Weil pairing $\mathrm{e}^{\lambda_\eta}$ (as in Proposition 5.2.2.1) can be rewritten as a pairing $\mathrm{V}^\Box T'_{\bar{\eta}}\times(Y'\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})\to\mathrm{V}^\Box\mathbf{G}_{\mathrm{m},\bar{\eta}}$, which we may denote by $\mathrm{e}^{f_X^{-1}\circ\phi\circ f_Y^{-1}}(\cdot,\cdot)$. The restriction of this pairing gives a pairing $\mathrm{T}^\Box T'_{\bar{\eta}}\times(Y'\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\to r\,\mathrm{T}^\Box\mathbf{G}_{\mathrm{m},\bar{\eta}}$, where $r\in\mathbb{Z}^\times_{(\Box),>0}$ is the unique number such that $\nu(g) = ru$ for some $u\in\hat{\mathbb{Z}}^{\Box,\times}$ in the approximation $\mathbb{A}^{\infty,\Box,\times} = \mathbb{Z}^\times_{(\Box),>0}\cdot\hat{\mathbb{Z}}^{\Box,\times}$. Comparing this with the canonical pairing $\mathrm{e}^{\phi'}:\mathrm{T}^\Box T'_{\bar{\eta}}\times(Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box)\to\mathrm{T}^\Box\mathbf{G}_{\mathrm{m},\bar{\eta}}$ induced by the λ'_η-Weil pairing $\mathrm{e}^{\lambda'_\eta}$, and taking into account the relation $\lambda'_\eta = r^{-1}(f^\vee_\eta)^{-1}\circ\lambda_\eta\circ f_\eta$, we see that we must have $\phi' = r^{-1}f_X^{-1}\circ\phi\circ f_Y^{-1}$, or equivalently $f_X\circ\phi'\circ f_Y = r\phi$.

Let us give a more intrinsic interpretation of f_X and f_Y via a comparison between $(\varphi_{-2,\mathcal{H}'},\varphi_{0,H'_{\mathcal{H}'}})$ and $(\varphi'_{-2,\mathcal{H}},\varphi'_{0,H_{\mathcal{H}}})$. First note that $\mathrm{Gr}(\hat{\alpha})$ determines a well-defined pair (φ_{-2},φ_0), such that $\varphi_{-2}:\mathrm{Gr}^{\mathtt{Z}}_{-2}\xrightarrow{\sim}\mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box,\hat{\mathbb{Z}}^\Box(1))$ differ from $\mathrm{Gr}_{-2}(\hat{\alpha})$ by the isomorphism $\nu(\hat{\alpha}):\hat{\mathbb{Z}}^\Box(1)\xrightarrow{\sim}\mathrm{T}^\Box\mathbf{G}_{\mathrm{m},\bar{\eta}}$, and such that $\varphi_0:\mathrm{Gr}^{\mathtt{Z}}_0\xrightarrow{\sim}Y\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^\Box$ is just $\mathrm{Gr}_0(\hat{\alpha})$. Similarly, $\mathrm{Gr}(\hat{\alpha}')$ determines a well-defined pair $(\varphi'_{-2},\varphi'_0)$. Then $(\varphi_{-2,H'_m},\varphi_{0,H'_m})$ is the H'_m-orbit of the reduction modulo m of (φ_{-2},φ_0), and $(\varphi'_{-2,H_n},\varphi'_{0,H_n})$ is the H_n-orbit of the reduction modulo n of $(\varphi'_{-2},\varphi'_0)$. Let us also denote by $\varphi_{-2}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}:\mathrm{Gr}^{\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}}_{-2}\xrightarrow{\sim}\mathrm{Hom}_{\hat{\mathbb{Z}}^\Box}(X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box},\mathbb{A}^{\infty,\Box}(1))$ etc., the induced morphisms between $\mathbb{A}^{\infty,\Box}$-modules. The isomorphism $g: L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}\xrightarrow{\sim}L\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}$ sends the filtration $\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}} = g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$ to $\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}$, which induces an isomorphism $\mathrm{Gr}_{-i}(g):\mathrm{Gr}^{\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}}}_{-i} := \mathtt{Z}'_{-i,\mathbb{A}^{\infty,\Box}}/\mathtt{Z}'_{-i-1,\mathbb{A}^{\infty,\Box}}\xrightarrow{\sim}\mathrm{Gr}^{\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}}_{-i} := \mathtt{Z}_{-i,\mathbb{A}^{\infty,\Box}}/\mathtt{Z}_{-i-1,\mathbb{A}^{\infty,\Box}}$ on each of the graded pieces. Then the pairs (φ_{-2},φ_0) and $(\varphi'_{-2},\varphi'_0)$ are related by

$$\varphi'_{-2}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box} = ({}^t f_X\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})\circ(\varphi_{-2}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})\circ(\nu(g)^{-1}\,\mathrm{Gr}_{-2}(g)) \quad (5.4.3.4)$$

and

$$\varphi'_0\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box} = (f_Y\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})\circ(\varphi_0\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box})\circ(\mathrm{Gr}_0(g)). \quad (5.4.3.5)$$

Lemma 5.4.3.6. *We have the approximations* $\mathrm{GL}_{\mathcal{O}}(X \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}) = (\mathrm{GL}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}) \cap \mathrm{GL}_{\mathcal{O}}(X \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box})) \cdot \mathrm{GL}_{\mathcal{O}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ *and* $\mathrm{GL}_{\mathcal{O}}(Y \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}) = (\mathrm{GL}(Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}) \cap \mathrm{GL}_{\mathcal{O}}(Y \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box})) \cdot \mathrm{GL}_{\mathcal{O}}(Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$.

Proof. Let us first note that we have the approximation $\mathrm{GL}(X \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}) = \mathrm{GL}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}) \cdot \mathrm{GL}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ (resp. $\mathrm{GL}(Y \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}) = \mathrm{GL}(Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}) \cdot \mathrm{GL}(Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$) by elementary lattice theory. Then the lemma follows by specializing this approximation to the subgroup $\mathrm{GL}_{\mathcal{O}}(X \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box})$ (resp. $\mathrm{GL}_{\mathcal{O}}(Y \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box})$) of $\mathcal{O}$-equivariant elements in $\mathrm{GL}(X \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box})$ (resp. $\mathrm{GL}(Y \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box})$). ☐

Lemma 5.4.3.7. *We have the relations* $\mathrm{GL}_{\mathcal{O}}(X) = (\mathrm{GL}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}) \cap \mathrm{GL}_{\mathcal{O}}(X \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box})) \cap \mathrm{GL}_{\mathcal{O}}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ *and* $\mathrm{GL}_{\mathcal{O}}(Y) = (\mathrm{GL}(Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}) \cap \mathrm{GL}_{\mathcal{O}}(Y \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box})) \cap \mathrm{GL}_{\mathcal{O}}(Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$.

Proof. These follow respectively from the relations $\mathrm{GL}(X) = \mathrm{GL}(X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}) \cap \mathrm{GL}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$ and $\mathrm{GL}(Y) = \mathrm{GL}(Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}) \cap \mathrm{GL}(Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box})$. ☐

Let us summarize the constructions above, together with an alternative (and more direct) interpretation of f_X and f_Y, in the following:

Proposition 5.4.3.8. *Suppose we have an element* $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$ *and suppose we have two open compact subgroups* $\mathcal{H}' \subset \mathcal{H}$ *in* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *such that* $g^{-1}\mathcal{H}'g \subset \mathcal{H}$*. Then* g *defines a map from the set of cusp labels at level* $\mathcal{H}'$ *to the set of cusp labels at level* $\mathcal{H}$*, as follows: Suppose we have a cusp label at level* $\mathcal{H}'$ *represented by some* $(\mathtt{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$.

1. *Let* $\hat{\alpha}$ *be any symplectic lifting of the level-*$\mathcal{H}'$ *structure* $\alpha_{\mathcal{H}'}$*. Then* $\hat{\alpha}$ *pulls back the filtration* $\mathtt{W}$ *on* $\mathrm{T}^{\Box}\, G_{\bar{\eta}}$ *to a fully symplectic filtration* $\mathtt{Z}$ *on* $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$*, together with a torus argument* $(X, Y, \phi, \varphi_{-2}, \varphi_0)$ *for* $\mathtt{Z}$*. Then* $\mathtt{Z}_{\mathcal{H}'}$ *is the* $\mathcal{H}'$*-orbit of* $\mathtt{Z}$.
2. *Let* $\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}$ *be any filtration on* $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}$ *that extends* $\mathtt{Z}$*. Then we obtain an admissible filtration* $\mathtt{Z}' := (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}) \cap g^{-1}(\mathtt{Z}_{\mathbb{A}^{\infty,\Box}})$ *on* $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$*, whose* $\mathcal{H}$*-orbit is independent of the choice of* $\mathtt{Z}$ *and determines* $\mathtt{Z}'_{\mathcal{H}}$.
3. *The submodule* $\mathrm{Gr}_{-2}^{\mathtt{Z}'}$ *of* $\mathrm{Gr}_{-2}^{\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}}}$ *is pulled back to the submodule* $(\nu(g)^{-1}\,\mathrm{Gr}_{-2}(g))(\mathrm{Gr}_{-2}^{\mathtt{Z}'})$ *of* $\mathrm{Gr}_{-2}^{\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}}$*. Under the isomorphism* $\varphi_{-2} \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} : \mathrm{Gr}_{-2}^{\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{A}^{\infty,\Box}}(X \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}, \mathbb{A}^{\infty,\Box}(1))$*, the difference between this submodule and the submodule* $\mathrm{Gr}_{-2}^{\mathtt{Z}}$ *of* $\mathrm{Gr}_{-2}^{\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}}$ *can be given by the transpose of an element in* $\mathrm{GL}_{\mathcal{O}}(X \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box})$*, which determines by Lemmas 5.4.3.6 and 5.4.3.7 the unique choices of an* $\mathcal{O}$*-lattice* X' *(up to isomorphism) and an isomorphism* $f_X : X' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$*. This choice of* f_X *also determines an isomorphism* φ'_{-2} *satisfying (5.4.3.4).*
4. *Similarly, the submodule* $\mathrm{Gr}_0^{\mathtt{Z}'}$ *of* $\mathrm{Gr}_0^{\mathtt{Z}'_{\mathbb{A}^{\infty,\Box}}}$ *is pulled back to the submodule* $\mathrm{Gr}_0(g)(\mathrm{Gr}_0^{\mathtt{Z}'})$ *of* $\mathrm{Gr}_0^{\mathtt{Z}_{\mathbb{A}^{\infty,\Box}}}$*, and this submodule determines the unique choice of an* $\mathcal{O}$*-lattice* Y' *in* $Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$*, which is equivalent to the unique choices of an* $\mathcal{O}$*-lattice* Y' *(up to isomorphism) and an isomorphism* $f_Y : Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Y' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$*. This choice of* f_Y *also determines an isomorphism* φ'_0 *satisfying (5.4.3.5).*
5. *Let* $r \in \mathbb{Z}^{\times}_{(\Box),>0}$ *be the unique number such that* $\nu(g) = ru$ *for some* $u \in \hat{\mathbb{Z}}^{\Box,\times}$ *in the approximation* $\mathbb{A}^{\infty,\Box,\times} = \mathbb{Z}^{\times}_{(\Box),>0} \cdot \hat{\mathbb{Z}}^{\Box,\times}$*. Then we set* $\phi' = r^{-1} f_X^{-1} \phi f_Y^{-1}$.
6. *Take any splitting* δ' *of the admissible filtration* $\mathtt{Z}'$*, and take* $\delta'_{\mathcal{H}}$ *to be the* $\mathcal{H}$*-orbit of* δ'.

The above steps determine a torus argument $\Phi' = (X', Y', \phi', \varphi'_{-2}, \varphi'_0)$ *up to equivalence, and hence a class of torus arguments* $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ *at level* $\mathcal{H}$ *depending only on the class of the torus argument* $\Phi_{\mathcal{H}'} = (X, Y, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$*. Since the cusp labels do not depend on the choice of the splittings* $\delta_{\mathcal{H}'}$ *and* $\delta'_{\mathcal{H}}$*, we obtain a well-defined map from the set of cusp labels at level* $\mathcal{H}'$ *to the set of cusp labels at level* $\mathcal{H}$.

Proof. Steps 1 and 2 have already been explained above. (The admissibility of $\mathtt{Z}'$ is automatic by Lemma 5.2.2.2.) Steps 3, 4, and 5 (involving Lemmas 5.4.3.6 and 5.4.3.7) are self-explanatory. ☐

Definition 5.4.3.9. *Suppose we have an element* $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$*, and suppose we have two open compact subgroups* $\mathcal{H}' \subset \mathcal{H}$ *such that* $g^{-1}\mathcal{H}'g \subset \mathcal{H}$*. We say that a triple* $(\mathtt{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *is* ***g-assigned*** *to* $(\mathtt{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$*, written as* $(\mathtt{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}) \to_g (\mathtt{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$*, if there are isomorphisms* $f_X : X' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$ *and* $f_Y : Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Y' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$ *as in Proposition 5.4.3.8 assigning some lifting* $\Phi' = (X', Y', \phi', \varphi'_{-2}, \varphi'_0)$ *of* $\Phi'_{\mathcal{H}}$ *to some lifting* $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$ *of* $\Phi_{\mathcal{H}'}$*. In this case we say that there is a* ***g-assignment*** $(f_X, f_Y) : (\mathtt{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}) \to_g (\mathtt{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$.

Remark 5.4.3.10. The pair (f_X, f_Y) of the two possible isomorphisms f_X and f_Y in Definition 5.4.3.9 is only unique up to multiplication by elements in $\mathrm{GL}_{\phi'}$ (defined analogously to GL_{ϕ} as in Definition 5.4.1.6) that leaves $\Phi'_{\mathcal{H}}$ invariant. (Later on, such a subgroup will be called $\Gamma_{\Phi'_{\mathcal{H}}}$ in Definition 6.2.4.1.)

The above result will be applied in Section 6.4.3 to the study of Hecke actions on towers of toroidal compactifications, after we have the meaning of cusp labels as part of the parameters of the stratifications on the toroidal compactifications we construct.

Chapter 6

Algebraic Constructions of Toroidal Compactifications

We will generalize the techniques in [42] and construct the toroidal compactifications of the moduli problems we considered in Chapter 1.

The main objective of this chapter is to state and prove Theorem 6.4.1, with by-products concerning Hecke actions given in Sections 6.4.2 and 6.4.3. Technical results worth noting are Propositions 6.2.2.4, 6.2.3.2, 6.2.3.18, and 6.2.5.18, Lemma 6.3.1.11, and Propositions 6.3.2.6, 6.3.2.10, 6.3.3.5, 6.3.3.13, and 6.3.3.17.

6.1 Review of Toroidal Embeddings

6.1.1 Rational Polyhedral Cone Decompositions

Let H be a group of multiplicative type of finite type over S, so that its character group $\underline{\mathbf{X}}(H) = \underline{\mathrm{Hom}}_S(H, \mathbf{G}_{\mathrm{m},S})$ is an étale sheaf of finitely generated commutative groups.

Definition 6.1.1.1. *The* ***cocharacter group*** $\underline{\mathbf{X}}(H)^\vee$ *of* H (*over* S) *is the étale sheaf of finitely generated* (*free*) *commutative groups* $\underline{\mathrm{Hom}}_S(\mathbf{G}_{\mathrm{m},S}, H) \cong \underline{\mathrm{Hom}}_S(\underline{\mathbf{X}}(H), \mathbb{Z})$.

For simplicity,

Assumption 6.1.1.2. *From now on, we shall assume that* H *is split.*

This assumption is harmless for our application to the construction of toroidal compactifications, because we will glue in the étale topology, in which every group of multiplicative type of finite type is split after localization.

Let us consider the $\mathbb{R}$-vector space $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}} := \underline{\mathbf{X}}(H)^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{R}$.

Definition 6.1.1.3. *A subset of* $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ *is called a* ***cone*** *if it is invariant under the natural multiplication action of* $\mathbb{R}^\times_{>0}$ *on the* $\mathbb{R}$*-vector space* $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$.

Definition 6.1.1.4. *A cone in* $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ *is* ***nondegenerate*** *if its closure does not contain any nonzero* $\mathbb{R}$*-vector subspace of* $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$.

Definition 6.1.1.5. *A* ***rational polyhedral cone*** *in* $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ *is a cone in* $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ *of the form* $\sigma = \mathbb{R}_{>0}v_1 + \cdots + \mathbb{R}_{>0}v_n$ *with* $v_1, \ldots, v_n \in \underline{\mathbf{X}}(H)^\vee_{\mathbb{Q}} = \underline{\mathbf{X}}(H)^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

Note that σ is an open subset in its closure $\overline{\sigma} = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_n$, and also in the smallest $\mathbb{R}$-vector subspace $\mathbb{R}v_1 + \cdots + \mathbb{R}v_n$ containing σ.

Definition 6.1.1.6. *A* ***supporting hyperplane*** P *of* σ *is a hyperplane in* $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ (*i.e., a translation of a codimension-one vector subspace of* $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$) *such that* σ *does not overlap with both sides of* P.

Definition 6.1.1.7. *A* ***face*** *of* σ *is a rational polyhedral cone* τ *such that* $\overline{\tau} = \overline{\sigma} \cap P$ *for some supporting hyperplane* P *of* σ.

The canonical pairing $\langle \cdot , \cdot \rangle : \underline{\mathbf{X}}(H) \times \underline{\mathbf{X}}(H)^\vee \to \mathbb{Z}$ defines by extension of scalars a canonical pairing
$$\langle \cdot , \cdot \rangle : \underline{\mathbf{X}}(H) \times \underline{\mathbf{X}}(H)^\vee_{\mathbb{R}} \to \mathbb{R}$$
whose restriction to $\underline{\mathbf{X}}(H) \times \underline{\mathbf{X}}(H)^\vee$ gives the original pairing. (We do not tensor $\underline{\mathbf{X}}(H)$ with $\mathbb{R}$, because the homomorphism $\underline{\mathbf{X}}(H) \to \underline{\mathbf{X}}(H) \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ is not injective when there exist nonzero torsion elements in $\underline{\mathbf{X}}(H)$.)

Definition 6.1.1.8. *If* σ *is a rational polyhedral cone in* $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$, *then* $\sigma^\vee$ *is the semisubgroup* (*with unit* 0) *of* $\underline{\mathbf{X}}(H)$ *defined by*
$$\sigma^\vee := \{x \in \underline{\mathbf{X}}(H) : \langle x, y \rangle \geq 0 \ \forall y \in \sigma\},$$
and $\sigma^\vee_0$ *is the semisubgroup* (*without unit* 0) *of* $\underline{\mathbf{X}}(H)$ *defined by*
$$\sigma^\vee_0 := \{x \in \underline{\mathbf{X}}(H) : \langle x, y \rangle > 0 \ \forall y \in \sigma\}.$$

Remark 6.1.1.9. The cone $\mathbb{R}^\times_{>0} \cdot \sigma^\vee_0$ is *not* the interior of the closed cone $\mathbb{R}^\times_{>0} \cdot \sigma^\vee$ in general: Try any top-dimensional nondegenerate rational polyhedral cone σ in $\mathbb{R}^2$.

Let Γ be any group acting on $\underline{\mathbf{X}}(H)$, which induces an action on H and hence also an action on $\underline{\mathbf{X}}(H)^\vee$. Let C be any cone in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$.

Definition 6.1.1.10. *A* Γ***-admissible rational polyhedral cone decomposition*** *of* C *is a collection* $\Sigma = \{\sigma_j\}_{j \in J}$ *with some indexing set* J *such that we have the following:*

1. *Each* σ_j *is a nondegenerate rational polyhedral cone.*
2. C *is the disjoint union of all the* σ_j*'s in* Σ. *For each* $j \in J$, *the closure of* σ_j *in* C *is a disjoint union of the* σ_k*'s with* $k \in J$. *In other words,* $C = \coprod\limits_{j \in J} \sigma_j$ *is a stratification of* C. (*Here "*$\coprod$*" only means a set-theoretic disjoint union. The geometric structure of* $\coprod\limits_{j \in J} \sigma_j$ *is still the one inherited from the ambient space* $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ *of* C.) *Moreover, each* σ_k *appearing in the closure of* σ_j *in* C *as above is a face of* σ_j.

3. *Σ is invariant under the action of Γ on $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$, in the sense that Γ permutes the cones in Σ. Under this action, the set Σ/Γ of Γ-orbits is finite.*

Definition 6.1.1.11. *A rational polyhedral cone σ in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$ is **smooth** with respect to the integral structure given by $\underline{\mathbf{X}}(H)^\vee$ if we have $\sigma = \mathbb{R}_{>0}v_1 + \cdots + \mathbb{R}_{>0}v_n$ with $v_1, \ldots, v_n$ forming part of a $\mathbb{Z}$-basis of $\underline{\mathbf{X}}(H)^\vee$.*

Definition 6.1.1.12. *A Γ**-admissible smooth rational polyhedral cone decomposition** of C is a Γ-admissible rational polyhedral cone decomposition $\{\sigma_j\}_{j\in J}$ of C in which every σ_j is **smooth**.*

6.1.2 Toroidal Embeddings of Torsors

Let $\mathcal{M}$ be an H-torsor over a scheme Z. Then $\mathcal{M}$ is relatively affine over Z, and the H-action on $\mathscr{O}_{\mathcal{M}}$ gives a decomposition $\mathscr{O}_{\mathcal{M}} = \underset{\chi\in\underline{\mathbf{X}}(H)}{\oplus} \mathscr{O}_{\mathcal{M},\chi}$, where $\mathscr{O}_{\mathcal{M},\chi}$ is the weight-χ subsheaf of $\mathscr{O}_{\mathcal{M}}$ under the H-action, together with isomorphisms

$$\mathscr{O}_{\mathcal{M},\chi} \underset{\mathscr{O}_Z}{\otimes} \mathscr{O}_{\mathcal{M},\chi'} \overset{\sim}{\to} \mathscr{O}_{\mathcal{M},\chi+\chi'} \tag{6.1.2.1}$$

giving the $\mathscr{O}_Z$-algebra structure of $\mathscr{O}_{\mathcal{M}}$ (see [39, I, Prop. 4.7.3]).

Remark 6.1.2.2. The specification of the isomorphisms in (6.1.2.1) is necessary because we are not assuming that Z satisfies Assumption 3.1.2.7 (which is needed for application of Theorem 3.1.3.3). We can not necessarily rigidify our H-torsor $\mathcal{M}$ or the invertible sheaves $\mathscr{O}_{\mathcal{M},\chi}$ in this general setting.

Definition 6.1.2.3. *For each nondegenerate rational polyhedral cone σ in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$, the **affine toroidal embedding** $\mathcal{M}(\sigma)$ along σ is the relatively affine scheme $\mathcal{M}(\sigma) := \underline{\mathrm{Spec}}_{\mathscr{O}_Z}\Big(\underset{\chi\in\sigma^\vee}{\oplus} \mathscr{O}_{\mathcal{M},\chi}\Big)$ over Z, where $\underset{\chi\in\sigma^\vee}{\oplus} \mathscr{O}_{\mathcal{M},\chi}$ has the structure of an $\mathscr{O}_Z$-algebra given by the isomorphisms in (6.1.2.1).*

By construction, the H-action on $\mathcal{M}$ extends naturally to $\mathcal{M}(\sigma)$.

Lemma 6.1.2.4. *In Definition 6.1.2.3, if τ is a face of σ, then τ is also nondegenerate, and there is an H-equivariant canonical embedding $\mathcal{M}(\tau) \hookrightarrow \mathcal{M}(\sigma)$ defined by the natural inclusion of structural sheaves.*

Definition 6.1.2.5. *For each nondegenerate rational polyhedral cone σ in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$, let $\sigma^\perp := \{x \in \underline{\mathbf{X}}(H) : \langle x, y\rangle = 0 \ \forall y \in \sigma\}$. This is a subgroup of $\underline{\mathbf{X}}(H)$ defining a (split) quotient (group scheme) H_σ of H (of multiplicative type of finite type) over Z.*

Lemma 6.1.2.6. *With the setting as above, the closed subscheme $\mathcal{M}_\sigma := \Big(\mathcal{M}(\sigma) - \underset{\substack{\tau \text{ is a face of } \sigma\\ \tau\neq\sigma}}{\cup} \mathcal{M}(\tau)\Big)_{\mathrm{red}}$ of $\mathcal{M}(\sigma)$ can be defined by the sheaf of ideals $\mathscr{I}_\sigma := \underset{\chi\in\sigma_0^\vee}{\oplus} \mathscr{O}_{\mathcal{M},\chi}$, which can be identified with $\mathcal{M}_\sigma = \underline{\mathrm{Spec}}_{\mathscr{O}_Z}\Big(\underset{\chi\in\sigma^\perp}{\oplus} \mathscr{O}_{\mathcal{M},\chi}\Big)$. This $\mathcal{M}_\sigma$ is an H_σ-torsor over Z (see Definition 6.1.2.5).*

Definition 6.1.2.7. *We call the closed subscheme $\mathcal{M}_\sigma$ of $\mathcal{M}(\sigma)$ defined in Lemma 6.1.2.6 the σ**-stratum** of $\mathcal{M}(\sigma)$. For each face τ of σ, the τ**-stratum** of $\mathcal{M}(\sigma)$ is the (locally closed) image of the τ-stratum $\mathcal{M}_\tau$ of $\mathcal{M}(\tau)$ under the canonical embedding $\mathcal{M}(\tau) \hookrightarrow \mathcal{M}(\sigma)$ in Lemma 6.1.2.4.*

Let C be a cone in $\underline{\mathbf{X}}(H)^\vee_{\mathbb{R}}$, let Γ be a group acting on $\underline{\mathbf{X}}(H)$, and let $\Sigma = \{\sigma_j\}_{j\in J}$ be a Γ-admissible rational polyhedral cone decomposition. Using Lemma 6.1.2.4, we can glue together affine toroidal embeddings $\mathcal{M}(\sigma_j)$ defined by various cones σ_j in $\Sigma = \{\sigma_j\}_{j\in J}$, which we denote by $\overline{\mathcal{M}}_\Sigma$, or simply by $\overline{\mathcal{M}}$ if Σ is clear from the context. This is the *toroidal embedding* of the H-torsor $\mathcal{M}$ defined by Σ.

Let us list the main properties of $\overline{\mathcal{M}}_\Sigma$ as follows:

Theorem 6.1.2.8 (cf. [42, Ch. IV, Thm. 2.5 and Rem. 2.6])**.** *The scheme $\overline{\mathcal{M}}_\Sigma$ constructed as above satisfies the following properties:*

1. *$\overline{\mathcal{M}}_\Sigma$ is separated and locally of finite type over the base scheme Z, which contains $\mathcal{M}$ as an open dense subscheme. The H-action on $\mathcal{M}$ extends naturally to $\overline{\mathcal{M}}_\Sigma$, and makes $\mathcal{M} \hookrightarrow \overline{\mathcal{M}}_\Sigma$ an H-equivariant embedding.*

2. *For each $\sigma_j \in \Sigma$, the affine toroidal embedding $\mathcal{M}(\sigma_j)$ embeds as a relatively affine H-invariant open dense subscheme of $\overline{\mathcal{M}}_\Sigma$ over Z, containing $\mathcal{M}$. We can view $\overline{\mathcal{M}}_\Sigma$ as the union of various $\mathcal{M}(\sigma_j)$'s. Then (by construction) we understand that if $\overline{\sigma}_l = \overline{\sigma}_j \cap \overline{\sigma}_k$, where σ_l is the largest common face of σ_j and σ_k, then $\mathcal{M}(\sigma_l) = \mathcal{M}(\sigma_j) \cap \mathcal{M}(\sigma_k)$.*

3. *$\overline{\mathcal{M}}_\Sigma$ has a natural stratification by locally closed subschemes $\mathcal{M}_{\sigma_j}$, for $j \in J$, as defined in Lemma 6.1.2.6. Moreover, by construction, $\overline{\mathcal{M}}_{\sigma_j} \supset \overline{\mathcal{M}}_{\sigma_k}$ if and only if $\overline{\sigma}_j \subset \overline{\sigma}_k$, where $\overline{\mathcal{M}}_{\sigma_j}$ (resp. $\overline{\mathcal{M}}_{\sigma_k}$) denotes the closures of $\mathcal{M}_{\sigma_j}$ (resp. $\mathcal{M}_{\sigma_k}$) in $\overline{\mathcal{M}}_\Sigma$.*

4. *The group Γ acts on $\overline{\mathcal{M}}_\Sigma$ by sending $\mathcal{M}(\sigma_j)$ (resp. $\mathcal{M}_{\sigma_j}$) to $\mathcal{M}(\gamma\sigma_j)$ (resp. $\mathcal{M}_{\gamma\sigma_j}$) if $\gamma \in \Gamma$ sends σ_j to $\gamma\sigma_j$ in Σ.*

5. *If σ_j is smooth, then $\mathcal{M}(\sigma_j)$ is smooth over Z. If the cone decomposition $\Sigma = \{\sigma_j\}_{j\in J}$ is smooth, then $\overline{\mathcal{M}}_\Sigma$ is smooth over Z, and the complement of $\mathcal{M}$ in each open set $\mathcal{M}(\sigma_j)$, for $\sigma_j \in \Sigma$, is a relative Cartier divisor with normal crossings.*

6.2 Construction of Boundary Charts

6.2.1 The Setting for This Section

In this section, we will focus on the following type of general construction:

Let B be a finite-dimensional semisimple algebra over $\mathbb{Q}$ with a positive involution * and center F, and let $\mathcal{O}$ be an order in B mapped to itself under *. Let $\mathrm{Disc} = \mathrm{Disc}_{\mathcal{O}/\mathbb{Z}}$ be the discriminant of $\mathcal{O}$ over $\mathbb{Z}$, and let $\mathrm{I}_{\mathrm{bad}} = 2$ or 1 depending on whether or not B involves any simple factor of type D (see Definitions 1.2.1.15 and 1.2.1.18).

Let $(L, \langle\cdot,\cdot\rangle, h)$ be a PEL-type $\mathcal{O}$-lattice (see Definition 1.2.1.3) that satisfies Condition 1.4.3.10 (see Remark 1.4.3.9). Let $n \geq 1$ be an integer that will be our level. Let $\Box$ be a set of good primes, in the sense that $\Box \nmid n\,\mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}[L^\# : L]$, so that we can define a moduli problem M_n over the base scheme $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ as in Definition 1.4.1.2. More generally, we shall consider an open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ for some $\Box \nmid \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}[L^\# : L]$, so that we can define a moduli problem $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{S}_0 == \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ as in Definition 1.4.1.4.

From now on, let us adopt the following convention:

Convention 6.2.1.1. *All morphisms between schemes, algebraic stacks, or their formal analogues over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ will be defined over S_0, unless otherwise specified.*

The first goal of this section is to construct formal schemes for each cusp label of $\mathsf{M}_{\mathcal{H}}$ (see Definition 5.4.2.4), over which we have the so-called Mumford families playing the role of Tate curves for modular curves along the cusps. Then we approximate these Mumford families by the so-called *good algebraic models* over algebraic schemes (instead of formal schemes), and glue them with our moduli problem $\mathsf{M}_{\mathcal{H}}$ in the étale topology (in Section 6.3) to form the arithmetic toroidal compactification.

Let us explain the setting for M_n, which will be tacitly assumed in Sections 6.2.2 and 6.2.3 below. (The setting for $\mathsf{M}_{\mathcal{H}}$ will be postponed until the beginning of Section 6.2.4.)

Let $(\mathsf{Z}_n, \Phi_n, \delta_n)$ be a representative of a cusp label at level n, where $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ is a torus argument at level n. Here X and Y are constant group schemes that will serve as the character groups of the torus parts, as usual. We know X and Y are $\mathcal{O}$-lattices of the same $\mathcal{O}$-multirank, and $\phi : Y \hookrightarrow X$ is an $\mathcal{O}$-equivariant embedding of $\mathcal{O}$-lattices.

Recall that we have defined in Definition 5.4.1.7 the group $\Gamma_\phi = \Gamma_{X,Y,\phi}$ of pairs of isomorphisms $(\gamma_X : X \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y)$ in $\mathrm{GL}_{\mathcal{O}}(X) \times \mathrm{GL}_{\mathcal{O}}(Y)$ such that $\phi = \gamma_X \phi \gamma_Y$.

Definition 6.2.1.2. *Let $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ be any torus argument as in Definition 5.4.1.4. The group Γ_{Φ_n} is the subgroup of elements (γ_X, γ_Y) in Γ_ϕ satisfying $\varphi_{-2,n} = {}^t\gamma_X \varphi_{-2,n}$ and $\varphi_{0,n} = \gamma_Y \varphi_{0,n}$. In particular, $\Gamma_{\Phi_1} = \Gamma_\phi$.*

As in Lemma 5.2.7.5, the information of Z_n alone allows one to define (up to isomorphism) a moduli problem $\mathsf{M}_n^{\mathsf{Z}_n}$ over S_0 as in Definition 1.4.1.2. Let $(A, \lambda_A, i_A, \varphi_{-1,n})$ be the tautological tuple over $\mathsf{M}_n^{\mathsf{Z}_n}$. Then,

1. A is a (relative) abelian scheme over $\mathsf{M}_n^{\mathsf{Z}_n}$;
2. $\lambda_A : A \xrightarrow{\sim} A^\vee$ is a prime-to-$\Box$ polarization of A;
3. $i_A : \mathcal{O} \to \mathrm{End}_{\mathsf{M}_n^{\mathsf{Z}_n}}(A)$ defines an $\mathcal{O}$-structure of (A, λ_A);
4. $\underline{\mathrm{Lie}}_{A/\mathsf{M}_n^{\mathsf{Z}_n}}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-module structure given naturally by i_A satisfies the determinantal condition in Definition 1.3.4.1 given by $(\mathrm{Gr}^{\mathsf{Z}}_{-1,\mathbb{R}}, \langle \cdot , \cdot \rangle_{11,\mathbb{R}}, h_{-1})$;
5. $\varphi_{-1,n} : (\mathrm{Gr}^{\mathsf{Z}}_{-1,n})_{\mathsf{M}_n^{\mathsf{Z}_n}} \xrightarrow{\sim} A[n]$ is an integral principal level-n structure for (A, λ_A, i_A) of type $(\mathrm{Gr}^{\mathsf{Z}}_{-1}, \langle \cdot , \cdot \rangle_{11})$ as in Definition 1.3.6.2.

Based on the above data Z_n, $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$, $(A, \lambda_A, i_A, \varphi_{-1,n})$, and δ_n, together with some additional combinatorial data (to be specified later), we would like to construct a formal algebraic stack $\mathfrak{X}_{\Phi_n,\delta_n}$ over which there is a tautological tuple

$$(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n))$$

like an object in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}(R, I)$ for some R and I as in Section 5.2.1 (see Definition 5.4.1.1). Note that we cannot really say that this is an object of $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}(R, I)$, because the base $\mathfrak{X}_{\Phi_n,\delta_n}$ is a formal algebraic stack which does not readily fit into the setting of Section 5.2.1. Over each affine formal scheme $\mathrm{Spf}(R, I)$ that is étale (i.e., formally étale and of finite type; see [59, I, 10.13.3]) over $\mathfrak{X}_{\Phi_n,\delta_n}$, it should induce an object in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}(R, I)$ and hence an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}(R, I)$. By Theorem 5.2.7.14, an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_n}(R, I)$ defines an object in $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_n}(R, I)$, which is in particular, a degenerating family of type M_n (see Definition 5.3.2.3). The degenerating families over various different affine formal schemes should glue together (by étale descent) and form a degenerating family called the *Mumford family* over $\mathfrak{X}_{\Phi_n,\delta_n}$. Therefore, stated more precisely, our goal in this section is to construct $\mathfrak{X}_{\Phi_n,\delta_n}$ and the Mumford family over it.

Following Convention 5.4.2.5, we shall not make Z_n explicit in notation such as $\mathfrak{X}_{\Phi_n,\delta_n}$.

6.2.2 Construction without the Positivity Condition or Level Structures

For simplicity, let us begin by constructions without any consideration of level-n or level-$\mathcal{H}$ structures. Note that this does not mean the construction for even the level-1 structures (see Remark 1.3.6.3). In fact, this section does not produce any space that we will need later. Nevertheless, understanding the naive construction in this section (which seems to be the only one available in special cases in the literature) might be helpful for understanding the role played by the symplectic-liftability condition in the construction in Section 6.2.3.

Proposition 6.2.2.1. *Let $n = 1$. Let us fix the choice of a representative $(\mathsf{Z}_1, \Phi_1, \delta_1)$ of a cusp label at level 1, which defines a moduli problem $\mathsf{M}_1^{\mathsf{Z}_1}$ as in Lemma 5.2.7.5.*

Let us consider the category fibered in groupoids over the category of schemes over $\mathsf{M}_1^{\mathsf{Z}_1}$, whose fiber over each scheme S (over $\mathsf{M}_1^{\mathsf{Z}_1}$) has objects the tuples

$$(A, \lambda_A, i_A, \varphi_{-1,1}, c, c^\vee, \tau)$$

describing degeneration data without the positivity condition over S. Explicitly, each tuple as above satisfies the following conditions:

1. *$(A, \lambda_A, i_A, \varphi_{-1,1})$ is the pullback of the tautological tuple over $\mathsf{M}_1^{\mathsf{Z}_1}$.*
2. *$c : X \to A^\vee$ and $c^\vee : Y \to A$ are $\mathcal{O}$-equivariant group homomorphisms satisfying the compatibility relation $\lambda_A c^\vee = c\phi$ with the prescribed $\phi : Y \hookrightarrow X$.*
3. *$\tau : \mathbf{1}_{Y \times X} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_A$ is a trivialization of biextensions over S, which satisfies the symmetry $\tau(y, \phi(y')) = \tau(y', \phi(y))$ as sections of $(c^\vee(y), c(\phi(y')))^* \mathcal{P}_A^{\otimes -1} \cong (c^\vee(y'), c(\phi(y)))^* \mathcal{P}_A^{\otimes -1}$, and satisfies the $\mathcal{O}$-compatibility $\tau(by, \chi) = \tau(y, b^\star \chi)$ as sections of $(c^\vee(by), c(\chi))^* \mathcal{P}_A^{\otimes -1} \cong (c^\vee(y), c(b^\star \chi))^* \mathcal{P}_A^{\otimes -1}$. (Here it makes sense to write equalities of sections because the isomorphisms are all canonical.)*

In this category, an isomorphism

$$(A, \lambda_A, i_A, \varphi_{-1,1}, c, c^\vee, \tau) \xrightarrow{\sim} (A', \lambda_{A'}, i_{A'}, \varphi'_{-1,1}, c', (c^\vee)', \tau')$$

is a collection of isomorphisms

$$(f_X : X \xrightarrow{\sim} X, f_Y : Y \xrightarrow{\sim} Y) \in \Gamma_{\Phi_1}$$

(see Definition 6.2.1.2) and

$$f_A : (A, \lambda_A, i_A, \varphi_{-1,1}) \xrightarrow{\sim} (A', \lambda_{A'}, i_{A'}, \varphi'_{-1,1})$$

over S, such that

1. *the homomorphisms $c : X \to A^\vee$ and $c' : X \to (A')^\vee$ are related by $cf_X = {f_A}^\vee c'$;*

2. *the homomorphisms $c^\vee : Y \to A$ and $(c^\vee)' : Y \to A'$ are related by $f_A c^\vee = (c^\vee)' f_Y$;*

3. *the trivializations $\tau : \mathbf{1}_{Y\times X} \xrightarrow{\sim} (c^\vee \times c)^*\mathcal{P}_A$ and $\tau' : \mathbf{1}_{Y\times X} \xrightarrow{\sim} ((c^\vee)' \times c')^*\mathcal{P}_{A'}$ are related by $(\mathrm{Id}_Y \times f_X)^*\tau = (f_Y \times \mathrm{Id}_X)^*\tau'$.*

Then there is an algebraic stack $\ddot{\Xi}_{\Phi_1}$ separated, smooth, and schematic over $\mathsf{M}_1^{\mathsf{Z}_1}$, together with a tautological tuple and a natural action of Γ_{Φ_1} on $\ddot{\Xi}_{\Phi_1}$, such that the quotient $\ddot{\Xi}_{\Phi_1}/\Gamma_{\Phi_1}$ is isomorphic to the category described above (as categories fibered in groupoids over $\mathsf{M}_1^{\mathsf{Z}_1}$). Equivalently, for each tuple $(A, \lambda_A, i_A, \varphi_{-1,1}, c, c^\vee, \tau)$ as above over a scheme S over $\mathsf{M}_1^{\mathsf{Z}_1}$, there is a morphism $S \to \ddot{\Xi}_{\Phi_1}$ (over $\mathsf{M}_1^{\mathsf{Z}_1}$), which is unique after we fix an isomorphism $(f_X : X \xrightarrow{\sim} X, f_Y : Y \xrightarrow{\sim} Y)$ in Γ_{Φ_1}, such that the tuple over S is the pullback of the tautological tuple over $\ddot{\Xi}_{\Phi_1}$ if we identify X by f_X and Y by f_Y.

Remark 6.2.2.2. We use clumsy notation such as $\ddot{\Xi}_{\Phi_1}$ (instead of the simpler Ξ_{Φ_1}) because we expect it to have unwanted additional components in general. Later in Section 6.2.3 we will learn how to extract the exact components we want; that is, to drop the *unwanted dots*.

The construction of the algebraic stack $\ddot{\Xi}_{\Phi_1}$ over $\mathsf{M}_1^{\mathsf{Z}_1}$ with a tautological tuple $(A, \lambda_A, i_A, \varphi_{-1,1}, c, c^\vee, \tau)$ can be described as follows:

Over $\mathsf{M}_1^{\mathsf{Z}_1}$, we have the tautological tuple $(A, \lambda_A, i_A, \varphi_{-1,1})$. Therefore it only remains to construct the tautological triple $(c, c^\vee, \tau)$ that satisfies the conditions we want.

Consider the group functors $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$ and $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)$ over $\mathsf{M}_1^{\mathsf{Z}_1}$, over which we have the tautological homomorphisms $c : X \to A^\vee$ and $c^\vee : Y \to A$, respectively. By Proposition 5.2.3.9, the four group functors $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A)$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$, and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)$ are all relatively representable by (relative) proper smooth schemes whose fiberwise geometric identity components (see Definition 5.2.3.8) are (relative) abelian schemes over $\mathsf{M}_1^{\mathsf{Z}_1}$. By composition with $\phi : X \to Y$ (resp. $\lambda_A : A \to A^\vee$), we obtain a homomorphism $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)$ (resp. $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)$) with finite étale kernel. By definition, the compatibility condition $\lambda_A c^\vee = c\phi$ can be tautologically achieved over the fiber product $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y,A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$. As a result, we have the following:

Corollary 6.2.2.3. *The fiber product*
$$\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y,A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$$
above is also relatively representable by a (relative) proper smooth group scheme $\ddot{C}_{\Phi_1}$ over $\mathsf{M}_1^{\mathsf{Z}_1}$.

Here $\ddot{C}_{\Phi_1}$ may not be an abelian scheme because its geometric fibers may not be connected.

Proposition 6.2.2.4. 1. *Let $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A)^\circ$ be the fiberwise geometric identity component of $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A)$ (see Definition 5.2.3.8). Then the canonical homomorphism*
$$\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A)^\circ \to \underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y,A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$$
over $\mathsf{M}_1^{\mathsf{Z}_1}$ has kernel the finite étale group scheme
$$\underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), \ker(\lambda_A)) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(X, A)^\circ$$
and image an abelian subscheme $\ddot{C}^\circ_{\Phi_1}$ of $\ddot{C}_{\Phi_1}$ (see Lemma 5.2.3.7 and Definition 5.2.3.8).

2. *There exists an integer $m \geq 1$ such that multiplication by m maps $\ddot{C}_{\Phi_1}$ scheme-theoretically to a subscheme of $\ddot{C}^\circ_{\Phi_1}$, so that the group scheme $\underline{\pi}_0(\ddot{C}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ of fiberwise geometric connected components of $\ddot{C}_{\Phi_1}$ is defined (see Lemma 5.2.3.7 and Definition 5.2.3.8). Moreover, the rank of $\underline{\pi}_0(\ddot{C}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ (as a local constant) has no prime factors other than those of* Disc, *$[X : \phi(Y)]$, and the rank of $\ker(\lambda_A)$. In particular, $\underline{\pi}_0(\ddot{C}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ is finite étale over $\mathsf{M}_1^{\mathsf{Z}_1}$.*

Proof. The first claim of the lemma is clear, because the finite flat group scheme
$$\underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), \ker(\lambda_A)) = \underline{\mathrm{Hom}}_{\mathcal{O}}(X, \ker(\lambda_A)) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), A)$$
is the kernel of
$$\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y,A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A).$$

For the second claim, let $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)^\circ$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)^\circ$, and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)^\circ$ denote the fiberwise geometric identity components of $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$, and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)$, respectively, and let $\ddot{C}^{\circ\circ\circ}_{\Phi_1}$ denote the proper smooth group scheme representing the fiber product $\underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)^\circ \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y,A^\vee)^\circ}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)^\circ$. By 4 of Proposition 5.2.3.9, the group schemes $\underline{\pi}_0(\ddot{C}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ and $\underline{\pi}_0(\ddot{C}^{\circ\circ\circ}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ are defined, and their ranks differ up to multiplication by numbers with no prime factors other than those of Disc. Therefore it suffices to show that the rank of $\underline{\pi}_0(\ddot{C}^{\circ\circ\circ}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ has no prime factors other than those of $[X : \phi(Y)]$ and the rank of $\ker(\lambda_A)$.

The kernel K of the canonical homomorphism $\ddot{C}^{\circ\circ\circ}_{\Phi_1} \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)^\circ$ is given by a fiber product $K_1 \underset{\mathsf{M}_1^{\mathsf{Z}_1}}{\times} K_2$, where $K_1 := \underline{\mathrm{Hom}}_{\mathcal{O}}(X/\phi(Y), A^\vee) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)^\circ$ and $K_2 := \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, \ker(\lambda_A)) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)^\circ$. Since $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)^\circ$ is an abelian scheme, the group $\underline{\pi}_0(\ddot{C}^{\circ\circ\circ}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ can be identified with a quotient of K. Since the rank of K is the product of the ranks of K_1 and of K_2, it has no prime factors other than those of $X/\phi(Y)$ and the rank of $\ker(\lambda_A)$, as desired. □

For each section (y, χ) of $Y \times X$ over $\ddot{C}_{\Phi_1}$, we can interpret $(c^\vee(y), c(\chi))$ as a morphism from $\ddot{C}_{\Phi_1}$ to $A \underset{\mathsf{M}_1^{\mathsf{Z}_1}}{\times} A^\vee$, and consider the pullback invertible sheaf $(c^\vee(y), c(\chi))^*\mathcal{P}_A$ over $\ddot{C}_{\Phi_1}$.

Lemma 6.2.2.5. *The assignment of $(c^\vee(y), c(\chi))^*\mathcal{P}_A$ to (y, χ) satisfies the following properties:*

1. *Additivity in the first variable:*
$$(c^\vee(y), c(\chi))^*\mathcal{P}_A \underset{\mathcal{O}_{\ddot{C}_{\Phi_1}}}{\otimes} (c^\vee(y'), c(\chi))^*\mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y + y'), c(\chi))^*\mathcal{P}_A.$$

2. *Additivity in the second variable:*
$$(c^\vee(y), c(\chi))^*\mathcal{P}_A \underset{\mathcal{O}_{\ddot{C}_{\Phi_1}}}{\otimes} (c^\vee(y), c(\chi'))^*\mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y), c(\chi + \chi'))^*\mathcal{P}_A.$$

3. *Symmetry with respect to ϕ:*

$$\begin{aligned}(c^\vee(y), c(\phi(y')))^*\mathcal{P}_A &= (c^\vee(y), \lambda_A c^\vee(y'))^*\mathcal{P}_A\\ &\overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y), c^\vee(y'))^*(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A\\ &\overset{\text{sym.}}{\xrightarrow{\sim}} (c^\vee(y'), c^\vee(y))^*(\mathrm{Id}_A \times \lambda_A)^*\mathcal{P}_A\\ &\overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y'), \lambda_A c^\vee(y))^*\mathcal{P}_A = (c^\vee(y'), c(\phi(y)))^*\mathcal{P}_A.\end{aligned}$$

4. *Hermitian with respect to $\star$:*

$$\begin{aligned}(c^\vee(by), c(\chi))^*\mathcal{P}_A &\overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y), c(\chi))^*(i_A(b) \times \mathrm{Id}_{A^\vee})^*\mathcal{P}_A\\ &\overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y), c(\chi))^*(\mathrm{Id}_A \times i_A(b)^\vee)^*\mathcal{P}_A\\ &= (c^\vee(y), c(\chi))^*(\mathrm{Id}_A \times i_{A^\vee}(b^\star))^*\mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c^\vee(y), c(b^\star\chi))^*\mathcal{P}_A.\end{aligned}$$

Proof. These follow formally from the biextension structure of $\mathcal{P}_A$. □

Let us consider the finitely generated commutative group (i.e., the $\mathbb{Z}$-module)

$$\ddot{\mathbf{S}}_{\Phi_1} := (Y \underset{\mathbb{Z}}{\otimes} X)/\begin{pmatrix} y\otimes\phi(y') - y'\otimes\phi(y)\\ (by)\otimes\chi - y\otimes(b^\star\chi)\end{pmatrix}_{\substack{y,y'\in Y,\\ \chi\in X, b\in\mathcal{O}}}.$$

Remark 6.2.2.6. By Proposition 1.2.2.3, the cardinality of the torsion subgroup of $\ddot{\mathbf{S}}_{\Phi_1}$ only has prime factors dividing $\mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}[X:\phi(Y)]$. In particular, it is prime-to-$\Box$, because $\Box \nmid \mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}[X:\phi(Y)]$ by assumption.

By Lemma 6.2.2.5, if we assign to each

$$\ell = \sum_{1\le i\le k}[y_i\otimes\chi_i] \in \ddot{\mathbf{S}}_{\Phi_1}$$

the invertible sheaf

$$\Psi_1(\ell) := \bigotimes_{\mathscr{O}_{\ddot{C}_{\Phi_1}}, 1\le i\le k}(c^\vee(y_i), c(\chi_i))^*\mathcal{P}_A$$

over $\ddot{C}_{\Phi_1}$, then this assignment is well defined (i.e., independent of the expression of ℓ we choose), and there exists a canonical isomorphism $\Delta^*_{\ell,\ell'}:\Psi_1(\ell)\underset{\mathscr{O}_{\ddot{C}_{\Phi_1}}}{\otimes}\Psi_1(\ell')\xrightarrow{\sim}\Psi_1(\ell+\ell')$ for each $\ell,\ell'\in\ddot{\mathbf{S}}_{\Phi_1}$. As a result, we can form an $\mathscr{O}_{\ddot{C}_{\Phi_1}}$-algebra $\underset{\ell\in\ddot{\mathbf{S}}_{\Phi_1}}{\oplus}\Psi_1(\ell)$ with algebra structure given by the isomorphisms $\Delta^*_{\ell,\ell'}$ above, and define

$$\ddot{\Xi}_{\Phi_1} := \underline{\mathrm{Spec}}_{\mathscr{O}_{\ddot{C}_{\Phi_1}}}\Big(\underset{\ell\in\ddot{\mathbf{S}}_{\Phi_1}}{\oplus}\Psi_1(\ell)\Big).$$

If we denote by $\ddot{E}_{\Phi_1} := \underline{\mathrm{Hom}}(\ddot{\mathbf{S}}_{\Phi_1}, \mathbf{G}_{\mathrm{m}})$ the group of multiplicative type of finite type with character group $\ddot{\mathbf{S}}_{\Phi_1}$ over $\mathrm{Spec}(\mathbb{Z})$, then we see that there exists an étale surjective morphism $S'\to\ddot{C}_{\Phi_1}$ over which the pullbacks of all the invertible sheaves $\Psi_1(\ell)$ are trivialized (which is possible because $\ddot{\mathbf{S}}_{\Phi_1}$ is finitely generated), so that there is an isomorphism $\ddot{\Xi}_{\Phi_1}\underset{\ddot{C}_{\Phi_1}}{\times}S'\cong\ddot{E}_{\Phi_1}\underset{\mathrm{Spec}(\mathbb{Z})}{\times}S'$. In particular, this shows that $\ddot{\Xi}_{\Phi_1}$ is an $\ddot{E}_{\Phi_1}$-torsor.

Remark 6.2.2.7. This is essentially the same argument behind Theorem 3.1.3.3. The only issue is that the base scheme $\ddot{C}_{\Phi_1}$ does not necessarily satisfy Assumption 3.1.2.7. Therefore we have to either weaken the assumption and make the statement more clumsy in Theorem 3.1.3.3, or include some ad hoc explanation here.

For the trivialization $\tau:\mathbf{1}_{Y\times X}\xrightarrow{\sim}(c^\vee\times c)^*\mathcal{P}_A^{\otimes-1}$ of biextensions over a scheme $S\to\ddot{C}_{\Phi_1}$ to define an $\mathcal{O}$-equivariant period homomorphism $\iota:\underline{Y}\to G^\natural$, we need to identify $\tau(y,\phi(y'))$ with $\tau(y',\phi(y))$ under the canonical isomorphism $(c^\vee(y), c(\phi(y')))^*\mathcal{P}_A^{\otimes-1}\overset{\text{can.}}{\xrightarrow{\sim}}(c^\vee(y'), c(\phi(y)))^*\mathcal{P}_A^{\otimes-1}$, and identify $\tau(by,\chi)$ with $\tau(y,b^\star\chi)$ under the canonical isomorphism $(c^\vee(by), c(\chi))^*\mathcal{P}_A^{\otimes-1}\overset{\text{can.}}{\xrightarrow{\sim}}(c^\vee(y), c(b^\star\chi))^*\mathcal{P}_A^{\otimes-1}$, for each $y,y'\in Y$, $\chi\in X$, and $b\in\mathcal{O}$. Therefore, we can interpret τ as depending only on the class of $y\otimes\chi$ in $\ddot{\mathbf{S}}_{\Phi_1}$. The sections $\tau(y\otimes\chi)$ of $\Psi_1(y\otimes\chi)^{\otimes-1}$ correspond by definition to isomorphisms $\Psi_1(y\otimes\chi)\xrightarrow{\sim}\mathscr{O}_S$, which extend by linearity to isomorphisms $\Psi_1(\ell)\xrightarrow{\sim}\mathscr{O}_S$ for all $\ell\in\ddot{\mathbf{S}}_{\Phi_1}$. Hence they define an $\mathscr{O}_S$-algebra homomorphism $\mathscr{O}_{\ddot{\Xi}_{\Phi_1}}\cong\underset{\ell\in\ddot{\mathbf{S}}_{\Phi_1}}{\oplus}\Psi_1(\ell)\to\mathscr{O}_S$, corresponding to an S-valued point $S\to\ddot{\Xi}_{\Phi_1}$.

Conversely, since the $\Psi_1(\ell)$'s all appear in the structural sheaf of $\ddot{\Xi}_{\Phi_1}$ over $\ddot{C}_{\Phi_1}$, we have natural trivializations $\mathscr{O}_{\ddot{\Xi}_{\Phi_1}}\xrightarrow{\sim}\mathscr{O}_{\ddot{\Xi}_{\Phi_1}}\underset{\mathscr{O}_{\ddot{C}_{\Phi_1}}}{\otimes}\Psi_1(\ell)^{\otimes-1}$ for all $\ell\in\ddot{\mathbf{S}}_{\Phi_1}$. That is, we have a tautological trivialization $\tau:\mathbf{1}_{Y\times X}\xrightarrow{\sim}(c^\vee\times c)^*\mathcal{P}_A^{\otimes-1}$ of biextensions over $\ddot{\Xi}_{\Phi_1}$. Hence each S-valued point of $\ddot{\Xi}_{\Phi_1}$ admits by pullback a tautological trivialization $\tau:\mathbf{1}_{Y\times X}\xrightarrow{\sim}(c^\vee\times c)^*\mathcal{P}_A^{\otimes-1}$ of biextensions.

Note that there is an ambiguity in the identification of (X,Y,ϕ): In our definition of isomorphism classes, we always allow homomorphisms involving X and Y to be compatibly twisted by a pair of isomorphisms (γ_X,γ_Y) in Γ_{Φ_1}. Therefore we should consider the natural action of Γ_{Φ_1} on $\ddot{\Xi}_{\Phi_1}\to\ddot{C}_{\Phi_1}\to\mathsf{M}_1^{\mathsf{Z}_1}$, and consider the quotient $\ddot{\Xi}_{\Phi_1}/\Gamma_{\Phi_1}$ as the universal parameter space, whose structural morphism can be factorized as $\ddot{\Xi}_{\Phi_1}/\Gamma_{\Phi_1}\to\ddot{C}_{\Phi_1}/\Gamma_{\Phi_1}\to\mathsf{M}_1^{\mathsf{Z}_1}$. (Note that $\ddot{\Xi}_{\Phi_1}/\Gamma_{\Phi_1}$ is not necessarily an algebraic stack according to our convention, as we only allow Deligne–Mumford stacks.)

As a result, if S is a scheme over $\mathsf{M}_1^{\mathsf{Z}_1}$ such that there is a tuple $(A,\lambda_A,i_A,\varphi_{-1,1},c,c^\vee,\tau)$ over S with the prescribed $\phi:Y\to X$ describing a degeneration datum without the positivity condition, then after a choice of an isomorphism in Γ_{Φ_1} giving the identification of (X,Y,ϕ) on S and on $\ddot{\Xi}_{\Phi_1}$, there is a unique morphism $S\to\ddot{\Xi}_{\Phi_1}$ over $\mathsf{M}_1^{\mathsf{Z}_1}$ such that the tuple is the pullback of the tautological tuple on $\ddot{\Xi}_{\Phi_1}$.

This finishes the construction of $\ddot{\Xi}_{\Phi_1}$ and proves Proposition 6.2.2.1.

6.2.3 Construction with Principal Level Structures

Let us take the level-n structures (see Definition 1.3.6.2) into consideration.

Let $n\ge1$ be any (positive) integer such that $\Box\nmid n$, and let $(\mathsf{Z}_n,\Phi_n,\delta_n)$ be a representative of a (principal) cusp label at level n (see Definition 5.4.1.9). Then Z_n alone defines a moduli problem $\mathsf{M}_n^{\mathsf{Z}_n}$ with tautological tuple $(A,\lambda_A,i_A,\varphi_{-1,n})$, as described in Section 6.2.1. We would like to construct an algebraic stack Ξ_{Φ_n,δ_n} separate, smooth, and schematic over $\mathsf{M}_n^{\mathsf{Z}_n}$ over which there is a tautological tuple

$$(\mathsf{Z}_n,(X,Y,\phi,\varphi_{-2,n},\varphi_{0,n}),(A,\lambda_A,i_A,\varphi_{-1,n}),\delta_n,(c_n,c_n^\vee,\tau_n))$$

like a tuple in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_n}$ for some R and I (see Definition 5.4.1.1), but (without any R and I, and) without the positivity condition. For this purpose, we need to have a tautological triple $(c_n,c_n^\vee,\tau_n)$ lifting $(c,c^\vee,\tau)$, and this tautological triple has to be liftable to some $(c_m,c_m^\vee,\tau_m)$ for each $n|m$, $\Box\nmid m$, as described in Definition 5.2.3.4

and Corollary 5.2.3.5.

The prescribed liftable splitting $\delta_n : \mathrm{Gr}^{\mathbf{Z}}_n \xrightarrow{\sim} L/nL$ defines two pairings
$$\langle\,\cdot\,,\,\cdot\,\rangle_{10,n} : \mathrm{Gr}^{\mathbf{Z}}_{-1,n} \times \mathrm{Gr}^{\mathbf{Z}}_{0,n} \to (\mathbb{Z}/n\mathbb{Z})(1)$$
and
$$\langle\,\cdot\,,\,\cdot\,\rangle_{00,n} : \mathrm{Gr}^{\mathbf{Z}}_{0,n} \times \mathrm{Gr}^{\mathbf{Z}}_{0,n} \to (\mathbb{Z}/n\mathbb{Z})(1).$$
The prescribed level-n structure $\varphi_{-1,n} : \mathrm{Gr}^{\mathbf{Z}}_{-1,n} \xrightarrow{\sim} A[n]_\eta$ defines a liftable isomorphism $f_{-1,n} := \varphi_{-1,n}$ by itself, and determines a necessarily unique liftable isomorphism $\nu(\hat{f}_{-1}) := \nu(\varphi_{-1,n}) : (\mathbb{Z}/n\mathbb{Z})(1) \xrightarrow{\sim} \boldsymbol{\mu}_{n,\eta}$. On the other hand, the prescribed liftable isomorphism $f_{0,n} := \varphi_{0,n} : \mathrm{Gr}^{\mathbf{Z}}_{0,n} \xrightarrow{\sim} Y/nY$ defines (by abuse of notation) a liftable isomorphism $\varphi_{0,n} : \mathrm{Gr}^{\mathbf{Z}}_{0,n} \xrightarrow{\sim} \frac{1}{n}Y/Y$ via the canonical isomorphism $Y/nY \xrightarrow{\sim} \frac{1}{n}Y/Y$, which we again denote by $f_{0,n}$. Therefore we have two liftable pairings
$$(f^{-1}_{-1,n} \times f^{-1}_{0,n})^*(\langle\,\cdot\,,\,\cdot\,\rangle_{10,n}) : A[n]_\eta \times \tfrac{1}{n}Y/Y \to (\mathbb{Z}/n\mathbb{Z})(1)$$
and
$$(f^{-1}_{0,n} \times f^{-1}_{0,n})^*(\langle\,\cdot\,,\,\cdot\,\rangle_{00,n}) : \tfrac{1}{n}Y \times \tfrac{1}{n}Y \to (\mathbb{Z}/n\mathbb{Z})(1).$$

Lemma 6.2.3.1. *The choice of a representative (Φ_n, δ_n) of a cusp label determines a liftable homomorphism $b_{\Phi_n,\delta_n} : \frac{1}{n}Y \to A^\vee[n]$ and a liftable pairing $a_{\Phi_n,\delta_n} : \frac{1}{n}Y \times \frac{1}{n}Y \to \mathbf{G}_{\mathrm{m}}$, by requiring*
$$(f^{-1}_{-1,n} \times f^{-1}_{0,n})^*(\langle\,\cdot\,,\,\cdot\,\rangle_{10,n})(a, \tfrac{1}{n}y) = \nu(f_{-1,n})^{-1} \circ \mathrm{e}_{A[n]}(a, b_{\Phi_n,\delta_n}(\tfrac{1}{n}y))$$
and
$$(f^{-1}_{0,n} \times f^{-1}_{0,n})^*(\langle\,\cdot\,,\,\cdot\,\rangle_{00,n})(\tfrac{1}{n}y, \tfrac{1}{n}y') = \nu(f_{-1,n})^{-1} \circ a_{\Phi_n,\delta_n}(\tfrac{1}{n}y, \tfrac{1}{n}y')$$
for all $a \in A[n]_\eta$ and $\frac{1}{n}y, \frac{1}{n}y' \in \frac{1}{n}Y$.

Following Definition 5.2.7.8 and Proposition 5.2.7.9, we shall require $\lambda_A c_n^\vee - c_n\phi_n$ to agree with $b_{\Phi_n,\delta_n} : \frac{1}{n}Y \to A^\vee[n]$, and require τ_n to define a pairing that agrees with $a_{\Phi_n,\delta_n} : \frac{1}{n}Y \times \frac{1}{n}Y \to \mathbf{G}_{\mathrm{m}}$ in the sense that
$$\tau_n(\tfrac{1}{n}y, \phi(y'))\tau_n(\tfrac{1}{n}y', \phi(y))^{-1} = a_{\Phi_n,\delta_n}(\tfrac{1}{n}y, \tfrac{1}{n}y')$$
for all $\frac{1}{n}y, \frac{1}{n}y' \in \frac{1}{n}Y$, on the algebraic stack Ξ_{Φ_n,δ_n} we will construct.

Let us consider the homomorphism $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A^\vee) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)$ (resp. $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$) defined by restriction from $\frac{1}{n}X$ to X (resp. $\frac{1}{n}Y$ to Y). Let $\ddot{C}_{\Phi_n}$ be the (relative) proper smooth group scheme over $\mathsf{M}^{\mathbf{Z}_n}_n$ representing the fiber product
$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y,A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A)$$
(see Proposition 5.2.3.9 and Corollary 6.2.2.3). Then there is a canonical homomorphism $\ddot{C}_{\Phi_n} \to \ddot{C}_{\Phi_1} \underset{\mathsf{M}^{\mathbf{Z}_1}_1}{\times} \mathsf{M}^{\mathbf{Z}_n}_n$ corresponding to the restriction homomorphisms $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A^\vee) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(X, A^\vee)$ and $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A)$ above.

The structure of $\ddot{C}_{\Phi_n}$ can be analyzed as in Proposition 6.2.2.4:

Proposition 6.2.3.2. *1. Let $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A)^\circ$ be the fiberwise geometric identity component of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A)$ (see 4 of Proposition 5.2.3.9). Then the canonical homomorphism*
$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A)^\circ \to \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y,A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A)$$
over $\mathsf{M}^{\mathbf{Z}_n}_n$ has kernel the finite étale group scheme
$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X/\phi_n(\tfrac{1}{n}Y), \ker(\lambda_A)) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A)^\circ$$
and image an abelian subscheme $\ddot{C}^\circ_{\Phi_n}$ of $\ddot{C}_{\Phi_n}$ (see Lemma 5.2.3.7 and Definition 5.2.3.8).

2. There exists an integer $m \geq 1$ such that multiplication by m maps $\ddot{C}_{\Phi_n}$ scheme-theoretically to a subscheme of $\ddot{C}^\circ_{\Phi_n}$, so that the group scheme $\underline{\pi}_0(\ddot{C}_{\Phi_n}/\mathsf{M}^{\mathbf{Z}_n}_n)$ of fiberwise connected components of $\ddot{C}_{\Phi_n}$ over $\mathsf{M}^{\mathbf{Z}_n}_n$ is defined (see Lemma 5.2.3.7 and Definition 5.2.3.8). Moreover, the rank of $\underline{\pi}_0(\ddot{C}_{\Phi_n}/\mathsf{M}^{\mathbf{Z}_n}_n)$ (as a local constant) has no prime factors other than those of Disc, *n, $[X : \phi(Y)]$, and the rank of $\ker(\lambda_A)$. (This implies that the rank of $\underline{\pi}_0(\ddot{C}_{\Phi_n}/\mathsf{M}^{\mathbf{Z}_n}_n)$ has no prime factors other than those of* Disc, *n and $[L^\# : L]$.) In particular, $\underline{\pi}_0(\ddot{C}_{\Phi_n}/\mathsf{M}^{\mathbf{Z}_n}_n)$ is finite étale over $\mathsf{M}^{\mathbf{Z}_n}_n$.*

Proof. The first claim of the lemma is clear, because the finite flat group scheme
$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X/\phi_n(\tfrac{1}{n}Y), \ker(\lambda_A)) = \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, \ker(\lambda_A)) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X/\phi_n(\tfrac{1}{n}Y), A)$$
is the kernel of
$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y,A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A).$$

For the second claim, let $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A^\vee)^\circ$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A)^\circ$, and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)^\circ$ denote the fiberwise geometric identity component of $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A^\vee)$, $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A)$, and $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)$, respectively, and let $\ddot{C}^{\circ\circ\circ}_{\Phi_n}$ denote the proper smooth group scheme representing the fiber product $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A^\vee)^\circ \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y,A^\vee)^\circ}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A)^\circ$. By 4 of Proposition 5.2.3.9, the group schemes $\underline{\pi}_0(\ddot{C}_{\Phi_n}/\mathsf{M}^{\mathbf{Z}_n}_n)$ and $\underline{\pi}_0(\ddot{C}^{\circ\circ\circ}_{\Phi_n}/\mathsf{M}^{\mathbf{Z}_n}_n)$ are defined, and their ranks differ up to multiplication by numbers having only prime factors of those of Disc. Therefore it suffices to show that the rank of $\underline{\pi}_0(\ddot{C}^{\circ\circ\circ}_{\Phi_n}/\mathsf{M}^{\mathbf{Z}_n}_n)$ has no prime factors other than those of n, $[X : \phi(Y)]$, and the rank of $\ker(\lambda_A)$.

The kernel K_n of the canonical homomorphism $\ddot{C}^{\circ\circ\circ}_{\Phi_n} \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)^\circ$ is given by a fiber product $K_{n,1} \underset{\mathsf{M}^{\mathbf{Z}_n}_n}{\times} K_{n,2}$, where $K_{n,1} := \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X/\phi(Y), A^\vee) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A^\vee)^\circ$ and $K_{n,2} := \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, \ker(n\lambda_A)) \cap \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A)^\circ$. Since $\underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)^\circ$ is an abelian scheme, the group $\underline{\pi}_0(\ddot{C}^{\circ\circ\circ}_{\Phi_n}/\mathsf{M}^{\mathbf{Z}_n}_n)$ can be identified with a quotient of K_n. Since the rank of K_n is the product of the ranks of $K_{n,1}$ and of $K_{n,2}$, it has no prime factors other than those of n, $X/\phi(Y)$, and the rank of $\ker(\lambda_A)$, as desired. $\square$

Let us consider the natural homomorphism
$$\partial^{(1)}_n : \ddot{C}_{\Phi_n} \to \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A^\vee)$$
defined by sending a pair $(c_n, c_n^\vee)$ to $\lambda_A c_n^\vee - c_n\phi_n \in \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A^\vee)$ as considered in Lemma 5.2.3.12. Note that for each pair $(c_n, c_n^\vee)$ in $\ddot{C}_{\Phi_n}$, the associated $\partial^{(1)}_n(c_n, c_n^\vee)$ actually lies in the kernel of the canonical restriction homomorphism $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A^\vee) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Y, A^\vee)$, or rather $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y/Y, A^\vee[n])$, because c_n and $c_n^\vee$ satisfy $\lambda_A(c_n^\vee|_Y) = \lambda_A c^\vee = c\phi = (c_n|_X)(\phi_n|_Y)$ when restricted to Y, or in other words when multiplied by n, by the definition of $\ddot{C}_{\Phi_n}$ as a fiber product. Therefore the natural homomorphism $\partial^{(1)}_n$ can be written as
$$\partial^{(1)}_n : \ddot{C}_{\Phi_n} \to \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y/Y, A^\vee[n]),$$
and it makes sense to talk about the *fiber* $\ddot{C}_{\Phi_n,b_n} := (\partial^{(1)}_n)^{-1}(b_n)$ of $\partial^{(1)}_n$ over a particular section $b_n \in \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y/Y, A^\vee[n])$.

Over $\ddot{C}_{\Phi_n}$, we have two tautological homomorphisms $c_n : \frac{1}{n}X \to A^\vee$ and $c_n^\vee : \frac{1}{n}Y \to A$, where the restriction of c_n to X (resp. $c_n^\vee$ to Y) is the pullback of $c : X \to A^\vee$ (resp. $c^\vee : Y \to A$) via $\ddot{C}_{\Phi_n} \to \ddot{C}_{\Phi_1}$. Then the upshot of defining $\ddot{C}_{\Phi_n,b_n}$ is that

$$\lambda_A c_n^\vee - c_n\phi_n = b_n \tag{6.2.3.3}$$

holds tautologically over $\ddot{C}_{\Phi_n,b_n}$.

Lemma 6.2.3.4. *The homomorphism $\partial_n^{(1)}$ induces a canonical homomorphism $\ddot{C}_{\Phi_n} \twoheadrightarrow \underline{\pi}_0(\ddot{C}_{\Phi_n}/\mathsf{M}_n^{\mathsf{Z}_n}) \to \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y/Y, A^\vee[n])$. In particular, the fibers $\ddot{C}_{\Phi_n,b_n}$ of $\partial_n^{(1)}$ are (possibly empty) proper smooth subschemes of $\ddot{C}_{\Phi_n}$ over $\mathsf{M}_n^{\mathsf{Z}_n}$.*

Note that $\ddot{C}_{\Phi_n,0}$, where 0 stands for the trivial homomorphism $\frac{1}{n}Y \to A^\vee[n]$ sending everything to the identity, is the proper smooth group scheme representing the fiber product

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}X, A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\tfrac{1}{n}Y, A).$$

Moreover, as soon as $\ddot{C}_{\Phi_n,b_n}$ has a section $\tilde{\partial^{(1)}}_n$ (whose image under $\partial_n^{(1)}$ is b_n), $\ddot{C}_{\Phi_n,b_n}$ is necessarily the translation of $\ddot{C}_{\Phi_n,0}$ by this section $\tilde{\partial^{(1)}}_n$ of $\ddot{C}_{\Phi_n}$.

Now we shall construct a scheme $\ddot{\Xi}_{\Phi_n}$ as in the case of $\ddot{\Xi}_{\Phi_1}$ in Proposition 6.2.2.1, which provides us with a tautological triple $(c_n^\vee, c_n, \tau_n)$ on top of the tautological tuple $(A, \lambda_A, i_A, \varphi_{-1,n}, X, Y, \phi, c, c^\vee, \tau)$ over $\ddot{\Xi}_{\Phi_1} \underset{\mathsf{M}_1^{\mathsf{Z}_1}}{\times} \mathsf{M}_n^{\mathsf{Z}_n}$. Here Z_1 is the "reduction modulo 1" of Z_n, in the sense that we still keep the liftability and the ranks as conditions (see Remark 5.2.2.8).

Let us consider the finitely generated commutative group (i.e., the $\mathbb{Z}$-module)

$$\ddot{\mathbf{S}}_{\Phi_n} := ((\tfrac{1}{n}Y) \underset{\mathbb{Z}}{\otimes} X) / \begin{pmatrix} y \otimes \phi(y') - y' \otimes \phi(y) \\ (b\frac{1}{n}y) \otimes \chi - (\frac{1}{n}y) \otimes (b^\star \chi) \end{pmatrix}_{\substack{y,y' \in Y, \\ \chi \in X, b \in \mathcal{O}}}. \tag{6.2.3.5}$$

Remark 6.2.3.6. By Proposition 1.2.2.3, the cardinality of the torsion subgroup of $\ddot{\mathbf{S}}_{\Phi_n}$ only has prime factors dividing $n\,\mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}[X : \phi(Y)]$ (cf. Remark 6.2.2.6). In particular, it is prime-to-$\Box$, because $\Box \nmid n\,\mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}[X : \phi(Y)]$ by assumption.

As in the construction of $\ddot{\Xi}_{\Phi_1}$, the formal properties of the pullbacks of the Poincaré biextension (as in Lemma 6.2.2.5) allow us to assign to each

$$\ell = \sum_{1 \le i \le k} [(\tfrac{1}{n}y_i) \otimes \chi_i] \in \ddot{\mathbf{S}}_{\Phi_n}$$

a well-defined invertible sheaf

$$\Psi_n(\ell) := \underset{\mathscr{O}_{\ddot{C}_{\Phi_n}}, 1 \le i \le k}{\otimes} (c_n^\vee(\tfrac{1}{n}y_i), c(\chi_i))^* \mathcal{P}_A$$

over $\ddot{C}_{\Phi_n}$, such that there exists a canonical isomorphism $\Delta^*_{n,\ell,\ell'} : \Psi_n(\ell) \underset{\mathscr{O}_{\ddot{C}_{\Phi_n}}}{\otimes} \Psi_n(\ell') \xrightarrow{\sim} \Psi_n(\ell + \ell')$ for each $\ell, \ell' \in \ddot{\mathbf{S}}_{\Phi_n}$. As a result, we can form an $\mathscr{O}_{\ddot{C}_{\Phi_n}}$-algebra $\underset{\ell \in \ddot{\mathbf{S}}_{\Phi_n}}{\oplus} \Psi_n(\ell)$ with algebra structure given by the isomorphisms $\Delta^*_{n,\ell,\ell'}$ above, and define

$$\ddot{\Xi}_{\Phi_n} := \underline{\mathrm{Spec}}_{\mathscr{O}_{\ddot{C}_{\Phi_n}}} \Big(\underset{\ell \in \ddot{\mathbf{S}}_{\Phi_n}}{\oplus} \Psi_n(\ell) \Big).$$

If we denote by $\ddot{E}_{\Phi_n} = \underline{\mathrm{Hom}}(\ddot{\mathbf{S}}_{\Phi_n}, \mathbf{G}_{\mathrm{m}})$ the group of multiplicative type of finite type with character group $\ddot{\mathbf{S}}_{\Phi_n}$ over $\mathrm{Spec}(\mathbb{Z})$, then $\ddot{\Xi}_{\Phi_n}$ is an $\ddot{E}_{\Phi_n}$-torsor, by the same argument as in the case of the $\ddot{E}_{\Phi_1}$-torsor $\ddot{\Xi}_{\Phi_1}$. Moreover, we have a tautological trivialization $\tau_n : \mathbf{1}_{(\frac{1}{n}Y) \times X} \xrightarrow{\sim} (c_n^\vee \times c)^* \mathcal{P}_A^{\otimes -1}$ of biextensions over $\ddot{\Xi}_{\Phi_n}$, which corresponds to a tautological homomorphism $\iota_n : \frac{1}{n}Y \to G^\natural$. Let $\tau : \mathbf{1}_{Y \times X} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_A^{\otimes -1}$ be the restriction of τ_n to $\mathbf{1}_{Y \times X}$, which corresponds to a tautological homomorphism $\iota : Y \to G^\natural$.

Let $\ddot{\mathbf{S}}_{\Phi_n,\mathrm{tor}}$ denote the subgroup of all torsion elements in $\ddot{\mathbf{S}}_{\Phi_n}$, and let $\ddot{\mathbf{S}}_{\Phi_n,\mathrm{free}}$ denote the quotient of $\ddot{\mathbf{S}}_{\Phi_n}$ by $\ddot{\mathbf{S}}_{\Phi_n,\mathrm{tor}}$, namely, the free commutative quotient group of $\ddot{\mathbf{S}}_{\Phi_n}$. Let $\ddot{E}_{\Phi_n,\mathrm{tor}} := \underline{\mathrm{Hom}}(\ddot{\mathbf{S}}_{\Phi_n,\mathrm{tor}}, \mathbf{G}_{\mathrm{m}})$ (resp. $\ddot{E}_{\Phi_n,\mathrm{free}} := \underline{\mathrm{Hom}}(\ddot{\mathbf{S}}_{\Phi_n,\mathrm{free}}, \mathbf{G}_{\mathrm{m}})$) be the group of multiplicative type of finite type with character group $\ddot{\mathbf{S}}_{\Phi_n,\mathrm{tor}}$ (resp. $\ddot{\mathbf{S}}_{\Phi_n,\mathrm{free}}$) over $\mathrm{Spec}(\mathbb{Z})$. Then the exact sequence

$$0 \to \ddot{\mathbf{S}}_{\Phi_n,\mathrm{tor}} \to \ddot{\mathbf{S}}_{\Phi_n} \to \ddot{\mathbf{S}}_{\Phi_n,\mathrm{free}} \to 0$$

induces an exact sequence

$$0 \to \ddot{E}_{\Phi_n,\mathrm{free}} \to \ddot{E}_{\Phi_n} \to \ddot{E}_{\Phi_n,\mathrm{tor}} \to 0$$

in the reversed direction. Note that $\ddot{E}_{\Phi_n,\mathrm{free}}$ is a torus (see Definition 3.1.1.5), because $\ddot{\mathbf{S}}_{\Phi_n,\mathrm{free}}$ is a finitely generated free commutative group.

Consider the subgroup $\dddot{\mathbf{S}}^{(n)}_{\Phi_n}$ of $\ddot{\mathbf{S}}_{\Phi_n}$ generated by $[(\frac{1}{n}y) \otimes \phi(y')] - [(\frac{1}{n}y') \otimes \phi(y)]$, where $\frac{1}{n}y$ and $\frac{1}{n}y'$ run through arbitrary elements of $\frac{1}{n}Y$. This subgroup is torsion, because the nth multiple of every element is part of the defining relation in $\ddot{\mathbf{S}}_{\Phi_n}$.

By the very definition of $\dddot{\mathbf{S}}^{(n)}_{\Phi_n}$,

Lemma 6.2.3.7. *The group $\dddot{E}^{(n)}_{\Phi_n} := \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}^{(n)}_{\Phi_n}, \mathbf{G}_{\mathrm{m}})$ is canonically isomorphic to the group of (alternating) pairings $a_n : \frac{1}{n}Y \times \frac{1}{n}Y \to \mathbf{G}_{\mathrm{m}}$ such that there exists some (not necessarily alternating) pairing $a'_n : \frac{1}{n}Y \times Y \to \mathbf{G}_{\mathrm{m}}$ satisfying $a_n(\frac{1}{n}y, \frac{1}{n}y') = a'_n(\frac{1}{n}y, y')a'_n(\frac{1}{n}y', y)^{-1}$, $a'_n(\frac{1}{n}y, y') = a'_n(\frac{1}{n}y', y)$, and $a'_n(b\frac{1}{n}y, y') = a'_n(\frac{1}{n}y, b^\star y')$ for all $y, y' \in Y$ and $b \in \mathcal{O}$.*

We shall henceforth identify $\dddot{E}^{(n)}_{\Phi_n} = \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}^{(n)}_{\Phi_n}, \mathbf{G}_{\mathrm{m}})$ with the group of such pairings, and write $a_n \in \dddot{E}^{(n)}_{\Phi_n}$ in this case.

Lemma 6.2.3.8. *The scheme $\dddot{\Xi}^{(n)}_{\Phi_n} := \underline{\mathrm{Spec}}_{\mathscr{O}_{\ddot{C}_{\Phi_n}}} \Big(\underset{\ell \in \dddot{\mathbf{S}}^{(n)}_{\Phi_n}}{\oplus} \Psi_n(\ell) \Big)$ is a canonically trivial $\dddot{E}^{(n)}_{\Phi_n}$-torsor over $\ddot{C}_{\Phi_n}$. Namely, there is a canonical isomorphism $\dddot{\Xi}^{(n)}_{\Phi_n} \xrightarrow{\sim} \dddot{E}^{(n)}_{\Phi_n} \underset{\mathrm{Spec}(\mathbb{Z})}{\times} \ddot{C}_{\Phi_n}$, which is $\ddot{E}_{\Phi_n}$-equivariant and defines a canonical $\ddot{E}_{\Phi_n}$-equivariant section*

$$\partial_n^{(0)} : \ddot{\Xi}_{\Phi_n} \to \dddot{E}^{(n)}_{\Phi_n}$$

by the composition $\ddot{\Xi}_{\Phi_n} \twoheadrightarrow \dddot{\Xi}^{(n)}_{\Phi_n} \twoheadrightarrow \dddot{E}^{(n)}_{\Phi_n}$. In particular, $\partial_n^{(0)}$ is surjective.

Proof. By definition of $\dddot{\Xi}^{(n)}_{\Phi_n}$, every $\ell \in \dddot{\Xi}^{(n)}_{\Phi_n}$ is of the form $\ell = [\tilde{\ell}]$, where

$$\tilde{\ell} = \sum_{1 \le i \le k} ((\tfrac{1}{n}y_i) \otimes \phi(y'_i) - (\tfrac{1}{n}y'_i) \otimes \phi(y_i))$$

for some $y_i, y'_i \in Y$. Then we have a canonical isomorphism

$$\Psi_n(\ell) \overset{\text{can.}}{\xrightarrow{\sim}} \underset{1 \le i \le k}{\otimes} \Big((c_n^\vee(\tfrac{1}{n}y_i), c(\phi(y'_i)))^* \mathcal{P}_A \otimes (c_n^\vee(\tfrac{1}{n}y'_i), c(\phi(y_i)))^* \mathcal{P}_A^{\otimes -1} \Big).$$

Since there is a canonical symmetry isomorphism

$$(c_n^\vee(\tfrac{1}{n}y_i), c(\phi(y'_i)))^* \mathcal{P}_A \overset{\text{can.}}{\xrightarrow{\sim}} (c_n^\vee(\tfrac{1}{n}y'_i), c(\phi(y_i)))^* \mathcal{P}_A$$

for each $1 \le i \le k$, we obtain a canonical trivialization

$$\Psi_n(\ell) \xrightarrow{\sim} \mathscr{O}_{\ddot{C}_{\Phi_n}}$$

for each $\ell \in \ddot{\Xi}^{(n)}_{\Phi_n}$. Moreover, the trivializations for various different $\ell, \ell' \in \ddot{\Xi}^{(n)}_{\Phi_n}$ are compatible under the canonical isomorphisms $\Delta^*_{n,\ell,\ell'} : \Psi_n(\ell) \underset{\mathscr{O}_{\ddot{C}^{\mathrm{com.}}_{\Phi_n}}}{\otimes} \Psi_n(\ell') \xrightarrow{\sim} \Psi_n(\ell+\ell')$ because they all involve the same canonical biextension properties of $\mathcal{P}_A$. Therefore we have an isomorphism between $\mathscr{O}_{\ddot{C}_{\Phi_n}}$-algebras, which defines an isomorphism $\dddot{E}^{(n)}_{\Phi_n} \underset{\mathrm{Spec}(\mathbb{Z})}{\times} \ddot{C}_{\Phi_n} \xrightarrow{\sim} \ddot{\Xi}^{(n)}_{\Phi_n}$, as desired. □

On the other hand, for each $\frac{1}{n}y$ and $\frac{1}{n}y'$, we can define a section $\ddot{\Xi}_{\Phi_n} \to \mathbf{G}_{\mathrm{m}}$ by taking the difference $\tau_n(\frac{1}{n}y, \phi(y'))\tau_n(\frac{1}{n}y', \phi(y))^{-1} \in \mathbf{G}_{\mathrm{m}}(\ddot{\Xi}_{\Phi_n})$ between the two tautological sections $\tau_n(\frac{1}{n}y, \phi(y'))$ and $\tau_n(\frac{1}{n}y', \phi(y))$ under the canonical symmetry isomorphism $(c_n^\vee(\frac{1}{n}y), c(\phi(y')))^*\mathcal{P}_A^{\otimes -1} \overset{\mathrm{can.}}{\xrightarrow{\sim}} (c_n^\vee(\frac{1}{n}y'), c(\phi(y)))^*\mathcal{P}_A^{\otimes -1}$ over $\ddot{\Xi}_{\Phi_n}$, which is a special case of the one we used in the proof of Lemma 6.2.3.8. Hence we obtain the following corollary:

Corollary 6.2.3.9. *For each $\frac{1}{n}y$ and $\frac{1}{n}y'$, the tautological section $\ddot{\Xi}_{\Phi_n} \to \mathbf{G}_{\mathrm{m}}$ defined by $\tau_n(\frac{1}{n}y, \phi(y'))\tau_n(\frac{1}{n}y', \phi(y))^{-1} \in \mathbf{G}_{\mathrm{m}}(\ddot{\Xi}_{\Phi_n})$ agrees with the evaluation of the canonical section $\partial^{(0)}_n : \ddot{\Xi}_{\Phi_n} \to \dddot{E}^{(n)}_{\Phi_n} = \underline{\mathrm{Hom}}(\ddot{\mathbf{S}}^{(n)}_{\Phi_n}, \mathbf{G}_{\mathrm{m}})$ in Lemma 6.2.3.8 at $[(\frac{1}{n}y) \otimes \phi(y')] - [(\frac{1}{n}y') \otimes \phi(y)] \in \ddot{\mathbf{S}}^{(n)}_{\Phi_n}$.*

Then it makes sense to talk about the fiber $\ddot{\Xi}_{\Phi_n,a_n} := (\partial^{(0)}_n)^{-1}(a_n)$ of $\partial^{(0)}_n$ over a particular section $a_n \in \dddot{E}^{(n)}_{\Phi_n}$, as in the case of $\ddot{C}_{\Phi_n,b_n}$. The upshot of defining $\ddot{\Xi}_{\Phi_n,a_n}$ is that we have a tautological relation

$$\tau_n(\tfrac{1}{n}y, \phi(y'))\tau_n(\tfrac{1}{n}y', \phi(y))^{-1} = a_n(\tfrac{1}{n}y, \tfrac{1}{n}y') \qquad (6.2.3.10)$$

for all $\frac{1}{n}y, \frac{1}{n}y' \in \frac{1}{n}Y$ over $\ddot{\Xi}_{\Phi_n,a_n}$. If we define

$$\ddot{\Xi}_{\Phi_n,(b_n,a_n)} := \ddot{\Xi}_{\Phi_n,a_n}|_{\ddot{C}_{\Phi_n,b_n}} := \ddot{\Xi}_{\Phi_n,a_n} \underset{\ddot{C}_{\Phi_n}}{\times} \ddot{C}_{\Phi_n,b_n},$$

then we have both the tautological relations (6.2.3.3) and (6.2.3.10) over $\ddot{\Xi}_{\Phi_n,(b_n,a_n)}$.

As a result, we have obtained a tautological triple $(c_n, c_n^\vee, \tau_n)$ over $\ddot{\Xi}_{\Phi_n,(b_n,a_n)}$, on top of the tautological tuple $(A, \lambda_A, i_A, \varphi_{-1,n}, X, Y, \phi, c, c^\vee, \tau)$ over $\ddot{\Xi}_\phi \underset{\mathsf{M}_1^{\mathsf{Z}_1}}{\times} \mathsf{M}_n^{\mathsf{Z}_n}$, satisfying the tautological relations (6.2.3.3) and (6.2.3.10). It remains to find a subalgebraic stack of $\ddot{\Xi}_{\Phi_n,(b_n,a_n)}$ over which the tautological triple $(c_n, c_n^\vee, \tau_n)$ is liftable to some system $(\hat{c}, \hat{c}^\vee, \hat{\tau}) := \{(c_m, c_m^\vee, \tau_m)\}_{n|m,\Box\nmid m}$ (as in Definition 5.2.3.4) which is compatible with some system $(\hat{b}, \hat{a}) := \{(b_m, a_m)\}_{n|m,\Box\nmid m}$ lifting (b_n, a_n) in the natural sense.

To achieve this, let us consider the common schematic image $\ddot{\Xi}^{\mathrm{com.}}_{\Phi_n,(b_n,a_n)}$ of the canonical morphisms $\ddot{\Xi}_{\Phi_m,(b_m,a_m)} \to \ddot{\Xi}_{\Phi_n,(b_n,a_n)}$ for m such that $n|m$ and $\Box \nmid m$. This common image $\ddot{\Xi}^{\mathrm{com.}}_{\Phi_n,(b_n,a_n)}$ will cover the common schematic image $\ddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}$ of the canonical morphisms $\ddot{C}_{\Phi_m,b_m} \to \ddot{C}_{\Phi_n,b_n}$.

Lemma 6.2.3.11. *Under the assumption that we have chosen a system $\hat{b} = \{b_m\}_{n|m,\Box\nmid m}$ lifting b_n, we have a recipe to produce a uniquely determined system of elements $\{\tilde{b}_m\}_{n|m,\Box\nmid m}$ such that $\tilde{b}_m \in \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{m}X/\phi(Y), A^\vee) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(Y,A^\vee)}{\times} \{e\} \subset \ddot{C}_{\Phi_m}$, such that $\partial^{(1)}_m(\tilde{b}_m) = b_m$, and such that $\tilde{b}_l$ is mapped to $\tilde{b}_m$ under the canonical homomorphism $\ddot{C}_{\Phi_l} \to \ddot{C}_{\Phi_m}$ for each $m|l$ in the system. Moreover, the choice of each $\tilde{b}_m$ is independent of those b_l with $l > m$.*

As a result, $\tilde{b}_m$ translates $\ddot{C}_{\Phi_m,0}$ to $\ddot{C}_{\Phi_m,b_m}$ for each $m \geq 1$ such that $n|m$ and $\Box \nmid m$, and the translations are compatible among different m's and l's.

Proof. Let $m \geq 1$ be an integer such that $n|m$ and $\Box \nmid m$, and let $l \geq 1$ be another integer such that $n|l$, $\Box \nmid l$, and $\frac{1}{m}X \subset \frac{1}{l}\phi(Y)$. Certainly, what is implicit in the choices of $\hat{b}$ is that there is a choice of a filtration Z lifting Z_n, which in particular, induces two filtrations Z_m and Z_l by reductions. By making an étale localization to $\mathsf{M}_l^{\mathsf{Z}_l}$, we may restrict the homomorphism $b_l : \frac{1}{l}Y \to A^\vee[l]$ to $\frac{1}{m}X$, and obtain a homomorphism $\frac{1}{m}X \to A^\vee$ over $\mathsf{M}_l^{\mathsf{Z}_l}$, whose restriction to $\phi(Y)$ is necessarily trivial. That is, we obtain an element in $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{m}X/\phi(Y), A^\vee)$. Let us define $\tilde{b}_m := (-b_l|_{\frac{1}{m}X}, 0)$. (The reason for the minus sign will be clear later.) Since the group $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{m}X/\phi(Y), A^\vee)$ is already defined over $\mathsf{M}_m^{\mathsf{Z}_m}$, this homomorphism $\tilde{b}_m$ is defined over $\mathsf{M}_m^{\mathsf{Z}_m}$ by étale descent.

By definition, $\partial^{(1)}_m(\tilde{b}_m) = \partial^{(1)}_m(-b_l|_{\frac{1}{m}X}, 0)$ is given by the restriction of $b_l|_{\frac{1}{m}X}$ to $\frac{1}{m}Y$, which is necessarily b_m. The same argument also shows that $\tilde{b}_m$ is compatible among different m's and l's.

Finally, the fact that $\tilde{b}_m$ is defined without making a base change to $\mathsf{M}_l^{\mathsf{Z}_l}$ with $l > m$ shows that $\tilde{b}_m$ must be independent of those b_l with $l > m$. □

Remark 6.2.3.12. The statement that $\tilde{b}_n$ translates $\ddot{C}_{\Phi_n,0}$ to $\ddot{C}_{\Phi_n,b_n}$ shows that the underlying geometric spaces of $\ddot{C}_{\Phi_n,0}$ and $\ddot{C}_{\Phi_n,b_n}$ are isomorphic. The essential difference is that we see potentially different tautological homomorphisms 0 and b_n over them, which might lead to different tautological tuples at the end of the construction (for suitable R and I; see Definition 5.4.1.1), and might produce different degenerating families that should not be glued to the same part of the same Shimura variety. (It is possible that they should be glued to some different Shimura varieties.)

Lemma 6.2.3.13. *The common schematic image $\ddot{C}^{\mathrm{com.}}_{\Phi_n,0}$ of the canonical homomorphisms $\ddot{C}_{\Phi_m,0} \to \ddot{C}_{\Phi_n,0}$ is $\ddot{C}^{\circ}_{\Phi_n}$.*

Proof. Note that we have a canonical isomorphism

$$\ddot{C}_{\Phi_n,0} \underset{\mathsf{M}_n^{\mathsf{Z}_n}}{\times} \mathsf{M}_m^{\mathsf{Z}_m} \xrightarrow{\sim} \ddot{C}_{\Phi_1} \underset{\mathsf{M}_1^{\mathsf{Z}_1}}{\times} \mathsf{M}_m^{\mathsf{Z}_m} \xrightarrow{\sim} \ddot{C}_{\Phi_m,0}$$

given by the canonical isomorphisms $\frac{1}{n}Y \xrightarrow{\sim} Y \xrightarrow{\sim} \frac{1}{m}Y$ and $\frac{1}{n}X \xrightarrow{\sim} X \xrightarrow{\sim} \frac{1}{m}X$ for each m we consider, because the $\ddot{C}_{\Phi_m,0}$'s are of the form $\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y, A) \underset{\underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}Y,A^\vee)}{\times} \underline{\mathrm{Hom}}_{\mathcal{O}}(\frac{1}{n}X, A^\vee)$. Therefore, the canonical homomorphisms $\ddot{C}_{\Phi_m,0} \to \ddot{C}_{\Phi_n,0}$ defined by the inclusions $\frac{1}{n}Y \hookrightarrow \frac{1}{m}Y$ and $\frac{1}{n}X \hookrightarrow \frac{1}{n}X$ have the same image as the multiplication by $\frac{m}{n}$ on $\ddot{C}_{\Phi_n,0}$ itself. Since we know from Proposition 6.2.2.4 that the rank of the finite group scheme $\underline{\pi}_0(\ddot{C}_{\Phi_n,0}/\mathsf{M}_n^{\mathsf{Z}_n})$ over $\mathsf{M}_n^{\mathsf{Z}_n}$ is prime-to-□, because it is the same as the pullback of the finite group scheme $\underline{\pi}_0(\ddot{C}_{\Phi_1}/\mathsf{M}_1^{\mathsf{Z}_1})$ over $\mathsf{M}_1^{\mathsf{Z}_1}$, we see that there is an integer $m \geq 1$ such that $n|m$ and $\Box \nmid m$, and such that the multiplication by $\frac{m}{n}$ annihilates the group scheme $\underline{\pi}_0(\ddot{C}_{\Phi_m,0}/\mathsf{M}_m^{\mathsf{Z}_m})$. In particular, the schematic image of $\ddot{C}_{\Phi_m,0} \to \ddot{C}_{\Phi_n,0}$ for this m is contained in $\ddot{C}^{\circ}_{\Phi_n}$. This forces the common schematic image $\ddot{C}^{\mathrm{com.}}_{\Phi_n,\{b_m\}}$ to agree with $\dddot{C}^{\circ}_{\Phi_n}$, because $\dddot{C}^{\circ}_{\Phi_n}$ is an abelian scheme by Proposition 6.2.3.2 and hence multiplications by integers are surjective on it. $\square$

Combining Lemmas 6.2.3.11 and 6.2.3.13,

Corollary 6.2.3.14. *The common schematic image $\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}$ is the translation of the geometric identity component $\dddot{C}^{\circ}_{\Phi_n}$ of $\dddot{C}_{\Phi_n}$ by $\tilde{b}_n$, as defined in Lemma 6.2.3.11. In particular, it is an abelian scheme. Moreover, it does not depend on the choice of the liftings $\{b_m\}_{n|m,\square\nmid m}$ of b_n.*

To find the common schematic image $\dddot{\Xi}^{\mathrm{com.}}_{\Phi_n,(b_n,a_n)}$ of the canonical morphisms $\dddot{\Xi}_{\Phi_m,(b_m,a_m)} \to \dddot{\Xi}_{\Phi_n,(b_n,a_n)}$ for m such that $n|m$ and $\square\nmid m$, it suffices to find the common schematic image of the canonical morphisms $\dddot{\Xi}_{\Phi_m,(b_m,a_m)}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_m,b_m}} \to \dddot{\Xi}_{\Phi_n,(b_n,a_n)}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$, or rather $\dddot{\Xi}_{\Phi_m,a_m}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_m,b_m}} \to \dddot{\Xi}_{\Phi_n,a_n}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$.

Lemma 6.2.3.15. *Under the assumption that we have chosen a system $\hat{a} = \{a_m\}_{n|m,\square\nmid m}$ lifting a_n, we have a recipe (depending on whether $2\in\square$ or not) to produce a system of elements $\{\tilde{a}_m\}_{n|m,\square\nmid m}$ such that $\tilde{a}_m \in \dddot{E}_{\Phi_m}$, such that $\tilde{a}_m$ is mapped to a_m under the canonical surjection $\dddot{E}_{\Phi_m} \twoheadrightarrow \dddot{E}^{(m)}_{\Phi_m}$, and such that $\tilde{a}_{m'}$ is mapped to $\tilde{a}_m$ under the canonical homomorphism $\dddot{E}_{\Phi_l} \to \dddot{E}_{\Phi_m}$ for each $m|l$ in the system. (This time the $\tilde{a}_m$ may depend on those a_l with $l > m$.)*

In particular, $\tilde{a}_m$ translates $\dddot{\Xi}_{\Phi_m,0}$ to $\dddot{\Xi}_{\Phi_m,a_m}$ for each $m \geq 1$ such that $n|m$ and $\square\nmid m$, and the translations are compatible among different m's and l's.

Proof. By Lemma 6.2.3.7, for each $m\geq 1$ such that $n|m$ and $\square\nmid m$, the chosen lifting a_m is alternating and satisfies (in particular) $a_m(b\frac{1}{m}y,y') = a_m(\frac{1}{m}y, b^\star y')$ for all $y,y'\in Y$ and $b\in\mathcal{O}$. On the other hand, each of the elements $\tilde{a}_m$ we would like to produce in $\dddot{E}_{\Phi_m}$ can be identified with a homomorphism $\tilde{a}_m : (\frac{1}{m}Y)\underset{\mathbb{Z}}{\otimes} X \to \mathbf{G}_{\mathrm{m}}$ satisfying $\tilde{a}_m(y,\phi(y')) = -\tilde{a}_m(y',\phi(y))$ and $\tilde{a}_m(b\frac{1}{m}y,\chi) = \tilde{a}(\frac{1}{m}y,b^\star\chi)$ for all $y,y'\in Y$, $\chi\in X$ and $b\in\mathcal{O}$. To verify that $\tilde{a}_m$ is mapped to a_m under $\dddot{E}_{\Phi_m}\twoheadrightarrow\dddot{E}^{(m)}_{\Phi_m}$, we need to verify the relation $\tilde{a}_m(\frac{1}{m}y,\phi(y')) - \tilde{a}_m(\frac{1}{m}y',\phi(y)) = a_m(\frac{1}{m}y,\frac{1}{m}y')$ for all $y,y'\in Y$.

Let us take an integer $l\geq 1$ such that $lX\subset\phi(Y)$. Since $\square\nmid[X:\phi(Y)]$ by the original choice of $\square$, we shall take this integer l to be prime-to-$\square$.

If $2\notin\square$, then a_{2lm} is defined for each $m\geq 1$ such that $n|m$ and $\square\nmid m$. Let us define $\tilde{a}_m$ by
$$\tilde{a}_m(\tfrac{1}{m}y,\chi) := a_{2lm}(\tfrac{1}{2lm}y,\phi^{-1}(l\chi)).$$
Then we have
$$\begin{aligned}
&\tilde{a}_m(\tfrac{1}{m}y,\phi(y')) - \tilde{a}_m(\tfrac{1}{m}y',\phi(y))\\
&= a_{2lm}(\tfrac{1}{2lm}y,\phi^{-1}(l\phi(y'))) - a_{2lm}(\tfrac{1}{2lm}y',\phi^{-1}(l\phi(y)))\\
&= a_{2lm}(\tfrac{1}{2lm}y,ly') - a_{2lm}(\tfrac{1}{2lm}y',ly) = a_{2m}(\tfrac{1}{2m}y,y') - a_{2m}(\tfrac{1}{2m}y',y)\\
&= a_{2m}(\tfrac{1}{2m}y,y') + a_{2m}(\tfrac{1}{2m}y,y') = 2a_{2m}(\tfrac{1}{2m}y,y') = a_m(\tfrac{1}{m}y,y').
\end{aligned}$$
The upshot of this argument is $\frac{1}{2}+\frac{1}{2}=1$.

If $2\in\square$, then every integer $m\geq 1$ such that $\square\nmid m$ has to be odd, and we can pick a compatible system $\{e_m\}_{\square\nmid m}$ of elements $e_m\in\mathbb{Z}/m\mathbb{Z}$, defining an element of $\hat{\mathbb{Z}}^{\square}$, such that $2e_m = 1$ in $\mathbb{Z}/m\mathbb{Z}$ for all indices m. Then we can define $\tilde{a}_m$ by
$$\tilde{a}_m(\tfrac{1}{m}y,\chi) := e_{lm}a_{lm}(\tfrac{1}{lm}y,\phi^{-1}(l\chi)).$$
Then we have
$$\begin{aligned}
&\tilde{a}_m(\tfrac{1}{m}y,\phi(y')) - \tilde{a}_m(\tfrac{1}{m}y',\phi(y))\\
&= e_{lm}a_{lm}(\tfrac{1}{lm}y,\phi^{-1}(l\phi(y'))) - e_{lm}a_{lm}(\tfrac{1}{lm}y',\phi^{-1}(l\phi(y)))\\
&= e_{lm}a_{lm}(\tfrac{1}{lm}y,ly') - e_{lm}a_{lm}(\tfrac{1}{lm}y',ly) = e_m a_m(\tfrac{1}{m}y,y') - e_m a_m(\tfrac{1}{m}y',y)\\
&= e_m a_m(\tfrac{1}{m}y,y') + e_m a_m(\tfrac{1}{m}y,e^\star y') = 2e_m a_m(\tfrac{1}{m}y,y') = a_m(\tfrac{1}{m}y,y').
\end{aligned}$$
The upshot of this argument is $2e_m = 1$.

This gives our recipes for producing a system $\{\tilde{a}_m\}_{n|m,\square\nmid m}$ explicitly for each system $\hat{a} = \{a_m\}_{n|m,\square\nmid m}$ lifting a_n. $\square$

Lemma 6.2.3.16. *For each $\hat{b}$ lifting b_n, the common schematic image $\dddot{\Xi}^{\mathrm{com.}}_{\Phi_n,(b_n,0)}$ of the morphisms $\dddot{\Xi}_{\Phi_m,0}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_m,b_m}} \to \dddot{\Xi}_{\Phi_n,0}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$ is a torsor under the torus $\dddot{E}_{\Phi_n,\mathrm{free}}$ over $\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}$.*

Proof. For each $m\geq 1$ such that $n|m$ and $\square\nmid m$, the fiber $\dddot{\Xi}_{\Phi_m,0} = (\partial^{(1)}_m)^{-1}(0)$ can be constructed explicitly as follows:

Let $\dddot{\mathbf{S}}_{\Phi_m,0}$ be the finitely generated commutative group defined by
$$\dddot{\mathbf{S}}_{\Phi_m,0} := ((\tfrac{1}{m}Y)\underset{\mathbb{Z}}{\otimes}X)/\begin{pmatrix}(\frac{1}{m}y)\otimes\phi(y') - (\frac{1}{m}y')\otimes\phi(y)\\ (b\frac{1}{m}y)\otimes\chi - (\frac{1}{m}y)\otimes(b^\star\chi)\end{pmatrix}_{\substack{y,y'\in Y,\\ \chi\in X,b\in\mathcal{O}}}.$$
As in the construction of $\dddot{\Xi}_{\Phi_n}$, the formal properties of the pullbacks of the Poincaré biextension (as in Lemma 6.2.2.5) allow us to assign to each $\ell = \sum_{1\leq i\leq k}[(\frac{1}{m}y_i)\otimes\chi_i] \in \dddot{\mathbf{S}}_{\Phi_m,0}$ a well-defined invertible sheaf $\Psi_{m,0}(\ell) := \underset{\dddot{C}_{\Phi_m},1\leq i\leq k}{\otimes}(c^\vee_m(\frac{1}{m}y_i),c(\chi_i))^*\mathcal{P}_A$, such that there exists a canonical isomorphism $\Delta^*_{m,0,\ell,\ell'} : \Psi_{m,0}(\ell)\underset{\mathscr{O}_{\dddot{C}_{\Phi_m}}}{\otimes}\Psi_{m,0}(\ell') \overset{\sim}{\to} \Psi_m(\ell+\ell')$ for each $\ell,\ell'\in\dddot{\mathbf{S}}_{\Phi_m,0}$. Here we are using a stronger relation than those used for $\dddot{\mathbf{S}}_{\Phi_m}$, namely, we are using the canonical symmetry isomorphisms $(c^\vee_m(\frac{1}{m}y),c(\phi(y')))^*\mathcal{P}_A \underset{\sim}{\overset{\text{can.}}{\to}} (c^\vee_m(\frac{1}{m}y'),c(\phi(y)))^*\mathcal{P}_A$ for all $y,y'\in Y$, instead of only $(c^\vee(y),c(\phi(y')))^*\mathcal{P}_A \underset{\sim}{\overset{\text{can.}}{\to}} (c^\vee(y'),c(\phi(y)))^*\mathcal{P}_A$. As a result, we can form an $\mathscr{O}_{\dddot{C}_{\Phi_m}}$-algebra $\underset{\ell\in\dddot{\mathbf{S}}_{\Phi_m,0}}{\oplus}\Psi_{m,0}(\ell)$ with algebra structure given by the isomorphisms $\Delta^*_{m,0,\ell,\ell'}$ above, and define $\dddot{\Xi}_{\Phi_m,0} := \underline{\mathrm{Spec}}_{\mathscr{O}_{\dddot{C}_{\Phi_m}}}\Big(\underset{\ell\in\dddot{\mathbf{S}}_{\Phi_m,0}}{\oplus}\Psi_{m,0}(\ell)\Big)$. This definition produces the same $\dddot{\Xi}_{\Phi_m,0}$ defined as the fiber $(\partial^{(0)}_m)^{-1}(0)$. Moreover, $\dddot{\Xi}_{\Phi_m,0}$ is naturally an $\dddot{E}_{\Phi_m,0}$-torsor under the group $\dddot{E}_{\Phi_m,0} := \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}_{\Phi_m,0},\mathbf{G}_{\mathrm{m}})$ of multiplicative type of finite type over $\mathrm{Spec}(\mathbb{Z})$ with character group $\dddot{\mathbf{S}}_{\Phi_m,0}$.

For each $m\geq 1$ we consider, we have
$$\dddot{\mathbf{S}}_{\Phi_m,0} \overset{\sim}{\to} \dddot{\mathbf{S}}_{\Phi_1} = (Y\underset{\mathbb{Z}}{\otimes}X)/\begin{pmatrix}y\otimes\phi(y') - y'\otimes\phi(y)\\ (by)\otimes\chi - y\otimes(b^\star\chi)\end{pmatrix}_{\substack{y,y'\in Y,\\ \chi\in X,b\in\mathcal{O}}}.$$
Therefore, the canonical homomorphism
$$\dddot{\mathbf{S}}_{\Phi_n,0}\to\dddot{\mathbf{S}}_{\Phi_m,0}$$
induced by $(\frac{1}{n}Y)\underset{\mathbb{Z}}{\otimes}X \hookrightarrow (\frac{1}{m}Y)\underset{\mathbb{Z}}{\otimes}X$ for each m can be viewed as the multiplication
$$\dddot{\mathbf{S}}_{\Phi_n,0}\overset{[\frac{m}{n}]}{\to}\dddot{\mathbf{S}}_{\Phi_n,0}.$$

Since the cardinality of $\dddot{\mathbf{S}}_{\Phi_1,\mathrm{tor}}$ only has prime factors dividing $\mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}[X:\phi(Y)]$, which is prime-to-$\Box$ (see Remark 6.2.2.6), there exists an integer $m \geq 1$ such that $n|m$, $\Box \nmid m$, and $\frac{m}{n}$ is divisible by the cardinality of $\dddot{\mathbf{S}}_{\Phi_1,\mathrm{tor}}$. Let us fix the choice of such an integer m.

Since the group schemes $\dddot{E}_{\Phi_m,0,\mathrm{tor}}$ and $\dddot{E}_{\Phi_n,0,\mathrm{tor}}$ are both isomorphic to $\dddot{E}_{\Phi_1,\mathrm{tor}} = \underline{\mathrm{Hom}}(\dddot{\mathbf{S}}_{\Phi_1,\mathrm{tor}}, \mathbf{G}_{\mathrm{m},\mathsf{M}_n^{\mathsf{Z}_n}})$, the canonical homomorphism $\dddot{E}_{\Phi_m,0,\mathrm{tor}} \to \dddot{E}_{\Phi_n,0,\mathrm{tor}}$ can be identified with the homomorphism $\dddot{E}_{\Phi_1,\mathrm{tor}} \to \dddot{E}_{\Phi_1,\mathrm{tor}}$ defined by multiplication by $\frac{m}{n}$, which has trivial schematic image because $\frac{m}{n}$ is divisible by the cardinality of $\dddot{\mathbf{S}}_{\Phi_1,\mathrm{tor}}$. Consequently, the bottom morphism of the $\dddot{E}_{\Phi_m,0}$-equivariant commutative diagram

$$\begin{tikzcd}
\dddot{\Xi}_{\Phi_m,0}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_m,b_m}} \arrow[r] \arrow[d] & \dddot{\Xi}_{\Phi_n,0}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}} \arrow[d] \\
\dddot{\Xi}_{\Phi_m,0,\mathrm{tor}}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_m,b_m}} \arrow[r] & \dddot{\Xi}_{\Phi_n,0,\mathrm{tor}}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}
\end{tikzcd}$$

induced by the canonical homomorphism $\dddot{\mathbf{S}}_{\Phi_n,0} \to \dddot{\mathbf{S}}_{\Phi_m,0}$, which is equivariant with respect to the canonical homomorphism $\dddot{E}_{\Phi_m,0,\mathrm{tor}} \to \dddot{E}_{\Phi_n,0,\mathrm{tor}}$, has schematic image isomorphic to the base $\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}$. Since $\dddot{\Xi}_{\Phi_n,0}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}} \to \dddot{\Xi}_{\Phi_n,0,\mathrm{tor}}$ is a torsor under the torus $\dddot{E}_{\Phi_n,\mathrm{free}}$, the common schematic image of $\dddot{\Xi}_{\Phi_m,0}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_m,b_m}} \to \dddot{\Xi}_{\Phi_n,0}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$ is also a torsor under $\dddot{E}_{\Phi_n,\mathrm{free}}$, as desired. $\Box$

Combining Lemmas 6.2.3.15 and 6.2.3.16,

Corollary 6.2.3.17. *The common schematic image $\dddot{\Xi}^{\mathrm{com.}}_{\Phi_n,(b_n,a_n)}$ is étale locally the translation of the subalgebraic stack $\dddot{\Xi}^{\mathrm{com.}}_{\Phi_n,(b_n,0)}$ of $\dddot{\Xi}_{\Phi_n}|_{\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}}$ by some (noncanonical) element $\tilde{a}_n \in \dddot{E}_{\Phi_n}$ in Lemma 6.2.3.15. (Because of the assumption of chosen liftings in Lemma 6.2.3.15, the element $\tilde{a}_n$ is only étale locally defined.) In particular, $\dddot{\Xi}^{\mathrm{com.}}_{\Phi_n,(b_n,a_n)}$ is a torsor under the torus $\dddot{E}_{\Phi_n,\mathrm{free}}$ over the abelian scheme $\dddot{C}^{\mathrm{com.}}_{\Phi_n,b_n}$.*

Proposition 6.2.3.18. *Let $n \geq 1$ be an integer prime-to-$\Box$ (in the setting of Section 6.2.1). Let us fix the choice of a representative $(\mathsf{Z}_n, \Phi_n, \delta_n)$ of a cusp label at level n, where $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$, which defines a moduli problem $\mathsf{M}_n^{\mathsf{Z}_n}$ (as in Lemma 5.2.7.5). Then such a choice of cusp label determines b_{Φ_n,δ_n} and a_{Φ_n,δ_n} as in Lemma 6.2.3.1.*

Let us consider the category fibered in groupoids over the category of schemes over $\mathsf{M}_n^{\mathsf{Z}_n}$ whose fiber over each scheme S (over $\mathsf{M}_n^{\mathsf{Z}_n}$) has objects the tuples
$$(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n))$$
describing degeneration data without the positivity condition over S. Explicitly, each tuple as above satisfies the following conditions:

1. *$(A, \lambda_A, i_A, \varphi_{-1,n})$ is the pullback of the tautological tuple over $\mathsf{M}_n^{\mathsf{Z}_n}$.*

2. *$c_n : \frac{1}{n}X \to A^\vee$ and $c_n^\vee : \frac{1}{n}Y \to A$ are $\mathcal{O}$-equivariant group homomorphisms satisfying the compatibility relation $\lambda_A c_n^\vee - c_n\phi_n = b_{\Phi_n,\delta_n}$ with $\phi_n : \frac{1}{n}Y \hookrightarrow \frac{1}{n}X$ induced by the prescribed $\phi : Y \hookrightarrow X$.*

3. *$\tau_n : \mathbf{1}_{(\frac{1}{n}Y)\times X} \xrightarrow{\sim} (c_n^\vee \times c)^*\mathcal{P}_A$ is a trivialization of biextensions over S which satisfies the relation*
$$\tau_n(\tfrac{1}{n}y, \phi(y'))\tau_n(\tfrac{1}{n}y', \phi(y))^{-1} = a_{\Phi_n,\delta_n}(\tfrac{1}{n}y, \tfrac{1}{n}y') \in \boldsymbol{\mu}_n(S)$$
for each $\frac{1}{n}y, \frac{1}{n}y' \in \frac{1}{n}Y$ under the canonical symmetry isomorphism $(c_n^\vee(\frac{1}{n}y), c(\phi(y')))^*\mathcal{P}_A \underset{\mathcal{O}_S}{\otimes} (c_n^\vee(\frac{1}{n}y')c(\phi(y)))^*\mathcal{P}_A^{\otimes -1} \cong \mathcal{O}_S$, and satisfies the $\mathcal{O}$-compatibility $\tau(b\frac{1}{n}y, \chi) = \tau(\frac{1}{n}y, b^\star\chi)$ for each $\frac{1}{n}y \in Y$ and $\chi \in X$ under the canonical isomorphism $(c_n^\vee(b\frac{1}{n}y), c(\chi))^*\mathcal{P}_A \cong (c_n^\vee(\frac{1}{n}y), c(b^\star\chi))^*\mathcal{P}_A$. (Here it makes sense to write equalities of sections because the isomorphisms are all canonical.)

4. *The triple $(c_n, c_n^\vee, \tau_n)$ is **liftable** in the following sense: For each $m \geq 1$ such that $n|m$ and $\Box \nmid m$, suppose we have lifted all the other data to some liftable tuple*
$$(\mathsf{Z}_m, (X, Y, \phi, \varphi_{-2,m}, \varphi_{0,m}), (A, \lambda_A, i_A, \varphi_{-1,m}), \delta_m)$$
at level m. Then the triple $(c_n, c_n^\vee, \tau_n)$ is also liftable to some $(c_m, c_m^\vee, \tau_m)$ that has the same kind of compatibility as $(c_n, c_n^\vee, \tau_n)$ does with other data.

In this category, an isomorphism
$$(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n))$$
$$\xrightarrow{\sim} (\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A', \lambda_{A'}, i_{A'}, \varphi'_{-1,n}), \delta_n, (c'_n, (c_n^\vee)', \tau'_n))$$
is a collection of isomorphisms
$$(f_X : X \xrightarrow{\sim} X, f_Y : Y \xrightarrow{\sim} Y) \in \Gamma_{\Phi_n}$$
(see Definition 6.2.1.2) and
$$f_A : (A, \lambda_A, i_A, \varphi_{-1,n}) \xrightarrow{\sim} (A', \lambda_{A'}, i_{A'}, \varphi'_{-1,n})$$
over S such that

1. *the homomorphisms $c_n : \frac{1}{n}X \to A^\vee$ and $c'_n : \frac{1}{n}X \to (A')^\vee$ are related by $cf_X = f_A{}^\vee c'$ (here f_X also stands for the isomorphism $\frac{1}{n}X \xrightarrow{\sim} \frac{1}{n}X$ canonically induced by f_X);*

2. *the homomorphisms $c_n^\vee : \frac{1}{n}Y \to A$ and $(c_n^\vee)' : \frac{1}{n}Y \to A'$ are related by $f_A c_n^\vee = (c_n^\vee)' f_Y$ (here f_Y also stands for the isomorphism $\frac{1}{n}Y \xrightarrow{\sim} \frac{1}{n}Y$ canonically induced by f_X);*

3. *the trivializations $\tau_n : \mathbf{1}_{(\frac{1}{n}Y)\times X} \xrightarrow{\sim} (c_n^\vee \times c)^*\mathcal{P}_A$ and $\tau'_n : \mathbf{1}_{(\frac{1}{n}Y)\times X} \xrightarrow{\sim} ((c_n^\vee)' \times c')^*\mathcal{P}_{A'}$ are related by $(\mathrm{Id}_{\frac{1}{n}Y} \times f_X)^*\tau = (f_Y \times \mathrm{Id}_X)^*\tau'$.*

Then there is an algebraic stack Ξ_{Φ_n,δ_n} separated, smooth, and schematic over $\mathsf{M}_n^{\mathsf{Z}_n}$, together with a tautological tuple and a natural action of Γ_{Φ_n} on Ξ_{Φ_n,δ_n}, such that the quotient $\Xi_{\Phi_n,\delta_n}/\Gamma_{\Phi_n}$ is isomorphic to the category described above (as categories fibered in groupoids over $\mathsf{M}_n^{\mathsf{Z}_n}$). Equivalently, for each tuple $(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n))$ as above over a scheme S over $\mathsf{M}_n^{\mathsf{Z}_n}$, there is a morphism $S \to \Xi_{\Phi_n,\delta_n}$ (over $\mathsf{M}_n^{\mathsf{Z}_n}$), which is unique after we fix an isomorphism $(f_Y : Y \xrightarrow{\sim} Y, f_X : X \xrightarrow{\sim} X)$ in Γ_{Φ_n}, such that the tuple over S is the pullback of the tautological tuple over Ξ_{Φ_n,δ_n} if we identify X by f_X and Y by f_Y.

Proof. Simply take $\Xi_{\Phi_n,\delta_n} := \dddot{\Xi}^{\mathrm{com.}}_{\Phi_n,\delta_n,(b_{\Phi_n,\delta_n},a_{\Phi_n,\delta_n})}$ as in Corollary 6.2.3.17. Then the expected universal properties, including the tautological relations (6.2.3.3) and (6.2.3.10), and the liftability, follow from the construction. As in the case of $\dddot{\Xi}_{\Phi_1}$, the ambiguity of identification of $(X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ necessitates the quotient by Γ_{Φ_n}. $\Box$

Remark 6.2.3.19. The tautological τ over Ξ_{Φ_n,δ_n} in Proposition 6.2.3.18 (induced by the tautological τ_n) does not satisfy the positivity condition (see Definition 4.2.1.10) needed for applying Theorem 5.2.7.14. Therefore, we cannot construct the desired degenerating family over Ξ_{Φ_n,δ_n} (or $\Xi_{\Phi_n,\delta_n}/\Gamma_{\Phi_n}$). We will learn in Section 6.2.5 how to construct degenerating families over formal completions along some nice (but noncanonical) partial compactifications of Ξ_{Φ_n,δ_n} given by toroidal embeddings.

For simplicity, we shall set up the following convention:

Convention 6.2.3.20. *We shall set up the following simplification of notation:*

1. $\mathbf{S}_{\Phi_n} := \ddot{\mathbf{S}}_{\Phi_n,\mathrm{free}}$.
2. $E_{\Phi_n} := \ddot{E}_{\Phi_n,\mathrm{free}}$.
3. $C_{\Phi_n,\delta_n} := \ddot{C}^{\mathrm{com.}}_{\Phi_n,b_{\Phi_n,\delta_n}}$.
4. $\Xi_{\Phi_n,\delta_n} := \ddot{\Xi}^{\mathrm{com.}}_{\Phi_n,\delta_n,(b_{\Phi_n,\delta_n},a_{\Phi_n,\delta_n})}$ *(already set up in Proposition 6.2.3.18).*

Moreover, as an E_{Φ_n}-torsor, we shall describe Ξ_{Φ_n,δ_n} using a decomposition
$$\mathscr{O}_{\Xi_{\Phi_n,\delta_n}} \cong \bigoplus_{\ell\in\mathbf{S}_{\Phi_n}} \Psi_{\Phi_n,\delta_n}(\ell)$$
into weight subsheaves under the E_{Φ_n}-action. Here the notation Ψ_{Φ_n,δ_n} means that, when we set up the equivalences between, for example, $(c_n^\vee(\frac{1}{n}y), c\phi(y'))^*\mathcal{P}_A$ and $(c_n^\vee(\frac{1}{n}y'), c\phi(y))^*\mathcal{P}_A$, the isomorphisms
$$(c_n^\vee(\tfrac{1}{n}y), c\phi(y'))^*\mathcal{P}_A \xrightarrow{\sim} (c_n^\vee(\tfrac{1}{n}y'), c\phi(y))^*\mathcal{P}_A$$
differ from the canonical symmetry isomorphism
$$(c_n^\vee(\tfrac{1}{n}y), c\phi(y'))^*\mathcal{P}_A \overset{\mathrm{can.}}{\xrightarrow{\sim}} (c_n^\vee(\tfrac{1}{n}y'), c\phi(y))^*\mathcal{P}_A$$
by the $a_{\Phi_n,\delta_n}(\frac{1}{n}y, \frac{1}{n}y')$ prescribed by Lemma 6.2.3.1.

Then the construction of Ξ_{Φ_n,δ_n} shows the following:

Proposition 6.2.3.21 (continuation of Proposition 6.2.3.18)**.** *The structural morphism $\Xi_{\Phi_n,\delta_n} \to \mathsf{M}_n^{\mathsf{Z}_n}$ factorizes as the composition $\Xi_{\Phi_n,\delta_n} \to C_{\Phi_n,\delta_n} \to \mathsf{M}_n^{\mathsf{Z}_n}$, where $\Xi_{\Phi_n,\delta_n} \to C_{\Phi_n,\delta_n}$ is a torsor under the torus $E_{\Phi_n} \cong \underline{\mathrm{Hom}}(\mathbf{S}_{\Phi_n}, \mathbf{G}_{\mathrm{m}})$, and where $C_{\Phi_n,\delta_n} \to \mathsf{M}_n^{\mathsf{Z}_n}$ is a (relative) abelian scheme. The E_{Φ_n}-torsor structure of Ξ_{Φ_n,δ_n} defines a canonical homomorphism*
$$\mathbf{S}_{\Phi_n} \to \underline{\mathrm{Pic}}_e(C_{\Phi_n,\delta_n}/\mathsf{M}_n^{\mathsf{Z}_n}) : \ell \mapsto \Psi_{\Phi_n,\delta_n}(\ell), \tag{6.2.3.22}$$
giving for each $\ell \in \mathbf{S}_{\Phi_n}$ a rigidified invertible sheaf $\Psi_{\Phi_n,\delta_n}(\ell)$ over C_{Φ_n,δ_n}.

6.2.4 Construction with General Level Structures

Let us take general level-$\mathcal{H}$ structures (see Definition 1.3.7.6) into consideration.

With the setting as in Section 6.2.1, let $\mathcal{H} \subset \mathrm{G}(\hat{\mathbb{Z}}^\square)$ be an open compact subgroup, and let $(\mathsf{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})$ be a representative of a cusp label at level $\mathcal{H}$ (see Definition 5.4.2.4), where $\Phi_\mathcal{H} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ is a torus argument at level $\mathcal{H}$. Here X and Y are constant group schemes that will serve as the character group of the torus parts, as usual. As explained in Definition 5.4.2.6, the information of $\mathsf{Z}_\mathcal{H}$ alone defines a moduli problem $\mathsf{M}_{\mathcal{H}_h}$ over S_0 as in Definition 1.4.1.2, which admits finite étale covers $\mathsf{M}_\mathcal{H}^{\Phi_\mathcal{H}} \to \mathsf{M}_\mathcal{H}^{\mathsf{Z}_\mathcal{H}} \to \mathsf{M}_{\mathcal{H}_h}$. Let $(A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}})$ be the tautological tuple over $\mathsf{M}_{\mathcal{H}_h}$. For simplicity, we shall denote its pullbacks by the same notation.

As the counterpart for Γ_{Φ_n} (see Definition 6.2.1.2),

Definition 6.2.4.1. *For each torus argument $\Phi_\mathcal{H} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$, the group $\Gamma_{\Phi_\mathcal{H}}$ is the subgroup of elements (γ_X, γ_Y) in Γ_ϕ satisfying $\varphi_{-2,\mathcal{H}} = {}^t\gamma_X\varphi_{-2,\mathcal{H}}$ and $\varphi_{0,\mathcal{H}} = \gamma_Y\varphi_{0,\mathcal{H}}$ (as collections of orbits). (This is the subgroup of Γ_ϕ appeared in Definition 5.4.2.6.)*

Essentially by definition (see Definition 5.4.2.6), $\Gamma_{\Phi_\mathcal{H}}$ acts on the finite étale cover $\mathsf{M}_\mathcal{H}^{\Phi_\mathcal{H}} \to \mathsf{M}_\mathcal{H}^{\mathsf{Z}_\mathcal{H}}$ and induces an isomorphism $\mathsf{M}_\mathcal{H}^{\Phi_\mathcal{H}}/\Gamma_{\Phi_\mathcal{H}} \xrightarrow{\sim} \mathsf{M}_\mathcal{H}^{\mathsf{Z}_\mathcal{H}}$.

For each integer $n \geq 1$ such that $\square \nmid n$ and $\mathcal{U}^\square(n) \subset \mathcal{H}$, set $H_n := \mathcal{H}/\mathcal{U}^\square(n)$ as usual. Then we can interpret $(\mathsf{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})$ as a collection $\{(\mathsf{Z}_{H_n}, \Phi_{H_n}, \delta_{H_n})\}_n$ indexed by n's as above, each $(\mathsf{Z}_{H_n}, \Phi_{H_n}, \delta_{H_n})$ being an H_n-orbit of some representative $(\mathsf{Z}_n, \Phi_n, \delta_n)$ of cusp label at level n. For each such representative $(\mathsf{Z}_n, \Phi_n, \delta_n)$, we have constructed in Section 6.2.3 the algebraic stacks $\Xi_{\Phi_n,\delta_n} \twoheadrightarrow C_{\Phi_n,\delta_n} \twoheadrightarrow \mathsf{M}_n^{\mathsf{Z}_n}$, such that the first morphism is a torsor under some torus E_{Φ_n} with character group $\mathbf{S}_{\Phi_n}$, such that the second morphism is an abelian scheme, and such that the quotient functor $\Xi_{\Phi_n,\delta_n}/\Gamma_{\Phi_n}$ is universal for tuples of the form
$$(\mathsf{Z}_n, (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n}), (A, \lambda_A, i_A, \varphi_{-1,n}), \delta_n, (c_n, c_n^\vee, \tau_n)).$$

Our goal in this section is to construct algebraic stacks
$$\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}} \twoheadrightarrow C_{\Phi_\mathcal{H},\delta_\mathcal{H}} \twoheadrightarrow \mathsf{M}_\mathcal{H}^{\Phi_\mathcal{H}}, \tag{6.2.4.2}$$
carrying compatible actions of $\Gamma_{\Phi_\mathcal{H}}$, such that the first morphism in (6.2.4.2) is a torsor under some torus $E_{\Phi_\mathcal{H}}$ with some character group $\mathbf{S}_{\Phi_\mathcal{H}}$, such that the second morphism in (6.2.4.2) is a abelian scheme torsor, and such that the quotient functor $\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}/\Gamma_{\Phi_\mathcal{H}}$ is universal for tuples of the form
$$(\mathsf{Z}_\mathcal{H}, (X, Y, \phi, \varphi^\sim_{-2,\mathcal{H}}, \varphi^\sim_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_\mathcal{H}, (c_\mathcal{H}, c_\mathcal{H}^\vee, \tau_\mathcal{H})),$$
where $(\varphi^\sim_{-2,\mathcal{H}}, \varphi^\sim_{0,\mathcal{H}})$ induces $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ as in Definition 5.4.2.8. (Following Convention 5.4.2.5, we shall not make $\mathsf{Z}_\mathcal{H}$ explicit in notation such as $\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}$.)

If we take any $(\mathsf{Z}_n, \Phi_n, \delta_n)$ in the H_n-orbit $(\mathsf{Z}_{H_n}, \Phi_{H_n}, \delta_{H_n})$, then we have the natural inclusions
$$H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}} \subset H_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}} \subset H_{n,\mathrm{Z}^{\mathrm{ess}}_{\mathsf{Z}_n}} \subset H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}} \subset H_n$$
as in Definition 5.3.1.11. Note that the quotient $H_n/H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}}$ describes elements in the orbit Z_{H_n}, and the fibers of $\Phi_{H_n} \to \mathsf{Z}_{H_n}$ are torsors under the image $H'_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n}}$ of $H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}}$ in $\mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n}$. Once we have fixed a choice of $(\mathsf{Z}_n, \Phi_n, \delta_n)$, by viewing the semidirect product $\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n} \ltimes \mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}$ as a subgroup of $\mathrm{G}^{\mathrm{ess}}(\mathbb{Z}/n\mathbb{Z})$ using the splitting δ_n, and by viewing $\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n} \ltimes \mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}$ as its quotient by $\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}$, we can define as in 5.3.1.11 the groups $H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}}$ and $H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}}$, fitting into short exact sequences
$$1 \to H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}} \to H_{n,\mathrm{P}^{\mathrm{ess}}_{\mathsf{Z}_n}} \to H'_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n}} \to 1$$
and
$$1 \to H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}} \to H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}} \to H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}} \to 1.$$
Let $H'_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$ denote the canonical image of $H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}}$ in $\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}$, so that we have an exact sequence
$$1 \to H_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}} \to H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}} \to H'_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}} \to 1.$$

Then we have the following commutative diagram

$$\begin{tikzcd}
\Xi_{\Phi_n,\delta_n} \arrow[r, two heads] \arrow[dr] & \Xi_{\Phi_n,\delta_n}/H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}} \arrow[r, two heads] \arrow[d] & \Xi_{\Phi_n,\delta_n}/H_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}} \arrow[r, two heads] \arrow[d] & \Xi_{\Phi_n,\delta_n}/H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}} \arrow[d] \\
& C_{\Phi_n,\delta_n} \arrow[r] \arrow[dr] & C_{\Phi_n,\delta_n}/H_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}} \arrow[r, two heads] \arrow[d] & C_{\Phi_n,\delta_n}/H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}} \arrow[d] \\
& & \mathsf{M}_n^{\mathsf{Z}_n} \arrow[r] \arrow[dr] & \mathsf{M}_n^{\mathsf{Z}_n}/H'_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}} \arrow[d] \\
& & & \mathsf{S}_0
\end{tikzcd} \tag{6.2.4.3}$$

in which the squares are all Cartesian by definition. Moreover, we have used the fact that $H_{n,\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}}$ (resp. $H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$) acts trivially on (objects parameterized by) $\mathsf{M}_n^{\mathsf{Z}_n}$ (resp. C_{Φ_n,δ_n}).

Lemma 6.2.4.4. *The finite group $H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$ can be canonically identified with a finite étale subgroup of the n-torsion elements in the torus E_{Φ_n}, and the action of $H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$ on the E_{Φ_n}-torsor Ξ_{Φ_n,δ_n} over C_{Φ_n,δ_n} can be canonically identified with the torsor-action of this subgroup of E_{Φ_n}, so that $\Xi_{\Phi_n,\delta_n}/H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$ is a torsor under the quotient torus $E_{\Phi_{H_n}} := E_{\Phi_n}/H_{n,\mathrm{U}^{\mathrm{ess}}_{2,\mathsf{Z}_n}}$. We shall denote the character group of $E_{\Phi_{H_n}}$ by $\mathbf{S}_{\Phi_{H_n}}$. The torus $E_{\Phi_{H_n}}$ and its character group $\mathbf{S}_{\Phi_{H_n}}$ are independent of the choice of n, and hence define a torus $E_{\Phi_{\mathcal{H}}}$ with character group $\mathbf{S}_{\Phi_{\mathcal{H}}}$.*

Lemma 6.2.4.5. *The finite group $H_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}}$ can be canonically identified with a finite étale subgroup of C_{Φ_n,δ_n}, and its action on C_{Φ_n,δ_n} can be canonically identified with the translation action of this subgroup. As a result, the quotient $C_{\Phi_{H_n},\delta_{H_n}} := C_{\Phi_n,\delta_n}/H_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}}$ is an abelian scheme over M_n.*

Lemma 6.2.4.6. *The finite groups $H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$ and $H'_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$ act on $\mathsf{M}_n^{\mathsf{Z}_n}$ by twisting the level-n structures, which induce isomorphisms $\mathsf{M}_n^{\mathsf{Z}_n}/H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H}_h}$ and $\mathsf{M}_n^{\mathsf{Z}_n}/H'_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$ by the moduli interpretations.*

Since the quotients in (6.2.4.3) all exist as algebraic stacks, the above lemmas show that the claim would follow if we identify (6.2.4.2) with the equivariant quotient of

$$\coprod \Xi_{\Phi_n,\delta_n}/H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}} \twoheadrightarrow \coprod C_{\Phi_n,\delta_n}/H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}} \twoheadrightarrow \coprod \mathsf{M}_n^{\mathsf{Z}_n}/H'_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$$

by $H'_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n}}$, where the morphisms

$$\Xi_{\Phi_n,\delta_n}/H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{\mathsf{Z}_n}} \twoheadrightarrow C_{\Phi_n,\delta_n}/H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}} \twoheadrightarrow \mathsf{M}_n^{\mathsf{Z}_n}/H'_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$$

are the vertical arrows in (6.2.4.3), and where the disjoint unions are over elements Φ_n in the fiber of $\Phi_{H_n} \to \mathsf{Z}_{H_n}$ above Z_n. (Here we fix the choice of Z_n and δ_n, but allow Φ_n to vary in its $H'_{n,\mathrm{G}^{\mathrm{ess}}_{l,\mathsf{Z}_n}}$-orbit.) Equivalently, we shall identify (6.2.4.2) with the equivariant quotient of

$$\coprod \Xi_{\Phi_n,\delta_n} \to \coprod C_{\Phi_n,\delta_n} \to \coprod \mathsf{M}_n^{\mathsf{Z}_n}$$

by H_n, where the disjoint unions are over representatives $(\mathsf{Z}_n,\Phi_n,\delta_n)$ (with the same (X,Y,ϕ)) in $(\mathsf{Z}_{\mathcal{H}},\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$, which carries the desired compatible actions of $\Gamma_{\Phi_{\mathcal{H}}}$. (Here we allow the whole $(\mathsf{Z}_n,\Phi_n,\delta_n)$ to vary.)

Note that the construction is independent of the n we choose, because the construction using any m such that $n|m$ and $\Box \nmid m$ will factor through the above quotient diagram (6.2.4.3) and displayed equations. This enables us to state the following analogue of Proposition 6.2.3.18 for $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$:

Proposition 6.2.4.7. *Let $\mathcal{H}$ be an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ as above. Let us fix the choice of a representative $(\mathsf{Z}_{\mathcal{H}},\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ of a cusp label at level $\mathcal{H}$, where $\Phi_{\mathcal{H}} = (X,Y,\phi,\varphi_{-2,\mathcal{H}},\varphi_{0,\mathcal{H}})$, which defines a finite étale cover $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ (as in Definition 5.4.2.6).*

Let us consider the category fibered in groupoids over the category of schemes over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ whose fiber over each scheme S has objects the tuples

$$(\mathsf{Z}_{\mathcal{H}},(X,Y,\phi,\varphi^{\sim}_{-2,\mathcal{H}},\varphi^{\sim}_{0,\mathcal{H}}),(A,\lambda_A,i_A,\varphi_{-1,\mathcal{H}}),\delta_{\mathcal{H}},(c_{\mathcal{H}},c^{\vee}_{\mathcal{H}},\tau_{\mathcal{H}}))$$

describing degeneration data without the positivity condition over S such that $(A,\lambda_A,i_A,\varphi_{-1,\mathcal{H}})$ is the pullback of the tautological tuple over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$, which is a collection of H_n-orbits of objects defined as in Proposition 6.2.3.18 for each integer $n \geq 1$ such that $\Box \nmid n$ and $\mathcal{U}^{\Box}(n) \subset \mathcal{H}$, and such that $(\varphi^{\sim}_{-2,\mathcal{H}},\varphi^{\sim}_{0,\mathcal{H}})$ induces the $(\varphi_{-2,\mathcal{H}},\varphi_{0,\mathcal{H}})$ in $\Phi_{\mathcal{H}}$ as in Definition 5.4.2.8.

Then there is an algebraic stack $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ separated, smooth, and schematic over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$, together with a tautological tuple and a natural action of $\Gamma_{\Phi_{\mathcal{H}}}$ on $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, such that the quotient $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$ is isomorphic to the category described above (as categories fibered in groupoids over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$). Equivalently, for each tuple $(\mathsf{Z}_{\mathcal{H}},(X,Y,\phi,\varphi^{\sim}_{-2,\mathcal{H}},\varphi^{\sim}_{0,\mathcal{H}}),(A,\lambda_A,i_A,\varphi_{-1,\mathcal{H}}),\delta_{\mathcal{H}},(c_{\mathcal{H}},c^{\vee}_{\mathcal{H}},\tau_{\mathcal{H}}))$ over a scheme S over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$, there is a morphism $S \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ (over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$), which is unique up to an isomorphism $(f_Y : Y \xrightarrow{\sim} Y, f_X : X \xrightarrow{\sim} X)$ in $\Gamma_{\Phi_{\mathcal{H}}}$, such that the tuple over S is the pullback of the tautological tuple over $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ if we identify X by f_X and Y by f_Y.

The structural morphism $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ factorizes as the composition $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ of morphisms compatible with the natural actions of $\Gamma_{\Phi_{\mathcal{H}}}$ (trivial on $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$), where $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is a torsor under the torus $E_{\Phi_{\mathcal{H}}} \cong \underline{\mathrm{Hom}}(\mathbf{S}_{\Phi_{\mathcal{H}}},\mathbf{G}_{\mathrm{m}})$; where $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$ is an abelian scheme torsor, which is an abelian scheme when, for some (and hence every) choice of a representative $(\mathsf{Z}_n,\Phi_n,\delta_n)$ in $(\mathsf{Z}_{\mathcal{H}},\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$, the splitting of the canonical homomorphism $\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n} \ltimes \mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n} \twoheadrightarrow \mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}$ defined by δ_n induces a splitting of the canonical homomorphism $H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}} \twoheadrightarrow H'_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}}$, and hence an isomorphism $H_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}\ltimes\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}} \cong H'_{n,\mathrm{G}^{\mathrm{ess}}_{h,\mathsf{Z}_n}} \ltimes H_{n,\mathrm{U}^{\mathrm{ess}}_{1,\mathsf{Z}_n}}$; and where $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ is as above, inducing an isomorphism $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}} \xrightarrow{\sim} \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. The $E_{\Phi_{\mathcal{H}}}$-torsor structure of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ defines a canonical homomorphism

$$\mathbf{S}_{\Phi_{\mathcal{H}}} \to \underline{\mathrm{Pic}}(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}) : \ell \mapsto \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell), \tag{6.2.4.8}$$

giving for each $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$ an invertible sheaf $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ (up to isomorphism), together with isomorphisms $\Delta^*_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\ell,\ell'} : \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell') \xrightarrow{\sim} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell+\ell')$ for all $\ell,\ell' \in \mathbf{S}_{\Phi_{\mathcal{H}}}$, satisfying the necessary compatibilities with each other making $\underset{\ell\in\mathbf{S}_{\Phi_{\mathcal{H}}}}{\oplus} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ an $\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-algebra, such that

$$\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \cong \underline{\mathrm{Spec}}_{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}\Big(\underset{\ell\in\mathbf{S}_{\Phi_{\mathcal{H}}}}{\oplus} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)\Big).$$

6.2.5 Construction with the Positivity Condition

We now construct toroidal embeddings $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over which the action of $\Gamma_{\Phi_{\mathcal{H}}}$ extends naturally, such that the period homomorphism vanishes along a suitable subalgebraic stack of the *boundary* $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} - \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. (In general, the period homomorphism does not vanish along the *whole* boundary of $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.)

Let $\mathbf{S}_{\Phi_{\mathcal{H}}}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_{\mathcal{H}}},\mathbb{Z})$ be the $\mathbb{Z}$-dual of $\mathbf{S}_{\Phi_{\mathcal{H}}}$, and let $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}} := \mathbf{S}_{\Phi_{\mathcal{H}}}^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{R} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_{\mathcal{H}}},\mathbb{R})$. By definition of $\mathbf{S}_{\Phi_{\mathcal{H}}}$ (in Lemma 6.2.4.4), the $\mathbb{R}$-vector space $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ is isomorphic to the space of Hermitian pairings $(\!|\cdot,\cdot|\!) : (Y\underset{\mathbb{Z}}{\otimes}\mathbb{R}) \times (Y\underset{\mathbb{Z}}{\otimes}\mathbb{R}) \to \mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R} = B\underset{\mathbb{Q}}{\otimes}\mathbb{R}$, by sending a Hermitian pairing $(\!|\cdot,\cdot|\!)$ to the function $y\otimes\phi(y') \mapsto \mathrm{Tr}_{B/\mathbb{Q}}(\!|y,y'|\!)$ in $\mathrm{Hom}_{\mathbb{R}}((Y\underset{\mathbb{Z}}{\otimes}\mathbb{R})\times(Y\underset{\mathbb{Z}}{\otimes}\mathbb{R}),\mathbb{R}) \cong (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ (see Lemma 1.1.4.5).

Definition 6.2.5.1. *An element b in $B\underset{\mathbb{Q}}{\otimes}\mathbb{R}$ is **symmetric** if $b^\star = b$.*

Definition 6.2.5.2. *An element b in $B\underset{\mathbb{Q}}{\otimes}\mathbb{R}$ is **positive** (resp. **semipositive**) if it is symmetric and if*
$$(B\underset{\mathbb{Q}}{\otimes}\mathbb{R})\times(B\underset{\mathbb{Q}}{\otimes}\mathbb{R}) \to \mathbb{R} : (x,y)\mapsto \mathrm{Tr}_{B/\mathbb{Q}}(ybx^\star)$$
*defines a **positive definite** (resp. **positive semidefinite**) symmetric $\mathbb{R}$-bilinear pairing. We denote this by $b>0$ (resp. $b\geq 0$).*

Definition 6.2.5.3. *A Hermitian pairing $(\!|\cdot,\cdot|\!) : (Y\underset{\mathbb{Z}}{\otimes}\mathbb{R}) \times (Y\underset{\mathbb{Z}}{\otimes}\mathbb{R}) \to B\underset{\mathbb{Q}}{\otimes}\mathbb{R}$ is **positive definite** (resp. **positive semidefinite**) if $(\!|y,y|\!)>0$ (resp. $(\!|y,y|\!)\geq 0$) for all nonzero $y\in Y$.*

Definition 6.2.5.4. *We say that the radical (namely, the annihilator of the whole space) of a positive semidefinite Hermitian pairing $(\!|\cdot,\cdot|\!) : (Y\underset{\mathbb{Z}}{\otimes}\mathbb{R}) \times (Y\underset{\mathbb{Z}}{\otimes}\mathbb{R}) \to B\underset{\mathbb{Q}}{\otimes}\mathbb{R}$ is **admissible** if it is the $\mathbb{R}$-span of some **admissible submodule** Y' of Y.*

Remark 6.2.5.5. An admissible radical is automatically *rational* in the sense that it is spanned by elements in $Y\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. When $\mathcal{O}$ is maximal, the two notions are identical.

Definition 6.2.5.6. *We define $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$) to be the subset of $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ corresponding to positive semidefinite (resp. positive definite) Hermitian pairings with admissible radicals.*

Then both $\mathbf{P}_{\Phi_{\mathcal{H}}}$ and $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$ are cones in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ (see Definition 6.1.1.3).

Lemma 6.2.5.7. *If $H\in\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $H\in\mathbf{P}^+_{\Phi_{\mathcal{H}}}$) then $H(y,\phi(y))\geq 0$ (resp. $H(y,\phi(y))>0$) for all nonzero $y\in Y$.*

Proof. By definition, there exists some positive definite (resp. positive semidefinite) Hermitian form $(\!|\cdot,\cdot|\!)$ such that $H(y,\phi(y')) = \mathrm{Tr}_{B/\mathbb{Q}}(\!|y,y'|\!)$ for all $y,y'\in Y$. Hence $(\!|y,y|\!)>0$ (resp. $(\!|y,y|\!)\geq 0$) for all nonzero $y\in Y$, which means we have $H(y,\phi(y))>0$ (resp. $H(y,\phi(y))\geq 0$) for all nonzero $y\in Y$ after applying $\mathrm{Tr}_{B/\mathbb{Q}}$. □

According to [76, §2], there is no loss of generality in identifying $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ with products of one of those standard examples $\mathrm{M}_r(\mathbb{R})$, $\mathrm{M}_r(\mathbb{C})$, and $\mathrm{M}_r(\mathbb{H})$, with the cone $\mathbf{P}_{\Phi_{\mathcal{H}}}$ identified with the positive semidefinite matrices with admissible radicals, a condition no stronger than the condition with rational radicals (see Remark 6.2.5.5). Hence, by [16, Ch. II], with a minor error corrected by Looijenga as remarked in [42, Ch. IV, §2], it is known that there exist $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decompositions of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ with respect to the integral structure given by $\mathbf{S}^\vee_{\Phi_{\mathcal{H}}}$ in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$, naturally directed partially ordered by refinements.

Let $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j\in J}$ be any such cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$. Let $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} = \overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ be the toroidal embedding of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ defined by $\Sigma_{\Phi_{\mathcal{H}}}$ as in Definition 6.1.2.3. Note that the choice of $\delta_{\mathcal{H}}$ is unrelated to the choice of $\Sigma_{\Phi_{\mathcal{H}}}$ (see also Definition 6.2.6.2 below; this is why we use the notation $\Sigma_{\Phi_{\mathcal{H}}}$ rather than $\Sigma_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$). By construction, $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ has the properties described in Theorem 6.1.2.8, with the following additional ones:

Proposition 6.2.5.8 (cf. [42, Ch. IV, p. 102])**.** *1. There are constructible $\Gamma_{\Phi_{\mathcal{H}}}$-equivariant étale constructible sheaves (of $\mathcal{O}$-lattices) $\underline{X}$ and $\underline{Y}$ on $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, together with an ($\mathcal{O}$-equivariant) embedding $\underline{\phi} : \underline{Y} \hookrightarrow \underline{X}$, which are defined as follows:*

Each admissible surjection $X \twoheadrightarrow X'$ of $\mathcal{O}$-lattices (see Definition 1.2.6.7) determines a surjection from $(\mathsf{Z}_{\mathcal{H}},\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ to some representative $(\mathsf{Z}'_{\mathcal{H}},\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ of a cusp label at level $\mathcal{H}$ by Lemma 5.4.2.11, where $\mathsf{Z}'_{\mathcal{H}}$ and $\Phi'_{\mathcal{H}} = (X',Y',\phi',\varphi'_{-2,\mathcal{H}},\varphi'_{0,\mathcal{H}})$ are uniquely determined by the construction. Consequently, it makes sense to define $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ and an embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ for each admissible surjection $X\twoheadrightarrow X'$.

Over the locally closed stratum $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_j}$, the sheaf $\underline{X}$ is the constant quotient sheaf X_{σ_j} of X, with the quotient $X\twoheadrightarrow X_{\sigma_j}$ an admissible surjection defining a pair $(\mathsf{Z}_{\mathcal{H},\sigma_j},\Phi_{\mathcal{H},\sigma_j} = (X_{\sigma_j},Y_{\sigma_j},\phi_{\sigma_j},\varphi_{-2,\mathcal{H},\sigma_j},\varphi_{0,\mathcal{H},\sigma_j}))$ such that σ_j is contained in the image of the embedding $\mathbf{P}^+_{\phi_{\sigma_j}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$. We shall interpret this as having a sheaf version of $\Phi_{\mathcal{H}}$, written as $\underline{\Phi}_{\mathcal{H}} = (\underline{X},\underline{Y},\underline{\phi},\underline{\varphi}_{-2,\mathcal{H}},\underline{\varphi}_{0,\mathcal{H}})$.

2. *The formation of $\mathbf{S}_{\Phi_{\mathcal{H}}}$ from $\Phi_{\mathcal{H}}$ applies to $\underline{\Phi}_{\mathcal{H}}$ and defines a sheaf $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}}$.*

3. *There is a tautological homomorphism $\underline{B} : \underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}} \to \underline{\mathrm{Inv}}(\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})$ of constructible sheaves of groups (see Definition 4.2.4.1) which sends the class of $\ell\in\mathbf{S}_{\Phi_{\mathcal{H},\sigma_j}}$ to the sheaf of ideals $\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma_j)} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ on $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma_j)$, such that*

   (a) this homomorphism $\underline{B}$ is $\Gamma_{\Phi_{\mathcal{H}}}$-equivariant (because it is compatible with twists of the identification of $\Phi_{\mathcal{H}}$) and $E_{\Phi_{\mathcal{H}}}$-invariant (because $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ corresponds to a weight subsheaf of the $\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-algebra $\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ under the action of $E_{\Phi_{\mathcal{H}}}$), and is trivial on the open subscheme $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ of $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$;

   (b) for each local section y of $\underline{Y}$, the support of $\underline{B}(y\otimes\phi(y))$ is effective, and is the same as the support of y. This is because $\sigma(y,\phi(y))\geq 0$ for all $\sigma\subset\mathbf{P}_{\Phi_{\mathcal{H}}}$ and $y\in Y$, and $\sigma(y,\phi(y))>0$ when $\sigma\subset\mathbf{P}^+_{\Phi_{\mathcal{H},\sigma}}$ and $0\neq y\in Y_\sigma$.

For each nondegenerate rational polyhedral cone σ in $\mathbf{P}_{\Phi_{\mathcal{H}}}\subset(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$, we can define the affine toroidal embedding $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$, which can be interpreted as the *moduli space* for certain degeneration data without the positivity condition, as follows:

Let R be a noetherian normal domain with fraction field K, and suppose we have a morphism $t_R : \mathrm{Spec}(R) \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ that is liftable over $\mathrm{Spec}(K)$ to a morphism $\tilde{t}_K : \mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. By abuse of notation, let us denote by $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)_R$ the R-invertible module defined by the pullback under t_R of the invertible sheaf $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, and denote $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)_R \underset{R}{\otimes} K$ by $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)_K$. Since $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \cong \underline{\mathrm{Spec}}_{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}\Big(\underset{\ell\in\mathbf{S}_{\Phi_{\mathcal{H}}}}{\oplus} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)\Big)$, the morphism $\tilde{t}_K$ defines isomorphisms $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)_K \xrightarrow{\sim} K$, which defines an embedding of $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)_R$ as an R-invertible submodule I_ℓ of K. Therefore, the pullback of the homomorphism (6.2.4.8) in Proposition 6.2.4.7 determines a homomorphism

$$B : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathrm{Inv}(R) : \ell \mapsto I_\ell \tag{6.2.5.9}$$

(see Definition 4.2.4.1). If $\ell = [y \otimes \chi]$ for some $y \in Y$ and $\chi \in X$, then $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \cong (c^\vee(y), c(\chi))^*\mathcal{P}_A$ by construction, and hence $I_\ell = I_{y,\chi}$ as R-invertible submodules of K (see Definition 4.2.4.6). For each discrete valuation $\upsilon : K^\times \to \mathbb{Z}$ of K, since I_ℓ is locally principal for every ℓ, it makes sense to consider the composition

$$\upsilon \circ B : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathbb{Z} : \ell \mapsto \upsilon(I_\ell), \tag{6.2.5.10}$$

which is an element in $\mathbf{S}_{\Phi_{\mathcal{H}}}^\vee$.

Proposition 6.2.5.11. *With assumptions and notation as above, the universal property of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ is as follows: The morphism $\tilde{t}_K : \mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ extends to a morphism $\tilde{t}_R : \mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ if and only if, for every discrete valuation $\upsilon : K^\times \to \mathbb{Z}$ of K such that $\upsilon(R) \geq 0$, the corresponding homomorphism $\upsilon \circ B : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathbb{Z}$ as in (6.2.5.10) (or rather its composition with $\mathbb{Z} \hookrightarrow \mathbb{R}$) lies in the closure $\overline{\sigma}$ of σ in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$.*

Proof. Since $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma) \cong \underline{\mathrm{Spec}}_{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}\Big(\underset{\ell\in\sigma^\vee}{\oplus} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)\Big)$ is relatively affine over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, the morphism $\tilde{t}_K$ extends to a morphism $\tilde{t}_R : \mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ if $I_\ell \subset R$ for every $\ell \in \sigma^\vee$. Since R is noetherian and normal, this is true if $(\upsilon \circ B)(\ell) \geq 0$ for every discrete valuation υ of K such that $\upsilon(R) \geq 0$ and for every $\ell \in \sigma^\vee$, or equivalently if $\upsilon\circ B$ pairs nonnegatively with $\sigma^\vee$ under the canonical pairing between $\mathbf{S}_{\Phi_{\mathcal{H}}}^\vee$ and $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$, or equivalently if $\upsilon \circ B$ lies in $\overline{\sigma}$, as desired. ☐

Remark 6.2.5.12. If $\tilde{t}_K$ extends to $\tilde{t}_R$, then the homomorphism $B : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathrm{Inv}(R)$ agrees with the pullback of the homomorphism $\underline{B} : \underline{\mathbf{S}_{\Phi_{\mathcal{H}}}} \to \underline{\mathrm{Inv}}(\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})$ under $\tilde{t}_R$. (Thus, the notation is consistent when B and $\underline{B}$ can be compared over R.)

Remark 6.2.5.13. Recall that the σ-stratum $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ is defined (see Lemma 6.1.2.6 and Definition 6.1.2.7) by the sheaf of ideals $\mathscr{I}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \cong \underset{\ell\in\sigma_0^\vee}{\oplus} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ in $\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)} \cong \underset{\ell\in\sigma^\vee}{\oplus} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ (see Convention 6.2.3.20). Since $\sigma \subset \mathbf{P}_{\Phi_{\mathcal{H}}}$ is positive semidefinite, we have $\sigma(\ell) \geq 0$ for every ℓ of the form $[y \otimes \phi(y)]$. As a result, the trivialization

$$\tau(y, \phi(y)) : \mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(y \otimes \phi(y)) \xrightarrow{\sim} \mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$$

over $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ extends to a section

$$\tau(y, \phi(y)) : \mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(y \otimes \phi(y)) \to \mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)} \tag{6.2.5.14}$$

over $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$. If $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$, then by Lemma 6.2.5.7, we have $\sigma(y \otimes \phi(y)) > 0$ for every $y \neq 0$. In this case, the section $\tau(y, \phi(y))$ over $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ as in (6.2.5.14) has image contained in $\mathscr{I}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. This is almost the positivity condition, except that the base scheme is not completed along $\mathscr{I}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$.

Since we have the tautological presence of $G^\natural$ and τ (defined by the tautological tuple $(A, \underline{X}, \underline{Y}, c, c^\vee, \tau)$) over the algebraic stack $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ separated, smooth, and locally of finite type over S_0, we can define as in Section 4.6.2 the Kodaira–Spencer morphism

$$\mathrm{KS}_{(G^\natural,\iota)/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0} : \underline{\mathrm{Lie}}^\vee_{G^\natural/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \to \Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}[d\log\infty]. \tag{6.2.5.15}$$

Let $\lambda^\natural : G^\natural \to G^{\vee,\natural}$ be the homomorphism defined by the tautological data $\lambda_A : A \to A^\vee$ and $\phi : Y \to X$. Then $\lambda^\natural$ induces an $\mathcal{O}$-equivariant morphism $(\lambda^\natural)^* : \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \to \underline{\mathrm{Lie}}^\vee_{G^\natural/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$. Let $i^\natural : \mathcal{O} \to \mathrm{End}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}(G^\natural)$ denote the tautological $\mathcal{O}$-action morphism on $G^\natural$.

Definition 6.2.5.16. *The $\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-module $\underline{\mathrm{KS}} = \underline{\mathrm{KS}}_{(G^\natural,\lambda^\natural,i^\natural)/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ is the quotient of $\underline{\mathrm{Lie}}^\vee_{G^\natural/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ by the $\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-submodule spanned by $(\lambda^\natural)^*(y)\otimes z - (\lambda^\natural)^*(z)\otimes y$ and $(i^\natural(b))^*(x)\otimes y - x\otimes (i^\natural(b)^\vee)^*(y)$, for $x \in \underline{\mathrm{Lie}}^\vee_{G^\natural/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$, $y, z \in \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$, and $b \in \mathcal{O}$ (cf. Definition 2.3.5.1).*

Remark 6.2.5.17. By Proposition 1.2.2.3 and the assumptions on the Lie algebra conditions, the formation of $\underline{\mathrm{KS}}$ does not produce torsion elements because $\mathrm{I}_{\mathrm{bad}}\,\mathrm{Disc}$ is invertible in the structural rings of the base schemes.

Proposition 6.2.5.18. *The Kodaira–Spencer morphism (6.2.5.15) factors through the sheaf $\underline{\mathrm{KS}}$ defined in Definition 6.2.5.16, and induces an isomorphism*

$$\underline{\mathrm{KS}} \xrightarrow{\sim} \Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}[d\log\infty]. \tag{6.2.5.19}$$

Proof. Let us analyze the structural morphism $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{S}_0$ as a composition of *smooth* morphisms,

$$\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \xrightarrow{\pi_0} C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \xrightarrow{\pi_1} \mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}} \xrightarrow{\pi_2} \mathsf{S}_0.$$

For simplicity, let us denote the composition $\pi_1\circ\pi_0$ by π_{10}. Then $\Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}[d\log\infty]$ has an *increasing filtration*

$$0 \subset \pi_{01}^*\,\Omega^1_{\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}/\mathsf{S}_0} \subset \pi_0^*\,\Omega^1_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0} \subset \Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}[d\log\infty],$$

with graded pieces given by $\pi_{01}^*\ \Omega^1_{\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}/\mathsf{S}_0}$, $\pi_0^*\ \Omega^1_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}}$, and $\Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}[d\log\infty]$, all of which are locally free of finite rank.

On the other hand, the sheaf $\underline{\mathrm{KS}} = \underline{\mathrm{KS}}_{(G^\natural,\lambda^\natural,i^\natural)/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ has an increasing filtration given by $\pi_{01}^*\ \underline{\mathrm{KS}}_{(A,\lambda_A,i_A)/\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}}$, the pullback (under π_0) of the quotient $\underline{\mathrm{KS}}_{(A,c,c^\vee,\lambda^\natural,i^\natural)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ of

$$\underline{\mathrm{Lie}}^\vee_{G^\natural/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} + \underline{\mathrm{Lie}}^\vee_{A/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$$

(as an $\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-submodule of $\underline{\mathrm{Lie}}^\vee_{G^\natural/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$) by relations as in Definition 6.2.5.16, and the whole sheaf $\underline{\mathrm{KS}}$. Hence it suffices to show that the morphism (6.2.5.19) respects the filtrations and induces isomorphisms between the graded pieces.

By Proposition 2.3.5.2, since $\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}$ is étale over $\mathsf{M}_{\mathcal{H}_h}$, the Kodaira–Spencer morphism $\mathrm{KS}_{A/\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}/\mathsf{S}_0}$ for A over $\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}$ induces an isomorphism $\underline{\mathrm{KS}}_{(A,\lambda_A,i_A)/\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}} \xrightarrow{\sim} \Omega^1_{\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}/\mathsf{S}_0}$, and hence the same remains true under pullback by π_{01}. Since the Kodaira–Spencer morphism $\mathrm{KS}_{A/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0} = \pi_{01}^* \mathrm{KS}_{A/\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}/\mathsf{S}_0}$ for A over $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is the restriction of the Kodaira–Spencer morphism KS in (6.2.5.15) (cf. Remark 4.6.2.7), we see that the first filtered pieces are respected.

By the deformation-theoretic interpretation of the Kodaira–Spencer morphisms $\mathrm{KS}_{(A,c)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$ and $\mathrm{KS}_{(A^\vee,c^\vee)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$ in Section 4.6.1 (see, in particular, Definition 4.6.1.2), we see that the restrictions of both of them to $\underline{\mathrm{Lie}}^\vee_{A/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ agree with $\mathrm{KS}_{A/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$, which induces a surjection onto $\pi_1^* \Omega^1_{\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}/\mathsf{S}_0}$. Hence they define a morphism from

$$\underline{\mathrm{Lie}}^\vee_{G^\natural/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} + \underline{\mathrm{Lie}}^\vee_{A/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^{\vee,\natural}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$$

to $\Omega^1_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$, which induces (after replacing the source of this morphism with its quotient by $\underline{\mathrm{Lie}}^\vee_{A/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$) a morphism from

$$\underline{\mathrm{Lie}}^\vee_{T/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} + \underline{\mathrm{Lie}}^\vee_{A/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{T^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$$

to $\Omega^1_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}}$. The realization of $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ as a finite étale quotient of some C_{Φ_n,δ_n} with the condition that $\lambda_A c_n^\vee - c_n\phi_n = b_{\Phi_n,\delta_n}$ for a tautological homomorphism $b_{\Phi_n,\delta_n} : \frac{1}{n}Y/Y \to A^\vee[n]$ defined over $\mathsf{M}_n^{\mathsf{Z}_n}$ implies that the above morphism factors through the (same) quotient image of either $\underline{\mathrm{Lie}}^\vee_{T/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{A^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ or $\underline{\mathrm{Lie}}^\vee_{A/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{T^\vee/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$, which can be identified with the quotient of the above $\underline{\mathrm{KS}}_{(A,c,c^\vee,\lambda^\natural,i^\natural)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ by $\underline{\mathrm{KS}}_{(A,\lambda_A,i_A)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} = \pi_1^* \underline{\mathrm{KS}}_{(A,\lambda_A,i_A)/\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}}$. By the fact that $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is the universal space for $(c_{\mathcal{H}}, c^\vee_{\mathcal{H}})$ over $\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}$, this defines an isomorphism

$$\underline{\mathrm{KS}}_{(A,c,c^\vee,\lambda^\natural,i^\natural)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} / \underline{\mathrm{KS}}_{(A,\lambda_A,i_A)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \xrightarrow{\sim} \Omega^1_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}},$$

and hence an isomorphism

$$\underline{\mathrm{KS}}_{(A,c,c^\vee,\lambda^\natural,i^\natural)/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \xrightarrow{\sim} \Omega^1_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}.$$

Since the pullback of this isomorphism (under π_0) to $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is induced by the restriction of the Kodaira–Spencer morphism KS in (6.2.5.15) (cf. Remark 4.6.2.7), we see that the second filtered pieces are also respected, with an induced isomorphism between the second graded pieces.

Finally, we arrive at the top filtered pieces, and the question is whether the induced morphism

$$\underline{\mathrm{Lie}}^\vee_{T/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{T^\vee/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \to \Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}[d\log\infty] \tag{6.2.5.20}$$

induces an isomorphism between the top graded pieces. For simplicity, let us denote by $\underline{\mathrm{KS}}_{(T,\lambda_T,i_T)/\mathsf{S}_0}$ the quotient of $\underline{\mathrm{Lie}}^\vee_{T/\mathsf{S}_0} \underset{\mathscr{O}_{\mathsf{S}_0}}{\otimes} \underline{\mathrm{Lie}}^\vee_{T^\vee/\mathsf{S}_0}$ by relations as in Definition 6.2.5.16, and by $\underline{\mathrm{KS}}_{(T,\lambda_T,i_T)/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ and $\underline{\mathrm{KS}}_{(T,\lambda_T,i_T)/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ their pullbacks to $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ and $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, respectively.

Let us first consider the restriction

$$\underline{\mathrm{Lie}}^\vee_{T/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \underset{\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{T^\vee/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \to \Omega^1_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \tag{6.2.5.21}$$

of (6.2.5.20) to $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, which is induced by the Kodaira–Spencer morphism $\mathsf{KS}_{(G^\natural,\iota)/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$ defined deformation-theoretically as in Definition 4.6.2.6. The realization of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ as a finite étale quotient of some Ξ_{Φ_n,δ_n} with the condition that $\iota_n(\frac{1}{n}y, \phi(y'))\iota_n(\frac{1}{n}y', \phi(y))^{-1} = a_{\Phi_n,\delta_n}(\frac{1}{n}y, \frac{1}{n}y')$ for a tautological homomorphism $a_{\Phi_n,\delta_n} : \frac{1}{n}Y \times \frac{1}{n}Y \to \mathbf{G}_{\mathrm{m}}$ over C_{Φ_n,δ_n} implies that the morphism (6.2.5.21) factors through the quotient $\underline{\mathrm{KS}}_{(T,\lambda_T,i_T)/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ of its source. By the fact that $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is the universal space for ι over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, we see that the induced morphism $\underline{\mathrm{KS}}_{(T,\lambda_T,i_T)/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \to \Omega^1_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$ is an isomorphism.

If we work over $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, then the morphism (6.2.5.20) is induced by the extended Kodaira–Spencer morphism $\mathsf{KS}_{(G^\natural,\iota)/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\mathsf{S}_0}$ defined as in Definition 4.6.2.12. Since its image in $\Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}[d\log\infty]$ contains $d\log(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))$ for all $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$, which are exactly the generators, we see that (6.2.5.20) induces an isomorphism $\underline{\mathrm{KS}}_{(T,\lambda_T,i_T)/\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \xrightarrow{\sim} \Omega^1_{\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}[d\log\infty]$ between the top graded pieces, as desired. □

Let us return to the context that we have the toroidal embedding $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \hookrightarrow \overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} = \overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ defined by a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j\in J}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}} \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$. Let $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} = \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ be the formal completion of $\overline{\Xi}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ along the union of the σ_j-strata $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_j}$ for $\sigma_j \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$. For each $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$, let $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ be the formal completion of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ along the σ-stratum $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. Then, using the language of relative schemes over formal algebraic stacks (see [61]), there are tautological tuples of the form

$$(\mathsf{Z}_{\mathcal{H}}, (X, Y, \phi, \varphi^\sim_{-2,\mathcal{H}}, \varphi^\sim_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c^\vee_{\mathcal{H}}, \tau_{\mathcal{H}})) \tag{6.2.5.22}$$

over the formal algebraic stacks $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ and $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (where $(\varphi^\sim_{-2,\mathcal{H}}, \varphi^\sim_{0,\mathcal{H}})$ induces the $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ in $\Phi_{\mathcal{H}}$ as in Definition 5.4.2.8), the one on the latter being the pullback of the one on the former under the canonical morphism $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.

Moreover, this tautological tuple over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ satisfies the positive condition in the following sense: We have a functorial assignment that, to each connected affine formal scheme $\mathfrak{U}$ with an étale (i.e., formally étale and of finite type; see [59, I, 10.13.3]) morphism $\mathfrak{U} = \mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, where R and I satisfy the setting in Section 5.2.1, assigns a tuple of the form (6.2.5.22) (with the positivity condition) over the (smooth) scheme $\mathrm{Spec}(R) = \mathrm{Spec}(\Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}}))$ over S_0. By Theorem 5.3.1.19 (see also Definition 5.4.2.8 and Remark 5.4.2.9), Mumford's construction defines an object

$$({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathrm{Spec}(\Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}}))$$

in $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R, I)$, which we call a degenerating family of type $\mathsf{M}_{\mathcal{H}}$ as in Definition 5.3.2.1. Moreover, the torus part of each fiber of ${}^\heartsuit G$ over the support of $\mathfrak{U}$ is split with character group X. If we have an étale (i.e., formally étale and of finite type) morphism $\mathrm{Spf}(R_1, I_1) \to \mathrm{Spf}(R_2, I_2)$ (with (R_1, I_1) and (R_2, I_2) as the (R, I) above), and if a degeneration datum over $\mathrm{Spec}(R_2)$ pulls back to a degeneration datum over $\mathrm{Spec}(R_1)$, then the degenerating family constructed by Mumford's construction using the degeneration datum over $\mathrm{Spec}(R_2)$ pulls back to a degenerating family over $\mathrm{Spec}(R_1)$. The functoriality in Theorem 4.4.16 over $\mathrm{Spec}(R_1)$ then assures that this pullback agrees with the degenerating family constructed using the degeneration datum over $\mathrm{Spec}(R_1)$. In particular, we see that the assignment of $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathrm{Spec}(R) = \mathrm{Spec}(\Gamma(\mathfrak{U}, \mathscr{O}_{\mathfrak{U}}))$ to $\mathfrak{U} = \mathrm{Spf}(R, I)$ is functorial. Hence the assignment defines a (relative) degenerating family

$$({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}.$$

Since the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ is $\Gamma_{\Phi_{\mathcal{H}}}$-admissible, the group $\Gamma_{\Phi_{\mathcal{H}}}$ acts naturally on all the objects involved in the degeneration data, and hence by functoriality on the degenerating family $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.

Definition 6.2.5.23. *Let σ be any nondegenerate rational polyhedral cone in $\mathbf{P}_{\Phi_{\mathcal{H}}}$. The group $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is defined as the subgroup of $\Gamma_{\Phi_{\mathcal{H}}}$ consisting of elements that map σ to itself under the natural action of $\Gamma_{\Phi_{\mathcal{H}}}$ on $\mathbf{P}_{\Phi_{\mathcal{H}}}$.*

Then similarly we have the degenerating family

$$({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma},$$

together with an equivariant action of $\Gamma_{\Phi_{\mathcal{H}},\sigma}$.

Definition 6.2.5.24. *An **admissible boundary component** of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is the image of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ under the embedding $(\mathbf{S}_{\Phi'_{\mathcal{H}}})^{\vee}_{\mathbb{R}} \hookrightarrow (\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ defined by some surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ (see Definition 5.4.2.12).*

Condition 6.2.5.25 (cf. [42, Ch. IV, Rem. 5.8(a)])**.** *The cone decomposition $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j\in J}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is chosen such that, for each $j \in J$, if $\gamma\overline{\sigma}_j \cap \overline{\sigma}_j \neq \{0\}$ for some $\gamma \in \Gamma_{\Phi_{\mathcal{H}}}$, then a power of γ acts as the identity on the smallest admissible boundary component of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ containing $\gamma\overline{\sigma}_j \cap \overline{\sigma}_j$.*

Remark 6.2.5.26. Suppose the image of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ under the embedding $(\mathbf{S}_{\Phi'_{\mathcal{H}}})^{\vee}_{\mathbb{R}} \hookrightarrow (\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ defined by some surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ is the smallest admissible boundary component of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ containing σ_j. Then the condition that a power of γ acts as the identity on the image of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ shows that the eigenvalues of the actions of γ on X' and on Y' are roots of unity. This forces the actions to be trivial if $\mathcal{H}$ is neat (see Definition 1.4.1.8).

Lemma 6.2.5.27. *Suppose that Condition 6.2.5.25 is satisfied. If $\mathcal{H}$ is neat (see Definition 1.4.1.8) and if $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$, then $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ acts trivially on $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ and $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. Hence $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma} = \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is a formal algebraic space when $\mathcal{H}$ is neat. As a consequence, $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is a formal (Deligne–Mumford) algebraic stack for general open compact subgroups $\mathcal{H}$ in $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$.*

Proof. Suppose Condition 6.2.5.25 is satisfied, and suppose $\mathcal{H}$ is neat, then $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is forced to be trivial as explained in Remark 6.2.5.26. The general case then follows from the existence of a surjection from a similar parameter space (with the same remaining data) defined by some neat open compact subgroup $\mathcal{H}'$ of $\mathcal{H}$. ☐

Let us assume from now on that the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j\in J}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ satisfies Condition 6.2.5.25. This is possible by refining any given cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$. (It suffices to make sure that, for each $j \in J$, no two one-dimensional faces of σ_j are in the same $\Gamma_{\Phi_{\mathcal{H}}}$-orbit.) Then the quotients $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ are formal (Deligne–Mumford) algebraic stacks, and the equivariant action of $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ on $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ implies that we have a descended family $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.

Definition 6.2.5.28. *All the degenerating families $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$, and $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ constructed above are called **Mumford families**.*

Remark 6.2.5.29. By abuse of notation, we will use the same notation $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}})$ for the Mumford families over various bases.

Remark 6.2.5.30. If $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ is in the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ defining $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, then $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is the pullback of $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ under the canonical morphism $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.

Remark 6.2.5.31. If $\sigma, \sigma' \in \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ are two smooth rational polyhedral cones such that $\sigma \subset \sigma'$, then $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is the pullback of $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma'}$ under the canonical morphism $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma'}$. However, the morphism $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma')$ between affine toroidal embeddings may map certain strata to smaller-dimensional ones, for example, when a face τ of σ is not contained in any face τ' of σ' of the same dimension as τ. Hence we cannot expect the morphism to be flat in general.

6.2.6 Identifications between Parameter Spaces

Definition 6.2.6.1. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, let $\sigma \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$, and let $\sigma' \subset (\mathbf{S}_{\Phi'_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$. We say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ are **equivalent** if there exists a pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ such that we have the following:*

1. *The two representatives $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent under $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ as in Definition 5.4.2.4. In other words, $\mathrm{Z}_{\mathcal{H}}$ and $\mathrm{Z}'_{\mathcal{H}}$ are identical, and $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ and $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ are equivalent under $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ as in Definition 5.4.2.2.*
2. *The isomorphism $(\mathbf{S}_{\Phi'_{\mathcal{H}}})^{\vee}_{\mathbb{R}} \xrightarrow{\sim} (\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ induced by $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ sends σ' to σ.*

In this case, we say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ are equivalent under the pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$.

Definition 6.2.6.2. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, and let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). We say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ are **equivalent** if $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent under some pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$, and if under one (and hence every) such (γ_X, γ_Y) the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is identified with the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$. In this case, we say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ are equivalent under the pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$.*

Definition 6.2.6.3. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, and let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). We say that the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ is a **refinement** of the triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ if $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent under some pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$, and if under one (and hence every) such (γ_X, γ_Y) the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is identified with a refinement of the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$. In this* *case, we say that the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ is a refinement of the triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ under the pair of isomorphisms $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$.*

Definition 6.2.6.4. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, and let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). A **surjection** $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ is given by a surjection $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y') : (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ (see Definition 5.4.2.12) that induces an embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ such that the restriction $\Sigma_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\Phi'_{\mathcal{H}}}}$ of the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ to $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ is the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$.*

Then we have the following formal observations:

Proposition 6.2.6.5. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, let $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$, and let $\sigma' \subset \mathbf{P}^+_{\Phi'_{\mathcal{H}}}$. Then the Mumford families $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ and $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma'}/\Gamma_{\Phi'_{\mathcal{H}},\sigma'}$ are isomorphic over $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ if and only if the triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ are equivalent (see Definition 6.2.6.1).*

Thus, up to isomorphism, the assignment of the Mumford family $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ to the equivalence class of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ is well defined.

Proposition 6.2.6.6. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels, and let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). Then the corresponding Mumford families $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ and $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\Sigma_{\Phi'_{\mathcal{H}}}}$ are isomorphic over $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ (up to an isomorphism that is unique up to actions of $\Gamma_{\Phi_{\mathcal{H}}}$ and $\Gamma_{\Phi'_{\mathcal{H}}}$) if and only if the triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ are equivalent (see Definition 6.2.6.2).*

Proposition 6.2.6.7. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels, and let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). Suppose the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ is a refinement of the triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ (see Definition 6.2.6.3). Then the corresponding Mumford family $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}}$ is the pullback of the Mumford family $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\Sigma_{\Phi'_{\mathcal{H}}}}$ via a surjection $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}}} \twoheadrightarrow \mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\Sigma_{\Phi'_{\mathcal{H}}}}$ (unique up to actions of $\Gamma_{\Phi_{\mathcal{H}}}$ and $\Gamma_{\Phi'_{\mathcal{H}}}$).*

6.3 Approximation and Gluing

Let us continue with the setting of Section 6.2.1 in this section (including especially Convention 6.2.1.1). Assume moreover that Condition 6.2.5.25 is satisfied for all the cone decompositions we choose.

For the ease of exposition we shall make the following definition:

Definition 6.3.1. *Let $(G, \lambda, i, \alpha_{\mathcal{H}})$ be a degenerating family of type $\mathsf{M}_{\mathcal{H}}$ over S (as defined in Definition 5.3.2.1) over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. Let $\underline{\mathrm{Lie}}^{\vee}_{G/S} := e^*_G\Omega^1_{G/S}$ be the dual of $\underline{\mathrm{Lie}}_{G/S}$, and let $\underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/S} := e^*_G\Omega^1_{G^{\vee}/S}$ be the dual of $\underline{\mathrm{Lie}}_{G^{\vee}/S}$. The homomorphism $\lambda : G \to G^{\vee}$ induces an $\mathcal{O}$-equivariant morphism $\lambda^* : \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/S} \to \underline{\mathrm{Lie}}^{\vee}_{G/S}$. Then we define the sheaf $\underline{\mathrm{KS}} = \underline{\mathrm{KS}}_{(G,\lambda,i)/S} = \underline{\mathrm{KS}}_{(G,\lambda,i,\alpha_{\mathcal{H}})/S}$ (cf. Definitions 2.3.5.1 and 6.2.5.16) to be*

$$(\underline{\mathrm{Lie}}^{\vee}_{G/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/S}) / \begin{pmatrix} \lambda^*(y) \otimes z - \lambda^*(z) \otimes y \\ i(b)^*(x) \otimes y - x \otimes (i(b)^{\vee})^*(y) \end{pmatrix}_{\substack{x \in \underline{\mathrm{Lie}}^{\vee}_{G/S}, \\ y,z \in \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/S}, \\ b \in \mathcal{O}}}.$$

6.3.1 Good Formal Models

Construction 6.3.1.1. Suppose that we are given a torus argument $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ at level $\mathcal{H}$ for some (split) $\mathsf{Z}_{\mathcal{H}}$. Let $(G, \lambda, i, \alpha_{\mathcal{H}})$ be any degenerating family of type $\mathsf{M}_{\mathcal{H}}$ over an excellent normal algebraic stack S. (The excellence assumption might be removed by inductive limit arguments, but we do not need this generality for our purpose.) We would like to define a homomorphism $\underline{B}(G)$, following [42, Ch. III, §10], that encodes the degeneration pattern of G in a convenient way. By abuse of notation, let us set $\underline{X}(G) := \underline{\mathbf{X}}(G)$ and $\underline{Y}(G) := \underline{\mathbf{X}}(G^{\vee})$, which are étale sheaves defined using Theorem 3.3.1.9.

Step 1. Let us begin with the case that $S = \mathrm{Spec}(R)$, where R is a noetherian normal complete *local* domain over $\mathcal{O}_{F_0,(\Box)}$ satisfying the setting of Section 5.2.1 with I the maximal ideal of R.

Suppose that the character group of the torus part of the special fiber of G (resp. $G^{\vee}$) is constant and identified with X (resp. Y). Then the $\underline{X}(G)$ (resp. $\underline{Y}(G)$) is a quotient sheaf of the constant sheaf X (resp. Y). For each point s of S, let $X(s)$ (resp. $Y(s)$) denote the pullback of $\underline{X}(G)$ (resp. $\underline{Y}(G)$) to s. Then we have quotient homomorphisms $X \twoheadrightarrow X(s)$ and $Y \twoheadrightarrow Y(s)$, which are compatible with the procedures in Lemma 5.4.2.11 and Proposition 6.2.5.8, and define sheaf objects

$$\underline{\Phi}_{\mathcal{H}}(G) = (\underline{X}(G), \underline{Y}(G), \underline{\phi}(G), \underline{\varphi}_{-2,\mathcal{H}}(G), \underline{\varphi}_{0,\mathcal{H}}(G))$$

and $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)}$ on S.

The degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ defines an object in $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R)$, and hence an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R)$ by Theorem 5.3.1.19. By Lemma 5.4.2.10, the choice of the representative $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of the cusp label determines a unique object

$$(\mathsf{Z}_{\mathcal{H}}, (X, Y, \phi, \varphi^{\sim}_{-2,\mathcal{H}}, \varphi^{\sim}_{0,\mathcal{H}}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, (c^{\vee}_{\mathcal{H}}), \tau_{\mathcal{H}}))$$

in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R)$ (up to isomorphisms inducing automorphisms of $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$, where $(\varphi^{\sim}_{-2,\mathcal{H}}, \varphi^{\sim}_{0,\mathcal{H}})$ induces $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ as in Definition 5.4.2.8). The entries other than $\tau_{\mathcal{H}}$ determine a morphism from $S = \mathrm{Spec}(R)$ to $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, while $\tau_{\mathcal{H}}$ (with its positivity) lifts this morphism to a morphism from the generic point of S to $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. Hence we can define as in (6.2.5.9) a homomorphism

$$B : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathrm{Inv}(S) : \ell \mapsto I_{\ell}. \tag{6.3.1.2}$$

Let us denote by X_s (resp. Y_s) the kernel of $X \twoheadrightarrow X(s)$ (resp. $Y \twoheadrightarrow Y(s)$). (The usage of Y_s here is consistent with that in Proposition 4.5.3.11 and Corollary 4.5.3.12.) By Proposition 4.5.3.11, $I_{[y \otimes \phi(y)]} = I_{y,\phi(y)}$ becomes trivial after localizing at a point s of S if and only if y lies in Y_s. By normality of R and by evaluating at height-one primes, the same argument as in the proof of Lemma 4.5.1.7 shows that $I_{y,\chi}$ is trivial after localizing at a point s if either $y \in Y_s$ or $\chi \in X_s$, and that I_{ℓ} and $I_{\ell'}$ agree in $\mathrm{Inv}(\mathrm{Spec}(\mathscr{O}_{S,s}))$ whenever ℓ and ℓ' have the same image in $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)}(\mathrm{Spec}(\mathscr{O}_{S,s}))$. That is, (6.3.1.2) induces a well-defined homomorphism $B_s : \underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)}(\mathrm{Spec}(\mathscr{O}_{S,s})) \to \mathrm{Inv}(\mathrm{Spec}(\mathscr{O}_{S,s}))$ for each point s. Hence (6.3.1.2) sheafifies

and defines a homomorphism
$$\underline{B}(G) : \underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)} \to \underline{\mathrm{Inv}}(S). \tag{6.3.1.3}$$

Suppose $S^\dagger = \mathrm{Spec}(R^\dagger)$, where $R^\dagger$ is the completion of R along a point $s \in S$. The pullback $(G^\dagger, \lambda^\dagger, i^\dagger, \alpha^\dagger_{\mathcal{H}})$ of $(G, \lambda, i, \alpha_{\mathcal{H}})$ to $S^\dagger$ defines an object in $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R^\dagger)$, and hence an object in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R^\dagger)$. The pullbacks of $\underline{\Phi}_{\mathcal{H}}(G)$ and $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)}$ determine the objects $\Phi^\dagger_{\mathcal{H}}$ and $\mathbf{S}_{\Phi^\dagger_{\mathcal{H}}}$ over $S^\dagger$, and hence by Lemma 5.4.2.10 an object
$$(\mathsf{Z}^\dagger_{\mathcal{H}}, (X^\dagger, Y^\dagger, \phi^\dagger, \varphi^{\dagger,\sim}_{-2,\mathcal{H}}, \varphi^{\dagger,\sim}_{0,\mathcal{H}}), (A^\dagger, \lambda^\dagger_A, i^\dagger_A, \varphi^\dagger_{-1,\mathcal{H}}), \delta^\dagger_{\mathcal{H}}, (c^\dagger_{\mathcal{H}}, (c^\vee_{\mathcal{H}})^\dagger, \tau^\dagger_{\mathcal{H}}))$$
in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R^\dagger)$ (up to isomorphisms inducing automorphisms of $\Phi^\dagger_{\mathcal{H}} = (X^\dagger, Y^\dagger, \phi^\dagger, \varphi^\dagger_{-2,\mathcal{H}}, \varphi^\dagger_{0,\mathcal{H}})$, where $(\varphi^{\dagger,\sim}_{-2,\mathcal{H}}, \varphi^{\dagger,\sim}_{0,\mathcal{H}})$ induces $(\varphi^\dagger_{-2,\mathcal{H}}, \varphi^\dagger_{0,\mathcal{H}})$ as in Definition 5.4.2.8). Then, similar to the B in (6.3.1.2) above, we have a homomorphism
$$B^\dagger : \mathbf{S}_{\Phi^\dagger_{\mathcal{H}}} \to \mathrm{Inv}(S^\dagger). \tag{6.3.1.4}$$
By Proposition 4.5.6.1, $B^\dagger$ coincides with $\underline{B}(G)(S^\dagger)$ where $\underline{B}(G)$ is the homomorphism in (6.3.1.3). Hence $\underline{B}(G)$ can be used to describe the degeneration of G along formal completions of the points of S.

Step 2. Now let us treat the case when S is the spectrum of a discrete valuation ring V with discrete valuation $\upsilon : \mathrm{Inv}(V) \to \mathbb{Z}$. Note that the desired homomorphism $\underline{B}(G)$ is determined by $B = \underline{B}(G)(S)$, and B is determined by the composition
$$\upsilon \circ \underline{B} : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathbb{Z}$$
in $\mathrm{Hom}(\mathbf{S}_{\Phi_{\mathcal{H}}}, \mathbb{Z}) = \mathbf{S}^\vee_{\Phi_{\mathcal{H}}}$. The pullback $(G, \lambda, i, \alpha_{\mathcal{H}})$ to the completion $S^\dagger$ of S along its closed point defines a homomorphism $B^\dagger$ as in (6.3.1.4) (or rather (6.3.1.2)), and the composition
$$\upsilon \circ B^\dagger : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathbb{Z}$$
is the desired homomorphism.

Step 3. Finally, let us treat the general case. Let S be an excellent normal algebraic stack. We have to define (compatibly) for each étale morphism $S' \to S$ (from a scheme S') a homomorphism
$$\underline{B}(G)(S') : \underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)}(S') \to \underline{\mathrm{Inv}}(S').$$
Since S is excellent and normal, so is S'. By étale descent, it suffices to define for those S' over which $\underline{X}(G)$ and $\underline{Y}(G)$ become quotients of constant sheaves. For each height-one point s' of S', the construction in Step 2 for discrete valuation rings (with suitable choices of representatives of cusp labels) defines a homomorphism
$$\underline{B}(G)(\mathrm{Spec}(\mathscr{O}_{S',s'})) : \underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)}(\mathrm{Spec}(\mathscr{O}_{S',s'})) \to \underline{\mathrm{Inv}}(\mathrm{Spec}(\mathscr{O}_{S',s'})).$$
Note that there are only finitely many s' such that $X(s')$ and $Y(s')$ are nontrivial. Over each open affine subscheme $U = \mathrm{Spec}(R)$ of S', these morphisms define a coherent R-submodule of $\mathrm{Frac}(R)$ for each $y \in \underline{Y}(U)$ and $\chi \in \underline{X}(U)$. The only question is whether these coherent R-submodules define compatible objects in $\underline{\mathrm{Inv}}(S')$, and we know that the answer is affirmative by making base changes to the complete local case in Step 1. This concludes the construction of the homomorphism
$$\underline{B}(G) : \underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G)} \to \underline{\mathrm{Inv}}(S) \tag{6.3.1.5}$$
in general. (This finishes Construction 6.3.1.1.)

Let us summarize the properties of $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ as follows:

Proposition 6.3.1.6. *Let $S_{\mathrm{for}} = \mathrm{Spf}(R, I)$ be an affine formal scheme, with an étale (i.e., formally étale and of finite type; see [59, I, 10.13.3]) morphism $\hat{f} : S_{\mathrm{for}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ inducing a morphism $f : S = \mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ mapping the support $\mathrm{Spec}(R/I)$ of S_{for} to the σ-stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. (In this case, the subscheme $\mathrm{Spec}(R/I)$ of S is the scheme-theoretic preimage of its image under f.) Let $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to S = \mathrm{Spec}(R)$ be the pullback of $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ to S_{for} under $\hat{f}$ (by abuse of language). Then R is an I-adically complete excellent ring, which is formally smooth over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, and hence also formally smooth over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. Let K be the fraction field of R, and let $\eta = \mathrm{Spec}(K)$ be the generic point of S.*

1. *The stratification of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ determines a stratification of $S = \mathrm{Spec}(R)$ parameterized by $\{\text{faces } \tau \text{ of } \sigma\}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ such that each stratum of S (with its reduced structure, namely, its structure as an open subscheme in a closed subscheme with reduced structure) is the scheme-theoretic preimage of the corresponding stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ under f.*

2. *The formal completion of ${}^\diamondsuit G$ along the preimage of $\mathrm{Spec}(R/I)$ is canonically isomorphic to the pullback of $G^\natural$ under $\hat{f}$ (as a formal algebraic stack, rather than a relative scheme).*

3. *The étale sheaf $\underline{\mathbf{X}}({}^\diamondsuit G)$ (see Theorem 3.3.1.9) is the quotient sheaf of the constant sheaf X such that, over the $(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\sigma})$-stratum, the sheaf $\underline{\mathbf{X}}({}^\diamondsuit G)$ is a constant quotient $X_{(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\sigma})}$ of X, with an admissible surjection $X \twoheadrightarrow X_{(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\sigma})}$ inducing a torus argument $\Phi_{\mathcal{H},(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\sigma})}$ from $\Phi_{\mathcal{H}}$ as in Lemma 5.4.2.11, such that τ is contained in the $\Gamma_{\Phi_{\mathcal{H}}}$-orbit of the image of the induced embedding $\mathbf{P}^+_{\Phi_{\mathcal{H},(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\sigma})}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$. (We know the surjection is admissible because of the existence of level-$\mathcal{H}$ structures; see Lemmas 5.2.2.2 and 5.2.2.4.) This produces a sheaf version $\underline{\Phi}_{\mathcal{H}}({}^\diamondsuit G)$ of $\Phi_{\mathcal{H}}$ over S.*

   The formation of $\mathbf{S}_{\Phi_{\mathcal{H}}}$ from $\Phi_{\mathcal{H}}$ applies to $\underline{\Phi}_{\mathcal{H}}({}^\diamondsuit G)$ and defines a sheaf $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}({}^\diamondsuit G)}$ (cf. Proposition 6.2.5.8).

   Then $\underline{\Phi}_{\mathcal{H}}({}^\diamondsuit G)$ is equivalent (see Definition 5.4.2.2) to the pullback of the tautological $\underline{\Phi}_{\mathcal{H}}$ on $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ (see Proposition 6.2.5.8) under f.

4. *Under the equivalence between $\underline{\Phi}_{\mathcal{H}}({}^\diamondsuit G)$ and the pullback of $\underline{\Phi}_{\mathcal{H}}$ above, the pullback $f^*(\underline{B}) : \underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}({}^\diamondsuit G)} \to \underline{\mathrm{Inv}}(S)$ of the tautological homomorphism $\underline{B}$ over $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ (see Proposition 6.2.5.8) under f agrees with the homomorphism $\underline{B}({}^\diamondsuit G)$ defined by $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to S$ as in Construction 6.3.1.1.*

5. *Let $\underline{\mathrm{KS}}_{({}^\diamondsuit G, {}^\diamondsuit Pol, {}^\diamondsuit i)/S}$ be the sheaf defined by $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to S$ as in Definition 6.3.1. As in Section 4.6.3, let $\widehat{\Omega}^1_{S/\mathsf{S}_0}$ denote the completion of $\Omega^1_{S/\mathsf{S}_0}$ with respect to the topology of R defined by I, which is locally free of finite rank over $\mathscr{O}_S$ (cf. [59, 0_{IV}, 20.4.9]), and let $\widehat{\Omega}^1_{S/\mathsf{S}_0}[d\log\infty]$ be the subsheaf of $(\eta \hookrightarrow S)_*(\eta \hookrightarrow S)^*\widehat{\Omega}^1_{S/\mathsf{S}_0}$ generated locally by $\widehat{\Omega}^1_{S/\mathsf{S}_0}$ and those $d\log q$ where q is a local generator of a component of the normal crossings divisor of $\mathrm{Spec}(R)$ induced by the corresponding normal crossings divisor of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ (cf. 5 of Theorem 6.1.2.8). Then the extended Kodaira–Spencer morphism (see Theorem 4.6.3.16) defines an isomorphism*
$$\mathrm{KS}_{{}^\diamondsuit G/S/\mathsf{S}_0} : \underline{\mathrm{KS}}_{({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i)/S} \xrightarrow{\sim} \widehat{\Omega}^1_{S/\mathsf{S}_0}[d\log\infty].$$

6. *The morphism $\hat{f} : S_{\mathrm{for}} = \mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, or rather the morphism $f : S = \mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, is tautological with respect to its universal property, in the following sense:*

   The setting is as follows: The base ring R and the ideal I satisfy the setting of Section 5.2.1. Let $(G, \lambda, i, \alpha_{\mathcal{H}}) \to S$ be any degenerating family of type $\mathsf{M}_{\mathcal{H}}$ (see Definition 5.3.2.1) that defines an object of $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R, I)$ (see Definition 5.3.1.17). Then the family determines an object of $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R, I)$ by Theorem 5.3.1.19, which determines, in particular, a cusp label. Suppose $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ is a representative of this cusp label. By Lemma 5.4.2.10, there exists a tuple $(A, \lambda_A, i_A, X, Y, \phi, c, c^\vee, \tau, [\alpha^\natural_{\mathcal{H}}])$ that defines the above object of $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R, I)$, together with a representative $\alpha^\natural_{\mathcal{H}} = (\mathsf{Z}_{\mathcal{H}}, \varphi^\sim_{-2,\mathcal{H}}, \varphi^\sim_{-1,\mathcal{H}}, \varphi_{0,\mathcal{H}}, \delta_{\mathcal{H}}, c_{\mathcal{H}}, c^\vee_{\mathcal{H}}, \tau_{\mathcal{H}})$ of $[\alpha^\natural_{\mathcal{H}}]$ (up to isomorphisms inducing automorphisms of $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$, where $(\varphi^\sim_{-2,\mathcal{H}}, \varphi^\sim_{0,\mathcal{H}})$ induces $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ as in Definition 5.4.2.8). By Proposition 6.2.4.7, this tuple without its positivity condition defines a morphism $\mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ that is unique up to an action of $\Gamma_{\Phi_{\mathcal{H}}}$ on the identification of $\Phi_{\mathcal{H}}$, whose composition with $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ extends to a morphism $\mathrm{Spec}(R) \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. Let $\underline{B}(G) : \mathbf{S}_{\underline{\Phi}_{\mathcal{H}}(G)} \to \underline{\mathrm{Inv}}(S)$ be the homomorphism defined as in Construction 6.3.1.1.

   Then the universal property is as follows: Suppose there exists an identification of $\Phi_{\mathcal{H}}$ such that, for each discrete valuation $\upsilon : \underline{\mathrm{Inv}}(S) \to \mathbb{Z}$ defined by a height-one prime of R, the composition $\upsilon \circ \underline{B}(G) : \mathbf{S}_{\underline{\Phi}_{\mathcal{H}}(G)} \to \mathbb{Z}$ defines an element in the closure $\overline{\sigma}$ of σ in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$. Such an identification of $\Phi_{\mathcal{H}}$ is unique up to an element in $\Gamma_{\Phi_{\mathcal{H}},\sigma}$, and all morphisms $\mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ as above induce the same morphism $\mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ if they respect such identifications of $\Phi_{\mathcal{H}}$. Then this morphism extends to a (necessarily unique) morphism $f : S = \mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, sending the subscheme $\mathrm{Spec}(R/I)$ to the σ-stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ and hence inducing a morphism $\hat{f} : S_{\mathrm{for}} = \mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ between formal algebraic stacks, such that $(G, \lambda, i, \alpha_{\mathcal{H}}) \to S$ is isomorphic to the pullback $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to S$ of $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ under $\hat{f}$.

Proof. The only nontrivial part that we have not explained is the statement on the extended Kodaira–Spencer morphism. By Theorem 4.6.3.16, the morphism

$$\mathrm{KS}_{{}^\diamondsuit G/S/\mathsf{S}_0} : \underline{\mathrm{Lie}}^\vee_{{}^\diamondsuit G/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{{}^\diamondsuit G^\vee/S} \to \widehat{\Omega}^1_{S/\mathsf{S}_0}[d\log\infty] \quad (6.3.1.7)$$

(between locally free sheaves) can be canonically identified with the morphism

$$\mathrm{KS}_{({}^\diamondsuit G^\natural, {}^\diamondsuit\iota)/S/\mathsf{S}_0} : \underline{\mathrm{Lie}}^\vee_{{}^\diamondsuit G^\natural/S} \underset{\mathscr{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{{}^\diamondsuit G^{\vee,\natural}/S} \to \widehat{\Omega}^1_{S/\mathsf{S}_0}[d\log\infty]$$

defined by the pair $({}^\diamondsuit G^\natural, {}^\diamondsuit\iota)$ in $\mathrm{DD}(R, I)$ underlying the degeneration datum associated with $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to S$. Since $\hat{f} : S_{\mathrm{for}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is étale, and since $({}^\diamondsuit G^\natural, {}^\diamondsuit\iota)$ is the pullback of the tautological $(G^\natural, \iota)$ over $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ under the underlying morphism $f : S \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ of $\hat{f}$, we can conclude the proof by applying Proposition 6.2.5.18. □

As a by-product of our usage of Proposition 6.2.5.18 in the proof,

Corollary 6.3.1.8. *Suppose $\hat{f} : S_{\mathrm{for}} = \mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is a morphism between noetherian formal schemes formally smooth over S_0, with induced morphism $f : S = \mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ such that the support $\mathrm{Spec}(R/I)$ of S_{for} is the scheme-theoretic preimage under f of some subalgebraic stack Z of the σ-stratum $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. Suppose moreover that the pullback of the stratification of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ induces a stratification of $S = \mathrm{Spec}(R)$ such that each stratum of $S = \mathrm{Spec}(R)$ (with its reduced structure, as in 1 of Proposition 6.3.1.6) is the scheme-theoretic preimage of a stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. Under these assumptions, we have an induced morphism $f_0 : \mathrm{Spec}(R/I) \to Z$, and we can define the extended Kodaira–Spencer morphism*

$$\mathrm{KS}_{{}^\diamondsuit G/S/\mathsf{S}_0} : \underline{\mathrm{KS}}_{({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i)/S} \to \widehat{\Omega}^1_{S/\mathsf{S}_0}[d\log\infty] \quad (6.3.1.9)$$

*(as in Theorem 4.6.3.16 and 5 of Proposition 6.3.1.6), where $\underline{\mathrm{KS}}_{({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i)/S}$ is the sheaf defined as in Definition 6.3.1 by the pullback $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to S$ of $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ in the sense of relative schemes. Then the morphism $\hat{f}$ is **formally étale** if and only if it satisfies the conditions that f is flat, that f_0 is formally smooth, and that the morphism $\mathrm{KS}_{{}^\diamondsuit G/S/\mathsf{S}_0}$ in (6.3.1.9) is surjective.*

Proof. By definition (see [59, IV-4, 17.1.1]), $\hat{f}$ is formally étale if and only if it is both formally smooth and formally unramified.

By [59, 0_{IV}, 19.10.3(ii)], localizations are formally étale. By [59, 0_{IV}, 19.7.1], a local homomorphism $(R_1, \mathfrak{m}_1) \to (R_2, \mathfrak{m}_2)$ between noetherian local rings is formally smooth if and only if R_2 is flat over R_1 and $R_2 \underset{R_1}{\otimes} (R_1/\mathfrak{m}_1)$ is formally smooth over $R_1/\mathfrak{m}_1$. By [59, 0_{IV}, 19.3.5(iii)], pullbacks of formally smooth morphisms are formally smooth. Therefore, since f_0 is the pullback of f to Z (on the target), $\hat{f}$ is formally smooth if and only if it satisfies the conditions that f is flat and that f_0 is formally smooth.

Hence it remains to show that $\hat{f}$ is formally unramified if and only if the induced morphism (6.3.1.9) is surjective. Since the formation of $\underline{\mathrm{KS}}$ commutes with base change and with Mumford's construction (cf. Section 4.6.3), we may pullback the isomorphism in Proposition 6.2.5.18 to an isomorphism $\underline{\mathrm{KS}}_{({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i)/S} \xrightarrow{\sim} f^*\Omega^1_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\mathsf{S}_0}[d\log\infty]$ over S. Then the surjectivity of (6.3.1.9) is equivalent to the surjectivity of the canonical morphism

$$f^*\Omega^1_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\mathsf{S}_0}[d\log\infty] \to \widehat{\Omega}^1_{S/\mathsf{S}_0}[d\log\infty]. \quad (6.3.1.10)$$

Since each stratum of $S = \mathrm{Spec}(R)$ is the scheme-theoretic preimage of a stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, we can identify $\widehat{\Omega}^1_{S/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)}$ as a submodule of the cokernel of (6.3.1.10), because the generators of the differentials with (nontrivial) log poles in $\widehat{\Omega}^1_{S/\mathsf{S}_0}[d\log\infty]$ are all in the image of (6.3.1.10). Hence (6.3.1.9) is surjective if and only if $\widehat{\Omega}^1_{S/\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)} = 0$, which holds if and only if $\hat{f}$ is formally unramified (by [59, 0_{IV}, 20.7.4]), as desired. □

Before moving on, let us review the following criterion for flatness, which we learned from [66, pp. 507–508, in Notes Added in Proof]:

Lemma 6.3.1.11. *Let $f : Z_1 \to Z_2$ be a morphism between equidimensional locally noetherian schemes of the same dimension. Suppose f is quasi-finite, Z_1 is Cohen–Macaulay, and Z_2 is regular. Then f is automatically flat.*

Here we follow [66] and define a scheme to be *equidimensional* if all its open subschemes have the same dimension. Note that this is stronger than the definition in [59, IV-3, §13] because it has a different purpose.

Proof of Lemma 6.3.1.11. Since Z_2 is regular and locally noetherian, it is the disjoint union of its irreducible components. Then we may assume that Z_2 is irreducible. Let Z'_1 be any irreducible component of Z_1. By [59, IV-2, 5.4.1(i)], the assumption on dimensions implies that $f|_{Z'_1} : Z'_1 \to Z_2$ is dominant. By [59, IV-2, 5.6.4 and 5.6.5.3], the quasi-finiteness of $f|_{Z'_1}$ implies that $\dim \mathscr{O}_{Z'_1,z} = \dim \mathscr{O}_{Z_2,f(z)}$ at each point z of Z'_1. Since Z_1 is Cohen–Macaulay, [59, 0_{IV}, 16.5.4] implies that $\dim \mathscr{O}_{Z'_1,z} = \dim \mathscr{O}_{Z_1,z}$ at each point z of Z'_1. Since Z'_1 is arbitrary, we have equivalently $\dim \mathscr{O}_{Z_1,z} = \dim \mathscr{O}_{Z_2,f(z)}$ at each point z of Z_1. Then we can conclude the proof by applying [59, IV-3, 15.4.2 e′)⇒b)]. □

Remark 6.3.1.12. If Z_1 and Z_2 are local and regular, then a similar criterion can be found in [3, Ch. V, Cor. 3.6].

Corollary 6.3.1.13. *Let $f : Z_1 \to Z_2$ be a morphism between equidimensional locally noetherian schemes of the same dimension. Suppose f is unramified, Z_1 is Cohen–Macaulay, and Z_2 is regular. Then f is automatically étale.*

Proof. By [59, IV-4, 17.4.1 a)⇒d′)], f is quasi-finite because it is unramified. Then the corollary follows from Lemma 6.3.1.11. □

Corollary 6.3.1.13 implies that an unramified morphism between regular local rings is automatically étale. This partially justifies the consideration of the following special case of Corollary 6.3.1.8:

Corollary 6.3.1.14. *In the context of Corollary 6.3.1.8, suppose that R is a strict local ring with (separably closed) residue field k, so that the morphism f induces a morphism $\tilde{f} : \mathrm{Spec}(R) \to \mathrm{Spec}(\tilde{R})$ mapping $\mathrm{Spec}(k)$ to $\mathrm{Spec}(\tilde{k})$, where $\tilde{R}$ is a strict local ring of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ with (separably closed) residue field $\tilde{k}$, and suppose that k is of finite type over $\tilde{k}$. Then $\tilde{f}$ is formally étale if and only if R and f satisfy the conditions that R is equidimensional and has the same dimension as $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, and that the induced canonical morphism (6.3.1.9) is surjective. (This last condition forces the induced homomorphism $\tilde{k} \to k$ to be an isomorphism.)*

Proof. We may replace R (resp. $\tilde{R}$) with their completions with respect to its maximal ideal $\mathfrak{m}$ (resp. $\tilde{\mathfrak{m}}$), so that R (resp. $\tilde{R}$) becomes a *regular complete local ring* with residue field k (resp. $\tilde{k}$). By [59, 0_{IV}, 21.7.4], if $R \to \tilde{R}$ is formally unramified, then the induced homomorphism $\tilde{k} \to k$ of separably closed residue fields is an isomorphism. Hence the homomorphism $\tilde{f}$ is formally étale (which is now equivalent to being an isomorphism) if and only if the corresponding local homomorphism $\tilde{R} \to R$ induces a surjective morphism $\tilde{\mathfrak{m}}/\tilde{\mathfrak{m}}^2 \to \mathfrak{m}/\mathfrak{m}^2$ of $\tilde{k}$-vector spaces of the same dimension. This last condition is equivalent to the conditions that R and $\tilde{R}$ have the same dimension, and that $\tilde{R} \to R$ is formally unramified (cf. [59, 0_{IV}, 20.7.5]). By the proof of Corollary 6.3.1.8, $R \to \tilde{R}$ is formally unramified if and only if the induced canonical morphism (6.3.1.9) is surjective. Hence the corollary follows, as desired. □

Now we are ready to define the so-called *good formal models*, as in [42, Ch. IV, §3].

Definition 6.3.1.15. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ be a representative of a cusp label at level $\mathcal{H}$, and let $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ be a nondegenerate smooth rational polyhedral cone. A* ***good formal*** *$(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-**model** is a degenerating family $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}})$ of type $\mathsf{M}_{\mathcal{H}}$ over $\mathrm{Spec}(R)$ (see Definition 5.3.2.1) where we have the following:*

1. *R is a strict local ring that is complete with respect to an ideal $I = \mathrm{rad}(I)$, together with a stratification of $\mathrm{Spec}(R)$ with strata parameterized by $\Gamma_{\Phi_{\mathcal{H}},\sigma}$-orbits of faces of σ.*
2. *There exists a morphism $f : \mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ such that $\mathrm{Spec}(R/I)$ is the scheme-theoretic preimage of the σ-stratum under f, satisfying the following properties:*
   (a) *The morphism f makes R isomorphic to the completion of a strict local ring of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ with respect to the ideal defining the σ-stratum.*
   (b) *The stratification of $\mathrm{Spec}(R)$ is strictly compatible with that of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ in the sense that each stratum of $\mathrm{Spec}(R)$ (with its reduced structure, as in 1 of Proposition 6.3.1.6) is the scheme-theoretic preimage of the corresponding stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.*
   (c) *The degenerating family $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}})$ defines an object of $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R, I)$ (see Definition 5.3.1.17), and (by abuse of language) $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to \mathrm{Spec}(R)$ is the pullback of the Mumford family $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ under the morphism $\hat{f} : \mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ induced by f.*

Remark 6.3.1.16. As in Proposition 6.3.1.6, the morphism $\hat{f} : \mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ in Definition 6.3.1.15 (making $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to \mathrm{Spec}(R)$ the pullback of the Mumford family $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$) is necessarily unique.

Remark 6.3.1.17. By the universal property of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ (see Proposition 6.2.5.11 and 6 of Proposition 6.3.1.6), the morphism $f : \mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ in Definition 6.3.1.15 (with the desired properties) is tautological for the induced morphism $\mathrm{Spec}(R) \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ and the homomorphism $\underline{B}({}^\diamondsuit G) : \underline{\mathbf{S}}_{\Phi_{\mathcal{H}}({}^\diamondsuit G)} \to \underline{\mathrm{Inv}}(\mathrm{Spec}(R))$.

Then Corollary 6.3.1.14 implies the following:

Corollary 6.3.1.18. *Suppose R is a regular strict local ring complete with respect to an ideal $I = \mathrm{rad}(I)$, together with a morphism $\hat{f} : \mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ inducing a morphism $f : \mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ such that $\mathrm{Spec}(R/I)$ is the scheme-theoretic preimage of the σ-stratum under f, and inducing an isomorphism between separable closures of residue fields. Then we can verify the statement that f makes R isomorphic to the completion of a strict local ring of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ with respect to the ideal defining the σ-stratum by verifying the following conditions:*

1. *The scheme $\mathrm{Spec}(R)$ has the same dimension as $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.*
2. *The stratification of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ induces a stratification of $\mathrm{Spec}(R)$ that is strictly compatible with that of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ in the sense that each stratum of $\mathrm{Spec}(R)$ (with its reduced structure, as in 1 of Proposition 6.3.1.6) is the scheme-theoretic preimage of the corresponding stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.*
3. *The extended Kodaira–Spencer morphism (see Theorem 4.6.3.16) induces an isomorphism*
$$\mathrm{KS}_{{}^\diamondsuit G/\mathrm{Spec}(R)/\mathsf{S}_0} : \underline{\mathrm{KS}}_{({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i)/\mathrm{Spec}(R)} \xrightarrow{\sim} \widehat{\Omega}^1_{\mathrm{Spec}(R)/\mathsf{S}_0}[d\log\infty],$$

where the degenerating family $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to \operatorname{Spec}(R)$ *is the pullback of the Mumford family* $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *under* $\hat{f}$ *(by abuse of language).*

Remark 6.3.1.19. The various morphisms from $\operatorname{Spec}(R/I)$ to the support of the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}}\sigma}$, for the various good formal $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$-models, *cover* the whole σ-stratum.

Remark 6.3.1.20. A good formal $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$-model $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}})$ over $\operatorname{Spec}(R)$ is a good formal $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$-model if and only if $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ is equivalent to $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ (see Definition 6.2.6.1).

Remark 6.3.1.21. For two smooth rational polyhedral cones $\sigma,\sigma' \in \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ such that $\sigma \subset \sigma'$, a good formal $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$-model is not necessarily a good formal $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma')$-model (see Remark 6.2.5.31).

6.3.2 Good Algebraic Models

Although good formal models have many nice properties, we cannot patch them together naively, because their supports might not even overlap with each other. We would like to construct so-called good algebraic models, namely, families over schemes (instead of formal schemes), which are approximate enough to good formal models that a gluing process for the purpose of compactification can still be performed.

Proposition 6.3.2.1. *Let* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ *be a representative of a cusp label at level* $\mathcal{H}$, *and let* $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ *be a nondegenerate smooth rational polyhedral cone. Let* R *be the strict local ring of a geometric point* $\bar{x}$ *of the* σ*-stratum of* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *for some* $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$, *let* $R^\wedge$ *be the completion of* R *with respect to the ideal* I *defining the* σ*-stratum, and let* $I^\wedge := I \cdot R^\wedge \subset R^\wedge$. *Suppose* $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to \operatorname{Spec}(R^\wedge)$ *defines a good formal* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$*-model over* $\operatorname{Spec}(R^\wedge)$. *Then we can find (noncanonically) a degenerating family* $(G,\lambda,i,\alpha_{\mathcal{H}})$ *over* $\operatorname{Spec}(R)$ *as in Definition 5.3.2.1, which approximates* $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}})$ *in the following sense:*

1. *Over* $\operatorname{Spec}(R/I)$, $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i) \underset{R}{\otimes} (R/I) \cong (G,\lambda,i) \underset{R}{\otimes} (R/I)$. *(We do not compare* ${}^\diamondsuit\alpha_{\mathcal{H}}$ *and* $\alpha_{\mathcal{H}}$ *here, because they are not defined over* $\operatorname{Spec}(R/I)$.*)*
2. *Under the canonical homomorphism* $R \hookrightarrow R^\wedge$, *the pullbacks of the objects* $\underline{\Phi}_{\mathcal{H}}(G)$, $\underline{\mathbf{S}}_{\underline{\Phi}(G)}$, *and* $\underline{B}(G)$ *defined as in Construction 6.3.1.1 are isomorphic to the objects* $\Phi_{\mathcal{H}}({}^\diamondsuit G)$, $\underline{\mathbf{S}}_{\underline{\Phi}({}^\diamondsuit G)}$, *and* $\underline{B}({}^\diamondsuit G)$ *defined as in Proposition 6.3.1.6, respectively.*
3. *The pullback* $(G,\lambda,i,\alpha_{\mathcal{H}}) \underset{R}{\otimes} R^\wedge \to \operatorname{Spec}(R^\wedge)$ *of* $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \operatorname{Spec}(R)$ *under the canonical homomorphism* $R \hookrightarrow R^\wedge$ *defines a good formal* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$*-model, and can be realized as the pullback of the Mumford family* $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *via a canonically defined morphism* $\operatorname{Spf}(R^\wedge,I^\wedge) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. *Comparing this isomorphism with the original morphism* $\operatorname{Spf}(R^\wedge,I^\wedge) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *making the good formal* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$*-model* $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to \operatorname{Spec}(R^\wedge)$ *a pullback of the Mumford family, we see that they are approximate in the sense that the induced morphisms from* $\operatorname{Spec}(R/I)$ *to the* σ*-stratum of* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *(between the supports of the formal schemes)* ***coincide***.
4. *The extended Kodaira–Spencer morphism for the above pullback* $(G,\lambda,i,\alpha_{\mathcal{H}}) \underset{R}{\otimes} R^\wedge \to \operatorname{Spec}(R^\wedge)$ *(see Theorem 4.6.3.16) induces (cf. the proof of Theorem 4.6.3.43) an isomorphism*
$$\mathrm{KS}_{G/\operatorname{Spec}(R)/\mathsf{S}_0} : \underline{\mathrm{KS}}_{(G,\lambda,i)/\operatorname{Spec}(R)} \xrightarrow{\sim} \widetilde{\Omega}^1_{\operatorname{Spec}(R)/\mathsf{S}_0}[d\log\infty],$$
where $\widetilde{\Omega}^1_{\operatorname{Spec}(R)/\mathsf{S}_0}[d\log\infty]$ *is defined by* $\widetilde{\Omega}^1_{\operatorname{Spec}(R)/\mathsf{S}_0}$, *the coherent sheaf associated with the module of universal finite differentials* $\widetilde{\Omega}^1_{R/\mathcal{O}_{F_0,(\Box)}}$ *(see [77, §§11–12]), and by the normal crossings divisor of* $\operatorname{Spec}(R)$ *induced by the one of* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *(as in 5 of Proposition 6.3.1.6, with* $\widehat{\Omega}^1_{S/\mathsf{S}_0}$ *there replaced with* $\widetilde{\Omega}^1_{\operatorname{Spec}(R)/\mathsf{S}_0}$ *here).*

The proof of this requires the approximation techniques of Artin:

Proposition 6.3.2.2 (cf. [5, Thm. 1.10])**.** *Let* R_0 *be a field or an excellent discrete valuation ring, and let* R *be the Henselization (see [59, IV-4, 18.6]) of an* R_0*-algebra of finite type at a prime ideal. Let* I *be a proper ideal of* R, *and let* $R^\wedge$ *be the* I*-adic completion of* R. *Suppose* R_1 *is a subalgebra of* $R^\wedge$ *of finite type over* R. *Then the natural inclusion* $R \hookrightarrow R_1$ *has (homomorphic) sections* $R_1 \to R$ *such that the compositions* $R_1 \to R \hookrightarrow R^\wedge$ *can be arbitrarily close to the natural inclusion* $R_1 \hookrightarrow R^\wedge$ *in the* I*-adic topology.*

Proof. By writing R_1 as a quotient of a polynomial ring over R by finitely many relations, the natural inclusion $R_1 \hookrightarrow R^\wedge$ can be interpreted as giving a solution in $R^\wedge$ to a system of polynomial equations with coefficients in R. Then the approximation result of [5, Thm. 1.10] implies that we can find approximate solutions in R arbitrarily close to the given solution in $R^\wedge$ in the I-adic topology. In other words, we have sections $R_1 \to R$ to the natural inclusion $R \hookrightarrow R_1$ with the desired properties. □

We will also have to repeat [42, Ch. IV, Lem. 4.2]:

Lemma 6.3.2.3. *Suppose* R *is a noetherian ring and* I *is an ideal of* R, *such that* $\mathrm{Gr}_I(R)$ *is an integral domain, and such that the* I*-adic topology on* R *is separated. Suppose moreover that* f *and* g *are nonzero elements of* R *such that* f/g *lies in* R. *If* $\{f_i\}_{i\geq 0}$ *(resp.* $\{g_i\}_{i\geq 0}$*) is a sequence of nonzero elements of* R *(indexed by positive integers* i*) converging* I*-adically to* f *(resp.* g*) as* $i \to \infty$, *and if all the quotients* f_i/g_i *lie in* R, *then the quotients* f_i/g_i *converge* I*-adically to* f/g *as* $i \to \infty$.

Proof. The separateness assumption implies that there is an injection from R to $\mathrm{Gr}_I(R)$. By the I-adic order $\mathrm{ord}_I(x)$ of an element x in R, we mean the degree of the first nonzero entry of its image in $\mathrm{Gr}_I(R)$. Then a sequence of elements $x_i \in R$ satisfies $x_i \to 0$ (as $i \to \infty$) in the I-adic topology if and only if $\mathrm{ord}_I(x_i) \to \infty$. Since $\mathrm{Gr}_I(R)$ is an integral domain, the I-adic order of a product is the sum of the orders of its terms.

By assumption, $g \cdot g_i \cdot (f/g - f_i/g_i) = f \cdot g_i - f_i \cdot g$ converges to 0. For sufficiently large i, the I-adic order of $g \cdot g_i$ is constant, which is twice the I-adic order of g. Then $\mathrm{ord}_I(f/g - f_i/g_i) = \mathrm{ord}_I(f \cdot g_i - f_i \cdot g) - \mathrm{ord}_I(g \cdot g_i) \to \infty$ as $i \to \infty$, as desired. □

Proof of Proposition 6.3.2.1. Let R and $R^\wedge$ be as in the statement of Proposition 6.3.2.1, both of which are excellent and normal by assumption. Note that $R^\wedge$ is the filtering direct union of its normal subalgebras R_1 of finite type over R. By Proposition 6.3.2.2, for each such algebra R_1, the natural inclusion $R \hookrightarrow R_1$ has sections $R_1 \to R$ such that the compositions $R_1 \to R \hookrightarrow R^\wedge$ can be made arbitrarily close to the natural inclusion $R_1 \hookrightarrow R^\wedge$ in the I-adic topology.

Since ${}^\heartsuit G \to \mathrm{Spec}(R^\wedge)$ is of finite presentation, by [59, IV-3, 8.8.2], we may take some R_1 as above such that $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to \mathrm{Spec}(R^\wedge)$ is already defined over $\mathrm{Spec}(R_1)$. By enlarging R_1 if necessary, we may assume that the equivalence between $(\underline{\Phi}_{\mathcal{H}}({}^\diamondsuit G), \underline{B}({}^\diamondsuit G))$ and the pullback of $(\underline{\Phi}_{\mathcal{H}}, \underline{B})$ (described in Proposition 6.3.1.6) is already defined over R_1. (Then the various character groups of torus parts are trivialized over R_1.) By pullback under a section $R_1 \to R$ of $R \hookrightarrow R_1$ such that $R_1 \to R \hookrightarrow R^\wedge$ is close to the natural inclusion $R_1 \hookrightarrow R^\wedge$, we obtain a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathrm{Spec}(R)$, together with an equivalence between $(\underline{\Phi}_{\mathcal{H}}(G), \underline{B}(G))$ and the pullback of $(\underline{\Phi}_{\mathcal{H}}, \underline{B})$. By further enlarging R_1 if necessary, we may assume that the extended Kodaira–Spencer morphism for the pullback $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R}{\otimes} R^\wedge \to \mathrm{Spec}(R^\wedge)$ and the section $R_1 \to R$ induce an isomorphism $\mathrm{KS}_{G/\mathrm{Spec}(R)/\mathsf{S}_0}$ as in 4 of Proposition 6.3.2.1.

Although the degenerating families $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R}{\otimes} R^\wedge \to \mathrm{Spec}(R^\wedge)$ and $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to \mathrm{Spec}(R^\wedge)$ are not isomorphic (in general), they can be made close in the I-adic topology. This means, for each prescribed integer $a > 0$, there exists $R_1 \to R$ as above such that $(G, \lambda, i) \underset{R}{\otimes} (R/I^a) \cong ({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i) \underset{R}{\otimes} (R/I^a)$ over $\mathrm{Spec}(R/I^a)$. We claim that the degeneration datum

$$(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}^{\sim} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}^{\sim}, \varphi_{0,\mathcal{H}}^{\sim}), (A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, (c_{\mathcal{H}}, c_{\mathcal{H}}^\vee, \tau_{\mathcal{H}}))$$

in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R^\wedge, I^\wedge)$ associated with $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R}{\otimes} R^\wedge \to \mathrm{Spec}(R^\wedge)$ can be made I-adically close to the degeneration datum

$$(\mathsf{Z}_{\mathcal{H}}, {}^\diamondsuit\Phi_{\mathcal{H}}^{\sim}, ({}^\diamondsuit A, {}^\diamondsuit\lambda_A, {}^\diamondsuit i_A, {}^\diamondsuit\varphi_{-1,\mathcal{H}}), \delta_{\mathcal{H}}, ({}^\diamondsuit c_{\mathcal{H}}, {}^\diamondsuit c_{\mathcal{H}}^\vee, {}^\diamondsuit\tau_{\mathcal{H}}))$$

in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R^\wedge, I^\wedge)$ associated with $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to \mathrm{Spec}(R^\wedge)$. Since $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ is prescribed, since the choices of i_A, $\varphi_{-1,\mathcal{H}}$, and $(\varphi_{-2,\mathcal{H}}^{\sim}, \varphi_{0,\mathcal{H}}^{\sim})$ (inducing the $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ in $\Phi_{\mathcal{H}}$ as in Definition 5.4.2.8) are discrete in nature, and since the determination of $(c_{\mathcal{H}}, c_{\mathcal{H}}^\vee, \tau_{\mathcal{H}})$ from $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}^{\sim}, \delta_{\mathcal{H}})$ and $\alpha_{\mathcal{H}}$ is discrete in nature (by Proposition 5.2.7.9 and Theorem 5.2.7.14), this is essentially a statement on $(A, \lambda_A, c, c^\vee, \tau)$.

The statement that $(A, \lambda_A, c, c^\vee)$ can be made I-adically close to $({}^\diamondsuit A, {}^\diamondsuit\lambda_A, {}^\diamondsuit c, {}^\diamondsuit c^\vee)$ means, for each prescribed integer $a > 0$, there exists $R_1 \to R$ as above such that $(A, \lambda_A, c, c^\vee) \underset{R}{\otimes} (R/I^a) \cong ({}^\diamondsuit A, {}^\diamondsuit\lambda_A, {}^\diamondsuit c, {}^\diamondsuit c^\vee) \underset{R}{\otimes} (R/I^a)$ over $\mathrm{Spec}(R/I^a)$. This is possible because $({}^\diamondsuit A, {}^\diamondsuit\lambda_A, {}^\diamondsuit c, {}^\diamondsuit c^\vee)$ is determined by the I-adic completion $({}^\diamondsuit G_{\mathrm{for}}, {}^\diamondsuit\lambda_{\mathrm{for}})$.

The corresponding statement for τ is more tricky, because τ as a trivialization of biextensions is (in general) not defined over $\mathrm{Spec}(R/I^a)$ for any a. However, for each $y \in Y$ and $\chi \in X$, we can interpret $\tau(y, \chi)$ as an $R^\wedge$-module isomorphism from $(c^\vee(y), c(\chi))^* \mathcal{P}_A^{\otimes -1}$ to an $R^\wedge$-invertible submodule $I_{y,\chi}$ of $K = \mathrm{Frac}(R^\wedge)$ (as in the proof of Lemma 4.2.1.12 in Section 4.2.4). Then the statement that τ can be made I-adically close to ${}^\diamondsuit\tau$ means, for each prescribed integer $a > 0$, the two $R^\wedge$-module isomorphisms $\tau(y, \chi) : (c^\vee(y), c(\chi))^* \mathcal{P}_A^{\otimes -1} \overset{\sim}{\to} I_{y,\chi}$ and ${}^\diamondsuit\tau(y, \chi) : ({}^\diamondsuit c^\vee(y), {}^\diamondsuit c(\chi))^* \mathcal{P}_{{}^\diamondsuit A}^{\otimes -1} \overset{\sim}{\to} {}^\diamondsuit I_{y,\chi}$ can be made identical modulo I^a (under the identification of $(c^\vee(y), c(\chi))^* \mathcal{P}_A^{\otimes -1}$ and $({}^\diamondsuit c^\vee(y), {}^\diamondsuit c(\chi))^* \mathcal{P}_{{}^\diamondsuit A}^{\otimes -1}$ over $\mathrm{Spec}(R/I^a)$ defined by the above isomorphism $(A, \lambda_A, c, c^\vee) \underset{R}{\otimes} (R/I^a) \cong ({}^\diamondsuit A, {}^\diamondsuit\lambda_A, {}^\diamondsuit c, {}^\diamondsuit c^\vee) \underset{R}{\otimes} (R/I^a)$). Therefore, the difference between their pullbacks to $\mathrm{Spec}(R/I^a)$ is measured a priori by an invertible element in R/I^a, and we need to show that this invertible element can be made the identity. To justify our claim above that the degeneration data can be made close, it remains to show that we can make τ close to ${}^\diamondsuit\tau$ in this sense.

Let us take ${}^\diamondsuit\mathcal{L}_\eta := (\mathrm{Id}_{{}^\diamondsuit G_\eta}, {}^\diamondsuit\lambda_\eta)^* \mathcal{P}_{{}^\diamondsuit G_\eta}$. Note that ${}^\diamondsuit\mathcal{L}_\eta$ is symmetric. Let ${}^\diamondsuit\mathcal{L}$ be the unique cubical extension of ${}^\diamondsuit\mathcal{L}_\eta$ (given by Theorem 3.3.2.3), and let ${}^\diamondsuit\mathcal{M} := (\mathrm{Id}_{{}^\diamondsuit A}, {}^\diamondsuit\lambda_{{}^\diamondsuit A})^* \mathcal{P}_{{}^\diamondsuit A}$. Then ${}^\diamondsuit\mathcal{L}^\natural = {}^\diamondsuit\pi^* {}^\diamondsuit\mathcal{M}$, where ${}^\diamondsuit\pi : {}^\diamondsuit G^\natural \to {}^\diamondsuit A$ is the structural morphism. We know that ${}^\diamondsuit\mathcal{L}$ induces $2\,{}^\diamondsuit\lambda$, and so ${}^\diamondsuit\tau$ is part of the degeneration datum associated with $({}^\diamondsuit G, {}^\diamondsuit\mathcal{L})$ by $\mathrm{F}_{\mathrm{ample}}(R^\wedge, I^\wedge)$ (by Theorem 4.2.1.14 and the statement in Remark 4.2.1.16 that ${}^\diamondsuit\lambda$ and $2\,{}^\diamondsuit\lambda$ have the same associated ${}^\diamondsuit\tau$). More precisely, there is a ${}^\diamondsuit\psi$ such that, for each section ${}^\diamondsuit s$ in $\Gamma({}^\diamondsuit G, {}^\diamondsuit\mathcal{L}) \underset{R^\wedge}{\otimes} \mathrm{Frac}(R^\wedge)$, we have

$$ {}^\diamondsuit\psi(y)\, {}^\diamondsuit\tau(y, \chi)\, T^*_{{}^\diamondsuit c^\vee(y)} \circ {}^\diamondsuit\sigma_\chi({}^\diamondsuit s) = {}^\diamondsuit\sigma_{\chi + 2\phi(y)}({}^\diamondsuit s) \tag{6.3.2.4}$$

for all $y \in Y$ and $\chi \in X$, where ${}^\diamondsuit\sigma : \Gamma({}^\diamondsuit G, {}^\diamondsuit\mathcal{L}) \to \Gamma({}^\diamondsuit A, {}^\diamondsuit\mathcal{M}_\chi)$ is defined as in Section 4.3.1. As in the case for τ, for each $y \in Y$ and $\chi \in X$, we can interpret $\psi(y)$ as an $R^\wedge$-module isomorphism from $c^\vee(y)^* \mathcal{M}^{\otimes -1}$ to an $R^\wedge$-invertible submodule I_y of $K = \mathrm{Frac}(R^\wedge)$ (see the proof of Lemma 4.2.1.12 in Section 4.2.4). Then the statement that ψ can be made I-adically close to ${}^\diamondsuit\psi$ means, for each prescribed integer $a > 0$, the two $R^\wedge$-module isomorphisms $\psi(y) : c^\vee(y)^* \mathcal{M}^{\otimes -1} \overset{\sim}{\to} I_y$ and ${}^\diamondsuit\psi(y) : {}^\diamondsuit c^\vee(y)^* \mathcal{M}^{\otimes -1} \overset{\sim}{\to} {}^\diamondsuit I_y$ can be made identical modulo I^a (under the identification of the pullbacks of $c^\vee(y)^* \mathcal{M}^{\otimes -1}$ and ${}^\diamondsuit c^\vee(y)^*\, {}^\diamondsuit\mathcal{M}^{\otimes -1}$ to $\mathrm{Spec}(R/I^a)$). (The situation for ψ is easier than for τ because all the I_y are actually invertible submodules of $R^\wedge$ by positivity of ψ; see Definition 4.2.1.11.)

Since $({}^\diamondsuit G, {}^\diamondsuit\lambda)$ is defined over R_1, we may take all the above objects (including a basis of the sections ${}^\diamondsuit s$) to be defined over R_1. If we take $\mathcal{L} := (R_1 \to R)^*({}^\diamondsuit\mathcal{L})$, $\mathcal{M} := (R_1 \to R)^*({}^\diamondsuit\mathcal{M})$, and $s := (R_1 \to R)^*({}^\diamondsuit s)$, then we see that the same relations as above are true for the objects defined by $(G, \mathcal{L})$, and for each prescribed integer $a > 0$, we can take a section $R_1 \to R$ such that, for each $\chi \in X$, the pullbacks to $\mathrm{Spec}(R/I^a)$ of the morphism σ_χ for $(G, \mathcal{L})$ and the morphism ${}^\diamondsuit\sigma_\chi$ for $({}^\diamondsuit G, {}^\diamondsuit\mathcal{L})$ coincide with each other. Take basis elements of sections ${}^\diamondsuit s$ of $\Gamma({}^\diamondsuit G, {}^\diamondsuit\mathcal{L}) \underset{R^\wedge}{\otimes} \mathrm{Frac}(R^\wedge)$ such that each of the elements lies in some weight-$\bar{\chi}$ subspace for some $\bar{\chi} \in X/\phi(Y)$ as in Section 4.3.2. Since ${}^\diamondsuit\psi$ is determined by comparisons between (translations of) images of these basis elements (by taking, for example, $\chi = 0$ in (6.3.2.4)), we see that ψ can be made close to ${}^\diamondsuit\psi$.

Now that we know ψ can be made close to ${}^\diamondsuit\psi$, the relation

$${}^\diamondsuit\tau(y_1, y_2) = {}^\diamondsuit\psi(y_1 + y_2)\, {}^\diamondsuit\psi(y_1)^{-1}\, {}^\diamondsuit\psi(y_2)^{-1}$$

and Lemma 6.3.2.3 show that τ can be made close to ${}^\diamondsuit\tau$ over the subgroup $Y \times \phi(Y)$ of $Y \times X$. By Lemma 4.3.1.18, and by using the same technique as in the proof of Lemma 4.3.4.1, this shows that τ can be made close to ${}^\diamondsuit\tau$ over the whole group $Y \times X$. This concludes the proof of the claim that the degeneration datum associated with $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R}{\otimes} R^\wedge \to \mathrm{Spec}(R^\wedge)$ can be made close to the degeneration datum associated with $({}^\diamondsuit G, {}^\diamondsuit\lambda, {}^\diamondsuit i, {}^\diamondsuit\alpha_{\mathcal{H}}) \to \mathrm{Spec}(R^\wedge)$.

The proposition now follows from Remarks 6.3.1.16 and 6.3.1.17, and Corollaries 6.3.1.8 and 6.3.1.18. □

Definition 6.3.2.5. *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ be a representative of a cusp label at level $\mathcal{H}$, and let $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ be a nondegenerate smooth rational polyhedral cone. A* ***good algebraic***

$(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$***-model*** *consists of the following data:*

1. *An affine scheme* $\mathrm{Spec}(R_{\mathrm{alg}})$*, together with a stratification of* $\mathrm{Spec}(R_{\mathrm{alg}})$ *with strata parameterized by* $\Gamma_{\Phi_{\mathcal{H}},\sigma}$*-orbits of faces of* σ*.*
2. *A strata-preserving morphism* $\mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *making* $\mathrm{Spec}(R_{\mathrm{alg}})$ *an étale neighborhood of some geometric point* $\bar{x}$ *of* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *at the* σ*-stratum.*

   Let $R^{\wedge}$ *be the completion of the strict local ring of* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *at* $\bar{x}$ *with respect to the ideal defining the* σ*-stratum. Then there is a "natural inclusion"* $\imath^{\mathrm{nat}} : R_{\mathrm{alg}} \hookrightarrow R^{\wedge}$*.*
3. *A degenerating family* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *over* $\mathrm{Spec}(R_{\mathrm{alg}})$ *as in Definition 5.3.2.1, together with an embedding* $\imath^{\mathrm{alg}} : R_{\mathrm{alg}} \hookrightarrow R^{\wedge}$*, such that we have the following:*
   (a) *There are isomorphisms between the objects* $\underline{\Phi}_{\mathcal{H}}(G)$*,* $\underline{\mathbf{S}}_{\underline{\Phi}(G)}$*, and* $\underline{B}(G)$ *(see Construction 6.3.1.1), and the pullbacks of the tautological objects* $\underline{\Phi}_{\mathcal{H}}$*,* $\underline{\mathbf{S}}$*, and* $\underline{B}$ *over* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *(see in Proposition 6.2.5.8) under* $\mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*.*
   (b) *The embedding* $\imath^{\mathrm{alg}} : R_{\mathrm{alg}} \hookrightarrow R^{\wedge}$ *is* ***close to the natural inclusion*** $\imath^{\mathrm{nat}}$ *in the sense that the following two morphisms* $\mathrm{Spf}(R^{\wedge}, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ ***coincide*** *over the* σ*-stratum:*
      i. *The pullback* $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R_{\mathrm{alg}},\imath^{\mathrm{nat}}}{\otimes} R^{\wedge} \to \mathrm{Spec}(R^{\wedge})$ *defines a good formal* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$*-model by the isomorphisms in 3a above, and hence defines a canonical morphism* $\mathrm{Spf}(R^{\wedge}, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*.*
      ii. *The embedding* $\imath^{\mathrm{alg}} : R_{\mathrm{alg}} \hookrightarrow R^{\wedge}$ *defines a composition*
      $$\mathrm{Spec}(R^{\wedge}) \overset{\mathrm{Spec}(\imath^{\mathrm{alg}})}{\to} \mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma},$$
      inducing a morphism $\mathrm{Spf}(R^{\wedge}, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*.*
   (c) *The extended Kodaira–Spencer morphism (see Definition 4.6.3.44) induces an isomorphism*
   $$\mathrm{KS}_{G/\mathrm{Spec}(R_{\mathrm{alg}})/\mathsf{S}_0} : \underline{\mathrm{KS}}_{(G,\lambda,i)/\mathrm{Spec}(R_{\mathrm{alg}})} \overset{\sim}{\to} \Omega^1_{\mathrm{Spec}(R_{\mathrm{alg}})/\mathsf{S}_0}[d\log\infty].$$

Proposition 6.3.2.6. *Good algebraic* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$*-models* $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathrm{Spec}(R_{\mathrm{alg}})$ *exist, and the morphisms from the various* $\mathrm{Spec}(R_{\mathrm{alg}})$*'s to* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ ***cover*** *the* σ*-stratum of* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*.*

Proof. In the statement of Proposition 6.3.2.1, the strict local ring R is the inductive limit of the coordinate rings of all affine étale neighborhoods of $\bar{x}$. Then we can conclude the proof by taking R_{alg} to be such an affine étale neighborhood over which all the objects, sheaves, and isomorphisms in the statement and proof of Proposition 6.3.2.1 are defined. □

Remark 6.3.2.7. What is implicit behind Proposition 6.3.2.6 is that, although we need to approximate the (possibly infinitely many) good formal models at all geometric points of the σ-stratum, we only need *finitely many* good algebraic models to cover it, by quasi-compactness of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$.

Remark 6.3.2.8. A good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathrm{Spec}(R)$ is also a good algebraic $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$-model if and only if $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ is equivalent to $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ (see Remark 6.3.1.20).

Remark 6.3.2.9. For two smooth rational polyhedral cones $\sigma, \sigma' \in \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ such that $\sigma \subset \sigma'$, a good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model is not necessarily a good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma')$-model (see Remarks 6.2.5.31 and 6.3.1.21).

Proposition 6.3.2.10 (*openness of versality*)**.** *Suppose* $\bar{x}$ *is any geometric point in the* $(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\mathcal{H},\sigma})$*-stratum of a good algebraic* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$*-model* $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*, where* τ *is a face of* σ*. By pulling back to the completion* $R^{\wedge}_{\bar{x}}$ *of the strict local ring of* R_{alg} *at* $\bar{x}$ *with respect to the ideal defining the* $(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\mathcal{H},\sigma})$*-stratum, we obtain a good formal* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \tau')$*-model, where*

1. $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ *is the pullback of* $\underline{\Phi}_{\mathcal{H}}$ *to* $\bar{x}$*, which comes equipped with a surjection* $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y')$ *(as in Definition 5.4.2.12) by definition of* $\underline{\Phi}_{\mathcal{H}}$*;*
2. $\delta'_{\mathcal{H}}$ *is any splitting that makes* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *a representative of a cusp label. Then there is a surjection* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *(the actual choice of* $\delta'_{\mathcal{H}}$ *does not matter);*
3. $\tau' \subset \mathbf{P}^+_{\Phi'_{\mathcal{H}}}$ *is any nondegenerate smooth rational polyhedral cone whose image under the embedding* $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ *induced by the surjection* (s_X, s_Y) *is the translation of* τ *by an element of* $\Gamma_{\Phi_{\mathcal{H}}}$*.*

Proof. With the choice of $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$, the pullback of G to $R_{\bar{x}}$ determines (by Lemma 5.4.2.10) an object in $\mathrm{DD}^{\text{fil.-spl.}}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R^{\wedge}_{\bar{x}}, I)$ up to isomorphisms inducing automorphisms of $\Phi'_{\mathcal{H}}$, where I is the ideal defining the $(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\mathcal{H},\sigma})$-stratum and satisfying $\mathrm{rad}(I) = I$. As we already know the pullback of the homomorphism $\underline{B}(G)$ (in Construction 6.3.1.1, with the help of Proposition 4.5.6.1), we obtain (by Proposition 6.2.5.11; cf. Remark 6.3.1.17) a canonical morphism from $\mathrm{Spf}(R^{\wedge}_{\bar{x}}, I)$ to $\mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\tau'}/\Gamma_{\Phi'_{\mathcal{H}},\tau'}$, where τ and the image of τ' under $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ have the same $\Gamma_{\Phi_{\mathcal{H}}}$-orbit. Since $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R_{\mathrm{alg}}}{\otimes} R^{\wedge}_{\bar{x}} \to \mathrm{Spec}(R^{\wedge}_{\bar{x}})$ is the pullback of the Mumford family over $\mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\tau'}/\Gamma_{\Phi'_{\mathcal{H}},\tau'}$, for each $\ell \in \mathbf{S}_{\Phi'_{\mathcal{H}}}$ the I_ℓ given by $\underline{B}(G)$, which is the same as the I_ℓ defined as in (6.2.5.9) using the degeneration data of $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R_{\mathrm{alg}}}{\otimes} R^{\wedge}_{\bar{x}}$, agrees with the pullback of $\Psi_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}}}(\ell)$ (as an element of $\mathrm{Inv}(R^{\wedge}_{\bar{x}})$). This verifies the strict compatibility of stratifications (and a fortiori the condition for the pullback of the $(\tau \mod \Gamma_{\Phi_{\mathcal{H}},\mathcal{H},\sigma})$-stratum) in Definition 6.3.1.15.

Since $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is a good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model, we have a canonical isomorphism
$$\mathrm{KS}_{G_{R^{\wedge}_{\bar{x}}}/\mathrm{Spec}(R^{\wedge}_{\bar{x}})/\mathsf{S}_0} : \underline{\mathrm{KS}}_{((G,\lambda,i) \underset{R_{\mathrm{alg}}}{\otimes} R^{\wedge}_{\bar{x}})/\mathrm{Spec}(R^{\wedge}_{\bar{x}})} \overset{\sim}{\to} \widehat{\Omega}^1_{\mathrm{Spec}(R^{\wedge}_{\bar{x}})/\mathsf{S}_0}[d\log\infty], \quad (6.3.2.11)$$
where $\widehat{\Omega}^1_{\mathrm{Spec}(R^{\wedge}_{\bar{x}})/\mathsf{S}_0}[d\log\infty]$ is defined (as in 5 of Proposition 6.3.1.6) by the normal crossings divisor of $\mathrm{Spec}(R^{\wedge}_{\bar{x}})$ induced by the one of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. Since (6.3.2.11) is an isomorphism, the $R^{\wedge}_{\bar{x}}$-module $\widehat{\Omega}^1_{\mathrm{Spec}(R^{\wedge}_{\bar{x}})/\mathsf{S}_0}[d\log\infty]$ is spanned by $\widehat{\Omega}^1_{\mathrm{Spec}(R^{\wedge}_{\bar{x}})/\mathsf{S}_0}$ and $d\log(I_\ell)$ for all $\ell \in \mathbf{S}_{\Phi'_{\mathcal{H}}}$.

On the other hand, the pullback of the extended Kodaira–Spencer morphism over $\mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\tau'}/\Gamma_{\Phi'_{\mathcal{H}},\tau'}$ is necessarily the extended Kodaira–Spencer morphism for $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R_{\mathrm{alg}}}{\otimes} R^{\wedge}_{\bar{x}} \to \mathrm{Spec}(R^{\wedge}_{\bar{x}})$. Thus, we obtain a second morphism
$$\underline{\mathrm{KS}}_{((G,\lambda,i) \underset{R_{\mathrm{alg}}}{\otimes} R^{\wedge}_{\bar{x}})/\mathrm{Spec}(R^{\wedge}_{\bar{x}})} \to \widehat{\Omega}^1_{\mathrm{Spec}(R^{\wedge}_{\bar{x}})/\mathsf{S}_0}[d\log\infty'], \quad (6.3.2.12)$$

where $\widehat{\Omega}^1_{\mathrm{Spec}(R^\wedge_{\bar{x}})/\mathsf{S}_0}[d\log\infty']$ is defined (again, as in 5 of Proposition 6.3.1.6) by the normal crossings divisor of $\mathrm{Spec}(R^\wedge_{\bar{x}})$ induced by the one of $\Xi_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}}}(\tau')/\Gamma_{\Phi'_{\mathcal{H}},\tau'}$. That is, the $R^\wedge_{\bar{x}}$-module $\widehat{\Omega}^1_{\mathrm{Spec}(\mathrm{Spec}(R^\wedge_{\bar{x}}))/\mathsf{S}_0}[d\log\infty']$ is spanned by $\widehat{\Omega}^1_{\mathrm{Spec}(\mathrm{Spec}(R^\wedge_{\bar{x}}))/\mathsf{S}_0}$ and the pullbacks of $d\log(\Psi_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}}}(\ell))$ for all $\ell\in\mathbf{S}_{\Phi'_{\mathcal{H}}}$. Since I_ℓ is canonically isomorphic to the pullback of $\Psi_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}}}(\ell)$ for each $\ell\in\mathbf{S}_{\Phi'_{\mathcal{H}}}$, this shows that (6.3.2.12) is an isomorphism as well.

Thus, Corollary 6.3.1.18 implies that the above canonical morphism identifies $R^\wedge_{\bar{x}}$ with the completion of a strict local ring of $\Xi_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}}}(\tau')/\Gamma_{\Phi'_{\mathcal{H}},\tau'}$ under this morphism, as desired. □

Inspired by Proposition 6.3.2.10,

Definition 6.3.2.13. *Let $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ be a representative of a cusp label at level $\mathcal{H}$ and let $\sigma\subset\mathbf{P}^+_{\Phi_{\mathcal{H}}}$ be a nondegenerate smooth rational polyhedral cone. We say that a triple $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ is a **face** of $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ if*

1. *$(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ is the representative of some cusp label at level $\mathcal{H}$, such that there exists a surjection $(s_X: X\twoheadrightarrow X', s_Y: Y\twoheadrightarrow Y'):(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})\twoheadrightarrow(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ as in Definition 5.4.2.12;*

2. *$\sigma'\subset\mathbf{P}^+_{\Phi'_{\mathcal{H}}}$ is a nondegenerate smooth rational polyhedral cone such that, for one (and hence every) surjection $(s_X: X\twoheadrightarrow X', s_Y: Y\twoheadrightarrow Y')$ as above, the image of σ' under the induced embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}}\hookrightarrow\mathbf{P}_{\Phi_{\mathcal{H}}}$ is contained in the $\Gamma_{\Phi_{\mathcal{H}}}$-orbit of a face of σ.*

Note that this definition is not sensitive to the choices of representatives in the classes $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ and $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$. This justifies the following:

Definition 6.3.2.14. *We say that the equivalence class $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$ of $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ is a **face** of the equivalence class $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ of $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ if some triple equivalent to $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ is a face of some triple equivalent to $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$.*

Remark 6.3.2.15. Suppose $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ is a *face* of $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$, so that σ' is identified with some face τ of σ under some surjection $(s_X: X\twoheadrightarrow X', s_Y: Y\twoheadrightarrow Y'):(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})\twoheadrightarrow(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$. Then there always exists some good algebraic $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$-model that has a nonempty (τ mod $\Gamma_{\Phi_{\mathcal{H}},\sigma}$)-stratum on the base scheme.

For later reference, we shall also make the following definition:

Definition 6.3.2.16. *Let $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ be a representative of a cusp label at level $\mathcal{H}$, and let $\sigma\subset\mathbf{P}^+_{\Phi_{\mathcal{H}}}$ be a nondegenerate smooth rational polyhedral cone. Suppose $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ is a face of $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ such that the image of σ' under the embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}}\hookrightarrow\mathbf{P}_{\Phi_{\mathcal{H}}}$ induced by some surjection $(s_X: X\twoheadrightarrow X', s_Y: Y\twoheadrightarrow Y'):(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})\twoheadrightarrow(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ is a $\Gamma_{\Phi_{\mathcal{H}}}$-translation of a face τ of σ (which can be σ itself). Then we shall call the (τ mod $\Gamma_{\Phi_{\mathcal{H}},\sigma}$)-stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ the $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\tau')]$-**stratum**. (In this case, $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\tau')]$ is a **face** of $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$; see Definition 6.3.2.14.) We shall also call the induced (τ mod $\Gamma_{\Phi_{\mathcal{H}},\sigma}$)-strata of good formal $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$-models and good algebraic good $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$-models (see Definitions 6.3.1.15 and 6.3.2.5) their $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\tau')]$-strata.*

6.3.3 Étale Presentation and Gluing

To construct the arithmetic toroidal compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ as an algebraic stack, it suffices to give an étale presentation $\mathsf{U}_{\mathcal{H}}\twoheadrightarrow\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ such that $\mathsf{R}_{\mathcal{H}}:=\mathsf{U}_{\mathcal{H}}\underset{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}{\times}\mathsf{U}_{\mathcal{H}}$ is étale over $\mathsf{U}_{\mathcal{H}}$ via the two projections (see Proposition A.7.1.1 and Definition A.7.1.3). Equivalently, it suffices to construct the $\mathsf{U}_{\mathcal{H}}$ and $\mathsf{R}_{\mathcal{H}}$ that satisfy the required groupoid relations, which then realizes $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ as the quotient of $\mathsf{U}_{\mathcal{H}}$ by $\mathsf{R}_{\mathcal{H}}$. Let us first explain our choices of $\mathsf{U}_{\mathcal{H}}$ and $\mathsf{R}_{\mathcal{H}}$, then show that they have the desired properties.

Construction 6.3.3.1. Take a finite number of good algebraic models, which cover our potential compactification, as follows:

1. Choose a complete set of (mutually inequivalent) representatives $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ of cusp labels at level $\mathcal{H}$ as in Definition 5.4.2.4.

   This is a finite set for the following reason: By the Jordan–Zassenhaus theorem (see, for example, [107, Thm. 26.4]), there are only finitely many isomorphism classes of $\mathcal{O}$-lattices of each given $\mathcal{O}$-multirank. Given any two $\mathcal{O}$-lattices X and Y, there can only be finitely many mutually inequivalent representatives $(\Phi_{\mathcal{H}}=(X,Y,\phi,\varphi_{-2,\mathcal{H}},\varphi_{0,\mathcal{H}}),\delta_{\mathcal{H}})$ of cusp labels at level $\mathcal{H}$ containing X and Y as part of the data, because the remaining data such as $\mathsf{Z}_{\mathcal{H}}$, $\varphi_{-2,\mathcal{H}}$, $\varphi_{0,\mathcal{H}}$, and $\delta_{\mathcal{H}}$ are defined by morphisms between objects of finite cardinality, and $\phi: Y\hookrightarrow X$ is an embedding with finite cokernel. (We do not need to know anything about the exact parameterization of these.)

2. Make *compatible choices* of a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ for each $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ chosen above. We assume that each $\Sigma_{\Phi_{\mathcal{H}}}$ satisfies Condition 6.2.5.25.

   The compatibility means the following:

   Condition 6.3.3.2. *For every surjection $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})\twoheadrightarrow(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ of representatives of cusp labels, we require the cone decompositions $\Sigma_{\Phi_{\mathcal{H}}}$ and $\Sigma_{\Phi'_{\mathcal{H}}}$ to define a surjection $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\Sigma_{\Phi_{\mathcal{H}}})\twoheadrightarrow(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\Sigma_{\Phi'_{\mathcal{H}}})$ as in Definition 6.2.6.4.*

   Let us proceed by assuming that such a compatible choice is possible (see Proposition 6.3.3.5 below).

3. For each $\Sigma_{\Phi_{\mathcal{H}}}$ above, choose a complete set of (mutually inequivalent) representatives σ in $\Sigma_{\Phi_{\mathcal{H}}}/\Gamma_{\Phi_{\mathcal{H}}}$. This gives a complete set of representatives $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ of equivalence classes $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ defined in Definition 6.2.6.1.

   This is a finite set by the $\Gamma_{\Phi_{\mathcal{H}}}$-admissibility (see Definition 6.1.1.10) of each $\Sigma_{\Phi_{\mathcal{H}}}$. (So the question is rather about the existence of $\Sigma_{\Phi_{\mathcal{H}}}$ in the previous step.)

4. For each representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ above that satisfies moreover $\sigma\subset\mathbf{P}^+_{\Phi_{\mathcal{H}}}$, choose *finitely many* good algebraic $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$-models $\mathrm{Spec}(R_{\mathrm{alg}})$ (see Definition 6.3.2.5) such that the corresponding étale morphisms from the various $\mathrm{Spec}(R_{\mathrm{alg}}/I)$'s to the σ-stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, where I denotes the ideal of R_{alg} defining the σ-stratum of $\mathrm{Spec}(R_{\mathrm{alg}})$, *cover* the whole σ-stratum (see Proposition 6.3.2.6 and Remark 6.3.2.7).

   This is possible by the quasi-compactness of the σ-stratum of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, which follows from the realization of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ as a toroidal embedding of a torus torsor over an abelian scheme torsor over a finite étale cover of $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ (see Section 6.2 and Theorem 1.4.1.11).

Remark 6.3.3.3. The compatible choices of $\Sigma_{\Phi_{\mathcal{H}}}$ for the chosen representatives $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ extend by the equivalence defined in Definition 6.2.6.2 to compatible choices for all possible pairs $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ representing cusp labels. In particular, there is only one choice of cone decomposition needed for each cusp label.

Definition 6.3.3.4. *A* ***compatible choice of admissible smooth rational polyhedral cone decomposition data*** *for* $\mathsf{M}_{\mathcal{H}}$ *is a complete set* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *of* ***compatible choices*** *of* $\Sigma_{\Phi_{\mathcal{H}}}$ *(satisfying Condition 6.2.5.25) in the sense of Condition 6.3.3.2, for* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *running through all representatives of cusp labels* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$.

Proposition 6.3.3.5. *A compatible choice* Σ *of admissible smooth rational polyhedral cone decomposition data for* $\mathsf{M}_{\mathcal{H}}$ ***exists****.*

Remark 6.3.3.6. This is a combinatorial question completely unrelated to the question of integral models. It is already needed in the existing works on complex analytic constructions of toroidal compactifications (such as [16], [102], or [62]). Since the argument is not complicated in our case, and since the translation of different notation and settings will require too much effort, we would like to give a direct treatment here.

For ease of exposition, let us introduce the following notions:

Definition 6.3.3.7. *Let* $r = (r_{[\tau]})$ *be the* $\mathcal{O}$*-multirank of an* $\mathcal{O}$*-lattice as in Definition 1.2.1.21. The* ***magnitude*** $|r|$ *of* r *is defined to be* $|r| := \sum_{[\tau]} r_{[\tau]}$.

Definition 6.3.3.8. *Let* $r = (r_{[\tau]})$ *and* $r' = (r'_{[\tau]})$ *be the* $\mathcal{O}$*-multirank of an* $\mathcal{O}$*-lattice as in Definition 1.2.1.21. We say* r *is* ***greater*** *than* r', *denoted* $r > r'$, *if* $|r| > |r'|$ *and* $r_{[\tau]} \geq r'_{[\tau]}$ *for every* $[\tau]$. *(We say* r *is* ***smaller*** *than* r', *denoted* $r < r'$, *if* $r' > r$. *We say* r *is* ***equal*** *to* r' *if* $r = r'$ *in the literal sense that* $r_{[\tau]} = r'_{[\tau]}$ *for every* $[\tau]$.*) These relations define a* ***partial order*** *on the set of all possible* $\mathcal{O}$*-multiranks.*

Proof of Proposition 6.3.3.5. It suffices to take a complete set $\{(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of (mutually inequivalent) representatives of cusp labels $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ of $\mathsf{M}_{\mathcal{H}}$ at level $\mathcal{H}$, and then to construct for each representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (satisfying Condition 6.2.5.25), such that Condition 6.3.3.2 is satisfied for surjections between objects in $\{(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. Then there is no compatibility to satisfy between cone decompositions of representatives of cusp labels of the same $\mathcal{O}$-multirank.

If we begin with the representatives of cusp labels $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of $\mathcal{O}$-multirank r of magnitude one (see Definition 6.3.3.7), then there is no compatibility condition to satisfy, and hence we can take any $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (satisfying Condition 6.2.5.25) for each representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$.

Let us consider any $\mathcal{O}$-multirank r with magnitude strictly greater than one. Suppose we have constructed a compatible collection of cone decompositions as above for each representative of cusp labels of $\mathcal{O}$-multirank (see Definition 6.3.3.8) strictly smaller than r. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ be any representative of a cusp label of $\mathcal{O}$-multirank r (if it exists). The *admissible boundary* $\mathbf{P}_{\Phi_{\mathcal{H}}} - \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is the cone in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ formed by the union of the *admissible boundary components* of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (see Definition 6.2.5.24), namely, the images of $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ of surjections $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ from $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ to representatives $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ of cusp labels of $\mathcal{O}$-multiranks with magnitude strictly smaller than r. The choices of cone decompositions we have made for such $\mathbf{P}_{\Phi'_{\mathcal{H}}}$'s are compatible with each other, and determine a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth polyhedral cone decomposition (satisfying Condition 6.2.5.25) for the cone $\mathbf{P}_{\Phi_{\mathcal{H}}} - \mathbf{P}^+_{\Phi_{\mathcal{H}}}$, which is independent of the choice of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ by the definition of admissibility (see Definitions 6.1.1.10 and 6.1.1.12). Then we can take any $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth polyhedral cone decomposition (satisfying Condition 6.2.5.25) for the cone $\mathbf{P}_{\Phi_{\mathcal{H}}}$ that extends the above cone decomposition, which is possible by construction. This enables us to construct a compatible collection of cone decompositions as above for each representative of cusp labels of $\mathcal{O}$-multirank r.

Now we can conclude the proof by repeating the above process until we have exhausted all representatives of cusp labels. □

When the $\mathcal{O}$-multirank of a cusp label is zero, there is only one possible $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) = (0, 0)$ representing this cusp label, and also only one possible $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma) = (0, 0, \{0\})$. In this case, the Mumford family is the tautological tuple over the moduli problem $\mathsf{M}_{\mathcal{H}}$ we want to compactify. Then the good algebraic models $\mathrm{Spec}(R_{\mathrm{alg}})$ are affine schemes with étale morphisms $\mathrm{Spec}(R_{\mathrm{alg}}) \to \mathsf{M}_{\mathcal{H}}$, which altogether cover $\mathsf{M}_{\mathcal{H}}$.

Construction 6.3.3.9 (continuation of Construction 6.3.3.1). Let us form the scheme

$$\mathsf{U}_{\mathcal{H}} = \begin{pmatrix} \text{disjoint union of the (finitely many)} \\ \text{good algebraic } (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)\text{-models} \\ \mathrm{Spec}(R_{\mathrm{alg}}) \text{ chosen above} \end{pmatrix},$$

(smooth over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\square)})$) which comes equipped with a natural stratification labeled as follows:

On a good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model $\mathrm{Spec}(R_{\mathrm{alg}})$ used in the construction of $\mathsf{U}_{\mathcal{H}}$ above, its stratification inherited from $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ can be relabeled using equivalence classes $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \tau')]$ (see Definition 6.2.6.1) following the recipe in Definition 6.3.2.16, which are *faces* of $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ (see Definition 6.3.2.14). Then we define the stratification on the disjoint union $\mathsf{U}_{\mathcal{H}}$ to be induced by those on the good algebraic models.

By the compatibility of the choice of Σ (given by Condition 6.3.3.2) in Definition 6.3.3.4, we know that in each representative $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \tau')$ of each face $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \tau')]$ of $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$, the cone τ' is in the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ we have in Σ. Hence we may label all the strata by the equivalence classes of triples we have taken in the construction of $\mathsf{U}_{\mathcal{H}}$. For simplicity, we call the $[(0, 0, \{0\})]$-stratum the $[0]$-stratum of $\mathsf{U}_{\mathcal{H}}$, which we denote by $\mathsf{U}_{\mathcal{H}}^{[0]}$.

The good algebraic models $(G, \lambda, i, \alpha_{\mathcal{H}})$ over the various $\mathrm{Spec}(R_{\mathrm{alg}})$'s define (by taking union) a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{U}_{\mathcal{H}}$, whose restriction to the $[0]$-stratum $\mathsf{U}_{\mathcal{H}}^{[0]}$ is a tuple $(G^{[0]}, \lambda^{[0]}, i^{[0]}, \alpha_{\mathcal{H}})$ parameterized by the moduli problem $\mathsf{M}_{\mathcal{H}}$. This determines a canonical morphism $\mathsf{U}_{\mathcal{H}}^{[0]} \to \mathsf{M}_{\mathcal{H}}$. This morphism is étale because $\mathsf{U}_{\mathcal{H}}^{[0]}$ is locally of finite presentation, and the morphism $\mathsf{U}_{\mathcal{H}}^{[0]} \to \mathsf{M}_{\mathcal{H}}$ is formally étale at every geometric point of $\mathsf{U}_{\mathcal{H}}^{[0]}$ by the calculation of Kodaira–Spencer morphisms (using 3c of Definition 6.3.2.5, Theorem 4.6.3.16, and Proposition 2.3.5.2). As a result, the morphism $\mathsf{U}_{\mathcal{H}}^{[0]} \to \mathsf{M}_{\mathcal{H}}$ (surjective by definition) defines an étale presentation of $\mathsf{M}_{\mathcal{H}}$. This identifies $\mathsf{M}_{\mathcal{H}}$ with the quotient of $\mathsf{U}_{\mathcal{H}}^{[0]}$ by the étale groupoid

$\mathsf{R}^{[0]}_{\mathcal{H}}$ over $\mathsf{U}^{[0]}_{\mathcal{H}}$ defined by the representable functor

$$\mathsf{R}^{[0]}_{\mathcal{H}} := \underline{\mathrm{Isom}}_{\mathsf{U}^{[0]}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{U}^{[0]}_{\mathcal{H}}}(\mathrm{pr}_1^*(G^{[0]}, \lambda^{[0]}, i^{[0]}, \alpha_{\mathcal{H}}), \mathrm{pr}_2^*(G^{[0]}, \lambda^{[0]}, i^{[0]}, \alpha_{\mathcal{H}})), \qquad (6.3.3.10)$$

where $\mathrm{pr}_1, \mathrm{pr}_2 : \mathsf{U}^{[0]}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{U}^{[0]}_{\mathcal{H}} \to \mathsf{U}^{[0]}_{\mathcal{H}}$ denote, respectively, the two projections.

Proposition 6.3.3.11. *Suppose R is a noetherian normal complete local domain with fraction field K and algebraically closed residue field k. Assume that we have a degenerating family $(G^\ddagger, \lambda^\ddagger, i^\ddagger, \alpha^\ddagger_{\mathcal{H}})$ of type $\mathsf{M}_{\mathcal{H}}$ over $\mathrm{Spec}(R)$ as in Definition 5.3.2.1. Then the following conditions are equivalent:*

1. *There exists a morphism $\mathrm{Spec}(R) \to \mathsf{U}_{\mathcal{H}}$ sending the generic point $\mathrm{Spec}(K)$ to the $[0]$-stratum such that $(G^\ddagger, \lambda^\ddagger, i^\ddagger, \alpha^\ddagger_{\mathcal{H}}) \to \mathrm{Spec}(R)$ is the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{U}_{\mathcal{H}}$.*

2. *The degenerating family $(G^\ddagger, \lambda^\ddagger, i^\ddagger, \alpha^\ddagger_{\mathcal{H}}) \to \mathrm{Spec}(R)$ is the pullback of the Mumford family $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}}) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ via a morphism $\mathrm{Spf}(R) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, or equivalently a morphism $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, for some $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ (which can be assumed to be a triple used in the construction of $\mathsf{U}_{\mathcal{H}}$).*

3. *The degenerating family $(G^\ddagger, \lambda^\ddagger, i^\ddagger, \alpha^\ddagger_{\mathcal{H}})$ over $\mathrm{Spec}(R)$ defines an object of $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R)$, which corresponds to a tuple $(A^\ddagger, \lambda_{A^\ddagger}, i_{A^\ddagger}, \underline{X}^\ddagger, \underline{Y}^\ddagger, \phi^\ddagger, c^\ddagger, (c^\vee)^\ddagger, \tau^\ddagger, [(\alpha^\natural_{\mathcal{H}})^\ddagger])$ in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R)$ under Theorem 5.3.1.19. Then we have a fully symplectic-liftable admissible filtration $\mathsf{Z}^\ddagger_{\mathcal{H}}$ determined by $[(\alpha^\natural_{\mathcal{H}})^\ddagger]$. Moreover, the étale sheaves $\underline{X}^\ddagger$ and $\underline{Y}^\ddagger$ are necessarily constant, because the base scheme R is strict local. Hence it makes sense to say we also have a uniquely determined torus argument $\Phi^\ddagger_{\mathcal{H}}$ at level $\mathcal{H}$ for $\mathsf{Z}^\ddagger_{\mathcal{H}}$.*

   On the other hand, we have objects $\underline{\Phi}_{\mathcal{H}}(G^\ddagger)$, $\mathbf{S}_{\underline{\Phi}_{\mathcal{H}}(G^\ddagger)}$, and $\underline{B}(G^\ddagger)$, which define objects $\Phi^\ddagger_{\mathcal{H}}$, $\mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}}$, and in particular, $B^\ddagger : \mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}} \to \mathrm{Inv}(R)$ over the special fiber.

   If $\upsilon : K^\times \to \mathbb{Z}$ is any discrete valuation defined by a height-one prime of R, then $\upsilon \circ B^\ddagger : \mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}} \to \mathbb{Z}$ makes sense and defines an element of $\mathbf{S}^\vee_{\Phi^\ddagger_{\mathcal{H}}}$. Then the condition is that, for some (and hence every) choice of $\delta^\ddagger_{\mathcal{H}}$ making $(\mathsf{Z}^\ddagger_{\mathcal{H}}, \Phi^\ddagger_{\mathcal{H}}, \delta^\ddagger_{\mathcal{H}})$ a representative of a cusp label, there is a cone $\sigma^\ddagger$ in the cone decomposition $\Sigma_{\Phi^\ddagger_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi^\ddagger_{\mathcal{H}}}$ (given by the choice of Σ; cf. Definition 6.3.3.4) such that the closure $\overline{\sigma}^\ddagger$ of $\sigma^\ddagger$ in $(\mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}})^\vee_{\mathbb{R}}$ contains all $\upsilon \circ B^\ddagger$ obtained in this way.

Proof. The implication from 1 to 2 is clear, as the morphism from $\mathrm{Spec}(R)$ to $\mathsf{U}_{\mathcal{H}}$ necessarily factors through the completion of some strict local ring of $\mathsf{U}_{\mathcal{H}}$.

The implication from 2 to 3 is analogous to Proposition 6.3.1.6.

For the implication from 3 to 1, suppose there exists a cone $\sigma^\ddagger$ in the cone decomposition $\Sigma_{\Phi^\ddagger_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi^\ddagger_{\mathcal{H}}}$ such that $\overline{\sigma}^\ddagger$ contains all the $\upsilon \circ B^\ddagger$'s. Up to replacing $\sigma^\ddagger$ with another cone in $\Sigma_{\Phi^\ddagger_{\mathcal{H}}}$, let us assume that $\sigma^\ddagger$ is a minimal one. Then some linear combination with positive coefficients of the $\upsilon \circ B^\ddagger$'s lie in $\sigma^\ddagger$, the interior of $\overline{\sigma}^\ddagger$. On the other hand, by the positivity condition of $\tau^\ddagger$, such a linear combination with positive coefficients must be positive definite on $Y^\ddagger$. Hence $\sigma^\ddagger \subset \mathbf{P}^+_{\Phi^\ddagger_{\mathcal{H}}}$. Then there exists a unique triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ chosen in the construction $\mathsf{U}_{\mathcal{H}}$ (see Constructions 6.3.3.1 and 6.3.3.9) such that $(\Phi^\ddagger_{\mathcal{H}}, \delta^\ddagger_{\mathcal{H}}, \sigma^\ddagger)$ and $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ are equivalent (see Definition 6.2.6.1).

Since $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ is finite étale over $\mathsf{M}_{\mathcal{H}_h}$, and since $A^\ddagger$ is defined over R, as pointed out in Remark 5.3.2.2, the tuple $(A^\ddagger, \lambda_{A^\ddagger}, i_{A^\ddagger}, \varphi^\ddagger_{-1,\mathcal{H}})$ and the $\Gamma_{\Phi^\ddagger_{\mathcal{H}}}$-orbit of $(\varphi^{\sim,\ddagger}_{-2,\mathcal{H}}, \varphi^{\sim,\ddagger}_{0,\mathcal{H}})$ define a morphism $\mathrm{Spec}(R) \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ as soon as its restriction to $\mathrm{Spec}(K)$ defines a morphism $\mathrm{Spec}(K) \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. By Proposition 6.2.4.7, the degeneration datum (without the positivity condition) associated with $(G^\ddagger, \lambda^\ddagger, i^\ddagger, \alpha^\ddagger_{\mathcal{H}}) \to \mathrm{Spec}(R)$ determines (by the universal properties of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ and $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$) a morphism $\mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, whose composition with the (relatively affine) structural morphism $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ extends to a morphism $\mathrm{Spec}(R) \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. By Proposition 6.2.5.11 and the assumption on the $\upsilon \circ B^\ddagger$'s, the morphism $\mathrm{Spec}(K) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ extends to a morphism $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$, which identifies $\underline{B}(G^\ddagger)$ with the pullback of $\underline{B}$ under an identification of $\underline{\Phi}_{\mathcal{H}}(G^\ddagger)$ with the pullback of $\underline{\Phi}_{\mathcal{H}}$. The ambiguity of the identifications can be removed (or rather intrinsically incorporated) if we form the quotient $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. Hence we have a uniquely determined strata-preserving morphism $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, which is independent of the identification of $\Phi^\ddagger_{\mathcal{H}}$ with $\Phi_{\mathcal{H}}$ we have chosen. This determines a morphism $\mathrm{Spf}(R) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ as in 2.

Let us denote the image of the closed point of $\mathrm{Spec}(R)$ by x, which necessarily lies in the σ-stratum (thanks to the minimality of the choice of $\sigma^\ddagger$). By construction, there is some good algebraic $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$-model $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathrm{Spec}(R_{\mathrm{alg}})$ used in the construction of $\mathsf{U}_{\mathcal{H}}$ such that the image of the structural morphism $\mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ contains x. Let $R_{\mathrm{alg}}{}^\wedge$ be the completion of R_{alg} with respect to the ideal defining the σ-stratum of $\mathrm{Spec}(R_{\mathrm{alg}})$, and let $I^\wedge$ be the induced ideal of definition. Then the étale morphism $\mathrm{Spec}(R_{\mathrm{alg}}) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ induces a formally étale morphism $\mathrm{Spf}(R_{\mathrm{alg}}{}^\wedge, I^\wedge) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. By formal étaleness, the morphism $\mathrm{Spf}(R) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ can be lifted uniquely to a morphism $\mathrm{Spf}(R) \to \mathrm{Spf}(R_{\mathrm{alg}}{}^\wedge, I^\wedge)$. The underlying morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(R_{\mathrm{alg}}{}^\wedge)$ identifies the degeneration datum associated with $(G^\ddagger, \lambda^\ddagger, i^\ddagger, \alpha^\ddagger_{\mathcal{H}}) \to \mathrm{Spec}(R)$ with the degeneration datum associated with the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}}) \underset{R_{\mathrm{alg}}}{\otimes} R_{\mathrm{alg}}{}^\wedge \to \mathrm{Spec}(R_{\mathrm{alg}}{}^\wedge)$. Hence the morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(R_{\mathrm{alg}}{}^\wedge) \to \mathrm{Spec}(R_{\mathrm{alg}})$ identifies $(G^\ddagger, \lambda^\ddagger, i^\ddagger, \alpha^\ddagger_{\mathcal{H}}) \to \mathrm{Spec}(R)$ with the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathrm{Spec}(R_{\mathrm{alg}})$, as desired. □

Remark 6.3.3.12. Condition 3 in Proposition 6.3.3.11 is *always fulfilled* if R is a complete discrete valuation ring. This is because there is only one choice of the discrete valuation υ, and hence only one homomorphism $\upsilon \circ B^\ddagger$ to be considered. Since the union of the cones in $\Sigma_{\Phi^\ddagger_{\mathcal{H}}}$ cover $\mathbf{P}_{\Phi^\ddagger_{\mathcal{H}}}$ by definition of *admissibility* (in Definition 6.1.1.10), $\upsilon \circ B^\ddagger$ must lie in one of the cones. We shall see in the proof of Proposition 6.3.3.17 how this observation will be applied in justifying the *properness* of the toroidal compactification to be constructed.

Now let $\mathsf{R}_{\mathcal{H}}$ be the *normalization* of $\mathsf{R}^{[0]}_{\mathcal{H}} \to \mathsf{U}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{U}_{\mathcal{H}}$. By definition, $\mathsf{R}_{\mathcal{H}}$ is noetherian and normal. By Proposition 3.3.1.5, the tautological isomorphism

$$\mathsf{h}^{[0]} : (\mathrm{pr}_1^*(G^{[0]}, \lambda^{[0]}, i^{[0]}, \alpha_{\mathcal{H}}))_{\mathsf{R}^{[0]}_{\mathcal{H}}} \xrightarrow{\sim} (\mathrm{pr}_2^*(G^{[0]}, \lambda^{[0]}, i^{[0]}, \alpha_{\mathcal{H}}))_{\mathsf{R}^{[0]}_{\mathcal{H}}}$$

over $\mathsf{R}_{\mathcal{H}}^{[0]}$ (see (6.3.3.10)) extends uniquely to an isomorphism

$$\mathsf{h} : (\mathrm{pr}_1^*(G, \lambda, i, \alpha_{\mathcal{H}}))_{\mathsf{R}_{\mathcal{H}}} \xrightarrow{\sim} (\mathrm{pr}_2^*(G, \lambda, i, \alpha_{\mathcal{H}}))_{\mathsf{R}_{\mathcal{H}}}$$

over $\mathsf{R}_{\mathcal{H}}$. (Here $\mathrm{pr}_1, \mathrm{pr}_2 : \mathsf{U}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{U}_{\mathcal{H}} \to \mathsf{U}_{\mathcal{H}}$ are the two projections.)

The key to the gluing process is the following:

Proposition 6.3.3.13. *The two projections from $\mathsf{R}_{\mathcal{H}}$ to $\mathsf{U}_{\mathcal{H}}$ are **étale**.*

Proof. The plan is as follows: Let z be an arbitrary geometric point of $\mathsf{R}_{\mathcal{H}}$ and let $x := \mathrm{pr}_1(z)$ and $y := \mathrm{pr}_2(z)$. Let R_{12} (resp. R_1, resp. R_2) be the completion of the strict local ring of $\mathsf{R}_{\mathcal{H}}$ (resp. $\mathsf{U}_{\mathcal{H}}$, resp. $\mathsf{U}_{\mathcal{H}}$) at the geometric point z (resp. x, resp. y). As both projections, pr_1 and pr_2, induce dominant morphisms from irreducible components of $\mathsf{R}_{\mathcal{H}}$ to irreducible components of $\mathsf{U}_{\mathcal{H}}$ (by restriction to $\mathsf{U}_{\mathcal{H}}^{[0]}$), R_1 (resp. R_2) can be naturally embedded into R_{12} via the local homomorphism $\mathrm{pr}_1^* : R_1 \to R_{12}$ (resp. $\mathrm{pr}_2^* : R_2 \to R_{12}$). Our goal is to show that $R_{12} = \mathrm{pr}_1^*(R_1) = \mathrm{pr}_2^*(R_2)$.

Step 1. There are triples $(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1}, \sigma_1)$ and $(\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2}, \sigma_2)$, with $\sigma_1 \subset \mathbf{P}_{\phi_1}^+$ and $\sigma_2 \subset \mathbf{P}_{\phi_2}^+$, such that x (resp. y) lies on the $[(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1}, \sigma_1)]$-stratum (resp. $[(\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2}, \sigma_2)]$-stratum) of $\mathsf{U}_{\mathcal{H}}$. By Proposition 6.3.2.10 (i.e., the *openness of versality*), there exists a unique isomorphism from R_1 (resp. R_2) to the completion of the strict local ring of $\mathfrak{X}_{\Phi_{\mathcal{H},1},\delta_{\mathcal{H},1},\sigma_1}/\Gamma_{\Phi_{\mathcal{H},1},\sigma_1}$ (resp. $\mathfrak{X}_{\Phi_{\mathcal{H},2},\delta_{\mathcal{H},2},\sigma_2}/\Gamma_{\Phi_{\mathcal{H},2},\sigma_2}$) at a (unique) geometric point in the σ_1-stratum (resp. σ_2-stratum), such that the pullback $(G_1, \lambda_1, i_1, \alpha_{\mathcal{H},1}) \to \mathrm{Spec}(R_1)$ (resp. $(G_2, \lambda_2, i_2, \alpha_{\mathcal{H},2}) \to \mathrm{Spec}(R_2)$) of $(G, \lambda, i, \alpha_{\mathcal{H}})$ via $\mathrm{Spec}(R_1) \to \mathsf{U}_{\mathcal{H}}$ (resp. $\mathrm{Spec}(R_2) \to \mathsf{U}_{\mathcal{H}}$) defines the completion of a good formal $(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1}, \sigma_1)$-model (resp. good formal $(\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2}, \sigma_2)$-model) via this isomorphism.

Step 2. By the theory of degeneration data, we have an object

$$(A_1, \lambda_{A,1}, i_{A,1}, X_1, Y_1, \phi_1, c_1, c_1^\vee, \tau_1, [\alpha_{\mathcal{H},1}^\natural])$$

in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R_1)$, where $[\alpha_{\mathcal{H},1}^\natural]$ is represented by some

$$\alpha_{\mathcal{H},1}^\natural = (\mathsf{Z}_{\mathcal{H},1}, \varphi_{-2,\mathcal{H},1}^\sim, \varphi_{-1,\mathcal{H},1}, \varphi_{0,\mathcal{H},1}^\sim, \delta_{\mathcal{H},1}, c_{\mathcal{H},1}, c_{\mathcal{H},1}^\vee, \tau_{\mathcal{H},1})$$

which corresponds to the family $(G_1, \lambda_1, i_1, \alpha_{\mathcal{H},1}) \to \mathrm{Spec}(R_1)$ under the functor $\mathrm{M}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R_1)$ (see Theorem 5.3.1.19). Here we may and we do take the datum $(\mathsf{Z}_{\mathcal{H},1}, (X_1, Y_1, \phi_1, \varphi_{-2,\mathcal{H},1}, \varphi_{0,\mathcal{H},1}), \delta_{\mathcal{H},1})$ to be the one given by $(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1})$ (where the $(\varphi_{-2,\mathcal{H},1}, \varphi_{0,\mathcal{H},1})$ in $\Phi_{\mathcal{H},1}$ is induced by $(\varphi_{-2,\mathcal{H},1}^\sim, \varphi_{0,\mathcal{H},1}^\sim)$ as in Definition 5.4.2.8), because of the above-mentioned isomorphism from $\mathrm{Spec}(R_1)$ to $\mathfrak{X}_{\Phi_{\mathcal{H},1},\delta_{\mathcal{H},1},\sigma_1}/\Gamma_{\Phi_{\mathcal{H},1},\sigma_1}$. Similarly, we have an object

$$(A_2, \lambda_{A,2}, i_{A,2}, X_2, Y_2, \phi_2, c_2, c_2^\vee, \tau_2, [\alpha_{\mathcal{H},2}^\natural])$$

in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(R_2)$, where $[\alpha_{\mathcal{H},2}^\natural]$ is represented by some

$$\alpha_{\mathcal{H},2}^\natural = (\mathsf{Z}_{\mathcal{H},2}, \varphi_{-2,\mathcal{H},2}^\sim, \varphi_{-1,\mathcal{H},2}, \varphi_{0,\mathcal{H},2}^\sim, \delta_{\mathcal{H},2}, c_{\mathcal{H},2}, c_{\mathcal{H},2}^\vee, \tau_{\mathcal{H},2})$$

which corresponds to the family $(G_2, \lambda_2, i_2, \alpha_{\mathcal{H},2}) \to \mathrm{Spec}(R_2)$. Again, we may and we do take $(\mathsf{Z}_{\mathcal{H},2}, (X_2, Y_2, \phi_2, \varphi_{-2,\mathcal{H},2}, \varphi_{0,\mathcal{H},2}), \delta_{\mathcal{H},2})$ to be the one given by $(\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2})$ (where the $(\varphi_{-2,\mathcal{H},2}, \varphi_{0,\mathcal{H},2})$ in $\Phi_{\mathcal{H},2}$ is induced by $(\varphi_{-2,\mathcal{H},2}^\sim, \varphi_{0,\mathcal{H},2}^\sim)$ as in Definition 5.4.2.8).

As $(G_1, \lambda_1, i_1, \alpha_{\mathcal{H},1}) \to \mathrm{Spec}(R_1)$ and $(G_2, \lambda_2, i_2, \alpha_{\mathcal{H},2}) \to \mathrm{Spec}(R_2)$ become isomorphic over $\mathrm{Spec}(R_{12})$ via a tautological isomorphism

$$\mathsf{h} : (\mathrm{pr}_1^*(G_1, \lambda_1, i_1, \alpha_{\mathcal{H},1}))_{R_{12}} \xrightarrow{\sim} (\mathrm{pr}_2^*(G_2, \lambda_2, i_2, \alpha_{\mathcal{H},2}))_{R_{12}},$$

we have a corresponding isomorphism between the degeneration data over $\mathrm{Spec}(R_{12})$. Therefore, h matches $[\alpha_{\mathcal{H},1}^\natural]$ with $[\alpha_{\mathcal{H},2}^\natural]$, and $\alpha_{\mathcal{H},1}^\natural$ and $\alpha_{\mathcal{H},2}^\natural$ are equivalent to each other (see Definition 5.3.1.16). In particular, $(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1})$ and $(\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2})$ are equivalent to each other as representatives of cusp labels (see Definition 5.4.2.4). Since we have taken only one representative in each cusp label, they must be identical. We shall henceforth assume that $(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1}) = (\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2}) = (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$, where $(\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}}), \delta_{\mathcal{H}})$ is some representative that we have chosen. In particular, the equivalence induced by h between $(\Phi_{\mathcal{H},1}, \delta_{\mathcal{H},1}) = (\Phi_{\mathcal{H},2}, \delta_{\mathcal{H},2})$ is now given by an element $(\mathsf{h}_X : X \xrightarrow{\sim} Y, \mathsf{h}_Y : Y \xrightarrow{\sim} Y)$ in $\Gamma_{\Phi_{\mathcal{H}}}$.

Step 3. We claim that the pair of isomorphisms $(\mathsf{h}_X : X \xrightarrow{\sim} X, \mathsf{h}_Y : Y \xrightarrow{\sim} Y)$ identifies the cone $\sigma_1 \subset \mathbf{P}_{\Phi_{\mathcal{H}}}^+$ with the cone $\sigma_2 \subset \mathbf{P}_{\Phi_{\mathcal{H}}}^+$ up to $\Gamma_{\Phi_{\mathcal{H}}}$.

This is because of the following: Up to replacing $(\mathsf{h}_X : X \xrightarrow{\sim} X, \mathsf{h}_Y : Y \xrightarrow{\sim} Y)$ with a twist by some element of $\Gamma_{\Phi_{\mathcal{H}}}$, we may assume that the two homomorphisms $B_1 : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathrm{Inv}(R_1)$ and $B_2 : \mathbf{S}_{\Phi_{\mathcal{H}}} \to \mathrm{Inv}(R_2)$ become identified with each other under $(\mathsf{h}_X, \mathsf{h}_Y)$ when extended to $\mathrm{Inv}(R_{12})$. If σ_1 and σ_2 were not identified with each other under $(\mathsf{h}_X, \mathsf{h}_Y)$, then we could find elements ℓ_1 and ℓ_2 of $\mathbf{S}_{\Phi_{\mathcal{H}}}$, identified with each other under $(\mathsf{h}_X, \mathsf{h}_Y)$, such that $B_1(\ell_1)$ is contained in the maximal ideal of R_1 but $B_2(-\ell_2) = B_2(\ell_2)^{-1} \subset R_2$. Then they could not become identified with each other after making the base change to R_{12}, because (by their very definitions) the maximal ideal of R_{12} intersects R_1 and R_2 in their maximal ideals. In other words, the homomorphisms B_1 and B_2 determine σ_1 and σ_2 respectively, and $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma_1)$ and $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma_2)$ are equivalent under the element $(\mathsf{h}_X : X \xrightarrow{\sim} X, \mathsf{h}_Y : Y \xrightarrow{\sim} Y)$ identifying B_1 with B_2.

Thus σ_1 and σ_2 are necessarily the same cone σ, because we have used only one representative in each equivalence class of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ in our construction of $\mathsf{U}_{\mathcal{H}}$. By Proposition 6.3.2.10, we may compare R_1 and R_2 by viewing them as completions of strict local rings (at geometric points of the σ-strata) of the same algebraic stack $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.

Step 4. Now we have two morphisms

$$\mathrm{Spec}(R_{12}) \xrightarrow{\mathrm{pr}_1} \mathrm{Spec}(R_1) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$$

and

$$\mathrm{Spec}(R_{12}) \xrightarrow{\mathrm{pr}_2} \mathrm{Spec}(R_2) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$$

defined by the two degeneration data, which actually *coincide*. As both $\mathrm{pr}_1^*(R_1)$ and $\mathrm{pr}_2^*(R_2)$ are given by the image of the completion of the strict local ring at the *same* image of the closed point of $\mathrm{Spec}(R_{12})$, they must *coincide* as subrings of R_{12}. Therefore, the identification $\mathrm{pr}_1^*(R_1) = \mathrm{pr}_2^*(R_2)$ identifies the two degeneration data as well. By the functoriality in the theory of degeneration data, the isomorphism between $(G_1, \lambda_1, i_1, \alpha_{\mathcal{H},1})_{R_{12}}$ and $(G_2, \lambda_2, i_2, \alpha_{\mathcal{H},2})_{R_{12}}$ should be *already defined* over $\mathrm{pr}_1^*(R_1) = \mathrm{pr}_2^*(R_2)$.

By definition of $\mathsf{R}_{\mathcal{H}}^{[0]}$, we get a morphism from the *generic point* of $\mathrm{Spec}(R_1)$ to $\mathsf{R}_{\mathcal{H}}^{[0]}$, which lies above the morphism to $\mathsf{U}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{U}_{\mathcal{H}}$ with two factors given respectively by $\mathrm{Spec}(R_1) \to \mathsf{U}_{\mathcal{H}}$ and the composition of the isomorphism $\mathrm{Spec}(R_1) \xrightarrow{\sim} \mathrm{Spec}(R_2)$ given by $\mathrm{pr}_1^*(R_1) = \mathrm{pr}_2^*(R_2)$ in R_{12} with $\mathrm{Spec}(R_2) \to \mathsf{U}_{\mathcal{H}}$. By definition of $\mathsf{R}_{\mathcal{H}}$ as a normalization, this extends to a morphism $\mathrm{Spec}(R_1) \to \mathsf{R}_{\mathcal{H}}$ (not just from the generic point), which sends the closed point x of $\mathrm{Spec}(R_1)$ to the closed point z of R_{12}. This gives a homomorphism $R_{12} \to R_1$, which is a left inverse of $\mathrm{pr}_1^* : R_1 \hookrightarrow R_{12}$. Hence $R_{12} = \mathrm{pr}_1^*(R_1)$. Consequently, $R_{12} = \mathrm{pr}_2^*(R_2)$ as well. □

Corollary 6.3.3.14. *The scheme $\mathsf{R}_{\mathcal{H}}$ over $\mathsf{U}_{\mathcal{H}}$ defines an étale groupoid space (see Definition A.5.1.2), which extends the étale groupoid space $\mathsf{R}_{\mathcal{H}}^{[0]}$ over $\mathsf{U}_{\mathcal{H}}^{[0]}$. The* *scheme $\mathsf{R}_{\mathcal{H}}$ is **finite** over $\mathsf{U}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{U}_{\mathcal{H}}$, and hence (by Lemma A.7.2.9) $\mathsf{U}_{\mathcal{H}}/\mathsf{R}_{\mathcal{H}}$ defines an algebraic stack **separated** over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$.*

Proof. Fiber products of $\mathsf{R}_{\mathcal{H}}$ over $\mathsf{U}_{\mathcal{H}}$ are étale over $\mathsf{U}_{\mathcal{H}}$ as well. Hence they are normal and equal to the respective normalizations of fiber products of $\mathsf{R}_{\mathcal{H}}^{[0]}$. The necessary morphisms between the fiber products defining the groupoid relation of $\mathsf{R}_{\mathcal{H}}^{[0]}$ over $\mathsf{U}_{\mathcal{H}}^{[0]}$ extend to the normalizations. Hence $\mathsf{R}_{\mathcal{H}}$ over $\mathsf{U}_{\mathcal{H}}$ is a groupoid as well.

Since $\mathsf{R}_{\mathcal{H}}^{[0]}$ is *finite* over $\mathsf{U}_{\mathcal{H}}^{[0]} \underset{\mathsf{S}_0}{\times} \mathsf{U}_{\mathcal{H}}^{[0]}$ by the property of the Isom functor of abelian schemes (see the proof of condition 2 in Section 2.3.4), so is the normalization $\mathsf{R}_{\mathcal{H}}$ over $\mathsf{U}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{U}_{\mathcal{H}}$. □

Definition 6.3.3.15. *The separated algebraic stack $\mathsf{U}_{\mathcal{H}}/\mathsf{R}_{\mathcal{H}}$ (see Proposition A.7.1.1 and Definition A.7.1.3) will be denoted by $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ (or $\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$, to emphasize the compatible choice $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of cone decompositions).*

Corollary 6.3.3.16. *The degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ and the stratification over $\mathsf{U}_{\mathcal{H}}$ descend to $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, which we again denote by the same notation. This realizes $\mathsf{M}_{\mathcal{H}}$ as the $[0]$-stratum in the stratification, and identifies the restriction of $(G, \lambda, i, \alpha_{\mathcal{H}})$ to $\mathsf{M}_{\mathcal{H}}$ with the tautological tuple over $\mathsf{M}_{\mathcal{H}}$.*

Proof. The degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{U}_{\mathcal{H}}$ has a descent datum over $\mathsf{R}_{\mathcal{H}}$ defined by $\mathsf{h} : (\mathrm{pr}_1^*(G, \lambda, i, \alpha_{\mathcal{H}}))_{\mathsf{R}_{\mathcal{H}}} \xrightarrow{\sim} (\mathrm{pr}_2^*(G, \lambda, i, \alpha_{\mathcal{H}}))_{\mathsf{R}_{\mathcal{H}}}$. Hence the degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ descends to a degenerating family over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$. The proof of Proposition 6.3.3.13 shows that the two finite étale projections from $\mathsf{R}_{\mathcal{H}}$ to $\mathsf{U}_{\mathcal{H}}$ respect the stratification on $\mathsf{U}_{\mathcal{H}}$. Hence the stratification on $\mathsf{U}_{\mathcal{H}}$ descends to $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ as well. The remaining claims follow from the definitions. □

Proposition 6.3.3.17. *$\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is proper over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$.*

Proof. The proof is based on the valuative criterion for algebraic stacks of finite type (see Theorem A.7.2.12 and Remark A.7.2.13): Let V be a discrete valuation ring with fraction field K and an algebraically closed residue field k. Let $\mathrm{Spec}(K) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ be a morphism. Since the separateness of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is known by Proposition 6.3.3.13 and Corollary 6.3.3.14, and since $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is of finite type over S_0 because $\mathsf{U}_{\mathcal{H}}$ is, it suffices to show the existence of an extension $\mathrm{Spec}(V) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ up to replacing V with a finite extension.

Since $\mathsf{M}_{\mathcal{H}}$ is open and dense in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, it suffices (see Remark A.7.2.13) to treat the special case where the morphism $\mathrm{Spec}(K) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ has image in $\mathsf{M}_{\mathcal{H}}$. The morphism $\mathrm{Spec}(K) \to \mathsf{M}_{\mathcal{H}}$ (the latter being an algebraic stack) gives an object $(G_K, \lambda_K, i_K, \alpha_{\mathcal{H},K})$ of $\mathsf{M}_{\mathcal{H}}(K)$. By Theorem 3.3.2.4, up to replacing V with a finite extension, we may assume that the abelian scheme G_K extends to a semi-abelian scheme $G_V \to \mathrm{Spec}(V)$. By Proposition 3.3.1.5, the polarization λ_K (resp. the $\mathcal{O}$-endomorphism structure i_K of (G_K, λ_K)) extends to a homomorphism $\lambda_V : G_V \to G_V^\vee$ (resp. a homomorphism $i_V : \mathcal{O} \to \mathrm{End}_V(G_V)$). Since the base scheme V is a discrete valuation ring with algebraically closed residue field k, by Proposition 6.3.3.11 and Remark 6.3.3.12, there always exists a morphism $\mathrm{Spec}(V) \to \mathsf{U}_{\mathcal{H}}$ such that $(G_V, \lambda_V, i_V, \alpha_{\mathcal{H},K}) \to \mathrm{Spec}(V)$ is the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{U}_{\mathcal{H}}$ under this morphism. This establishes (up to replacing V with a finite extension) the existence of a morphism $\mathrm{Spec}(V) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ extending the given $\mathrm{Spec}(K) \to \mathsf{M}_{\mathcal{H}}$, as desired. □

6.4 Arithmetic Toroidal Compactifications

With the same setting as in Definition 1.4.1.2, assume moreover that $(L, \langle \cdot, \cdot \rangle, h)$ satisfies Condition 1.4.3.10 (see Remark 1.4.3.9). Let us also adopt Convention 6.2.1.1 for simplicity.

6.4.1 Main Results on Toroidal Compactifications

Theorem 6.4.1.1 (arithmetic toroidal compactifications)**.** *To each compatible choice $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of admissible smooth rational polyhedral cone decomposition data as in Definition 6.3.3.4, there is associated an algebraic stack $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} = \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ proper and smooth over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, which is an algebraic space when $\mathcal{H}$ is neat (see Definition 1.4.1.8), containing $\mathsf{M}_{\mathcal{H}}$ as an open dense subalgebraic stack, together with a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ (see Definition 5.3.2.1) such that we have the following:*

1. *The restriction $(G_{\mathsf{M}_{\mathcal{H}}}, \lambda_{\mathsf{M}_{\mathcal{H}}}, i_{\mathsf{M}_{\mathcal{H}}}, \alpha_{\mathcal{H}})$ of the degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ to $\mathsf{M}_{\mathcal{H}}$ is the tautological tuple over $\mathsf{M}_{\mathcal{H}}$.*

2. *$\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ has a stratification by locally closed subalgebraic stacks*
$$\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]},$$
with $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ running through a complete set of equivalence classes of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ (as in Definition 6.2.6.1) with $\sigma \subset \mathbf{P}_{\Phi_{\mathcal{H}}}^+$ and $\sigma \in \Sigma_{\Phi_{\mathcal{H}}} \in \Sigma$. (Here $\mathsf{Z}_{\mathcal{H}}$ is suppressed in the notation by Convention 5.4.2.5. The notation "$\coprod$" only means a set-theoretic disjoint union. The algebro-geometric structure is still that of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$.)

   In this stratification, the $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$-stratum $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}$ lies in the closure of the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ if and only if $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ is a face of $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$ as in Definition 6.3.2.14 (see also Remark 6.3.2.15).

   The $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is smooth over S_0 and isomorphic to the support of the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ for any representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ of $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$. The formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (before quotient by $\Gamma_{\Phi_{\mathcal{H}},\sigma}$) admits a canonical structure as the completion of an affine toroidal embedding $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ (along its σ-stratum $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$) of a torus torsor $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over an abelian scheme torsor $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over a finite étale cover $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$ of the algebraic stack $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ (separated, smooth, and of finite over S_0) in Definition 5.4.2.6. (Note that $\mathsf{Z}_{\mathcal{H}}$ and the isomorphism class of $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ depend only on the class $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$, but not on the choice of the representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$.)

   In particular, $\mathsf{M}_{\mathcal{H}}$ is an open dense stratum in this stratification.

3. *The complement of $\mathsf{M}_{\mathcal{H}}$ in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ (with its reduced structure) is a relative Cartier divisor $\mathsf{D}_{\infty,\mathcal{H}}$ with normal crossings, such that each irreducible component of a stratum of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} - \mathsf{M}_{\mathcal{H}}$ is open dense in an intersection of irreducible components of $\mathsf{D}_{\infty,\mathcal{H}}$ (including possible self-intersections). When $\mathcal{H}$ is neat, the irreducible components of $\mathsf{D}_{\infty,\mathcal{H}}$ have no self-intersections (cf. Condition 6.2.5.25, Remark 6.2.5.26, and [42, Ch. IV, Rem. 5.8(a)]).*

4. *The extended Kodaira–Spencer morphism (see Definition 4.6.3.44) for* $G \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *induces an isomorphism*
$$\underline{\mathrm{KS}}_{(G,\lambda,i)/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \overset{\sim}{\to} \Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log \mathsf{D}_{\infty,\mathcal{H}}]$$
(see Definition 6.3.1). Here the sheaf $\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log \mathsf{D}_{\infty,\mathcal{H}}]$ *is the sheaf of modules of log* 1*-differentials over* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$*, with respect to the relative Cartier divisor* $\mathsf{D}_{\infty,\mathcal{H}}$ *with normal crossings.*

5. *The formal completion* $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ *of the algebraic stack* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *along its* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ *is canonically isomorphic to the formal algebraic stack* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *for any representative* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ *of* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$*. (To form the formal completion along a given locally closed stratum, we first remove the other strata appearing in the closure of this stratum from the total space, and then form the formal completion of the remaining space along this stratum.)*

This isomorphism respects stratifications in the sense that, given any étale (i.e., formally étale and of finite type; see [59, I, 10.13.3]) morphism $\mathrm{Spf}(R,I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *inducing a morphism* $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$*, the stratification of* $\mathrm{Spec}(R)$ *inherited from* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *(see Proposition 6.3.1.6 and Definition 6.3.2.16) makes the induced morphism* $\mathrm{Spec}(R) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *a strata-preserving morphism.*

The pullback of the degenerating family $(G,\lambda,i,\alpha_{\mathcal{H}})$ *over* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *to* $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ *is the Mumford family* $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}})$ *over* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *(see Definition 6.2.5.28) after we identify the bases using the isomorphism.*

6. *Let* S *be an irreducible noetherian normal scheme over* S_0*, and suppose that we have a degenerating family* $(G^\dagger,\lambda^\dagger,i^\dagger,\alpha^\dagger_{\mathcal{H}})$ *of type* $\mathsf{M}_{\mathcal{H}}$ *over* S *as in Definition 5.3.2.1. Then* $(G^\dagger,\lambda^\dagger,i^\dagger,\alpha^\dagger_{\mathcal{H}}) \to S$ *is the pullback of* $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *via a (necessarily unique) morphism* $S \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *(over* S_0*) if and only if the following condition is satisfied:*

Consider any dominant morphism $\mathrm{Spec}(V) \to S$ *centered at a geometric point* $\bar{s}$ *of* S*, where* V *is a complete discrete valuation ring with quotient field* K*, algebraically closed residue field* k*, and discrete valuation* υ*. Let* $(G^\ddagger,\lambda^\ddagger,i^\ddagger,\alpha^\ddagger_{\mathcal{H}}) \to \mathrm{Spec}(V)$ *be the pullback of* $(G^\dagger,\lambda^\dagger,i^\dagger,\alpha^\dagger_{\mathcal{H}}) \to S$*. This pullback family defines an object of* $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(V)$*, which corresponds to a tuple* $(A^\ddagger,\lambda_{A^\ddagger},i_{A^\ddagger},\underline{X}^\ddagger,\underline{Y}^\ddagger,\phi^\ddagger,c^\ddagger,(c^\vee)^\ddagger,\tau^\ddagger,[(\alpha^\natural_{\mathcal{H}})^\ddagger])$ *in* $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}(V)$ *under Theorem 5.3.1.19. Then we have a fully symplectic-liftable admissible filtration* $\mathsf{Z}^\ddagger_{\mathcal{H}}$ *determined by* $[(\alpha^\natural_{\mathcal{H}})^\ddagger]$*. Moreover, the étale sheaves* $\underline{X}^\ddagger$ *and* $\underline{Y}^\ddagger$ *are necessarily constant, because the base ring* V *is strict local. Hence it makes sense to say we also have a uniquely determined torus argument* $\Phi^\ddagger_{\mathcal{H}}$ *at level* $\mathcal{H}$ *for* $\mathsf{Z}^\ddagger_{\mathcal{H}}$*.*

On the other hand, we have objects $\underline{\Phi}_{\mathcal{H}}(G^\ddagger)$*,* $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G^\ddagger)}$*, and* $\underline{B}(G^\ddagger)$ *(see Construction 6.3.1.1) that define objects* $\Phi^\ddagger_{\mathcal{H}}$*,* $\mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}}$*, and in particular,* $B^\ddagger : \mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}} \to \mathrm{Inv}(V)$ *over the special fiber. Then* $\upsilon \circ B^\ddagger : \mathbf{S}_{\Phi^\ddagger_{\mathcal{H}}} \to \mathbb{Z}$ *defines an element of* $\mathbf{S}^\vee_{\Phi^\ddagger_{\mathcal{H}}}$ *where* $\upsilon : \mathrm{Inv}(V) \to \mathbb{Z}$ *is the homomorphism induced by the discrete valuation of* V*.*

Then the condition is that, for each $\mathrm{Spec}(V) \to S$ *as above, and for some (and hence every) choice of* $\delta^\ddagger_{\mathcal{H}}$ *making* $(\mathsf{Z}^\ddagger_{\mathcal{H}},\Phi^\ddagger_{\mathcal{H}},\delta^\ddagger_{\mathcal{H}})$ *a representative of a cusp label, there is a cone* $\sigma^\ddagger$ *in the cone decomposition* $\Sigma_{\Phi^\ddagger_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi^\ddagger_{\mathcal{H}}}$ *(given by the choice of* Σ*; cf. Definition 6.3.3.4) such that* $\overline{\sigma}^\ddagger$ *contains all* $\upsilon \circ B^\ddagger$ *obtained in this way (for the same given geometric point* $\bar{s}$*).*

This is essentially a restatement of Proposition 6.3.3.11, which characterizes $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *uniquely for each choice of* Σ*.*

Proof. Let $\mathsf{U}_{\mathcal{H}}$ and $\mathsf{R}_{\mathcal{H}}$ be constructed (noncanonically) as in Section 6.3.3, and let us take the separated algebraic stack $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} = \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to be the groupoid quotient $\mathsf{U}_{\mathcal{H}}/\mathsf{R}_{\mathcal{H}}$ (see Proposition A.7.1.1 and Definition A.7.1.3) as in Definition 6.3.3.15.

Statements 1 and 2 follow from Corollaries 6.3.3.14 and 6.3.3.16, and Proposition 6.3.3.17. Statements 3 and 4 are étale local in nature, and hence are inherited from the étale presentation $\mathsf{U}_{\mathcal{H}}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ (with descent data over $\mathsf{R}_{\mathcal{H}}$) by construction.

Let us prove statement 6 by explaining why it is essentially a restatement of Proposition 6.3.3.11. Suppose we have a degenerating family $(G^\dagger,\lambda^\dagger,i^\dagger,\alpha^\dagger_{\mathcal{H}}) \to S$ as in the statement. Then there is an open dense subscheme S_1 of S such that the restriction of the family defines an object of $\mathsf{M}_{\mathcal{H}}$. Hence we have a morphism $S_1 \to \mathsf{M}_{\mathcal{H}}$ by the universal property of $\mathsf{M}_{\mathcal{H}}$. The question is whether this morphism extends to a morphism $S \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. If this is the case, then by Proposition 3.3.1.5, $(G^\dagger,\lambda^\dagger,i^\dagger,\alpha^\dagger_{\mathcal{H}}) \to S$ is isomorphic to the pullback of the tautological tuple $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ under this morphism, and the condition in the statement certainly holds. Conversely, assume that the condition holds. Since all objects involved are locally of finite presentation over S, we can apply Theorem 1.3.1.3 and assume that S is excellent. Since extendability is a local question (because $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is separated over S_0), we can work with $\mathsf{U}_{\mathcal{H}}$ and apply Proposition 6.3.3.11 (to pullbacks of $(G^\dagger,\lambda^\dagger,i^\dagger,\alpha^\dagger_{\mathcal{H}}) \to S$ to completions of local rings of S).

Next, let us prove statement 5. By statement 6 we have just proved, we know that there is a unique morphism from $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. (More precisely, we apply statement 6 to an étale covering of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ by affine formal schemes with descent data.) This induces a canonical morphism $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma} \to (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$. For an inverse morphism, note that by construction there is a canonical morphism from the formal completion of $\mathsf{U}_{\mathcal{H}}$ along its $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$-stratum to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. Since this canonical morphism is determined by the degeneration data associated with the pullback of the tautological tuple $(G,\lambda,i,\alpha_{\mathcal{H}})$ to the completion, and since the two pullbacks of the tautological tuple to $\mathsf{R}_{\mathcal{H}}$ are tautologically isomorphic by definition of $\mathsf{R}_{\mathcal{H}}$, we see that the morphism from the completion of $\mathsf{U}_{\mathcal{H}}$ along its $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$-stratum to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ descends to a canonical morphism $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. Then it follows from the constructions that these two canonical morphisms are inverses of each other.

Finally, suppose that $\mathcal{H}$ is neat. Then $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is a formal algebraic space by Lemma 6.2.5.27 because we have assumed in Definition 6.3.3.4 that each cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ in Σ satisfies Condition 6.2.5.25. By statements 2 and 5, it follows that points of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ have no nontrivial automorphisms. Since the diagonal 1-morphism $\Delta_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is finite (by Corollary 6.3.3.14), it must be a closed immersion. Hence $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is an algebraic space when $\mathcal{H}$ is neat, as desired. □

Following [42, Ch. IV, 5.10], we may deduce from Theorem 6.4.1.1 the following formal consequence:

Corollary 6.4.1.2. *All geometric fibers of* $\mathsf{M}_{\mathcal{H}} \to \mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ *have the same number of connected components.*

Proof. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is proper and smooth over S_0, all geometric fibers of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathsf{S}_0$ have the same number of connected components, by the analogue of Zariski's connectedness theorem in [36, Thm. 4.17]. (The precise assumption we need is that $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathsf{S}_0$ is proper flat and has geometrically normal fibers.) Then the corollary follows because $\mathsf{M}_{\mathcal{H}}$ is *fiberwise* dense in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ (by 2 of Theorem 6.4.1.1). □

Remark 6.4.1.3. As an application of Corollary 6.4.1.2, the connected components of the geometric fiber over a finite field can be matched with the connected components of the complex fiber, the latter of which can be understood using the complex uniformization by unions of Hermitian symmetric spaces (with the help of some Galois cohomology computations; see [76]).

6.4.2 Towers of Toroidal Compactifications

Definition 6.4.2.1. *Let* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *be two representatives of cusp labels. Let* σ *(resp.* σ'*) be any nondegenerate rational polyhedral cone in* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *(resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*). We say that the triple* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *is a* ***refinement*** *of the triple* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *if* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *are equivalent as in Definition 5.4.2.4, and if for one (and hence every) pair of isomorphisms* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ *that identifies* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *with* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$*, the cone* σ *is contained in a* $\Gamma_{\Phi'_{\mathcal{H}}}$*-translation of the cone* σ' *under the identification between* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *and* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ *defined by* (γ_X, γ_Y)*. In this case, we say that the triple* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *is a refinement of the triple* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *under the pair of isomorphisms* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$*.*

Definition 6.4.2.2. *Let* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ *and* $\Sigma' = \{\Sigma'_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ *be two compatible choices of admissible smooth rational polyhedral cone decomposition data for* $\mathsf{M}_{\mathcal{H}}$*. We say that* Σ *is a* ***refinement*** *of* Σ' *if the triple* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ *is a refinement of the triple* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma'_{\Phi_{\mathcal{H}}})$*, as in Definition 6.2.6.3, for* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *running through all representatives of cusp labels.*

Proposition 6.4.2.3. *Suppose* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ *and* $\Sigma' = \{\Sigma'_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ *are two compatible choices of admissible smooth rational polyhedral cone decomposition data for* $\mathsf{M}_{\mathcal{H}}$ *such that* Σ *is a refinement of* Σ' *as in Definition 6.4.2.2. Then the family* $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *is the pullback of the family* $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ *via a (unique) surjection* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$*. This surjection is proper, and is an isomorphism over* $\mathsf{M}_{\mathcal{H}}$*. (This is a global algebraized version of Proposition 6.2.6.7.)*

Moreover, the surjection maps the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ *of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ *to the* $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$*-stratum* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]}$ *of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ *if and only if there are representatives* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *of* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ *and* $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$*, respectively, such that* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *is a refinement of* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma')$ *as in Definition 6.4.2.1.*

Proof. The first statement follows from 6 of Theorem 6.4.1.1: The pullbacks of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to étale local charts of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ admit unique morphisms to $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$. These morphisms patch uniquely, and hence descend to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. Therefore there exists a unique morphism from $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, in the sense of relative schemes. The restriction of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to $\mathsf{M}_{\mathcal{H}}$ is the tautological tuple over $\mathsf{M}_{\mathcal{H}}$, which is mapped isomorphically to the restriction of $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ to $\mathsf{M}_{\mathcal{H}}$. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is proper and $\mathsf{M}_{\mathcal{H}}$ is dense in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, the morphism $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ is surjective and proper, as desired.

The second statement can be verified along the completions of strict local rings, which then follows from Proposition 6.3.3.11. □

Remark 6.4.2.4. Proposition 6.4.2.3 shows that there is a *tower* of toroidal compactifications labeled by the compatible choices Σ of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$. The (directed) partial order of refinements on the set of all possible Σ is translated into the (directed) partial order on the toroidal compactifications given by surjections that are proper and are isomorphisms over $\mathsf{M}_{\mathcal{H}}$. When properly interpreted, this tower can be viewed as a canonical compactification of $\mathsf{M}_{\mathcal{H}}$, without emphasizing the choice of Σ.

Let us also vary the level $\mathcal{H}$:

Definition 6.4.2.5. *Suppose* $\mathcal{H}$ *and* $\mathcal{H}'$ *are two open compact subgroups of* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *such that* $\mathcal{H}' \subset \mathcal{H}$*. Suppose* $(\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *and* $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *are representatives of cusp labels at levels* $\mathcal{H}'$ *and* $\mathcal{H}$*, respectively, where* $\Phi_{\mathcal{H}'} = (Y, X, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$ *and* $\Phi'_{\mathcal{H}} = (Y', X', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$*. We say that* $(\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *is a* ***lifting*** *of* $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *if the* $\mathcal{H}$*-orbit determined by* $(\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *in its natural sense (by Convention 5.3.1.15) is equivalent to* $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *as in Definition 5.4.2.4. In other words, the* $\mathcal{H}$*-orbit determined by* $\mathsf{Z}_{\mathcal{H}'}$ *is identical to* $\mathsf{Z}'_{\mathcal{H}}$*, and there exists a pair of isomorphisms* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ *which satisfies* $\phi = \gamma_X \phi' \gamma_Y$ *and identifies* $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ *with the* $\mathcal{H}$*-orbit determined by* $(\varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$*. In this case we say that the triple* $(\mathsf{Z}_{\mathcal{H}'}, \Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *is a lifting of the triple* $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *under the pair of isomorphisms* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$*. For simplicity, we shall suppress* $\mathsf{Z}_{\mathcal{H}'}$ *and* $\mathsf{Z}'_{\mathcal{H}}$ *from the notation as in Convention 5.4.2.5.*

Definition 6.4.2.6. *Suppose* $\mathcal{H}$ *and* $\mathcal{H}'$ *are two open compact subgroups of* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *such that* $\mathcal{H}' \subset \mathcal{H}$*. Suppose* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *are representatives of cusp labels at levels* $\mathcal{H}'$ *and* $\mathcal{H}$*, respectively, where* $\Phi_{\mathcal{H}'} = (Y, X, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$ *and* $\Phi'_{\mathcal{H}} = (Y', X', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$*. Let* σ *(resp.* σ'*) be any nondegenerate rational polyhedral cone in* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *(resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*). We say that the triple* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)$ *is a* ***refinement*** *of the triple* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *if* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *is a lifting of* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *as in Definition 6.4.2.5, and if for one (and hence every) pair of isomorphisms* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ *that identifies* $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ *with the* $\mathcal{H}$*-orbit determined by* $(\varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$*, the cone* σ *is contained in a* $\Gamma_{\Phi'_{\mathcal{H}}}$*-translation of the cone* σ' *under the identification between* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *and* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ *defined by* (γ_X, γ_Y)*. In this case, we say that the triple* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)$ *is a refinement of the triple* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *under the pair of isomorphisms* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$*.*

Definition 6.4.2.7. *Suppose* $\mathcal{H}$ *and* $\mathcal{H}'$ *are two open compact subgroups of* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *such that* $\mathcal{H}' \subset \mathcal{H}$*. Suppose* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *are representatives of cusp labels at levels* $\mathcal{H}'$ *and* $\mathcal{H}$*, respectively, where* $\Phi_{\mathcal{H}'} = (Y, X, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$ *and* $\Phi'_{\mathcal{H}} = (Y', X', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$*. Suppose* $\Sigma_{\Phi_{\mathcal{H}'}}$ *(resp.* $\Sigma'_{\Phi'_{\mathcal{H}}}$*) is a* $\Gamma_{\Phi_{\mathcal{H}'}}$*-admissible (resp.* $\Gamma_{\Phi'_{\mathcal{H}}}$*-admissible) smooth rational polyhedral cone decomposition of* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *(resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*). We say that the triple* $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \Sigma_{\Phi_{\mathcal{H}'}})$ *is a* ***refinement*** *of the triple*

$(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\Sigma'_{\Phi'_{\mathcal{H}}})$ *if* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})$ *is a lifting of* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ *as in Definition 6.4.2.5, and if for one (and hence every) pair of isomorphisms* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ *that identifies* $(\varphi'_{-2,\mathcal{H}},\varphi'_{0,\mathcal{H}})$ *with the* $\mathcal{H}$*-orbit determined by* $(\varphi_{-2,\mathcal{H}'},\varphi_{0,\mathcal{H}'})$*, the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}'}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *is a refinement of the cone decomposition* $\Sigma'_{\Phi'_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*. In this case, we say that the triple* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\Sigma_{\Phi_{\mathcal{H}'}})$ *is a refinement of the triple* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\Sigma'_{\Phi'_{\mathcal{H}}})$ *under the pair of isomorphisms* $(\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$*.*

Definition 6.4.2.8. *Suppose* $\mathcal{H}$ *and* $\mathcal{H}'$ *are two open compact subgroups of* $\mathrm{G}(\hat{\mathbb{Z}}^\square)$ *such that* $\mathcal{H}' \subset \mathcal{H}$*. Let* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ *and* $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}}}\}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ *be compatible choices of admissible smooth rational polyhedral cone decomposition data for* $\mathsf{M}_{\mathcal{H}'}$ *and* $\mathsf{M}_{\mathcal{H}}$*, respectively. We say that* Σ *is a* ***refinement*** *of* Σ' *if, for each* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})$ *refining* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$*, both running through all possible pairs representing cusp labels at levels* $\mathcal{H}'$ *and* $\mathcal{H}$*, respectively, the triple* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\Sigma_{\Phi_{\mathcal{H}'}})$ *is a refinement of* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\Sigma'_{\Phi'_{\mathcal{H}}})$ *as in Definition 6.4.2.7.*

Proposition 6.4.2.9. *Suppose* $\mathcal{H}$ *and* $\mathcal{H}'$ *are two open compact subgroups of* $\mathrm{G}(\hat{\mathbb{Z}}^\square)$ *such that* $\mathcal{H}' \subset \mathcal{H}$*. Suppose* $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ *and* $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}}}\}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ *are two compatible choices of admissible smooth rational polyhedral cone decomposition data for* $\mathsf{M}_{\mathcal{H}'}$ *and* $\mathsf{M}_{\mathcal{H}}$*, respectively, such that* Σ *is a refinement of* Σ' *as in Definition 6.4.2.8. Then the family* $(G,\lambda,i,\alpha_{\mathcal{H}'}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ *is the pullback of the family* $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ *via a (unique) surjection* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$*. This surjection is proper and is the canonical surjection* $\mathsf{M}_{\mathcal{H}'} \to \mathsf{M}_{\mathcal{H}}$ *over* $\mathsf{M}_{\mathcal{H}}$ *determined by the* $\mathcal{H}$*-orbit of the level-*$\mathcal{H}'$ *structure* $\alpha_{\mathcal{H}'}$*.*

Moreover, the surjection maps the $[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\sigma)]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\sigma)]}$ *of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ *to the* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$*-stratum* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}$ *of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ *if and only if there are representatives* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\sigma)$ *and* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ *of* $[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\sigma)]$ *and* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$*, respectively, such that* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\sigma)$ *is a refinement of* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ *as in Definition 6.4.2.6.*

Proof. The first statement again follows from 6 of Theorem 6.4.1.1: Consider the restriction of $(G,\lambda,i,\alpha_{\mathcal{H}'}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ to $\mathsf{M}_{\mathcal{H}'}$. Then there is a canonical surjection from $\mathsf{M}_{\mathcal{H}'}$ to $\mathsf{M}_{\mathcal{H}}$ identifying the $\mathcal{H}$-orbit of $\alpha_{\mathcal{H}'}$ with the pullback of $\alpha_{\mathcal{H}}$, as in the statement of the proposition. By abuse of notation, let us denote the $\mathcal{H}$-orbit of $\alpha_{\mathcal{H}'}$ by $\alpha_{\mathcal{H}}$. By 6 of Theorem 6.4.1.1, the pullbacks of $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ to étale local charts of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ admit unique morphisms to $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$. These morphisms patch uniquely, and hence descend to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$. Therefore there exists a unique morphism from $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ to $(G,\lambda,i,\alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ in the sense of relative schemes. By construction, the restriction of the morphism to $\mathsf{M}_{\mathcal{H}'}$ is the canonical morphism from $\mathsf{M}_{\mathcal{H}'}$ to $\mathsf{M}_{\mathcal{H}}$ determined by the $\mathcal{H}$-orbit of $\alpha_{\mathcal{H}'}$. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ is proper and $\mathsf{M}_{\mathcal{H}}$ is dense in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, the morphism $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ is surjective and proper, as desired.

The second statement can be verified along the completions of strict local rings, which then follows from Proposition 6.3.3.11. $\square$

Remark 6.4.2.10. Proposition 6.4.2.3 is now a special case of Proposition 6.4.2.9.

6.4.3 Hecke Actions on Toroidal Compactifications

Suppose we have an element $g \in \mathrm{G}(\mathbb{A}^{\infty,\square})$, and suppose we have two open compact subgroups $\mathcal{H}$ and $\mathcal{H}'$ of $\mathrm{G}(\hat{\mathbb{Z}}^\square)$ such that $\mathcal{H}' \subset g\mathcal{H}g^{-1}$. Then the *Hecke action* defined by g induces a canonical surjection from $\mathsf{M}_{\mathcal{H}'}$ to $\mathsf{M}_{\mathcal{H}}$. More precisely, it is determined as follows: Let us consider the tautological tuple $(G_{\mathsf{M}_{\mathcal{H}'}},\lambda_{\mathsf{M}_{\mathcal{H}'}},i_{\mathsf{M}_{\mathcal{H}'}},\alpha_{\mathcal{H}'})$ over $\mathsf{M}_{\mathcal{H}'}$, and consider the rational version $(G_{\mathsf{M}_{\mathcal{H}'}},\lambda_{\mathsf{M}_{\mathcal{H}'}},i_{\mathsf{M}_{\mathcal{H}'}},[\hat{\alpha}]_{\mathcal{H}'})$ over $\mathsf{M}_{\mathcal{H}'}$ (see Construction 1.3.8.4 and Definition 1.3.8.7). Let us denote by $(G'_{\mathsf{M}_{\mathcal{H}'}},\lambda'_{\mathsf{M}_{\mathcal{H}'}},i'_{\mathsf{M}_{\mathcal{H}'}},\alpha'_{\mathcal{H}})$ over $\mathsf{M}_{\mathcal{H}'}$ the tuple associated with $(G_{\mathsf{M}_{\mathcal{H}'}},\lambda_{\mathsf{M}_{\mathcal{H}'}},i_{\mathsf{M}_{\mathcal{H}'}},[\hat{\alpha}\circ g]_{\mathcal{H}})$ under Proposition 1.4.3.4, and call $(G'_{\mathsf{M}_{\mathcal{H}'}},\lambda'_{\mathsf{M}_{\mathcal{H}'}},i'_{\mathsf{M}_{\mathcal{H}'}},\alpha'_{\mathcal{H}})$ the *Hecke twist* of $(G_{\mathsf{M}_{\mathcal{H}'}},\lambda_{\mathsf{M}_{\mathcal{H}'}},i_{\mathsf{M}_{\mathcal{H}'}},\alpha_{\mathcal{H}'})$ by g over $\mathsf{M}_{\mathcal{H}'}$. This determines a canonical morphism $[g] : \mathsf{M}_{\mathcal{H}'} \to \mathsf{M}_{\mathcal{H}}$. This morphism $[g]$ is surjective, because it is surjective over all geometric points. We say that this is the natural surjection defined by the *Hecke action* of g on $\mathsf{M}^\square$ (see Remark 1.4.3.11; cf. Construction 5.4.3.1).

The argument at the beginning of Section 5.4.3 shows that $G'_{\mathsf{M}_{\mathcal{H}'}}$ can be realized as a quotient of $G_{\mathsf{M}_{\mathcal{H}'}}$ by a finite étale group scheme $K_{\mathsf{M}_{\mathcal{H}'}}$ over $\mathsf{M}_{\mathcal{H}'}$. Since the rank of this finite étale group scheme $K_{\mathsf{M}_{\mathcal{H}'}}$ over $\mathsf{M}_{\mathcal{H}'}$ is prime-to-$\square$, for the degenerating family $(G,\lambda,i,\alpha_{\mathcal{H}'})$ over each smooth toroidal compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ of $\mathsf{M}_{\mathcal{H}'}$ as in Theorem 6.4.1.1, the schematic closure K of $K_{\mathsf{M}_{\mathcal{H}'}}$ in G is a quasi-finite étale group scheme over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$. By Lemma 3.4.3.1, we can form the quotient $G' := G/K$, with additional structures λ' and i' uniquely extending $\lambda'_{\mathsf{M}_{\mathcal{H}'}}$ and $i'_{\mathsf{M}_{\mathcal{H}'}}$, respectively, by Proposition 3.3.1.5. We call the tuple $(G',\lambda',i',\alpha'_{\mathcal{H}})$ the *Hecke twist* of $(G,\lambda,i,\alpha_{\mathcal{H}'})$ by g over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$. Therefore the tuple $(G'_{\mathsf{M}_{\mathcal{H}'}},\lambda'_{\mathsf{M}_{\mathcal{H}'}},i'_{\mathsf{M}_{\mathcal{H}'}},\alpha'_{\mathcal{H}})$ over $\mathsf{M}_{\mathcal{H}'}$ extends to a degenerating family $(G',\lambda',i',\alpha'_{\mathcal{H}})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$, and the question is whether the canonical surjection $[g] : \mathsf{M}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}$ extends to some canonical surjection $[g]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ when Σ and Σ' satisfy some suitable condition.

Definition 6.4.3.1. *Suppose we have an element* $g \in \mathrm{G}(\mathbb{A}^{\infty,\square})$*, and suppose we have two open compact subgroups* $\mathcal{H}$ *and* $\mathcal{H}'$ *of* $\mathrm{G}(\hat{\mathbb{Z}}^\square)$ *such that* $\mathcal{H}' \subset g\mathcal{H}g^{-1}$*. Suppose* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})$ *and* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ *are representatives of cusp labels at levels* $\mathcal{H}'$ *and* $\mathcal{H}$*, respectively, where* $\Phi_{\mathcal{H}'} = (Y,X,\phi,\varphi_{-2,\mathcal{H}'},\varphi_{0,\mathcal{H}'})$ *and* $\Phi'_{\mathcal{H}} = (Y',X',\phi',\varphi'_{-2,\mathcal{H}},\varphi'_{0,\mathcal{H}})$*. Let* σ *(resp.* σ'*) be any nondegenerate rational polyhedral cone in* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *(resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*). We say that the triple* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\sigma)$ *is a* g*-****refinement*** *of the triple* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ *if there is a pair of isomorphisms* $(f_X : X'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\square)} \xrightarrow{\sim} X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\square)}, f_Y : Y\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\square)} \xrightarrow{\sim} Y'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\square)})$ *defining a* g*-assignment* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'}) \to_g (\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ *as in Definition 5.4.3.9, and if for one such pair of isomorphisms* (f_X,f_Y)*, the cone* σ *is contained in a* $\Gamma_{\Phi'_{\mathcal{H}}}$*-translation of the cone* σ' *under the identification between* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *and* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ *defined by* (f_X,f_Y)*. In this case, we say that the triple* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\sigma)$ *is a* g*-refinement of the triple* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ *under the pair of isomorphisms* (f_X,f_Y)*.*

Definition 6.4.3.2. *Suppose we have an element* $g \in \mathrm{G}(\mathbb{A}^{\infty,\square})$ *and suppose we have two open compact subgroups* $\mathcal{H}$ *and* $\mathcal{H}'$ *of* $\mathrm{G}(\hat{\mathbb{Z}}^\square)$ *such that* $\mathcal{H}' \subset g\mathcal{H}g^{-1}$*. Suppose* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})$ *and* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ *are representatives of cusp labels at levels* $\mathcal{H}'$ *and* $\mathcal{H}$*, respectively, where* $\Phi_{\mathcal{H}'} = (Y,X,\phi,\varphi_{-2,\mathcal{H}'},\varphi_{0,\mathcal{H}'})$ *and* $\Phi'_{\mathcal{H}} = (Y',X',\phi',\varphi'_{-2,\mathcal{H}},\varphi'_{0,\mathcal{H}})$*. Suppose* $\Sigma_{\Phi_{\mathcal{H}'}}$ *(resp.* $\Sigma'_{\Phi'_{\mathcal{H}}}$*) is a* $\Gamma_{\Phi_{\mathcal{H}'}}$*-admissible (resp.* $\Gamma_{\Phi'_{\mathcal{H}}}$*-admissible) smooth rational polyhedral cone decomposition of* $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ *(resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*). We say that the triple* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\Sigma_{\Phi_{\mathcal{H}'}})$ *is a* g*-****refinement*** *of the triple* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\Sigma'_{\Phi'_{\mathcal{H}}})$ *if there is a pair of isomorphisms* $(f_X : X'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\square)} \xrightarrow{\sim} X\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\square)}, f_Y : Y\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\square)} \xrightarrow{\sim} Y'\underset{\mathbb{Z}}{\otimes}\mathbb{Z}_{(\square)})$ *defining a* g*-assignment* $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'}) \to_g (\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ *as in Definition 5.4.3.9, and if for one (and hence every) such pair of isomorphisms* (f_X,f_Y)*, the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}'}}$ *of*

$\mathbf{P}_{\Phi_{\mathcal{H}'}}$ is a refinement of the cone decomposition $\Sigma'_{\Phi'_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ under the identification between $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ and $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ defined by (f_X, f_Y). In this case, we say that the triple $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \Sigma_{\Phi_{\mathcal{H}'}})$ is a g-refinement of the triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma'_{\Phi'_{\mathcal{H}}})$ under the pair of isomorphisms (f_X, f_Y).

Definition 6.4.3.3. *Suppose we have an element $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$ and suppose we have two open compact subgroups $\mathcal{H}$ and $\mathcal{H}'$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ such that $\mathcal{H}' \subset g\mathcal{H}g^{-1}$. Let $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ and $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}}}\}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ be compatible choices of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}'}$ and $\mathsf{M}_{\mathcal{H}}$, respectively. We say that Σ is a g-**refinement** of Σ' if, for each g-assignment $(f_X, f_Y) : (\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}) \to_g (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ of a representative $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ of cusp label at level $\mathcal{H}$ to a representative $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ of cusp label at level $\mathcal{H}'$ as in Definition 5.4.3.9, the triple $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \Sigma_{\Phi_{\mathcal{H}'}})$ is a g-refinement of $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma'_{\Phi'_{\mathcal{H}}})$ (under the pair of isomorphisms (f_X, f_Y)) as in Definition 6.4.3.2.*

Proposition 6.4.3.4. *Suppose we have an element $g \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$, and suppose we have two open compact subgroups $\mathcal{H}$ and $\mathcal{H}'$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ such that $\mathcal{H}' \subset g\mathcal{H}g^{-1}$. Suppose $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ and $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}}}\}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ are two compatible choices of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}'}$ and $\mathsf{M}_{\mathcal{H}}$, respectively, such that Σ is a g-refinement of Σ' as in Definition 6.4.3.3. Then the Hecke twist of the family $(G, \lambda, i, \alpha_{\mathcal{H}'}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ by g is the pullback of the family $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ via a (unique) surjection $[g]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$. This surjection is proper, and its restriction to $\mathsf{M}_{\mathcal{H}}$ is the canonical surjection $[g] : \mathsf{M}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}$ over $\mathsf{M}_{\mathcal{H}}$ defined by the **Hecke action** of g on $\mathsf{M}^{\Box}$ (see Remark 1.4.3.11; cf. Construction 5.4.3.1).*

Moreover, the surjection maps the $[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'},\sigma)]}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ to the $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$-stratum $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ if and only if there are representatives $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ of $[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)]$ and $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$, respectively, such that $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'}, \sigma)$ is a g-refinement of $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ as in Definition 6.4.3.1.

Proof. The first statement follows from a combination of Proposition 5.4.3.8 and 6 of Theorem 6.4.1.1: Let $(G', \lambda', i', \alpha'_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ be the Hecke twist of $(G, \lambda, i, \alpha_{\mathcal{H}'}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ by g. Then the restriction of $(G', \lambda', i', \alpha'_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ to $\mathsf{M}_{\mathcal{H}'}$ determines the canonical surjection $[g] : \mathsf{M}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}$ as in the statement of the proposition. By Proposition 5.4.3.8 and 6 of Theorem 6.4.1.1, the pullbacks of $(G', \lambda', i', \alpha'_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ to étale local charts of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ admit unique morphisms to $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ by our assumption that Σ is a g-refinement of Σ'. (Concretely, the cones containing pairings of the form $\upsilon \circ B' : Y' \times X' \to \mathbb{Z}$ are carried to cones containing pairings of the form $\upsilon \circ B : Y \times X \to \mathbb{Z}$ under the identification between $\mathbf{P}_{\Phi_{\mathcal{H}'}}$ and $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ defined by $(f_X : X' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} X \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}, f_Y : Y \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Y' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)})$, when we have the objects as in the context of Definition 6.4.3.3.) These morphisms patch uniquely, and hence descend to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$. Therefore there exists a unique morphism $[g]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ extending $[g]$, which pulls $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ back to $(G', \lambda', i', \alpha'_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ is proper and $\mathsf{M}_{\mathcal{H}}$ is dense in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$, the morphism $[g]^{\mathrm{tor}}$ is surjective and proper, as desired.

The second statement can be verified along the completions of strict local rings, which then follows from Proposition 6.3.3.11. $\Box$

Remark 6.4.3.5. Propositions 6.4.2.3 and 6.4.2.9 are now special cases of Proposition 6.4.3.4.

Chapter 7

Algebraic Constructions of Minimal Compactifications

In this chapter we explain the construction of arithmetic toroidal compactifications and several other useful results as a by-product of the construction of arithmetic toroidal compactifications.

Although all results to be stated have their analytic analogues over the complex numbers, the analytic techniques in [17] and [16] do not carry over naively. We need the arithmetic toroidal compactifications and the positivity of Hodge invertible sheaves (to be reviewed in Section 7.2.1, based on the theory of theta constants) to establish the finite generation of certain natural sheaves of graded algebras. This should be considered as the main technical input of this chapter.

The main objective is to state and prove Theorem 7.2.4.1, with Proposition 7.1.2.14, Corollary 7.2.4.13, Proposition 7.2.5.1, and Theorem 7.3.3.4 as important by-products. Technical results worth noting are Propositions 7.2.1.1, 7.2.1.2, 7.2.2.3, 7.2.3.3, 7.2.3.7, 7.2.3.13, 7.2.4.3, and 7.3.1.4.

Throughout this chapter, we shall assume the same setting as in Section 6.4.

7.1 Automorphic Forms and Fourier–Jacobi Expansions

7.1.1 Automorphic Forms of Naive Parallel Weights

Let $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ be a compatible choice of admissible smooth rational polyhedral cone decomposition data as in Definition 6.3.3.4, and let $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ be the proper smooth algebraic stack associated with Σ as in Theorem 6.4.1.1. Let $(G, \lambda, i, \alpha_{\mathcal{H}})$ be the degenerating family described in Theorem 6.4.1.1. For ease of later exposition, we shall change our notation and denote them by $(G^{\mathrm{tor}}, \lambda^{\mathrm{tor}}, i^{\mathrm{tor}}, \alpha_{\mathcal{H}}^{\mathrm{tor}})$. Let $\omega^{\mathrm{tor}} := \omega_{G^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{G^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \cong \wedge^{\mathrm{top}} e^*_{G^{\mathrm{tor}}} \Omega^1_{G^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ be the *Hodge invertible sheaf*.

Definition 7.1.1.1. *Let M be a module over $\mathcal{O}_{F_0,(\Box)}$, and let $k \geq 0$ be an integer. An **(arithmetic) automorphic form** over $\mathsf{M}_{\mathcal{H}}$ of naive parallel weight k and with coefficients in M is an element of $\mathrm{AF}(k, M) := \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)$. For simplicity, when the context is clear, we shall call such an element an automorphic form of naive weight k.*

Remark 7.1.1.2. We say the weight is *naive* because, in general, this is not the right weight to use (for scalar-valued automorphic forms) in number-theoretic applications.

Remark 7.1.1.3. For most applications, it suffices to consider those M that are algebras over $\mathcal{O}_{F_0,(\Box)}$. However, the theory becomes most systematic if we allow M to be arbitrary modules: As a functor in M (as modules over $\mathcal{O}_{F_0,(\Box)}$), $\mathrm{AF}(k, -)$ is left exact and commutes with filtering direct limits for each $k \geq 0$.

To justify our definition,

Lemma 7.1.1.4. *Let Σ' be a compatible choice of admissible smooth rational polyhedral cone decomposition data such that Σ' is a refinement of Σ (see Definition 6.4.2.2), and let $p : \mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}} \twoheadrightarrow \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ be the surjection given by Proposition 6.4.2.3. Let $\mathcal{E}$ be a quasi-coherent sheaf over $\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ of the form $\mathcal{E} = \mathcal{E}_0 \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M$, where $\mathcal{E}_0$ is a locally free sheaf over $\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$, and where M is a module over $\mathcal{O}_{F_0,(\Box)}$. Then the canonical morphism $H^i(\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}, \mathcal{E}) \to H^i(\mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}}, p^*\mathcal{E})$ is an isomorphism for each $i \geq 0$.*

Proof. Let us assume that $\mathcal{E} = \mathcal{E}_0 \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M$ as in the statement of the lemma. Since M is a direct limit of finitely generated $\mathcal{O}_{F_0,(\Box)}$-modules, and since taking cohomology and tensor products commutes with direct limits, we may assume that M is a finitely generated $\mathcal{O}_{F_0,(\Box)}$-module. Since $\mathcal{O}_{F_0,(\Box)}$ is either a field or a Dedekind domain, the (finitely generated) torsion submodule M_{tor} of M can be written as a (possibly zero) direct sum of modules of the form $\mathcal{O}_{F_0,(\Box)}/\mathfrak{n}$, where $\mathfrak{n}$ is a nonzero ideal of $\mathcal{O}_{F_0,(\Box)}$ (see [23, Ch. VII, §4, 10, Prop. 23]). On the other hand, the (finitely generated) torsion-free quotient M_{free} of M is automatically projective, and hence M is (noncanonically) isomorphic to $M_{\mathrm{tor}} \oplus M_{\mathrm{free}}$. Since M_{free} is automatically flat over $\mathcal{O}_{F_0,(\Box)}$, it is a limit of its free submodules. By the same fact that taking cohomology and tensor products commutes with direct limits, we may assume that it is free. In any case, we are reduced (by additivity of taking cohomology) to the case that M is of the form $\mathcal{O}_{F_0,(\Box)}/\mathfrak{n}$ for some (possibly zero) ideal $\mathfrak{n}$ of $\mathcal{O}_{F_0,(\Box)}$, and work over M after making the base change from $\mathcal{O}_{F_0,(\Box)}$.

According to the construction of arithmetic toroidal compactifications, this morphism p can be étale locally identified with a proper morphism between toric varieties, which is equivariant under the action of the same torus. By the arguments in [69, Ch. I, §3, especially p. 44, Cor. 2], this shows that $R^i p_* \mathscr{O}_{\mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}}} = 0$ for all $i > 0$ and that the canonical morphism $\mathscr{O}_{\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}} \to p_* \mathscr{O}_{\mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}}}$ is an isomorphism; both statements remain valid after making the base change from $\mathcal{O}_{F_0,(\Box)}$ to $\mathcal{O}_{F_0,(\Box)}/\mathfrak{n}$

for some ideal $\mathfrak{n}$. Since $\mathcal{E}_0$ is locally free over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, the lemma follows from the projection formula [59, 0_{I}, 5.4.10.1], as desired. □

Lemma 7.1.1.5. *Definition 7.1.1.1 is independent of the choice of* Σ *in the construction of* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$.

Proof. Since every two compatible choices of admissible smooth rational polyhedral cone decomposition data have a common refinement, it suffices to consider the case that we have a Σ' refining Σ as in Lemma 7.1.1.4. Let $\omega^{\mathrm{tor}}_{\mathcal{H},\Sigma'} := \omega_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}}$ and $\omega^{\mathrm{tor}}_{\mathcal{H},\Sigma} := \omega_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}}$ denote, respectively, the Hodge invertible sheaves over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. By construction, $\omega^{\mathrm{tor}}_{\mathcal{H},\Sigma'} \cong p^*\omega^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. Applying Lemma 7.1.1.4 to $\mathcal{E} := (\omega^{\mathrm{tor}}_{\mathcal{H},\Sigma})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M$, we see that Definition 7.1.1.1 is independent of the choice of Σ, as desired. □

7.1.2 Fourier–Jacobi Expansions

Let us take any *nonempty* stratum of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ labeled by some $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$, with some choice of a representative $(\Phi_{\mathcal{H}} = (X,Y,\phi,\varphi_{-2,\mathcal{H}},\varphi_{0,\mathcal{H}}),\delta_{\mathcal{H}},\sigma)$. According to 5 of Theorem 6.4.1.1, the formal completion $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ along the $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is isomorphic to the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, and the pullback of $(G^{\mathrm{tor}},\lambda^{\mathrm{tor}},i^{\mathrm{tor}},\alpha^{\mathrm{tor}}_{\mathcal{H}})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is the Mumford family $({}^{\heartsuit}G, {}^{\heartsuit}\lambda, {}^{\heartsuit}i, {}^{\heartsuit}\alpha_{\mathcal{H}})$ over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$. Moreover, the pullback of $\omega^{\mathrm{tor}} = \omega_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ is ${}^{\heartsuit}\omega := \omega_{{}^{\heartsuit}G/(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma})} := \wedge^{\mathrm{top}}\underline{\mathrm{Lie}}^{\vee}_{{}^{\heartsuit}G/(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma})}$. Note that there is a morphism from the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ to the abelian scheme torsor $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over the finite étale cover $\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}$ of $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ defined in Definition 5.4.2.6. Let $(A,\lambda_A,i_A,\varphi_{-1,\mathcal{H}})$ denote the tautological tuple over $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. Let $\omega_A := \omega_{A/\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}} := \wedge^{\mathrm{top}}\underline{\mathrm{Lie}}^{\vee}_{A/\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}$ denote the Hodge invertible sheaf over $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. We shall often use the same notation for the pullbacks of ω_A over other bases. Let T denote the torus over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ with character group X. Then we have $T \cong \underline{\mathrm{Hom}}_{\mathsf{S}_0}(X,\mathbf{G}_{\mathrm{m},\mathsf{S}_0})$, $\mathrm{Lie}_{T/\mathsf{S}_0} \cong \underline{\mathrm{Hom}}_{\mathsf{S}_0}(X,\mathscr{O}_{\mathsf{S}_0})$, $\underline{\mathrm{Lie}}^{\vee}_{T/\mathsf{S}_0} \cong X \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{S}_0}$, and $\omega_T := \omega_{T/\mathsf{S}_0} := \wedge^{\mathrm{top}}\underline{\mathrm{Lie}}^{\vee}_{T/\mathsf{S}_0} \cong (\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{S}_0}$. Similarly to the case of ω_A, we shall often use the same notation for the pullbacks of ω_T over other bases.

Lemma 7.1.2.1. *There is a canonical isomorphism* ${}^{\heartsuit}\omega \cong (\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A$ *over the formal algebraic stack* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$.

Proof. By étale descent, it suffices to verify this statement over each étale (i.e., formally étale and of finite type; see [59, I, 10.13.3]) morphism $S_{\mathrm{for}} = \mathrm{Spf}(R,I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, such that R and I satisfy the setting as in Section 5.2.1. Let us also denote the pullback of various objects by the same notation. Then it makes sense to talk about the Raynaud extension ${}^{\heartsuit}G^{\natural}$ associated with ${}^{\heartsuit}G$ over $S = \mathrm{Spec}(R)$, so that we have a canonical isomorphism ${}^{\heartsuit}G^{\natural}_{\mathrm{for}} \cong {}^{\heartsuit}G_{\mathrm{for}}$ along the I-adic completion S_{for} of S. Since $\underline{\mathrm{Lie}}^{\vee}_{{}^{\heartsuit}G_{\mathrm{for}}/S_{\mathrm{for}}} \cong \underline{\mathrm{Lie}}^{\vee}_{{}^{\heartsuit}G^{\natural}_{\mathrm{for}}/S_{\mathrm{for}}}$ over S_{for}, there is a canonical isomorphism $\underline{\mathrm{Lie}}^{\vee}_{{}^{\heartsuit}G/S} \cong \underline{\mathrm{Lie}}^{\vee}_{{}^{\heartsuit}G^{\natural}/S}$ (see Theorem 2.3.1.2). As a result, there are canonical isomorphisms $\omega_{{}^{\heartsuit}G/S} \cong \omega_{{}^{\heartsuit}G^{\natural}/S} := \wedge^{\mathrm{top}}\underline{\mathrm{Lie}}^{\vee}_{{}^{\heartsuit}G^{\natural}/S} \cong \wedge^{\mathrm{top}}\underline{\mathrm{Lie}}^{\vee}_{T/S} \underset{\mathscr{O}_S}{\otimes} \wedge^{\mathrm{top}}\underline{\mathrm{Lie}}^{\vee}_{A/S} \cong \omega_T \underset{\mathscr{O}_S}{\otimes} \omega_A \cong (\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A$, as desired. □

Recall that $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is by construction the formal completion of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ along its σ-stratum, and that $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ is by definition $\underline{\mathrm{Spec}}_{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}\Big(\underset{\ell\in\sigma^{\vee}}{\oplus}\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)\Big)$ (cf. Proposition 6.2.4.7). As described in Remark 6.2.5.13, the σ-stratum $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ is defined by the sheaf of ideals $\mathscr{I}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \cong \underset{\ell\in\sigma_0^{\vee}}{\oplus}\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ in $\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)} \cong \underset{\ell\in\sigma^{\vee}}{\oplus}\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$. Therefore we may write symbolically

$$\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}} \cong \underset{\ell\in\sigma^{\vee}}{\hat{\oplus}}\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$$

and

$$\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}} \cong \Big(\underset{\ell\in\sigma^{\vee}}{\hat{\oplus}}\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)\Big)^{\Gamma_{\Phi_{\mathcal{H}},\sigma}}.$$

Let us denote the structural morphism $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ by $\mathrm{p}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, which is proper and smooth because it is an abelian scheme torsor over the finite étale cover $\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}$ of $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. For simplicity of notation,

Definition 7.1.2.2. $\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} := (\mathrm{p}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_*(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))$.

Now it makes sense to consider the following composition of canonical morphisms:

$$\begin{aligned}
&\mathrm{AF}(k,M) = \Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}},(\omega^{\mathrm{tor}})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)\\
&\to \Gamma((\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}},(\omega^{\mathrm{tor}})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)\\
&\cong \Gamma(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma},{}^{\heartsuit}\omega^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)\\
&\to \Big[\prod_{\ell\in\sigma^{\vee}}\Gamma(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}},\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} ((\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A)^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)\Big]^{\Gamma_{\Phi_{\mathcal{H}},\sigma}}\\
&\cong \Big[\prod_{\ell\in\sigma^{\vee}}\Gamma(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}},\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}}{\otimes} ((\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A)^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)\Big]^{\Gamma_{\Phi_{\mathcal{H}},\sigma}}.
\end{aligned} \tag{7.1.2.3}$$

Definition 7.1.2.4. *The above composition (7.1.2.3) is called the* ***Fourier–Jacobi morphism*** *along* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$, *which we denote by* $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. *The image of an element* $f \in \mathrm{AF}(k,M)$ *has a natural expansion*

$$\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}(f) = \sum_{\ell\in\sigma^{\vee}}\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}(f)$$

where the sum can be infinite and where each $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}(f)$ *lies in*

$$\mathrm{FJC}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M) := \Gamma(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}},\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}}}{\otimes} ((\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A)^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M).$$

The expansion $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}(f)$ *is called the* ***Fourier–Jacobi expansion*** *of* f *along* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$, *with* ***Fourier–Jacobi coefficients*** $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}(f)$ *of each degree* $\ell \in \sigma^{\vee}$.

Remark 7.1.2.5. These are generalizations of the q-expansions and Fourier–Jacobi expansions for modular, Hilbert modular, or Siegel modular forms, with which the readers might be familiar.

Suppose that we have two cones σ_1 and σ_2 in $\Sigma_{\Phi_{\mathcal{H}}}$ such that $\sigma_1 \subset \overline{\sigma}_2$ and such that they are both in $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$. In this case we have $\sigma_2^\vee \subset \sigma_1^\vee$, and therefore an open embedding $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma_1) \hookrightarrow \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma_2)$, equivariant with respect to the torus action of $E_{\Phi_{\mathcal{H}}}$. This induces a canonical morphism $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_1}/\Gamma_{\Phi_{\mathcal{H}},\sigma_1} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_2}/\Gamma_{\Phi_{\mathcal{H}},\sigma_2}$ under which the Mumford family $({}^\heartsuit G, {}^\heartsuit\lambda, {}^\heartsuit i, {}^\heartsuit\alpha_{\mathcal{H}})$ over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_1}/\Gamma_{\Phi_{\mathcal{H}},\sigma_1}$ is the pullback of the Mumford family over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_2}/\Gamma_{\Phi_{\mathcal{H}},\sigma_2}$. This shows that, for $f \in \mathrm{AF}(k,M)$, the expansion $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_2}(f)$ is mapped to the expansion $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_1}(f)$ under the canonical morphism

$$\begin{aligned}&\Gamma(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_2}/\Gamma_{\Phi_{\mathcal{H}},\sigma_1}, {}^\heartsuit\omega^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)\\ &\to \Gamma(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_1}/\Gamma_{\Phi_{\mathcal{H}},\sigma_2}, {}^\heartsuit\omega^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M),\end{aligned}$$

which maps $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_2}(f)$ to $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_1}(f)$ as long as both of them are defined. In particular, we must have $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma_2}(f) = 0$ for those $\ell \in \sigma_2^\vee - \sigma_1^\vee$. That is, we only need the Fourier–Jacobi coefficients of degrees $\ell \in \sigma_1^\vee$. As every two cones in $\Sigma_{\Phi_{\mathcal{H}}}$ that are both in $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$ can be related by a sequence of inclusions (in either direction) of (closures of) cones, we see that each of the Fourier–Jacobi morphisms (defined by cones in $\Sigma_{\Phi_{\mathcal{H}}}$) has degrees (of nonzero coefficients) supported on

$$\Sigma^\vee_{\Phi_{\mathcal{H}}} := \bigcap_{\sigma\in\Sigma_{\Phi_{\mathcal{H}}}} \sigma^\vee.$$

Here it is harmless to also take those $\sigma \in \Sigma_{\Phi_{\mathcal{H}}}$ that are in $\mathbf{P}_{\Phi_{\mathcal{H}}}$ but might not be in $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$, because they are necessarily faces of some cones in $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$. Since $\Sigma_{\Phi_{\mathcal{H}}}$ is a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$, which means in particular, that the union of the cones in $\Sigma_{\Phi_{\mathcal{H}}}$ is $\mathbf{P}_{\Phi_{\mathcal{H}}}$, we see that $\Sigma^\vee_{\Phi_{\mathcal{H}}}$ is simply $\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}$, which is independent of the choice of $\Sigma_{\Phi_{\mathcal{H}}}$. Thus we have a canonical morphism

$$\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} : \mathrm{AF}(k,M) \to \mathrm{FJE}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M) := \prod_{\ell\in\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}} \mathrm{FJC}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M),$$

which is invariant under each of the groups $\Gamma_{\Phi_{\mathcal{H}},\sigma}$. It is also invariant under the full group $\Gamma_{\Phi_{\mathcal{H}}}$ because the Mumford family over each $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is the pullback of the Mumford family over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, and because the latter is invariant under the action of $\Gamma_{\Phi_{\mathcal{H}}}$. Since the pullback objects are naturally invariant under $\Gamma_{\Phi_{\mathcal{H}}}$, we may redefine $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ as

$$\begin{aligned}\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} : \mathrm{AF}(k,M) &\to \mathrm{FJE}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M)^{\Gamma_{\Phi_{\mathcal{H}}}}\\ &:= \Big[\prod_{\ell\in\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}} \mathrm{FJC}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M)\Big]^{\Gamma_{\Phi_{\mathcal{H}}}}.\end{aligned} \tag{7.1.2.6}$$

Definition 7.1.2.7. *The above morphism (7.1.2.6) is called the* ***Fourier–Jacobi morphism*** *along $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$, which we denote by $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. The image of an element $f \in \mathrm{AF}(k,M)$ has a natural expansion*

$$\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f) = \sum_{\ell\in\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}} \mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f),$$

where each $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$ lies in $\mathrm{FJC}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M)$. The expansion $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$ is called the ***Fourier–Jacobi expansion*** *of f along $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$, with* ***Fourier–Jacobi coefficients*** *$\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$ of each degree $\ell \in \mathbf{P}^\vee_{\Phi_{\mathcal{H}}}$.*

Let us record the above argument as follows:

Proposition 7.1.2.8. *The Fourier–Jacobi morphism $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ can be computed by any $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ as in Definition 7.1.2.4. The definition is independent of the σ we use.*

Moreover,

Proposition 7.1.2.9. *The Fourier–Jacobi morphism $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is independent of the $\Gamma_{\Phi_{\mathcal{H}}}$-admissible smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ we use.*

Proof. This is a consequence of Lemma 7.1.1.5, Proposition 6.4.2.3, and the construction of $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ using Mumford families. □

Definition 7.1.2.10. *The* ***constant term*** *of a Fourier–Jacobi expansion $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$ of an element $f \in \mathrm{AF}(k,M)$ is the Fourier–Jacobi coefficient $\mathrm{FJ}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f) \in \mathrm{FJC}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M)$ in degree zero.*

Lemma 7.1.2.11. *For every element ℓ of the semisubgroup $\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}$ of $\mathbf{S}_{\Phi_{\mathcal{H}}}$, there exists an integer $N \geq 1$ such that $N\ell$ is a finite sum $\sum_{1\leq i\leq k} [y_i \otimes \phi(y_i)]$ for some elements $y_i \in Y$.*

Proof. It suffices to check that $(\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}) \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is contained in the $\mathbb{Q}_{>0}$-span of elements of the form $[y \otimes \phi(y)]$ for some $y \in Y$. But this is equivalent to the definition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ as the cone of positive semidefinite Hermitian pairings in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ whose radicals are admissible (and hence rational) subspaces. □

Corollary 7.1.2.12. *The set $\mathbf{P}^\vee_{\Phi_{\mathcal{H}}} - \{0\}$ is the semisubgroup of $\mathbf{S}_{\Phi_{\mathcal{H}}}$ consisting of elements pairing positively with elements in $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$.*

Proposition 7.1.2.13. *The value of each element $f \in \mathrm{AF}(k,M)$ along the $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is determined by its constant term $\mathrm{FJ}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$, which is a $\Gamma_{\Phi_{\mathcal{H}}}$-invariant element in $\mathrm{FJC}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,M)$. In particular, since $\underline{\mathrm{FJ}}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \cong (\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})_* \mathscr{O}_{\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}}$ and $\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}/\Gamma_{\Phi_{\mathcal{H}}} \cong \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$, the value of f is* ***constant along the fibers*** *of the structural morphism $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. We say in this case that it depends only on the* ***abelian part*** *of $(G^{\mathrm{tor}}, \lambda^{\mathrm{tor}}, i^{\mathrm{tor}}, \alpha^{\mathrm{tor}}_{\mathcal{H}})$ over $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$.*

Proof. As described in Remark 6.2.5.13, the σ-stratum $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ of $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ is defined by the sheaf of ideals $\mathscr{I}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \cong \bigoplus_{\ell\in\sigma_0^\vee} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ in $\mathscr{O}_{\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)} \cong \bigoplus_{\ell\in\sigma^\vee} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$. In particular, the Fourier–Jacobi coefficient $\mathrm{FJ}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$ vanishes along $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ if $\ell \in \sigma_0^\vee$. By Corollary 7.1.2.12, every element ℓ of $\mathbf{P}^\vee_{\Phi_{\mathcal{H}}} - \{0\}$ lies in $\sigma_0^\vee$ (because $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$). This shows that $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f) = \mathrm{FJ}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$ along $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$. Since the value of f along $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is determined, in particular, by its pullback to the formal completion $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$, it is also determined by $\mathrm{FJ}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$. In other words, it is determined by its ($\Gamma_{\Phi_{\mathcal{H}}}$-invariant) constant term $\mathrm{FJ}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$, as desired. □

Proposition 7.1.2.14. *Let $\{\mathsf{Z}_{[(\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}},\sigma^{(i)})]}\}_{i\in I}$ be a finite collection of strata of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ such that the union of members in the collection intersects all irreducible components of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. For each $i \in I$, let $(\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}},\sigma^{(i)})$ be some representative of $[(\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}},\sigma^{(i)})]$. Let M be an $\mathcal{O}_{F_0,(\Box)}$-module and let $k \geq 0$ be an integer. Let f be an automorphic form over $\mathsf{M}_{\mathcal{H}}$, of naive parallel weight k, with coefficients in M, and regular at infinity. Then the following are true:*

1. *If $\mathrm{FJ}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}}}(f) = \sum\limits_{\ell\in\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}} \mathrm{FJ}^{(\ell)}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}}}(f) = 0$ for all $i \in I$, then $f = 0$.*

2. (***Fourier–Jacobi expansion principle***) *Suppose M_1 is an $\mathcal{O}_{F_0,(\Box)}$-submodule of M, and suppose the Fourier–Jacobi expansions $\mathrm{FJ}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}}}(f) \in \mathrm{FJE}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}}}(k,M)$ lie in the image of $\mathrm{FJE}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}}}(k,M_1) \hookrightarrow \mathrm{FJE}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}}}(k,M)$ for all $i \in I$. Then f lies in the image of $\mathrm{AF}(k,M_1) \hookrightarrow \mathrm{AF}(k,M)$.*

Proof. Note that the association of $\mathrm{AF}(k,-)$, $\mathrm{FJC}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,-)$ and $\mathrm{FJE}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(k,-)$ are left exact because they are defined by taking global sections of sheaves.

Let us prove the first statement. By the same reduction steps in the proof of Lemma 7.1.1.4, we may assume that M is an $\mathcal{O}_{F_0,(\Box)}$-algebra and work after making the base change from $\mathcal{O}_{F_0,(\Box)}$ to M. Since $\{\mathsf{Z}_{[(\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}},\sigma^{(i)})]}\}_{i\in I}$ intersects all irreducible components of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, the (finite) direct product of Fourier–Jacobi morphisms $\mathrm{FJ}_I := \prod\limits_{i\in I}\mathrm{FJ}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}}} = \prod\limits_{i\in I}\mathrm{FJ}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}},\sigma^{(i)}}$ is injective because each Fourier–Jacobi morphism $\mathrm{FJ}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}},\sigma^{(i)}}$ is defined by pullback of a global section of $(\omega^{\mathrm{tor}})^{\otimes k}$ to the completion along $\mathsf{Z}_{[(\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}},\sigma^{(i)})]}$, over which ω^{tor} is trivialized.

Let us prove the second statement. For simplicity of notation, let us define $\mathrm{FJE}_I(k,-) := \prod\limits_{i\in I}\mathrm{FJE}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}}}(k,-)$. Consider the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{AF}(k,M_1) \arrow[r] \arrow[d, hook, "\mathrm{FJ}_I"'] & \mathrm{AF}(k,M) \arrow[r] \arrow[d, hook, "\mathrm{FJ}_I"'] & \mathrm{AF}(k,M/M_1) \arrow[d, hook, "\mathrm{FJ}_I"'] \\ 0 \arrow[r] & \mathrm{FJE}_I(k,M_1) \arrow[r] & \mathrm{FJE}_I(k,M) \arrow[r] & \mathrm{FJE}_I(k,M/M_1) \end{tikzcd}$$

with exact rows. If $f \in \mathrm{AF}(k,M)$ and $\mathrm{FJ}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}}}(f)$ is in the image of $\mathrm{FJE}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}}}(k,M_1) \hookrightarrow \mathrm{FJE}_{\Phi^{(i)}_{\mathcal{H}},\delta^{(i)}_{\mathcal{H}}}(k,M)$ for all $i \in I$, then $\mathrm{FJ}_I(f) \in \mathrm{FJE}_I(k,M)$ is sent to zero in $\mathrm{FJE}_I(k,M/M_1)$. By injectivity of the morphism $\mathrm{FJ}_I : \mathrm{AF}(k,M/M_1) \to \mathrm{FJE}_I(k,M/M_1)$ and the commutativity of the diagram, this means f is sent to zero in $\mathrm{AF}(k,M/M_1)$. Then f lies in the image of $\mathrm{AF}(k,M_1) \hookrightarrow \mathrm{AF}(k,M)$, as desired. $\square$

7.2 Arithmetic Minimal Compactifications

7.2.1 Positivity of Hodge Invertible Sheaves

Proposition 7.2.1.1 ([42, Ch. V, Prop. 2.1])**.** *Let $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ be any (smooth) arithmetic toroidal compactification of $\mathsf{M}_{\mathcal{H}}$ as in Theorem 6.4.1.1, with a degenerating family $(G^{\mathrm{tor}},\lambda^{\mathrm{tor}},i^{\mathrm{tor}},\alpha^{\mathrm{tor}}_{\mathcal{H}})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ extending the tautological tuple $(G,\lambda,i,\alpha_{\mathcal{H}})$ over $\mathsf{M}_{\mathcal{H}}$. Consider the invertible sheaf $\omega^{\mathrm{tor}} := \omega_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} := \wedge^{\mathrm{top}}\,\underline{\mathrm{Lie}}^\vee_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \cong \wedge^{\mathrm{top}}\,e^*_{G^{\mathrm{tor}}}\Omega^1_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. Then there exists an integer $N_0 \geq 1$ such that $(\omega^{\mathrm{tor}})^{\otimes N_0}$ is generated by its global sections.*

Proof. This is a special case of [93, Ch. IX, Thm. 2.1] if we replace $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ with a normal excellent scheme. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is of finite type over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, its étale charts are excellent and normal. Hence the proposition follows. $\square$

Proposition 7.2.1.2 ([42, Ch. V, Prop. 2.2])**.** *Let C be a proper smooth irreducible curve over an algebraically closed field k. Let $f : G \to C$ be a semi-abelian scheme over C, and let $\omega_{G/C} := \wedge^{\mathrm{top}}\,\underline{\mathrm{Lie}}^\vee_{G/C} \cong \wedge^{\mathrm{top}}\,e^*_G\Omega^1_{G/C}$. Suppose that $\deg(\omega_{G/C}) \leq 0$. Then,*

1. *$\omega_{G/C}$ is a torsion bundle, that is, some positive power of it is trivial; hence $\deg(\omega_{G/C}) = 0$;*

2. *G is an extension of an isotrivial abelian scheme by a torus over C.*

Here an abelian scheme $A \to C$ is *isotrivial* if it becomes constant over a *finite étale* covering $\tilde{C}$ of C.

Proof. First let us note that if $A \underset{C}{\times} C'$ is constant for some proper smooth curve C' finite over C, then A is also isotrivial. To see this, take any $n \geq 3$ that is prime to $\mathrm{char}(k)$, and take $\tilde{C} := A[n]$, which is finite étale over C. Then $A \underset{C}{\times} \tilde{C} \to \tilde{C}$ is constant because $A \underset{C}{\times} \tilde{C} \underset{C}{\times} C' \to \tilde{C} \underset{C}{\times} C'$ is. Hence we may assume that C is projective.

By the theory of torus parts in Section 3.3.1, in particular Proposition 3.3.1.7 and Theorem 3.3.1.9, we may write $G \to C$ as an extension of a semi-abelian scheme G' by a torus H over C, such that G'_η is an abelian scheme over the generic point η of C. By replacing G with G', it suffices to treat the case when G_η is an abelian scheme.

By assumption, $\deg(\omega^{\otimes n}_{G/C}) \leq 0$ for every $n \geq 0$. In other words, all global sections of $\omega^{\otimes n}_{G/C}$ are constant, which means that they are either zero or nowhere zero. Combining with Proposition 7.2.1.1, this shows that $\deg(\omega^{\otimes n}_{G/C}) = 0$ for every $n \geq 0$.

By [93, Ch. XI, Thm. 4.5(b), (v bis)$\Longrightarrow$(iv)] and [93, Ch. X, Prop. 4.4, (i)$\Longleftrightarrow$(iii)], which uses implicitly the fact that *theta constants determine the moduli*, we see that G_η has potential good reduction everywhere, and that G is isotrivial over S. $\square$

7.2.2 Stein Factorizations and Finite Generation

In this section, we include several standard results that we will need for our main construction in Section 7.2.3 below. (For simplicity, we shall often omit subscripts for tensor products.)

Fix a noetherian base ring R and let $S = \mathrm{Spec}(R)$. Suppose W is a proper algebraic stack over S. Suppose $\mathcal{L}$ is an invertible sheaf over W such that there is an integer $N_0 \geq 1$ such that $\mathcal{L}^{\otimes N_0}$ is generated by its global sections. Then the global sections of $\mathcal{L}^{\otimes N_0}$ define a morphism

$$f : W \to \mathbb{P}^{r_0}_S$$

for some integer $r_0 \geq 0$. This is a proper morphism from an algebraic stack to a scheme, both of which are noetherian. The push-forward $f_*\mathscr{O}_W$ is a finite $\mathscr{O}_{\mathbb{P}^{r_0}_S}$-algebra, and it determines the *Stein factorization* (see [59, III-1, 4.3.3])

$$f^{\mathrm{st}} : W \to W^{\mathrm{st}} := \underline{\mathrm{Spec}}_{\mathscr{O}_{\mathbb{P}^{r_0}_S}}(f_*\mathscr{O}_W)$$

of f, such that the canonical morphism $\mathscr{O}_{W^{\mathrm{st}}} \to (f^{\mathrm{st}})_*\mathscr{O}_W$ is an isomorphism. In this case, the pullback of $\mathcal{O}(1)$ over $\mathbb{P}^{r_0}_S$ to W^{st} is an ample invertible sheaf, which we also denote by $\mathcal{O}(1)$ if there is no confusion, and its further pullback to W is the original $\mathcal{L}^{\otimes N_0}$.

Lemma 7.2.2.1. *For each locally free sheaf $\mathcal{E}$ of finite rank over W^{st}, we have a canonical isomorphism $f^{\mathrm{st}}_*(f^{\mathrm{st}})^*\mathcal{E} \cong \mathcal{E}$. As a by-product, we have $\Gamma(W, (f^{\mathrm{st}})^*\mathcal{E}) \cong \Gamma(W^{\mathrm{st}}, \mathcal{E})$.*

Proof. By the projection formula (see [59, 0_{I}, 5.4.10.1]), we have $f^{\mathrm{st}}_*(f^{\mathrm{st}})^*\mathcal{E} \cong (f^{\mathrm{st}}_*\mathscr{O}_W) \underset{\mathscr{O}_{W^{\mathrm{st}}}}{\otimes} \mathcal{E} \cong \mathcal{E}$. □

By [59, II, 4.6.3], we have an isomorphism

$$W^{\mathrm{st}} \cong \mathrm{Proj}\Big(\underset{k\geq 0}{\oplus} \Gamma(W^{\mathrm{st}}, \mathcal{O}(1)^{\otimes k})\Big).$$

By Lemma 7.2.2.1, this implies that we have

$$W^{\mathrm{st}} \cong \mathrm{Proj}\Big(\underset{k\geq 0}{\oplus} \Gamma(W, \mathcal{L}^{\otimes N_0 k})\Big).$$

It is desirable to explain the finite generation of algebras such as $\underset{k\geq 0}{\oplus} \Gamma(W, \mathcal{L}^{\otimes N_0 k})$ over R (as they appear in the above construction of Proj). The fundamental tool is *Serre's vanishing theorem*:

Theorem 7.2.2.2 (see [59, III-1, 2.2.1])**.** *Let Z be a proper scheme over $S = \mathrm{Spec}(R)$, where R is a noetherian ring. Then an invertible sheaf $\mathcal{M}$ is ample over W if and only if, for each coherent sheaf $\mathcal{E}$ over W, there is an integer k_0 (depending on $\mathcal{E}$) such that $H^i(Z, \mathcal{E} \otimes \mathcal{M}^{\otimes k}) = 0$ for all $i > 0$ and $k \geq k_0$.*

Proposition 7.2.2.3 (cf. [84, Exer. 1.2.22])**.** *Let Z be a scheme projective and flat over $S = \mathrm{Spec}(R)$, where R is a noetherian ring, and let $\mathcal{M}$ be an ample invertible sheaf over Z. For coherent sheaves $\mathcal{E}$ and $\mathcal{F}$ over Z such that at least one of them is flat over S, there is an integer k_0 (depending on $\mathcal{E}$ and $\mathcal{F}$) such that the cup product morphism*

$$\Gamma(Z, \mathcal{E} \otimes \mathcal{M}^{\otimes a}) \otimes \Gamma(Z, \mathcal{F} \otimes \mathcal{M}^{\otimes b}) \to \Gamma(Z, \mathcal{E} \otimes \mathcal{F} \otimes \mathcal{M}^{\otimes (a+b)})$$

*is **surjective** for all $a, b \geq k_0$.*

Proof. Let $S := \mathrm{Spec}(R)$. Consider the diagonal embedding $\Delta : Z \to Z \underset{S}{\times} Z$ and denote its image by $\Delta(Z)$. Let $\mathscr{I}$ be the sheaf of ideals fitting into the exact sequence

$$0 \to \mathscr{I} \to \mathrm{pr}_1^*\mathcal{E} \otimes \mathrm{pr}_2^*\mathcal{F} \to \mathscr{O}_{\Delta(Z)} \otimes \mathrm{pr}_1^*\mathcal{E} \otimes \mathrm{pr}_2^*\mathcal{F} \to 0$$

over $Z \underset{S}{\times} Z$. If we tensor the whole sequence with $\mathrm{pr}_1^*(\mathcal{M}^{\otimes a}) \otimes \mathrm{pr}_2^*(\mathcal{M}^{\otimes b})$, and take cohomology over $Z \underset{S}{\times} Z$, then we obtain (by the *Künneth formula* and the flatness of at least one of $\mathcal{E}$ and $\mathcal{F}$ over S; see [59, III-2, 6.7.8]) the exact sequence

$$\begin{aligned}\Gamma(Z, \mathcal{E} \otimes \mathcal{M}^{\otimes a}) \otimes \Gamma(Z, \mathcal{F} \otimes \mathcal{M}^{\otimes b}) &\to \Gamma(Z, \mathcal{E} \otimes \mathcal{F} \otimes \mathcal{M}^{\otimes (a+b)}) \\ &\to H^1(Z \underset{S}{\times} Z, \mathscr{I} \otimes \mathrm{pr}_1^*(\mathcal{M}^{\otimes a}) \otimes \mathrm{pr}_2^*(\mathcal{M}^{\otimes b})).\end{aligned}$$

The point is the vanishing of the last term. Let us denote the ample invertible sheaf $\mathrm{pr}_1^*\mathcal{M} \otimes \mathrm{pr}_2^*\mathcal{M}$ over $Z \underset{S}{\times} Z$ by $\mathcal{O}(1)$, and its mth tensor power by $\mathcal{O}(m)$. Using [59, III-1, 2.2.2 (iv)], there is a resolution

$$\cdots \to \mathcal{O}(-m_i)^{\oplus d_i} \to \cdots \to \mathcal{O}(-m_1)^{\oplus d_1} \to \mathcal{O}(-m_0)^{\oplus d_0} \to \mathscr{I} \to 0$$

over $Z \underset{S}{\times} Z$. Therefore the question is whether there is an integer k_0 such that

$$H^i(Z \underset{S}{\times} Z, \mathrm{pr}_1^*(\mathcal{M}^{\otimes(a-m_{i-1})}) \otimes \mathrm{pr}_2^*(\mathcal{M}^{\otimes(b-m_{i-1})})) = 0$$

for all $i > 0$ and $a, b \geq k_0$. It suffices to verify this for $0 < i \leq \dim(Z \underset{S}{\times} Z)$, which involves only finitely many terms. Using the Künneth formula again, this reduces the question to the existence of some integer k_0 such that

$$H^j(Z, \mathcal{M}^{\otimes a} \otimes \mathcal{M}^{\otimes -m_{i-1}}) \otimes H^{i-j}(Z, \mathcal{M}^{\otimes b} \otimes \mathcal{M}^{\otimes -m_{i-1}}) = 0$$

for $a, b \geq k_0$, which has a positive answer by Theorem 7.2.2.2. □

Corollary 7.2.2.4 (cf. [84, Exer. 2.1.30])**.** *Let Z be a scheme projective and flat over $S = \mathrm{Spec}(R)$, where R is a noetherian ring, and let $\mathcal{M}$ be an ample invertible sheaf over Z. Then the algebra $\underset{k\geq 0}{\oplus} \Gamma(Z, \mathcal{M}^{\otimes k})$ is finitely generated over R.*

Proof. Apply Proposition 7.2.2.3 with $\mathcal{E} = \mathcal{F} = \mathscr{O}_Z$. □

Corollary 7.2.2.5. *Let Z be a scheme projective and flat over $S = \mathrm{Spec}(R)$, where R is a noetherian ring, and let $\mathcal{E}$ be a coherent sheaf over Z. Then the module $\underset{k\geq 0}{\oplus} \Gamma(Z, \mathcal{E} \otimes \mathcal{M}^{\otimes k})$ is finitely generated over the algebra $\underset{k\geq 0}{\oplus} \Gamma(Z, \mathcal{M}^{\otimes k})$.*

Proof. Apply Proposition 7.2.2.3 with $\mathcal{F} = \mathscr{O}_Z$. □

Corollary 7.2.2.6. *Let us return to the context of W and $\mathcal{L}$ above. Suppose moreover that W^{st} is flat over S. Then the algebra $\underset{k\geq 0}{\oplus} \Gamma(W, \mathcal{L}^{\otimes N_1 k})$ is finitely generated over R for every integer $N_1 \geq 1$.*

Proof. For the purpose of proving this corollary, we may replace N_0 with its multiple (and replacing $f : Z \to \mathbb{P}^{r_0}_S$, f^{st}, Z^{st}, etc. accordingly) and assume that it is a multiple of N_1. Let $\mathcal{E} := f^{\mathrm{st}}_*\Big(\overset{(N_0/N_1)-1}{\underset{k=0}{\oplus}} \mathcal{L}^{\otimes N_1 k}\Big)$, which is a coherent sheaf over W^{st}. Applying Corollary 7.2.2.5 to $Z := W^{\mathrm{st}}$, $\mathcal{M} := \mathcal{O}(1)$, and $\mathcal{E}$ just defined above, we see that $\underset{k\geq 0}{\oplus} \Gamma(W, \mathcal{L}^{\otimes N_1 k}) \cong \underset{k\geq 0}{\oplus} \Gamma(W^{\mathrm{st}}, \mathcal{E} \otimes \mathcal{O}(1)^{\otimes k})$ is a finitely generated module over $\underset{k\geq 0}{\oplus} \Gamma(W, \mathcal{L}^{\otimes N_0 k}) \cong \underset{k\geq 0}{\oplus} \Gamma(W^{\mathrm{st}}, \mathcal{O}(1)^{\otimes k})$, the latter being finitely generated over R by Corollary 7.2.2.4. □

7.2.3 Main Construction of Minimal Compactification

With the same setting as in Section 7.1, let $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ be any (smooth) arithmetic toroidal compactification of $\mathsf{M}_{\mathcal{H}}$ as in Theorem 6.4.1.1, with a degenerating family $(G^{\mathrm{tor}}, \lambda^{\mathrm{tor}}, i^{\mathrm{tor}}, \alpha^{\mathrm{tor}}_{\mathcal{H}})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ extending the tautological tuple $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{M}_{\mathcal{H}}$. Let $\omega^{\mathrm{tor}} := \omega_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \cong \wedge^{\mathrm{top}} e^*_{G^{\mathrm{tor}}}\Omega^1_{G^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ be the invertible sheaf over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ which extends the invertible sheaf $\omega := \omega_{G/\mathsf{M}_{\mathcal{H}}} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{G/\mathsf{M}_{\mathcal{H}}} \cong \wedge^{\mathrm{top}} e^*_G\Omega^1_{G/\mathsf{M}_{\mathcal{H}}}$ over $\mathsf{M}_{\mathcal{H}}$ naturally. According to Proposition 7.2.1.1, there is an integer $N_0 \geq 1$ such that $(\omega^{\mathrm{tor}})^{\otimes N_0}$ is generated by its global sections. Let us fix a choice of such an integer N_0.

As in Section 7.2.2, the global sections of $(\omega^{\mathrm{tor}})^{\otimes N_0}$ define a morphism $\int_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathbb{P}_{\mathsf{S}_0}^{r_0}$ to some projective r_0-space $\mathbb{P}_{\mathsf{S}_0}^{r_0}$ for some integer $r_0 \geq 0$, together with a *Stein factorization*

$$\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min} := \underline{\mathrm{Spec}}_{\mathscr{O}_{\mathbb{P}_{\mathsf{S}_0}^{r_0}}}(\textstyle\int_{\mathcal{H},*} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})$$

of $\int_{\mathcal{H}}$, such that the canonical morphism $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\min}} \to \oint_{\mathcal{H},*} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ is an isomorphism. The induced morphism $\bar{\int}_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\min} \to \mathbb{P}_{\mathsf{S}_0}^{r_0}$ is finite, and $\mathsf{M}_{\mathcal{H}}^{\min}$ is projective and of finite type over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. In this case, we have an isomorphism

$$\mathsf{M}_{\mathcal{H}}^{\min} \cong \mathrm{Proj}\Big(\bigoplus_{k\geq 0} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes N_0 k})\Big).$$

Note that this is independent of the choice of $N_0 \geq 1$ because we have a canonical isomorphism (by [59, II, 2.4.7])

$$\mathrm{Proj}\Big(\bigoplus_{k\geq 0} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes N_0 k})\Big) \cong \mathrm{Proj}\Big(\bigoplus_{k\geq 0} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k})\Big)$$

for every $N_0 \geq 1$. Moreover, the right-hand side is independent of the choice of the cone decomposition Σ by Lemma 7.1.1.5.

According to [59, III-1, 4.3.1, 4.3.3, 4.3.4], with its natural generalization to the context of algebraic stacks, we see that the first factor $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ has nonempty connected geometric fibers. Since $(\omega^{\mathrm{tor}})^{\otimes N_0}$ is the pullback of $\mathcal{O}(1)$ (of $\mathbb{P}_{\mathsf{S}_0}^{r_0}$) to $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, the restriction of $(\omega^{\mathrm{tor}})^{\otimes N_0}$ to each fiber of $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ is trivial. Therefore we may apply Proposition 7.2.1.2 to morphisms of proper smooth irreducible curves to the geometric fibers of $\oint_{\mathcal{H}}$. Since these geometric fibers are all connected, the isomorphism class of the abelian part of G^{tor} is constant on each of the fibers. In particular, if a geometric fiber of $\oint_{\mathcal{H}}$ meets $\mathsf{M}_{\mathcal{H}}$, then it has only one closed point.

Lemma 7.2.3.1. *Let $f : Z_1 \twoheadrightarrow Z_2$ be a quasi-compact surjection from a locally noetherian algebraic stack to a locally noetherian scheme, such that f induces dominant morphisms from irreducible components of Z_1 to irreducible components of Z_2, and such that the canonical morphism $\mathscr{O}_{Z_2} \to f_* \mathscr{O}_{Z_1}$ is an isomorphism. Suppose Z_1 is normal. Then Z_2 is also normal.*

Proof. Since $\mathscr{O}_{Z_2} \xrightarrow{\sim} f_*\mathscr{O}_{Z_1}$, the local rings of Z_2 are integral domains. Take $\tilde{Z}_2$ to be the normalization of Z_2. By the universal property of $\tilde{Z}_2$ and the normality of Z_1, the surjection $f : Z_1 \twoheadrightarrow Z_2$ factors as a composition of surjections $Z_1 \overset{\tilde{f}}{\twoheadrightarrow} \tilde{Z}_2 \overset{\bar{f}}{\twoheadrightarrow} Z_2$, corresponding to a composition of canonical injections $\mathscr{O}_{Z_2} \hookrightarrow \bar{f}_*\mathscr{O}_{\tilde{Z}_2} \hookrightarrow \bar{f}_*\tilde{f}_*\mathscr{O}_{Z_1} \cong f_*\mathscr{O}_{Z_1}$, the latter composition being an isomorphism by assumption. This forces $\mathscr{O}_{Z_2} \xrightarrow{\sim} \bar{f}_*\mathscr{O}_{\tilde{Z}_2}$, or rather $\tilde{Z}_2 \xrightarrow{\sim} Z_2$, which implies that Z_2 is normal. □

Remark 7.2.3.2. In our context, one can also refer to [22, §6.7, Lem. 2].

Proposition 7.2.3.3. $\mathsf{M}_{\mathcal{H}}^{\min}$ *is normal.*

Proof. $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is normal because it is smooth over the normal base scheme $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ (see [59, IV-4, 17.5.7]). Moreover, the canonical morphism $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\min}} \to \oint_{\mathcal{H},*}\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ is an isomorphism by construction. Hence the proposition follows from Lemma 7.2.3.1. □

Since $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ is a localization of the rings of integers of a number field, $\mathsf{M}_{\mathcal{H}}^{\min} \to \mathsf{S}_0$ is flat because $\mathsf{M}_{\mathcal{H}}^{\min}$ is normal and all its maximal points (see [60, 0, 2.1.2]) are of characteristic zero. By Corollary 7.2.2.6, this shows (for reassurance) that the algebra $\bigoplus_{k\geq 0} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k})$ is finitely generated over $\mathcal{O}_{F_0,(\Box)}$.

Recall (see Section A.7.5) that each algebraic stack Z has an associated *coarse moduli space* $[Z]$, which is an algebraic space together with a canonical morphism $Z \to [Z]$ such that each morphism $f : Z \to Z'$ from Z to an algebraic space Z' factors uniquely as a composition $Z \to [Z] \overset{[f]}{\to} Z'$. The formation of $[Z]$ commutes with flat base change. In particular, taking étale neighborhoods and forming completions commute with such a process. If Z is representable by an algebraic space, then its coarse moduli space is just itself. Thus, if $\mathcal{H}$ is *neat* (see Definition 1.4.1.8), then the canonical morphism $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to [\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}]$ is an isomorphism.

Let us quote the following version of Zariski's main theorem:

Proposition 7.2.3.4 (Zariski's main theorem)**.** *A proper morphism of locally noetherian algebraic spaces is finite over the set of points over which the morphism has discrete fibers. Moreover, such a set of points is open.*

Proof. The statement for schemes can be found in [59, III-1, 4.4.3, 4.4.11]. A weaker statement for algebraic spaces can be found in [73, V, 4.2], whose proof also explains how to translate stronger statements for schemes into statements for algebraic spaces. □

As mentioned above, the restriction of $\oint_{\mathcal{H}}$ to $\mathsf{M}_{\mathcal{H}}$ is a morphism $\oint_{\mathcal{H}}|_{\mathsf{M}_{\mathcal{H}}} : \mathsf{M}_{\mathcal{H}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ from an algebraic stack to a scheme, each of whose geometric fibers has only one single closed point. Since $\mathsf{M}_{\mathcal{H}}$ is open in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, and since the formation of coarse moduli spaces commutes with flat base change, we see that $[\mathsf{M}_{\mathcal{H}}]$ is an open subalgebraic space of $[\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}]$. The morphism $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ factors as $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to [\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}] \overset{[\oint_{\mathcal{H}}]}{\to} \mathsf{M}_{\mathcal{H}}^{\min}$, whose restriction to $\mathsf{M}_{\mathcal{H}}$ is the factorization $\mathsf{M}_{\mathcal{H}} \to [\mathsf{M}_{\mathcal{H}}] \overset{[\oint_{\mathcal{H}}|_{\mathsf{M}_{\mathcal{H}}}]}{\to} \mathsf{M}_{\mathcal{H}}^{\min}$. Applying Zariski's main theorem (Proposition 7.2.3.4) to $[\oint_{\mathcal{H}}]$, and taking into account the fact (Proposition 7.2.3.3) that $\mathsf{M}_{\mathcal{H}}^{\min}$ is normal, we see that $[\oint_{\mathcal{H}}]$ is an isomorphism over an open subscheme of $\mathsf{M}_{\mathcal{H}}^{\min}$ containing the image of $[\mathsf{M}_{\mathcal{H}}]$. (We will see below that the image of $[\mathsf{M}_{\mathcal{H}}]$ is actually open, with complements given by closed subschemes, and hence $[\oint_{\mathcal{H}}|_{\mathsf{M}_{\mathcal{H}}}]$ is an open immersion.)

More generally, suppose that a fiber of $\oint_{\mathcal{H}}$ meets the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ be any representative of the class $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$. By Proposition 7.1.2.13, the restriction of each element $f \in \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k})$ to $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ depends only on its constant term $\mathrm{FJ}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(f)$, which is constant along each fiber of the structural morphism $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$, where $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ is the moduli problem defined by the cusp label represented by $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ (see Definition 5.4.2.6). Applying this to those $k \geq 0$ divisible by N_0, we see that $\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}} : \mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]} \to \mathsf{M}_{\mathcal{H}}^{\min}$ factors through $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. This induces a morphism

$$\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\min} \tag{7.2.3.5}$$

from an algebraic stack to a scheme, each of whose geometric fibers has only one single point.

The argument used in proving Proposition 7.1.2.8 shows the following:

Lemma 7.2.3.6. *Two restrictions* $\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} : \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to \mathsf{M}_{\mathcal{H}}^{\min}$ *and* $\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}} : \mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]} \to \mathsf{M}_{\mathcal{H}}^{\min}$ *have the same image and induce the same morphism as in* (*7.2.3.5*) (*up to canonical identification between the sources*) *when there exist representatives* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ *and* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ *of* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ *and* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$, *respectively, such that* $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})$ *are equivalent as in Definition 5.4.2.4 and represent the same cusp label* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})] = [(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$.

Let us denote this common image by $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} = \mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$. We claim that the converse is also true:

Proposition 7.2.3.7. *If* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} := \mathrm{image}(\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}})$ *and* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]} := \mathrm{image}(\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}})$ *have a nonempty intersection, then the two cusp labels* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ *and* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$ *are the same.* (*In this case, we saw above that* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} = \mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$.)

Proof. Suppose there exists a geometric point $\bar{x}$ in the intersection of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ and $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$. Let C be any proper irreducible curve in the fiber of $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ over $\bar{x}$. By the same argument (using Proposition 7.2.1.2) as before, we see that the pullback of G^{tor} to C is globally an extension of an isotrivial abelian scheme by a torus. If we take any geometric point $\bar{z}$ of C, and take the pullback of $(G^{\mathrm{tor}},\lambda^{\mathrm{tor}},i^{\mathrm{tor}},\alpha_{\mathcal{H}}^{\mathrm{tor}})$ to the strict local ring of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ at $\bar{z}$ completed along the curve C, then we obtain a degenerating family of type $\mathsf{M}_{\mathcal{H}}$ over a base ring $R_{\bar{z}}$ that fits into the setting of Section 5.2.1. (The key point here is that the pullback of G^{tor} to C is globally an extension of an abelian scheme by a *split* torus.) Therefore, it makes sense to consider the degeneration datum associated with such a family, and in particular, the equivalence class of the discrete data $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ associated with it. In other words, there is a locally constant association of a cusp label $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ over each such proper irreducible curve C. Since the fiber of $\oint_{\mathcal{H}}$ over $\bar{x}$ is connected, we see that the associated cusp label $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ must be globally constant over the whole fiber. This forces $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})] = [(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$, as desired. $\square$

Corollary 7.2.3.8. *The subschemes* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *form a stratification*

$$\mathsf{M}_{\mathcal{H}}^{\min} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \tag{7.2.3.9}$$

of $\mathsf{M}_{\mathcal{H}}^{\min}$ *by locally closed subscheme, with* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ *running through a complete set of cusp labels, such that the* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$*-stratum* $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ *lies in the closure of the* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$*-stratum* $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *if and only if there is a surjection from the cusp label* $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$ *to the cusp label* $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ *as in Definition 5.4.2.13.* (*The notation "*$\coprod$*" only means a set-theoretic disjoint union. The algebro-geometric structure is still that of* $\mathsf{M}_{\mathcal{H}}^{\min}$.)

Proof. According to 2 of Theorem 6.4.1.1, the closure of the $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is the union of the $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$-strata $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}$ such that $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$ is a face of $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ as in Definition 6.3.2.14. Since the morphism $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ is proper, we see that the closure of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is mapped to the closure of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ in $\mathsf{M}_{\mathcal{H}}^{\min}$, which is by definition the union of those $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ such that there is a surjection from $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ to $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$. By Proposition 7.2.3.7, this union is disjoint. Hence we may conclude (by induction on the incidence relations in the stratification of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$) that (7.2.3.9) is indeed a stratification of $\mathsf{M}_{\mathcal{H}}^{\min}$. $\square$

As a by-product, we have shown the following complement to Theorem 1.4.1.11, promised in Remark 1.4.1.13:

Corollary 7.2.3.10. *The coarse moduli space* $[\mathsf{M}_{\mathcal{H}}]$ *of* $\mathsf{M}_{\mathcal{H}}$ *is a quasi-projective scheme over* $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$. *In particular,* $\mathsf{M}_{\mathcal{H}}$ *is a quasi-projective scheme over* S_0 *when* $\mathcal{H}$ *is neat.*

Proof. The stratification (7.2.3.9) shows that $[\mathsf{M}_{\mathcal{H}}] \cong \mathsf{Z}_{[(0,0)]}$ is an open subalgebraic space in $\mathsf{M}_{\mathcal{H}}^{\min}$. Then the corollary follows from the fact that a subalgebraic space of a projective scheme (over S_0) is a scheme (see [73, II, 3.8]). $\square$

As another by-product,

Corollary 7.2.3.11. *If* σ *is top-dimensional in* $\mathbf{P}^+_{\Phi_{\mathcal{H}}} \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$, *then the restriction* $\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} : \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *is proper.*

Proof. Since σ is a top-dimensional cone, $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ can be a face of another $[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]$ (see Definition 6.3.2.14) only when $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})] \neq [(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]$. Then 2 of Theorem 6.4.1.1 and Proposition 7.2.3.7 imply that $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is a closed subalgebraic stack of the preimage $\oint_{\mathcal{H}}^{-1}(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})$. Since $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is proper over S_0, this shows that $\oint_{\mathcal{H}}|_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} : \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ is proper, as desired. $\square$

Combining this with Lemma 7.2.3.6 and with Zariski's main theorem (Proposition 7.2.3.4),

Corollary 7.2.3.12. *The morphism* $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}] \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ *induced by* (*7.2.3.5*) *is finite and induces a bijection on geometric points.*

Proposition 7.2.3.13. *Let* $\mathsf{M}_{\mathcal{H}}^1$ *be the open subscheme of* $\mathsf{M}_{\mathcal{H}}^{\min}$ *formed by the strata in* (*7.2.3.9*) *of codimension at most one. Then the pullback to* $\mathsf{M}_{\mathcal{H}}^1$ *of the canonical surjection* $[\oint_{\mathcal{H}}] : [\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}] \twoheadrightarrow \mathsf{M}_{\mathcal{H}}^{\min}$ *induced by* $\oint_{\mathcal{H}}$ *is an isomorphism* (*regardless of the choice of* Σ *in the construction of* $\mathsf{M}_{\mathcal{H}}^{\min}$).

Proof. Let $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ be any codimension-one stratum of $\mathsf{M}_{\mathcal{H}}^{\min}$. By Corollary 7.2.3.12, the codimension of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ in $\mathsf{M}_{\mathcal{H}}^{\min}$ is the difference between the dimensions of $\mathsf{M}_{\mathcal{H}}$ and of $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. By the description of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ in 2 of Theorem 6.4.1.1, if this difference is one, then the group $\mathbf{S}_{\Phi_{\mathcal{H}}}$ has $\mathbb{Z}$-rank one, and the proper smooth morphism $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ is of relative dimension zero, in which case the induced morphism $[\oint_{\mathcal{H}}] : [\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}] \twoheadrightarrow \mathsf{M}_{\mathcal{H}}^{\min}$ is quasi-finite over $\mathsf{M}_{\mathcal{H}}^1$. Since $\mathsf{M}_{\mathcal{H}}^{\min}$ is normal, and since the canonical morphism $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\min}} \to \oint_{\mathcal{H},*}\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ is an isomorphism, Zariski's main theorem (Proposition 7.2.3.4) shows that the pullback of the proper morphism $[\oint_{\mathcal{H}}]$ to $\mathsf{M}_{\mathcal{H}}^1$ is an isomorphism, as desired. $\square$

Corollary 7.2.3.14 (of the proof of Proposition 7.2.3.13)**.** *Suppose* $B \cong \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *is simple. Then the canonical surjection* $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}^{\min}$ *induces an isomorphism* $[\oint_{\mathcal{H}}] : [\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}] \xrightarrow{\sim} \mathsf{M}_{\mathcal{H}}^{\min}$ (*for one and hence every choice of* Σ *in the construction of* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$) *if and only if* $\mathsf{M}_{\mathcal{H}}$ *is either proper or of relative dimension at most one over* S_0.

Proof. Following the proof of Proposition 7.2.3.13, the induced morphism $[\oint_{\mathcal{H}}]$ is an isomorphism if and only if $\oint_{\mathcal{H}}$ is quasi-finite, which is the case if, over each stratum of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\mathsf{M}^{\min}_{\mathcal{H}}$ of codimension at least one, the structural morphism $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ is of relative dimension zero for every stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ in $\oint_{\mathcal{H}}^{-1}(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})$. Let us assume that $\mathsf{M}_{\mathcal{H}}$ is not proper over S_0, so that such a stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ does exist.

Since B is simple, and since $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ is a torsor under an abelian scheme $\mathbb{Z}^{\times}_{(\Box)}$-isogenous to $\mathrm{Hom}_{\mathcal{O}}(Y,A)$ of a finite étale cover of $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$, the condition that $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ is of relative dimension zero implies that $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ is of relative dimension zero over S_0. Since at least one stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ above $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ is of codimension one in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, this forces $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ (and hence $\mathsf{M}_{\mathcal{H}}$) to be of relative dimension at most one over S_0, as desired. □

Remark 7.2.3.15. In general (without assuming the simpleness of B), by decomposing the PEL-type $\mathcal{O}$-lattice $(L,\langle\cdot,\cdot\rangle,h)$ according to the decomposition (1.2.1.10) of B, and by decomposing the corresponding geometric objects up to finite morphisms, we see that the canonical morphism $[\oint_{\mathcal{H}}]:[\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}] \to \mathsf{M}^{\min}_{\mathcal{H}}$ is an isomorphism exactly when the moduli problem $\mathsf{M}_{\mathcal{H}}$ decomposes up to finite morphisms as a fiber product of moduli problems that are either proper or of relative dimensions at most one over S_0.

Proposition 7.2.3.16. *Let $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$ be a cusp label, and let $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ be a representative of $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$. Let $\bar{x}$ be a geometric point of $\mathsf{M}^{\min}_{\mathcal{H}}$ over the $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$, which by abuse of notation we also identify as a geometric point of $[\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}]$ by Corollary 7.2.3.12. Let $\mathrm{Aut}(\bar{x})$ be the group of automorphisms of $\bar{x} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ (cf. Section A.7.5). Let $(\mathsf{M}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}$ denote the completion of the strict localization of $\mathsf{M}^{\min}_{\mathcal{H}}$ at $\bar{x}$. Let $([\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}])^{\wedge}_{\bar{x}}$ denote the completion of the strict localization of $[\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}]$ at $\bar{x}$ (as a geometric point of $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$), and let $(\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$ denote the pullback of $\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ under the canonical morphism $(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^{\wedge}_{\bar{x}} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. For convenience, let us also use the notation of the various sheaves supported on $\bar{x}$ to denote their underlying rings or modules. Then we have a canonical isomorphism*

$$\mathscr{O}_{(\mathsf{M}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}} \cong \Big[\prod_{\ell\in\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}} (\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}\Big]^{\mathrm{Aut}(\bar{x})\times\Gamma_{\Phi_{\mathcal{H}}}} \tag{7.2.3.17}$$

of rings, which is adic if we interpret the product on the right-hand side as the completion of the elements that are finite sums with respect to the ideal generated by the elements without constant terms (i.e., with trivial projection to $(\underline{\mathrm{FJ}}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}}$). Let us denote by $(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^{\wedge}_{\bar{x}}$ the completion of the strict localization of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ at $\bar{x}$. Then (7.2.3.17) induces a ***structural morphism*** *from $(\mathsf{M}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}$ to $([\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}])^{\wedge}_{\bar{x}}$, whose precomposition with the canonical morphism $(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^{\wedge}_{\bar{x}} \to (\mathsf{M}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}$ defines a canonical isomorphism $(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^{\wedge}_{\bar{x}} \xrightarrow{\sim} ([\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}])^{\wedge}_{\bar{x}}$.*

Proof. By [59, III-1, 4.1.5 and 4.3.3], with natural generalizations to the context of algebraic stacks, the ring $\mathscr{O}_{(\mathsf{M}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}}}$ is isomorphic to the $\mathrm{Aut}(\bar{x})$-invariants in the ring of regular functions over the completion of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ along the fiber of $\oint_{\mathcal{H}}:\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathsf{M}^{\min}_{\mathcal{H}}$ at $\bar{x}$. By Proposition 7.2.3.7, the preimage $\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} := \oint_{\mathcal{H}}^{-1}(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})$ of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ under $\oint_{\mathcal{H}}$ is the union

$$\tilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} = \bigcup_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$$

of those strata $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ over $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. According to 5 of Theorem 6.4.1.1 and Lemma 6.2.5.27, there is a canonical isomorphism $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \cong \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ for each representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ of $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$. Therefore, the ring of regular functions over the completion of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ along the fiber of $\oint_{\mathcal{H}}$ at $\bar{x}$ is isomorphic to the *common intersection* of the rings of regular functions over the various completions of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ along the fibers of the structural morphisms $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. In other words, it is isomorphic to the common intersection of the $\Gamma_{\Phi_{\mathcal{H}},\sigma}$-invariants in the completions of $\hat{\bigoplus}_{\ell\in\sigma^{\vee}} \underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ along $\bar{x}$. Note that the identifications $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \cong \mathfrak{X}_{\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma'}$ for equivalent triples $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ and $(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')$ involve the canonical actions of $\Gamma_{\Phi_{\mathcal{H}}}$ on the structural sheaves. Hence the process of taking a common intersection also involves the process of taking $\Gamma_{\Phi_{\mathcal{H}}}$-invariants. This shows the existence of (7.2.3.17).

The claim that (7.2.3.17) is adic and that the composition $(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^{\wedge}_{\bar{x}} \to (\mathsf{M}^{\min}_{\mathcal{H}})^{\wedge}_{\bar{x}} \to ([\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}])^{\wedge}_{\bar{x}}$ is an isomorphism follows from the fact that the support $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ of each formal completion $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \cong \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ is defined by the vanishing of the ideal $\hat{\bigoplus}_{\ell\in\sigma_0^{\vee}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ of $\hat{\bigoplus}_{\ell\in\sigma^{\vee}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$, and that $\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}-\{0\} = \bigcap_{\sigma\in\Sigma_{\Phi_{\mathcal{H}}},\sigma\subset\mathbf{P}^{+}_{\Phi_{\mathcal{H}}}} \sigma_0^{\vee}$ (because $\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}}-\{0\}\subset\sigma_0^{\vee}$ for every $\sigma\subset\mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ and because $\mathbf{P}^{\vee}_{\Phi_{\mathcal{H}}} = \Sigma^{\vee}_{\Phi_{\mathcal{H}}} = \bigcap_{\sigma\in\Sigma_{\Phi_{\mathcal{H}}}} \sigma^{\vee}$ as explained in Section 7.1.2). Then we can conclude the proof by taking $\mathrm{Aut}(\bar{x})\times\Gamma_{\Phi_{\mathcal{H}}}$-invariants and by noting that $((\underline{\mathrm{FJ}}^{(0)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^{\wedge}_{\bar{x}})^{\mathrm{Aut}(\bar{x})\times\Gamma_{\Phi_{\mathcal{H}}}} \cong (\mathscr{O}_{(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^{\wedge}_{\bar{x}}})^{\mathrm{Aut}(\bar{x})} \cong \mathscr{O}_{[(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^{\wedge}_{\bar{x}}]} \cong \mathscr{O}_{([\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}])^{\wedge}_{\bar{x}}}$. □

Corollary 7.2.3.18. *The canonical finite surjection $[\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}] \twoheadrightarrow \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ defined by $\oint_{\mathcal{H}}$ is an isomorphism.*

Proof. The proof of Proposition 7.2.3.16 shows that the composition of the completion $([\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}])^{\wedge}_{\bar{x}} \to (\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^{\wedge}_{\bar{x}}$ of the finite surjection $[\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}] \twoheadrightarrow \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ defined by $\oint_{\mathcal{H}}$ (described in Corollary 7.2.3.12) with the canonical *structural isomorphism* $(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]})^{\wedge}_{\bar{x}} \xrightarrow{\sim} ([\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}])^{\wedge}_{\bar{x}}$ is the identity isomorphism. This forces $[\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}] \twoheadrightarrow \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ to be an isomorphism as the property of being an isomorphism can be verified over formal completions of the target. □

7.2.4 Main Results on Minimal Compactifications

Theorem 7.2.4.1 (arithmetic minimal compactification)**.** *There exists a normal scheme $\mathsf{M}^{\min}_{\mathcal{H}}$ projective and flat over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, such that we have the following:*

1. *$\mathsf{M}^{\min}_{\mathcal{H}}$ contains the* ***coarse moduli space*** *$[\mathsf{M}_{\mathcal{H}}]$ of $\mathsf{M}_{\mathcal{H}}$ as an open dense* ***subscheme****.*
2. *Let $(G,\lambda,i,\alpha_{\mathcal{H}})$ be the tautological tuple over $\mathsf{M}_{\mathcal{H}}$. Let us define the invertible sheaf $\omega := \omega_{G/\mathsf{M}_{\mathcal{H}}} := \wedge^{\mathrm{top}}\underline{\mathrm{Lie}}^{\vee}_{G/\mathsf{M}_{\mathcal{H}}} = \wedge^{\mathrm{top}} e_G^*\Omega^1_{G/\mathsf{M}_{\mathcal{H}}}$ over $\mathsf{M}_{\mathcal{H}}$. Then there is*

a smallest integer $N_0 \geq 1$ such that $\omega^{\otimes N_0}$ is the pullback of an ample invertible sheaf $\mathcal{O}(1)$ over $\mathsf{M}_{\mathcal{H}}^{\min}$.

If $\mathcal{H}$ is neat (see Definition 1.4.1.8), then $\mathsf{M}_{\mathcal{H}} \to [\mathsf{M}_{\mathcal{H}}]$ is an isomorphism, and embeds $\mathsf{M}_{\mathcal{H}}$ as an open dense subscheme of $\mathsf{M}_{\mathcal{H}}^{\min}$. Moreover, we have $N_0 = 1$ with a canonical choice of $\mathcal{O}(1)$, and the restriction of $\mathcal{O}(1)$ to $\mathsf{M}_{\mathcal{H}}$ is isomorphic to ω. We shall denote $\mathcal{O}(1)$ by $\omega^{\min}$, and interpret it as an extension of ω to $\mathsf{M}_{\mathcal{H}}^{\min}$.

By abuse of notation, for each integer k divisible by N_0, we shall denote $\mathcal{O}(1)^{\otimes k/N_0}$ by $(\omega^{\min})^{\otimes k}$ even when $\omega^{\min}$ itself is not defined.

3. *For any (smooth) arithmetic toroidal compactification $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ of $\mathsf{M}_{\mathcal{H}}$ as in Theorem 6.4.1.1, with a degenerating family $(G^{\mathrm{tor}}, \lambda^{\mathrm{tor}}, i^{\mathrm{tor}}, \alpha_{\mathcal{H}}^{\mathrm{tor}})$ over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ extending the tautological tuple $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{M}_{\mathcal{H}}$, let $\omega^{\mathrm{tor}} := \omega_{G^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^{\vee}_{G^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} = \wedge^{\mathrm{top}} e^*_{G^{\mathrm{tor}}} \Omega^1_{G^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ be the invertible sheaf over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ extending ω naturally. Then the graded algebra $\underset{k \geq 0}{\oplus} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k})$, with its natural algebra structure induced by tensor products, is finitely generated over $\mathcal{O}_{F_0,(\Box)}$, and is independent of the choice (of the Σ used in the definition) of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$.*

   The normal scheme $\mathsf{M}_{\mathcal{H}}^{\min}$ (projective and flat over S_0) is canonically isomorphic to $\mathrm{Proj}\Big(\underset{k \geq 0}{\oplus} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k})\Big)$, and there is a canonical morphism $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ determined by ω^{tor} and the universal property of Proj, such that $\oint_{\mathcal{H}}^* \mathcal{O}(1) \cong (\omega^{\mathrm{tor}})^{\otimes N_0}$ over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ and such that the canonical morphism $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\min}} \to \oint_{\mathcal{H},*} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ is an isomorphism. Moreover, when we vary the choices of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$'s, the morphisms $\oint_{\mathcal{H}}$'s are compatible with the canonical morphisms among the $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$'s as in Proposition 6.4.2.3.

   When $\mathcal{H}$ is neat, we have $\oint_{\mathcal{H}}^* \omega^{\min} \cong \omega^{\mathrm{tor}}$ and $\oint_{\mathcal{H},*} \omega^{\mathrm{tor}} \cong \omega^{\min}$.

4. *$\mathsf{M}_{\mathcal{H}}^{\min}$ has a natural stratification by locally closed subschemes*
$$\mathsf{M}_{\mathcal{H}}^{\min} = \coprod_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]} \mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]},$$
   with $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ running through a complete set of cusp labels as in Definition 5.4.2.4, such that the $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})]$-stratum $\mathsf{Z}_{[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})]}$ lies in the closure of the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ if and only if there is a surjection from the cusp label $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})]$ to the cusp label $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ as in Definition 5.4.2.13. (The notation "$\coprod$" only means a set-theoretic disjoint union. The algebro-geometric structure is still that of $\mathsf{M}_{\mathcal{H}}^{\min}$.)

   Each $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ is canonically isomorphic to the coarse moduli space $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}]$ (which is a scheme) of the corresponding algebraic stack $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ (separated, smooth, and of finite type over S_0) as in Definition 5.4.2.6.

   Let us define the $\mathcal{O}$-multirank of a stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ to be the $\mathcal{O}$-multirank of the cusp label represented by $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ (see Definition 5.4.2.7). The only stratum with $\mathcal{O}$-multirank zero is the open stratum $\mathsf{Z}_{[(0,0)]} \cong [\mathsf{M}_{\mathcal{H}}]$, and those strata $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ with nonzero $\mathcal{O}$-multiranks are called ***cusps***. *(This explains the name of the cusp labels.)*

5. *The restriction of $\oint_{\mathcal{H}}$ to the stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is a surjection to the stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ of $\mathsf{M}_{\mathcal{H}}^{\min}$. This surjection is smooth when $\mathcal{H}$ is neat, and is proper if σ is top-dimensional in $\mathbf{P}^+_{\Phi_{\mathcal{H}}} \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$.*

   Under the above-mentioned identification $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}] \xrightarrow{\sim} \mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ on the target, this surjection can be viewed as the quotient by $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ (see Definition 6.2.5.23) of a torsor under a torus $E_{\Phi_{\mathcal{H}},\sigma}$ over an abelian scheme torsor $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ (as in the construction) over the finite étale cover $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$ over the algebraic stack $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ over the coarse moduli space $[\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}]$ (which is a scheme). More precisely, this torus $E_{\Phi_{\mathcal{H}},\sigma}$ is the quotient of the torus $E_{\Phi_{\mathcal{H}}} := \underline{\mathrm{Hom}}(\mathbf{S}_{\Phi_{\mathcal{H}}}, \mathbf{G}_{\mathrm{m}})$ corresponding to the subgroup $\mathbf{S}_{\Phi_{\mathcal{H}},\sigma} := \{x \in \mathbf{S}_{\Phi_{\mathcal{H}}} : \langle x, y\rangle = 0 \; \forall y \in \sigma\}$ of $\mathbf{S}_{\Phi_{\mathcal{H}}}$ (see the definition of $E_{\Phi_{\mathcal{H}}}$ in Lemma 6.2.4.4, and definition of σ-stratum in Definition 6.1.2.7).

Proof. Let us take $\mathsf{M}_{\mathcal{H}}^{\min}$ to be the normal scheme (projective and flat over S_0) constructed in Section 7.2.3. The first concern is whether its properties as described by the theorem depend on the toroidal compactifications we choose. It is clear from the construction that statements 1, 4, and 5 are satisfied regardless of the choices. Let us verify that this is also the case for statements 2 and 3.

Suppose Σ' is a refinement of Σ as in Definition 6.4.2.2, and suppose the morphism $p : \mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}} \twoheadrightarrow \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ and the invertible sheaves $\omega_{\mathcal{H},\Sigma}^{\mathrm{tor}}$ and $\omega_{\mathcal{H},\Sigma'}^{\mathrm{tor}}$ are defined as in the proof of Lemma 7.1.1.5. Let $\oint_{\mathcal{H},\Sigma} : \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ and $\oint_{\mathcal{H},\Sigma'} : \mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ be the two canonical morphisms. Then $\oint_{\mathcal{H},\Sigma'} = \oint_{\mathcal{H},\Sigma} \circ p$ and $p_* \mathscr{O}_{\mathsf{M}_{\mathcal{H},\Sigma'}^{\mathrm{tor}}} \cong \mathscr{O}_{\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}}$ implies that $\oint_{\mathcal{H},\Sigma}^* \mathcal{O}(1) \cong (\omega_{\mathcal{H},\Sigma}^{\mathrm{tor}})^{\otimes N_0}$ if and only if $\oint_{\mathcal{H},\Sigma'}^* \mathcal{O}(1) \cong (\omega_{\mathcal{H},\Sigma'}^{\mathrm{tor}})^{\otimes N_0}$ (for the same $\mathcal{O}(1)$ and N_0). In other words, we can move freely between different choices of Σ by taking pullbacks or push-forwards; there is a choice of $\mathcal{O}(1)$ with the smallest value of $N_0 \geq 1$ that works for all Σ.

From now on, let us fix a choice of Σ and suppress it from the notation. We would like to show that ω extends to an ample invertible sheaf over $\mathsf{M}_{\mathcal{H}}^{\min}$ when $\mathcal{H}$ is *neat*.

By Proposition 7.2.3.13, the pullback of $\oint_{\mathcal{H}} : \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\min}$ to $\mathsf{M}_{\mathcal{H}}^1$ is an *isomorphism* because the canonical morphism $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} \to [\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}]$ is an isomorphism when $\mathcal{H}$ is neat. Therefore, we can view $\mathsf{M}_{\mathcal{H}}^1$ as an open subspace of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ and consider the restriction $\omega^{\mathrm{tor}}|_{\mathsf{M}_{\mathcal{H}}^1}$, where ω^{tor} is defined as in statement 3. Since the complement of $\mathsf{M}_{\mathcal{H}}^1$ in $\mathsf{M}_{\mathcal{H}}^{\min}$ has codimension at least two (by definition of $\mathsf{M}_{\mathcal{H}}^1$) and since $\mathsf{M}_{\mathcal{H}}^{\min}$ is noetherian and normal, it suffices to show that the coherent sheaf (see [53, VIII, Prop. 3.2])
$$\omega^{\min} := (\mathsf{M}_{\mathcal{H}}^1 \hookrightarrow \mathsf{M}_{\mathcal{H}}^{\min})_*(\omega^{\mathrm{tor}}|_{\mathsf{M}_{\mathcal{H}}^1})$$
is an invertible sheaf. By fpqc descent (see [56, VIII, 1.11]), it suffices to verify this statement over the completions of strict localizations of $\mathsf{M}_{\mathcal{H}}^{\min}$.

Let $\bar{x}$ be a geometric point over some $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ in $\mathsf{M}_{\mathcal{H}}^{\min}$, and consider any $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}$ in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ that maps surjectively to $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$. Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ be any representative of $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$. Since $\mathcal{H}$ is neat, our choice of Σ (see Definition 6.3.3.4) forces $\Gamma_{\Phi_{\mathcal{H}},\sigma}$ to act trivially on $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (by Lemma 6.2.5.27). Therefore we have $(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}} \cong \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (by 5 of Theorem 6.4.1.1). Let $(\mathsf{M}_{\mathcal{H}}^{\min})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}}$ denote the formal completion of $\mathsf{M}_{\mathcal{H}}^{\min}$ along the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$. Then we have a composition of canonical morphisms $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \cong (\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]}} \to (\mathsf{M}_{\mathcal{H}}^{\min})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}}$. By abuse of notation, let us denote the pullback of this composition from $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}}$ to the completion $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}$ of the strict localization of $\mathsf{M}^{\min}_{\mathcal{H}}$ at $\bar{x}$ by $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}} \cong (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}} \to (\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}$. According to Proposition 7.2.3.16, there is a *structural morphism* $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}} \to (\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^\wedge_{\bar{x}}$ such that the further composition $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}} \cong (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\bar{x}} \to (\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}} \to (\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^\wedge_{\bar{x}}$ agrees with the morphism $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}} \to (\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^\wedge_{\bar{x}}$ induced by the structural morphism $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$. Over $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}}$, the pullback ${}^\heartsuit\omega$ of ω^{tor} from $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is isomorphic to $(\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A$ by Lemma 7.1.2.1, which does descend to $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}$, because the pullback of ω_A from $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ also makes sense there. Since the complement of $\mathsf{M}^1_{\mathcal{H}}$ in the normal scheme $\mathsf{M}^{\min}_{\mathcal{H}}$ has codimension at least two, the pullback of $\omega^{\min}$ (from $\mathsf{M}^{\min}_{\mathcal{H}}$ to $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}$) has to agree with the pullback of $(\wedge^{\mathrm{top}}_{\mathbb{Z}} X) \underset{\mathbb{Z}}{\otimes} \omega_A$ to $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}$. In particular, it is invertible, as desired.

Since $\oint_{\mathcal{H}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathsf{M}^{\min}_{\mathcal{H}}$ satisfies $\mathscr{O}_{\mathsf{M}^{\min}_{\mathcal{H}}} \xrightarrow{\sim} \oint_{\mathcal{H},*} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ by construction as a Stein factorization, we see that two locally free sheaves $\mathcal{E}$ and $\mathcal{F}$ of finite rank over $\mathsf{M}^{\min}_{\mathcal{H}}$ are isomorphic if and only if $\oint^*_{\mathcal{H}} \mathcal{E} \cong \oint^*_{\mathcal{H}} \mathcal{F}$. Indeed, for the nontrivial implication we just need $\mathcal{E} \cong \oint_{\mathcal{H},*} \oint^*_{\mathcal{H}} \mathcal{E} \cong \oint_{\mathcal{H},*} \oint^*_{\mathcal{H}} \mathcal{F} \cong \mathcal{F}$ (by Lemma 7.2.2.1). Since $\oint^*_{\mathcal{H}} \omega^{\min} \cong \omega^{\mathrm{tor}}$, we have $\oint_{\mathcal{H},*} \omega^{\mathrm{tor}} \cong \omega^{\min}$, and the $\mathcal{O}(1)$ above such that $\oint^*_{\mathcal{H}} \mathcal{O}(1) \cong (\omega^{\mathrm{tor}})^{\otimes N_0}$ has to satisfy $\mathcal{O}(1) \cong (\omega^{\min})^{\otimes N_0}$. This shows that $\omega^{\min}$ is ample and finishes the verification of statements 2 and 3. $\square$

Remark 7.2.4.2. In general, $\mathsf{M}^{\min}_{\mathcal{H}}$ is not smooth over S_0.

Proposition 7.2.4.3 (base change properties)**.** *We can repeat the construction of $\mathsf{M}^{\min}_{\mathcal{H}}$ with $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ replaced with each (quasi-separated) locally noetherian* ***normal*** *scheme S over S_0, and obtain a normal scheme $\mathsf{M}^{\min}_{\mathcal{H},S}$ projective and flat over S, with analogous characterizing properties described as in Theorem 7.2.4.1 (with $\mathrm{Proj}(\cdot)$ replaced with $\underline{\mathrm{Proj}}_S(\cdot)$, and with $\Gamma(\cdot)$ replaced with direct images over S), together with a canonical* ***finite*** *morphism*

$$\mathsf{M}^{\min}_{\mathcal{H},S} \to \mathsf{M}^{\min}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} S. \tag{7.2.4.4}$$

If $S' \to S$ is a morphism between locally noetherian normal schemes, then we also have a canonical finite morphism

$$\mathsf{M}^{\min}_{\mathcal{H},S'} \to \mathsf{M}^{\min}_{\mathcal{H},S} \underset{S}{\times} S'. \tag{7.2.4.5}$$

Moreover, these finite morphisms satisfy the following properties:

1. *If $S \to \mathsf{S}_0$ (resp. $S' \to S$) is flat, then (7.2.4.4) (resp. (7.2.4.5)) is an isomorphism.*

2. *If $\mathsf{M}^{\min}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} S$ (resp. $\mathsf{M}^{\min}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} S'$) is noetherian and normal, then (7.2.4.4) (resp. (7.2.4.5)) is an isomorphism (by Zariski's main theorem; cf. Proposition 7.2.3.4).*

3. *Suppose $\bar{s}$ is a geometric point of S. Then (7.2.4.5) (with S' replaced with $\bar{s}$) is an isomorphism if the following condition is satisfied:*
$$\forall \text{ geometric points } \bar{x} \text{ of } \mathsf{M}^{\min}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \bar{s},\ \mathrm{char}(\bar{s}) \nmid \#\,\mathrm{Aut}(\bar{x}). \tag{7.2.4.6}$$
(As in Proposition 7.2.3.16, $\mathrm{Aut}(\bar{x})$ is the group of automorphisms of $\bar{x} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} \bar{s}$, or equivalently that of $\bar{x} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} S$ or $\bar{x} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$, if $\bar{x}$ is over the $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\mathsf{M}^{\min}_{\mathcal{H}}$.) In this case, the geometric fiber $\mathsf{M}^{\min}_{\mathcal{H},S} \underset{S}{\times} \bar{s}$ is normal because $\mathsf{M}^{\min}_{\mathcal{H},\bar{s}}$ is.

4. *Suppose (7.2.4.6) is satisfied by all geometric points $\bar{s}$ of S. (This is the case, for example, if $\mathcal{H}$ is neat. In general, there is a nonzero constant c depending only on the linear algebraic data defining $\mathsf{M}_{\mathcal{H}}$ such that $\#\,\mathrm{Aut}(\bar{x})|c$ for all geometric points $\bar{x}$ of $\mathsf{M}^{\min}_{\mathcal{H}}$). Then the scheme $\mathsf{M}^{\min}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} S$ is normal, (7.2.4.4) is an isomorphism (by property 2 above), and the morphism $\mathsf{M}^{\min}_{\mathcal{H},S} \to S$ is* ***normal*** *(i.e., flat with geometrically normal fibers; see [59, IV-2, 6.8.1 and 6.7.8]). Moreover, for every locally noetherian normal scheme S' over S, the scheme $\mathsf{M}^{\min}_{\mathcal{H},S} \underset{S}{\times} S'$ is normal, (7.2.4.5) is an isomorphism (again, by property 2 above), and the morphism $\mathsf{M}^{\min}_{\mathcal{H},S'} \to S'$ is normal.*

Proof. We may assume that S and S' are affine, noetherian normal, and connected, because property 1 (and the convention that all schemes are quasi-separated) allows us to patch the construction of $\mathsf{M}^{\min}_{\mathcal{H},S}$ along intersections of affine open subschemes of S.

Let us take any $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ as in the construction of $\mathsf{M}^{\min}_{\mathcal{H}}$, so that we have the canonical surjection

$$\oint_{\mathcal{H}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \twoheadrightarrow \mathsf{M}^{\min}_{\mathcal{H}} \cong \mathrm{Proj}\Big(\underset{k\geq 0}{\oplus} \Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}, (\omega^{\mathrm{tor}})^{\otimes k})\Big).$$

If we repeat the construction of $\mathsf{M}^{\min}_{\mathcal{H}}$ over S, then we obtain a normal scheme $\mathsf{M}^{\min}_{\mathcal{H},S}$ projective over S, together with a canonical surjection

$$\oint_{\mathcal{H},S} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} S \twoheadrightarrow \mathsf{M}^{\min}_{\mathcal{H},S} \cong \mathrm{Proj}\Big(\underset{k\geq 0}{\oplus} \Gamma(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} S, (\omega^{\mathrm{tor}} \underset{\mathscr{O}_{\mathsf{S}_0}}{\otimes} \mathscr{O}_S)^{\otimes k})\Big).$$

By the descriptions of the projective spectra, we obtain a canonical proper morphism as in (7.2.4.4), and we know it is an isomorphism when $S \to \mathsf{S}_0$ is flat.

The morphism $S \to \mathsf{S}_0$ either is flat or factors through a closed point s of S_0. In the former case, the morphism $\mathsf{M}^{\min}_{\mathcal{H},S} \to S$ is the pullback of $\mathsf{M}^{\min}_{\mathcal{H}} \to \mathsf{S}_0$, which is flat by Theorem 7.2.4.1. In the latter case, the morphism $\mathsf{M}^{\min}_{\mathcal{H},S} \to S$ is the pullback of $\mathsf{M}^{\min}_{\mathcal{H},s} \to s$, which is automatically flat. Thus $\mathsf{M}^{\min}_{\mathcal{H},S} \to S$ is always flat.

For each stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of $\mathsf{M}^{\min}_{\mathcal{H}}$, any surjection $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \twoheadrightarrow \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ that defines it factors through a canonical isomorphism $[\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}] \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. Consider the analogous construction over S: we may decompose the above morphism as a composition

$$[\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} S] \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})],S} \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]} \underset{\mathsf{S}_0}{\times} S, \tag{7.2.4.7}$$

which forces the second morphism in (7.2.4.7) to be quasi-finite. Since the second morphism is necessarily the restriction of (7.2.4.4) to $\mathsf{Z}_{[\Phi_{\mathcal{H}},\delta_{\mathcal{H}}],S}$, it forces (7.2.4.4) to be a finite morphism by Zariski's main theorem (Proposition 7.2.3.4).

The case of (7.2.4.5) is similar, with S_0 (resp. S) replaced with S (resp. S').

Now, property 1 has already been explained. Property 2 is self-explanatory, because $\mathsf{M}^{\min}_{\mathcal{H},S}$ and $\mathsf{M}^{\min}_{\mathcal{H},S'}$ are noetherian normal by construction.

Let us prove property 3. Suppose that the condition (7.2.4.6) is satisfied. Since (7.2.4.4) is an isomorphism if it is so over the completions of strict local rings at geometric points of the target, and since the formation of $\mathrm{Aut}(\bar{x})$-invariants commutes with the base change from S to $\bar{s}$ because $\mathrm{char}(\bar{s}) \nmid \#\,\mathrm{Aut}(\bar{x})$, by Proposition 7.2.3.16 (see, in particular, (7.2.3.17)), it suffices to show that, for each

$\ell_0 \in \mathbf{P}^\vee_{\Phi_\mathcal{H}}$ with stabilizer $\Gamma_{\Phi_\mathcal{H},\ell_0}$ in $\Gamma_{\Phi_\mathcal{H}}$, the formation of $\Gamma_{\Phi_\mathcal{H},\ell_0}$-invariants in $(\underline{\mathrm{FJ}}^{(\ell_0)}_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{x}} \cong \Gamma((C_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{x}}, (\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell_0))^\wedge_{\bar{x}})$ also commutes with the base change from S to $\bar{s}$. By the construction of $\mathsf{M}^{\Phi_\mathcal{H}}_\mathcal{H}$ (see Definition 5.4.2.6 and Proposition 6.2.4.7), there exists a finite index normal subgroup $\Gamma'_{\Phi_\mathcal{H}}$ of $\Gamma_{\Phi_\mathcal{H}}$ such that $\Gamma'_{\Phi_\mathcal{H}}$ acts trivially on $\mathsf{M}^{\Phi_\mathcal{H}}_\mathcal{H}$, and such that the induced action of $\Gamma_{\Phi_\mathcal{H}}/\Gamma'_{\Phi_\mathcal{H}}$ on $\mathsf{M}^{\Phi_\mathcal{H}}_\mathcal{H}$ makes $\mathsf{M}^{\Phi_\mathcal{H}}_\mathcal{H} \to \mathsf{M}^{\mathsf{Z}_\mathcal{H}}_\mathcal{H}$ an étale $(\Gamma_{\Phi_\mathcal{H}}/\Gamma'_{\Phi_\mathcal{H}})$-torsor. Hence, it suffices to show that, for each geometric point $\bar{y} \to \mathsf{M}^{\Phi_\mathcal{H}}_\mathcal{H}$ lifting $\bar{x} \to \mathsf{M}^{\mathsf{Z}_\mathcal{H}}_\mathcal{H}$, the formation of invariants of $\Gamma'_{\Phi_\mathcal{H},\ell_0} = \Gamma_{\Phi_\mathcal{H},\ell_0} \cap \Gamma'_{\Phi_\mathcal{H}}$ in $\Gamma((C_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{y}}, (\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell_0))^\wedge_{\bar{y}})$, where $(\,\cdot\,)^\wedge_{\bar{y}}$ denote the pullback to the completion of the strict localization of $\mathsf{M}^{\Phi_\mathcal{H}}_\mathcal{H}$ at $\bar{y}$, commutes with the base change from S to $\bar{s}$, for each $\ell_0 \in \mathbf{P}^\vee_{\Phi_\mathcal{H}}$.

Let X' and Y' be admissible sub-$\mathcal{O}$-lattices of X and Y, respectively, such that $\phi(Y') \subset X'$, such that ℓ' lies in the subgroup $\mathbf{S}'_{\Phi_\mathcal{H}}$ of $\mathbf{S}_{\Phi_\mathcal{H}}$ defined by the same construction of $\mathbf{S}_{\Phi_\mathcal{H}}$ using the embedding $\phi' : Y' \to X'$ induced by ϕ, and such that ℓ_0 is positive in $\Phi'_\mathcal{H}$ in the sense that, up to choosing a $\mathbb{Z}$-basis $y_1, \ldots, y_r$ of Y', and by completion of squares for quadratic forms, there exists some integer $N \geq 1$ such that $N \cdot \ell_0$ can be represented as a positive definite matrix of the form $ue\,{}^t u$, where e and u are matrices with integer coefficients, and where $e = \mathrm{diag}(e_1, \ldots, e_r)$ is diagonal with positive entries. In this case, $\Gamma'_{\Phi_\mathcal{H},\ell_0}$ acts on $\Phi'_\mathcal{H}$ via a discrete subgroup $\overline{\Gamma}'_{\Phi_\mathcal{H},\ell_0}$ of the compact orthogonal subgroup of $\mathrm{GL}_\mathbb{R}(Y' \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ preserving the above-mentioned positive definite matrix by conjugation, which is necessarily finite. Consider the abelian scheme torsor $C'_{\Phi_\mathcal{H},\delta_\mathcal{H}} \to \mathsf{M}^{\Phi_\mathcal{H}}_\mathcal{H}$ defined by the same construction of $C_{\Phi_\mathcal{H},\delta_\mathcal{H}}$, using the embedding $\phi' : Y' \to X'$ instead of $\phi : Y \to X$, with a canonical morphism $C_{\Phi_\mathcal{H},\delta_\mathcal{H}} \to C'_{\Phi_\mathcal{H},\delta_\mathcal{H}}$ over $\mathsf{M}^{\Phi_\mathcal{H}}_\mathcal{H}$ which is also an abelian scheme torsor, under which the $\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell_0)$ descends to an invertible sheaf $\Psi'_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell_0)$, which is relatively ample over $\mathsf{M}^{\Phi_\mathcal{H}}_\mathcal{H}$ because some positive tensor power of $\Psi'_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell_0)$ is isomorphic to the pullback of the line bundle $\underset{1 \leq i \leq r}{\otimes} (\mathrm{pr}_i^*(\mathrm{Id}_A, \lambda_A)^*\mathcal{P}_A)^{\otimes e_i}$ over A under the finite morphism given by the composition $C'_{\Phi_\mathcal{H},\delta_\mathcal{H}} \overset{\mathrm{can.}}{\to} \underline{\mathrm{Hom}}_\mathbb{Z}(Y, A) \overset{u^*}{\to} \underline{\mathrm{Hom}}_\mathbb{Z}(Y, A)$ over $\mathsf{M}^{\Phi_\mathcal{H}}_\mathcal{H}$, because λ_A is a polarization (cf. Definition 1.3.2.16), and because all the e_i's are positive. Then $\Gamma'_{\Phi_\mathcal{H},\ell_0}$ acts via the finite quotient $\overline{\Gamma}'_{\Phi_\mathcal{H},\ell_0}$ introduced above on $\Gamma((C_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{y}}, (\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell_0))^\wedge_{\bar{y}}) \cong \Gamma((C'_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{y}}, (\Psi'_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell_0))^\wedge_{\bar{y}})$.

If $\mathcal{H}$ is neat, then $\overline{\Gamma}'_{\Phi_\mathcal{H},\ell_0}$ is also neat and must be trivial. More generally, since $\Psi'_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell_0)$ is relatively ample over $\mathsf{M}^{\Phi_\mathcal{H}}_\mathcal{H}$, the action of $\overline{\Gamma}'_{\Phi_\mathcal{H},\ell_0}$ on $\underset{N \geq 0}{\oplus} \Gamma((C'_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{y}}, (\Psi'_{\Phi_\mathcal{H},\delta_\mathcal{H}}(N \cdot \ell_0))^\wedge_{\bar{y}})$ induces a faithful action of $\overline{\Gamma}'_{\Phi_\mathcal{H},\ell_0}$ on $(C'_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{y}}$ (cf. [94, §21, Thm. 5]). By construction, $(C'_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{y}}$ appears (up to some identification) in the toroidal boundary construction of $\mathsf{M}^{\mathsf{Z}''_\mathcal{H}}_\mathcal{H}$, where $\mathsf{M}^{\mathsf{Z}''_\mathcal{H}}_\mathcal{H}$ is isomorphic to the stratum of $\mathsf{M}^{\min}_\mathcal{H}$ labeled by the cusp label $[(\mathsf{Z}''_\mathcal{H}, \Phi''_\mathcal{H}, \delta''_\mathcal{H})]$ induced by $[(\mathsf{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})]$ by the admissible surjections $X \to X'' := X/X'$ and $Y \to Y'' := Y/Y'$ (see Lemmas 5.4.1.13 and 5.4.2.11). Therefore, there exists a degeneration (over a complete discrete valuation ring with fraction field $k(z)$) of an object parameterized by some functorial point $z \to \mathsf{M}^{\mathsf{Z}''_\mathcal{H}}_\mathcal{H}$ such that $\overline{\Gamma}'_{\Phi_\mathcal{H},\ell_0}$ is a subquotient of $\mathrm{Aut}(\bar{z})$ for any geometric point $\bar{z} \to \mathsf{M}^{\mathsf{Z}''_\mathcal{H}}_\mathcal{H}$ above z. Since $\mathrm{char}(\bar{s}) \nmid \#\mathrm{Aut}(\bar{z})$ by the assumption that the condition (7.2.4.6) is satisfied, it follows that $\mathrm{char}(\bar{s}) \nmid \#\overline{\Gamma}'_{\Phi_\mathcal{H},\ell_0}$. Therefore, the formation of $\overline{\Gamma}'_{\Phi_\mathcal{H},\ell_0}$-invariants in $\Gamma((C'_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{y}}, (\Psi'_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell_0))^\wedge_{\bar{y}})$ commutes with the base change from S to $\bar{s}$, for each $\ell_0 \in \mathbf{P}^\vee_{\Phi_\mathcal{H}}$, and property 3 follows.

It remains to prove property 4. Note that the assertions involving S' (in the last sentence) follow from the assertions involving only S, by [59, IV-2, 6.8.2 and 6.14.1] and property 2. To prove the assertions involving only S, we may replace S with its localizations, and assume that it is local. Let S_1 be the localization of S_0 at the image under $S \to \mathsf{S}_0$ of the closed point of S. Since S_1 is a localization of S_0, we know by property 1 that the canonical morphism $\mathsf{M}^{\min}_{\mathcal{H},\mathsf{S}_1} \to \mathsf{M}^{\min}_\mathcal{H} \underset{\mathsf{S}_0}{\times} \mathsf{S}_1$ is an isomorphism, so that the morphism $\mathsf{M}^{\min}_{\mathcal{H},\mathsf{S}_1} \to \mathsf{S}_1$ is the pullback of $\mathsf{M}^{\min}_\mathcal{H} \to \mathsf{S}_0$. Since the geometric points of S_1 are either of characteristic zero or dominated by those of S, by [59, IV-2, 6.7.7], the normality of fibers of $\mathsf{M}^{\min}_{\mathcal{H},\mathsf{S}_1} \to \mathsf{S}_1$ follows from the normality of the geometric fibers of $\mathsf{M}^{\min}_{\mathcal{H},\mathsf{S}_1} \underset{\mathsf{S}_1}{\times} S \cong \mathsf{M}^{\min}_\mathcal{H} \underset{\mathsf{S}_0}{\times} S \to S$, the latter of which follows from property 3 (and from the assumption that (7.2.4.6) is satisfied by all geometric points $\bar{s}$ of S). Thus, the morphism $\mathsf{M}^{\min}_{\mathcal{H},\mathsf{S}_1} \to \mathsf{S}_1$ is normal. By [59, IV-2, 6.8.2], the pullback $\mathsf{M}^{\min}_{\mathcal{H},\mathsf{S}_1} \underset{\mathsf{S}_1}{\times} S \to S$ is also a normal morphism. By [59, IV-2, 6.14.1], the scheme $\mathsf{M}^{\min}_{\mathcal{H},\mathsf{S}_1} \underset{\mathsf{S}_1}{\times} S \cong \mathsf{M}^{\min}_\mathcal{H} \underset{\mathsf{S}_0}{\times} S$ is normal. By property 2, this implies that (7.2.4.4) is an isomorphism, and hence that the morphism $\mathsf{M}^{\min}_{\mathcal{H},S} \to S$ is normal, as desired. □

Corollary 7.2.4.8. *Let M be a module over $\mathcal{O}_{F_0,(\Box)}$. Suppose there is a noetherian normal $\mathcal{O}_{F_0,(\Box)}$-algebra M_0 over which M is flat. Let $S := \mathrm{Spec}(M_0)$. Let $k \geq 0$ be an integer divisible by the smallest values of $N_0 \geq 1$ as in 2 of Theorem 7.2.4.1 for $\mathsf{M}^{\min}_{\mathcal{H},S}$ (rather than $\mathsf{M}^{\min}_\mathcal{H}$). (When $\mathcal{H}$ is neat, $N_0 = 1$ and there is no restriction on k other than being nonnegative.) Let us denote by $(\omega^{\min}_S)^{\otimes k}$ the invertible sheaf $\mathcal{O}(1)^{\otimes k/N_0}$ over $\mathsf{M}^{\min}_{\mathcal{H},S}$. Then the canonical morphism*

$$\Gamma(\mathsf{M}^{\min}_{\mathcal{H},S}, (\omega^{\min}_S)^{\otimes k} \underset{M_0}{\otimes} M) \to \Gamma(\mathsf{M}^{\mathrm{tor}}_\mathcal{H} \underset{\mathsf{S}_0}{\times} S, (\omega^{\mathrm{tor}} \underset{\mathscr{O}_{\mathsf{S}_0}}{\otimes} \mathscr{O}_S)^{\otimes k} \underset{M_0}{\otimes} M)$$
$$\cong \Gamma(\mathsf{M}^{\mathrm{tor}}_\mathcal{H}, (\omega^{\mathrm{tor}})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M) = \mathrm{AF}(k, M)$$

induced by the relation $\oint^*_{\mathcal{H},S}(\omega^{\min}_S)^{\otimes k} \cong (\omega^{\mathrm{tor}} \underset{\mathscr{O}_{\mathsf{S}_0}}{\otimes} \mathscr{O}_S)^{\otimes k}$ in 3 of Theorem 7.2.4.1 (cf. the proof of Proposition 7.2.4.3) is an isomorphism.

Moreover, if condition (7.2.4.6) in Proposition 7.2.4.3 is satisfied by all geometric points $\bar{s}$ of S (which is the case, for example, if $\mathcal{H}$ is neat), and if k is also divisible by the smallest values of $N_0 \geq 1$ as in 2 of Theorem 7.2.4.1 for $\mathsf{M}^{\min}_\mathcal{H}$, then we may replace $\Gamma(\mathsf{M}^{\min}_{\mathcal{H},S}, (\omega^{\min}_S)^{\otimes k} \underset{M_0}{\otimes} M)$ with $\Gamma(\mathsf{M}^{\min}_\mathcal{H}, (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\Box)}}{\otimes} M)$ in the above canonical morphism.

Proof. The first paragraph follows immediately from another canonical isomorphism $\oint_{\mathcal{H},S,*} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_\mathcal{H} \underset{\mathsf{S}_0}{\times} S} \cong \mathscr{O}_{\mathsf{M}^{\min}_{\mathcal{H},S}}$ in 3 of Theorem 7.2.4.1. (The flatness of M over M_0 implies that we also have $\oint_{\mathcal{H},S,*}(\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_\mathcal{H} \underset{\mathsf{S}_0}{\times} S} \underset{M_0}{\otimes} M) \cong \mathscr{O}_{\mathsf{M}^{\min}_{\mathcal{H},S}} \underset{M_0}{\otimes} M$.)

The second paragraph follows from the construction of $\mathsf{M}^{\min}_{\mathcal{H},S}$ (cf. Proposition

7.2.4.3), from which one deduces that $(\omega_S^{\min})^{\otimes k}$ is canonically isomorphic to the pullback of $(\omega^{\min} \underset{\mathscr{O}_{\mathsf{S}_0}}{\otimes} \mathscr{O}_S)^{\otimes k}$ under (7.2.4.4), the last of which is an isomorphism by 4 of Proposition 7.2.4.3. $\square$

Now we may redefine automorphic forms intrinsically:

Definition 7.2.4.9 (fake reformulation of Definition 7.1.1.1)**.** *With the assumptions as in the first paragraph of Corollary 7.2.4.8, an* ***(arithmetic) automorphic form*** *over $\mathsf{M}_{\mathcal{H}}$, of naive parallel weight k, with coefficients in M, and regular at infinity, is an element of $\Gamma(\mathsf{M}_{\mathcal{H},S}^{\min}, (\omega_S^{\min})^{\otimes k} \underset{M_0}{\otimes} M)$. For simplicity, when the context is clear, we shall call such an element an* ***automorphic form*** *of naive weight k.*

If the assumptions in the second paragraph of Corollary 7.2.4.8 are also satisfied, then we may replace $\Gamma(\mathsf{M}_{\mathcal{H},S}^{\min}, (\omega_S^{\min})^{\otimes k} \underset{M_0}{\otimes} M)$ with $\Gamma(\mathsf{M}_{\mathcal{H}}^{\min}, (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} M)$ in Definition 7.2.4.9.

For example, we have the following:

Corollary 7.2.4.10 (liftability of sections)**.** *Let R be a normal ring over $\mathcal{O}_{F_0,(\square)}$ such that condition (7.2.4.6) in Proposition 7.2.4.3 is satisfied by all geometric points $\bar{s}$ of $S := \mathrm{Spec}(R)$. Let I be an ideal of R such that R/I is also normal. (These hypotheses apply, for example, when $\mathcal{H}$ is neat, R is a discrete valuation ring flat over $\mathcal{O}_{F_0,(\square)}$, and I is the maximal ideal of R.) Then the canonical morphism $\mathrm{AF}(k,R) \to \mathrm{AF}(k,R/I)$ induced by the canonical surjection $R \twoheadrightarrow R/I$ is surjective for all sufficiently large k.*

Proof. By Corollary 7.2.4.8, or rather by Definition 7.2.4.9, the canonical morphism $\mathrm{AF}(k,R) \to \mathrm{AF}(k,R/I)$ can be canonically identified with the canonical morphism

$$\Gamma(\mathsf{M}_{\mathcal{H}}^{\min}, (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} R) \to \Gamma(\mathsf{M}_{\mathcal{H}}^{\min}, (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} (R/I)). \tag{7.2.4.11}$$

Since $\mathsf{M}_{\mathcal{H}}^{\min} \to \mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\square)})$ is flat, the invertible sheaf $\omega^{\min}$ is flat over S_0, and hence we have a short exact sequence

$$0 \to (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} I \to (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} R \to (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} (R/I) \to 0$$

of coherent sheaves over $\mathsf{M}_{\mathcal{H}}^{\min}$. By considering the attached cohomology long exact sequence, we see that (7.2.4.11) is surjective if $H^1(\mathsf{M}_{\mathcal{H}}^{\min}, (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} I) = 0$. Since $\omega^{\min}$ is ample over $\mathsf{M}_{\mathcal{H}}^{\min}$, by Serre's vanishing theorem (see Theorem 7.2.2.2), this last condition is true for all sufficiently large k, as desired. $\square$

Remark 7.2.4.12 (for readers familiar with the geometric construction of p-adic modular forms, as in the works of Hida and many others). The most well-known application of Corollary 7.2.4.10, at least in the Siegel modular case treated in [42], is the construction of liftings of sufficiently large powers of the *Hasse invariant*. In the context, R is some p-adic discrete valuation ring with maximal ideal I, and R/I is a finite field of some characteristic p. A delicate point in the construction of such liftings is that the powers of the Hasse invariant, a priori, are defined as sections of $\mathrm{AF}(k,R/I) = \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} (R/I))$ for some multiple k of $p-1$ (which agrees with $\Gamma(\mathsf{M}_{\mathcal{H},\mathrm{Spec}(R/I)}^{\min}, (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} (R/I))$ by Corollary 7.2.4.8, and with $\Gamma(\mathsf{M}_{\mathcal{H}}, \omega^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} (R/I))$ when Koecher's principle applies; see Corollary 7.2.4.13 below). But it is difficult to locate an explanation in the literature as to why they can also be identified with sections of $\Gamma(\mathsf{M}_{\mathcal{H}}^{\min}, (\omega^{\min})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} (R/I))$ (under suitable hypotheses, such as those in Corollary 7.2.4.10). (The constructions of liftings of Hasse invariants in the literature seldom emphasize their logical dependence on [42, especially Ch. V, Thm. 2.7(i)].) Without this last identification, Serre's vanishing theorem does not apply as in the proof of Corollary 7.2.4.10.

Corollary 7.2.4.13 (Koecher's principle)**.** *With the assumptions as in the first paragraph of Corollary 7.2.4.8, suppose that $\mathsf{M}_{\mathcal{H}}^1 = [\mathsf{M}_{\mathcal{H}}]$ as open subschemes of $\mathsf{M}_{\mathcal{H}}^{\min}$ (cf. Proposition 7.2.3.13), but no longer suppose that there exists some noetherian normal $\mathcal{O}_{F_0,(\square)}$-algebra M_0 over which M is flat. Then the canonical restriction morphism*

$$\Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, (\omega^{\mathrm{tor}})^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} M) \to \Gamma(\mathsf{M}_{\mathcal{H}}, \omega^{\otimes k} \underset{\mathcal{O}_{F_0,(\square)}}{\otimes} M) \tag{7.2.4.14}$$

is a bijection. In other words, automorphic forms of naive parallel weight k with coefficients in M are ***automatically regular at infinity****.*

Proof. By the same reduction step as in the proof of Lemma 7.1.1.4, we may assume that M is an $\mathcal{O}_{F_0,(\square)}$-algebra and work after making the base change from $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\square)})$ to $S = \mathrm{Spec}(M)$. By Corollary 7.2.4.8, (7.2.4.14) can be identified with

$$\Gamma(\mathsf{M}_{\mathcal{H},S}^{\min}, (\omega_S^{\min})^{\otimes k}) \to \Gamma([\mathsf{M}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} S], ((\omega_S^{\min})^{\otimes k}|_{[\mathsf{M}_{\mathcal{H}}] \underset{\mathsf{S}_0}{\times} S})). \tag{7.2.4.15}$$

By 1 of Theorem 7.2.4.1, $[\mathsf{M}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} S]$ is embedded as a subscheme of $\mathsf{M}_{\mathcal{H},S}^{\min}$, while the (rather linear-algebraic) assumption that $\mathsf{M}_{\mathcal{H}}^1 = [\mathsf{M}_{\mathcal{H}}]$ shows that the complement of $[\mathsf{M}_{\mathcal{H}} \underset{\mathsf{S}_0}{\times} S]$ in $\mathsf{M}_{\mathcal{H},S}^{\min}$ has codimension at least two. Therefore, the noetherian normality of $\mathsf{M}_{\mathcal{H},S}^{\min}$ (see Proposition 7.2.4.3) forces the bijectivity of (7.2.4.15). $\square$

Remark 7.2.4.16. It is not necessary to know whether (7.2.4.4) is an isomorphism for $S = \mathrm{Spec}(M)$ in this proof.

Remark 7.2.4.17. When $\mathsf{M}_{\mathcal{H}}^1 \neq [\mathsf{M}_{\mathcal{H}}]$, it is still possible to state a variant of Corollary 7.2.4.13 by introducing the open subscheme $\mathsf{M}_{\mathcal{H},S}^1$ of $\mathsf{M}_{\mathcal{H},S}^{\min}$ formed by strata of codimension at most one.

7.2.5 Hecke Actions on Minimal Compactifications

Let us state the following analogue of Proposition 6.4.3.4 for arithmetic minimal compactifications.

Proposition 7.2.5.1. *Suppose we have an element $g \in \mathrm{G}(\mathbb{A}^{\infty,\square})$, and suppose we have two open compact subgroups $\mathcal{H}$ and $\mathcal{H}'$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\square})$ such that $\mathcal{H}' \subset g\mathcal{H}g^{-1}$. Then there is a canonical morphism $[g]^{\min} : \mathsf{M}_{\mathcal{H}'}^{\min} \to \mathsf{M}_{\mathcal{H}}^{\min}$ extending the canonical morphism $[[g]] : [\mathsf{M}_{\mathcal{H}'}] \to [\mathsf{M}_{\mathcal{H}}]$ induced by the canonical morphism $[g] : \mathsf{M}_{\mathcal{H}'} \to \mathsf{M}_{\mathcal{H}}$ defined by the Hecke action of g, such that $(\omega^{\min})^{\otimes k}$ over $\mathsf{M}_{\mathcal{H}}^{\min}$ is pulled back to $(\omega^{\min})^{\otimes k}$ over $\mathsf{M}_{\mathcal{H}'}^{\min}$ whenever both are defined.*

Moreover, the surjection $[g]^{\min}$ maps the $[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})]}$ of $\mathsf{M}_{\mathcal{H}'}^{\min}$ to the $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})]$-stratum $\mathsf{Z}_{[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})]}$ of $\mathsf{M}_{\mathcal{H}}^{\min}$ if and only if there are representatives $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ of $[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})]$ and $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})]$, respectively, such that $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ is g-assigned to $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ as in Definition 5.4.3.9.

If $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ and $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}}}\}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ are two compatible choices of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}'}$ and $\mathsf{M}_{\mathcal{H}}$, respectively, such that Σ is a g-refinement of Σ' as in Definition 6.4.3.3, then the canonical surjection $[g]^{\min} : \mathsf{M}^{\min}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}^{\min}_{\mathcal{H}}$ is compatible with the surjection $[g]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ given by Proposition 6.4.3.4.

Proof. Let $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ and $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}}}\}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}})]}$ be any two compatible choices of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}'}$ and $\mathsf{M}_{\mathcal{H}}$, respectively, such that Σ is a g-refinement of Σ' as in Definition 6.4.3.3. Let $\oint_{\mathcal{H}'} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \twoheadrightarrow \mathsf{M}^{\min}_{\mathcal{H}'}$ and $\oint_{\mathcal{H}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'} \twoheadrightarrow \mathsf{M}^{\min}_{\mathcal{H}}$ be the surjections given by 3 of Theorem 7.2.4.1. Let $[g]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ be the canonical surjection given by Proposition 6.4.3.4 extending the canonical morphism $[g] : \mathsf{M}_{\mathcal{H}'} \to \mathsf{M}_{\mathcal{H}}$ defined by the Hecke action of g. The composition of $[g]^{\mathrm{tor}}$ with $\oint_{\mathcal{H}}$ gives a morphism $\oint_{\mathcal{H}} \circ [g]^{\mathrm{tor}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma} \to \mathsf{M}^{\min}_{\mathcal{H}}$, which pulls $(\omega^{\min})^{\otimes k}$ over $\mathsf{M}^{\min}_{\mathcal{H}}$ (whenever it is defined) back to $(\omega^{\mathrm{tor}})^{\otimes k}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$. By the universal property stated in 3 of Theorem 7.2.4.1, this composition morphism factors through $\oint_{\mathcal{H}'}$, and induces a morphism $[g]^{\min} : \mathsf{M}^{\min}_{\mathcal{H}'} \to \mathsf{M}^{\min}_{\mathcal{H}}$ pulling $(\omega^{\min})^{\otimes k}$ over $\mathsf{M}^{\min}_{\mathcal{H}}$ back to $(\omega^{\min})^{\otimes k}$ over $\mathsf{M}^{\min}_{\mathcal{H}'}$ whenever both are defined. By the fact that the restriction of $\oint_{\mathcal{H}'}$ to $\mathsf{M}_{\mathcal{H}'}$ is the canonical morphism $\mathsf{M}_{\mathcal{H}'} \to [\mathsf{M}_{\mathcal{H}'}]$, we see that the restriction of $[g]^{\min}$ to $[\mathsf{M}_{\mathcal{H}'}]$ is the canonical surjection $[[g]] : [\mathsf{M}_{\mathcal{H}'}] \to [\mathsf{M}_{\mathcal{H}}]$ induced by the canonical surjection $[g] : \mathsf{M}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}$ defined by the Hecke action of g. Since $\mathsf{M}^{\min}_{\mathcal{H}'}$ is proper over S_0, and since $[\mathsf{M}_{\mathcal{H}}]$ is dense in $\mathsf{M}^{\min}_{\mathcal{H}}$, we see that $[g]^{\min}$ is a surjection.

Since the stratification of $\mathsf{M}^{\min}_{\mathcal{H}}$ (resp. $\mathsf{M}^{\min}_{\mathcal{H}'}$) is induced by that of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma'}$ (resp. $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$), the statements about the images of the strata of $\mathsf{M}^{\min}_{\mathcal{H}'}$ under $[g]^{\min}$ follow from the corresponding statements in Proposition 6.4.3.4 about the images of the strata of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma}$ under $[g]^{\mathrm{tor}}$. □

Corollary 7.2.5.2. *Suppose we have two open compact subgroups $\mathcal{H}$ and $\mathcal{H}'$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ such that $\mathcal{H}'$ is a normal subgroup of $\mathcal{H}$. Then the canonical morphisms defined in Proposition 7.2.5.1 induce an action of the finite group $\mathcal{H}/\mathcal{H}'$ on $\mathsf{M}^{\min}_{\mathcal{H}'}$. The canonical surjection $[1]^{\min} : \mathsf{M}^{\min}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}^{\min}_{\mathcal{H}}$ defined by Proposition 7.2.5.1 can be identified with the quotient of $\mathsf{M}^{\min}_{\mathcal{H}'}$ by this action.*

Proof. The existence of such an action is clear. Since $\mathsf{M}^{\min}_{\mathcal{H}'}$ is projective over S_0 and normal, the quotient $\mathsf{M}^{\min}_{\mathcal{H}'}/(\mathcal{H}/\mathcal{H}')$ exists as a scheme (cf. [39, V, 4.1]). Then it follows from Zariski's main theorem (Proposition 7.2.3.4) that the induced morphism $\mathsf{M}^{\min}_{\mathcal{H}'}/(\mathcal{H}/\mathcal{H}') \to \mathsf{M}^{\min}_{\mathcal{H}}$ (with noetherian normal target) is an isomorphism, because it is generically so (over $[\mathsf{M}_{\mathcal{H}}]$ on the target, by the moduli interpretations of $\mathsf{M}_{\mathcal{H}'}$ and $\mathsf{M}_{\mathcal{H}}$, and by the characterization of coarse moduli spaces as geometric and uniform categorical quotients in the category of algebraic spaces; see Section A.7.5). □

7.3 Projectivity of Toroidal Compactifications

Assume that $\mathcal{H}$ is neat (see Definition 1.4.1.8). By Corollary 7.2.3.10 (and the fact that $\mathsf{M}^{\min}_{\mathcal{H}}$ is projective over S_0), the algebraic space $\mathsf{M}_{\mathcal{H}}$ is a quasi-projective scheme over S_0. However, the arithmetic toroidal compactifications $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$, which depend on the choices of the admissible smooth rational polyhedral cone decomposition data $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ we take in Theorem 6.4.1.1, are not schemes in general. It is a natural question whether there exists a nice condition for Σ that guarantees the projectivity of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ over S_0.

In the complex analytic case, this question is solved by Tai in [16, Ch. IV, §2]. With suitable reinterpretations, the same technique has an algebraic analogue: Assuming that Σ satisfies certain convexity conditions (to be defined in Section 7.3.1), the toroidal compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ can be realized as the normalization of a blowup of the minimal compactification $\mathsf{M}^{\min}_{\mathcal{H}}$ along a certain sheaf of ideals that vanishes to some sufficiently high power along the boundary of $\mathsf{M}^{\min}_{\mathcal{H}}$. Such an algebraic analogue is first provided in [25, Ch. IV] for Siegel moduli schemes over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$, and then in [42, Ch. V, §5] for Siegel moduli schemes over $\mathrm{Spec}(\mathbb{Z})$. (In fact, the toroidal compactifications in [25, Ch. IV] are constructed only using this approach, while those in [42, Ch. IV] are constructed by gluing good algebraic models as in Chapter 6. Thus, it is fair to say that the theory in [42, Ch. V, §5] is not only more complete, but also closer in spirit, to Tai's original work.)

The goal of this section is to show that the theory in [42, Ch. V, §5] can be fully generalized to our setting.

7.3.1 Convexity Conditions on Cone Decompositions

The following definition follows Tai's original one [16, Ch. IV, §2] very closely:

Definition 7.3.1.1. *Let $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ be a representative of a cusp label at level $\mathcal{H}$, and let $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j\in J}$ be any $\Gamma_{\Phi_{\mathcal{H}}}$-admissible rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (with respect to the integral structure given by $\mathbf{S}^{\vee}_{\Phi_{\mathcal{H}}}$ in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$). An* ***(invariant) polarization function*** *on $\mathbf{P}_{\Phi_{\mathcal{H}}}$ for the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ is a $\Gamma_{\Phi_{\mathcal{H}}}$-invariant continuous piecewise linear function $\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}$ such that we have the following:*

1. *$\mathsf{pol}_{\Phi_{\mathcal{H}}}$ is linear (i.e., coincides with a linear function) on each cone σ_j in $\Sigma_{\Phi_{\mathcal{H}}}$. (In particular, $\mathsf{pol}_{\Phi_{\mathcal{H}}}(tx) = t\mathsf{pol}_{\Phi_{\mathcal{H}}}(x)$ for all $x \in \mathbf{P}_{\Phi_{\mathcal{H}}}$ and $t \in \mathbb{R}_{\geq 0}$.)*
2. *$\mathsf{pol}_{\Phi_{\mathcal{H}}}((\mathbf{P}_{\Phi_{\mathcal{H}}} \cap \mathbf{S}^{\vee}_{\Phi_{\mathcal{H}}}) - \{0\}) \subset \mathbb{Z}_{>0}$. (In particular, $\mathsf{pol}_{\Phi_{\mathcal{H}}}(x) > 0$ for all nonzero x in $\mathbf{P}_{\Phi_{\mathcal{H}}}$.)*
3. *$\mathsf{pol}_{\Phi_{\mathcal{H}}}$ is linear (in the above sense) on a rational polyhedral cone σ in $\mathbf{P}_{\Phi_{\mathcal{H}}}$ if and only if σ is contained in some cone σ_j in $\Sigma_{\Phi_{\mathcal{H}}}$.*
4. *For all $x, y \in \mathbf{P}_{\Phi_{\mathcal{H}}}$, we have $\mathsf{pol}_{\Phi_{\mathcal{H}}}(x+y) \geq \mathsf{pol}_{\Phi_{\mathcal{H}}}(x) + \mathsf{pol}_{\Phi_{\mathcal{H}}}(y)$. This is called the* ***convexity*** *of $\mathsf{pol}_{\Phi_{\mathcal{H}}}$. (These functions are called* ***concave*** *in the usual context of mathematics.)*

If such a polarization function exists, then we say that the $\Gamma_{\Phi_{\mathcal{H}}}$-admissible rational polyhedral cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ is ***projective****.*

Following [16, Ch. II], with a minor error corrected by Looijenga, as mentioned in [42, Ch. V, §5] (cf. a similar remark in Section 6.2.5), we summarize the information we need as follows:

Proposition 7.3.1.2 (cf. [42, p. 173]). *1. Given any $\Gamma_{\Phi_{\mathcal{H}}}$-admissible rational polyhedral cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$, there exist refinements $\Sigma'_{\Phi_{\mathcal{H}}}$ of $\Sigma_{\Phi_{\mathcal{H}}}$ that are either projective, smooth, or both projective and smooth.*

2. *Let $\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}$ be a polarization function of a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible rational polyhedral cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$. Let*
$$K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} := \{x \in \mathbf{P}_{\Phi_{\mathcal{H}}} : \mathsf{pol}_{\Phi_{\mathcal{H}}}(x) \geq 1\}.$$
*This is a convex subset of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ not containing $\{0\}$ such that $\mathbb{R}_{\geq 1} \cdot K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} = K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ and $\mathbb{R}_{\geq 0} \cdot K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} \supset \mathbf{P}_{\Phi_{\mathcal{H}}}$, whose closure $\overline{K}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$ is a **cocore** in the context of [16, Ch. II, §5]. For simplicity, we shall also call $K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ a cocore in what follows.*

3. *The **dual** of $K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ is defined as*
$$K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} := \{x \in \mathbf{S}_{\Phi_{\mathcal{H}}} \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \langle x, y\rangle \geq 1 \ \forall y \in K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}\}$$
$$= \{x \in \mathbf{S}_{\Phi_{\mathcal{H}}} \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \langle x, y\rangle \geq 1 \ \forall y \in \overline{K}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}\}.$$
*This is a convex subset in $(\mathbb{R}_{\geq 0} \cdot \mathbf{P}_{\Phi_{\mathcal{H}}})^{\circ}$, the interior of $\mathbb{R}_{\geq 0} \cdot \mathbf{P}_{\Phi_{\mathcal{H}}}$, such that $\mathbb{R}_{\geq 1} \cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} = K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ and $\mathbb{R}_{>0} \cdot K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} = (\mathbb{R}_{\geq 0} \cdot \mathbf{P}_{\Phi_{\mathcal{H}}})^{\circ}$, which is a **core** in the context of [16, Ch. II, §5].*

4. *The top-dimensional cones σ in the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ in 2 correspond bijectively to the vertices ℓ of the core $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, which are linear forms whose restrictions to each σ coincide with the restriction of $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ to σ.*

5. *Suppose we have a surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ as in Definition 6.2.6.4, and suppose $\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}$ is a polarization function for $\Sigma_{\Phi_{\mathcal{H}}}$. By definition of a surjection, there is a surjection $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y') : (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ (see Definition 5.4.2.12) that induces an embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ such that the restriction $\Sigma_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\Phi'_{\mathcal{H}}}}$ of the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ to $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ is the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$. Then the restriction of $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ to $\Sigma_{\Phi'_{\mathcal{H}}}$ (via any choice of (s_X, s_Y)) is an (invariant) polarization function for $\Sigma_{\Phi'_{\mathcal{H}}}$.*

Definition 7.3.1.3. *We say that a compatible choice $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$ (see Definition 6.3.3.4) is **projective** if it satisfies the following condition: There is a collection $\mathsf{pol} = \{\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}\}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ of polarization functions labeled by representatives $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of cusp labels, each $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ being a polarization function of the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ in Σ (see Definition 7.3.1.1), which are **compatible** in the following sense: For each surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ of representatives of cusp labels (see Definition 5.4.2.12) inducing an embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$, we have $\mathsf{pol}_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\Phi'_{\mathcal{H}}}} = \mathsf{pol}_{\Phi'_{\mathcal{H}}}$.*

Proposition 7.3.1.4. *There exists a compatible choice $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$ (see Definition 6.3.3.4) that is **projective** in the sense of Definition 7.3.1.3.*

Remark 7.3.1.5. As in Remark 6.3.3.6, this is a combinatorial question unrelated to the question of compactifying integral models. It is already needed in existing works on complex analytic or rational models of toroidal compactifications.

Proof of Proposition 7.3.1.4. Following exactly the same induction steps as in the proof of Proposition 6.3.3.5, we simply have to impose projectivity on all the cone decompositions that we construct. □

Let us fix a choice of a projective smooth $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]}$ as in Proposition 7.3.1.4, with a polarization function $\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}$ for each projective cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ in Σ.

Construction 7.3.1.6. Given any open compact subgroup $\mathcal{H}'$ of $\mathcal{H}$, we can define an induced cone decomposition $\Sigma^{(\mathcal{H}')} = \{\Sigma_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})]}$, with an induced compatible collection $\mathsf{pol}^{(\mathcal{H}')} = \{\mathsf{pol}_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})]}$ of polarization functions for $\mathsf{M}_{\mathcal{H}'}$, as follows: For each representative $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ of cusp labels at level $\mathcal{H}'$ whose $\mathcal{H}$-orbit determines a representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of cusp labels at level $\mathcal{H}$ in its natural sense (by Convention 5.3.1.15), we have a canonical isomorphism $\mathbf{P}_{\Phi_{\mathcal{H}'}} \cong \mathbf{P}_{\Phi_{\mathcal{H}}}$ induced by the canonical isogeny $E_{\Phi_{\mathcal{H}'}} \twoheadrightarrow E_{\Phi_{\mathcal{H}}}$ (defined naturally by the construction of $E_{\Phi_{\mathcal{H}'}}$ and $E_{\Phi_{\mathcal{H}}}$ in Lemma 6.2.4.4). Then we define $\Sigma^{(\mathcal{H}')}$ (resp. $\mathsf{pol}^{(\mathcal{H}')}$) by taking the cone decomposition $\Sigma_{\Phi_{\mathcal{H}'}}$ (resp. polarization function $\mathsf{pol}_{\Phi_{\mathcal{H}'}}$) labeled by $(\Phi_{\mathcal{H}'}, \delta_{\mathcal{H}'})$ to be the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. polarization function $\mathsf{pol}_{\Phi_{\mathcal{H}}}$) labeled by $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$. (Since the definition of smoothness depends on the integral structure $\mathbf{S}^{\vee}_{\Phi_{\mathcal{H}'}}$, which can a priori be different from $\mathbf{S}^{\vee}_{\Phi_{\mathcal{H}}}$, we do not claim that $\Sigma^{(\mathcal{H}')}$ is smooth.)

Let us quote the following useful combinatorial results from [42, Ch. V, §5]:

Lemma 7.3.1.7 (cf. [42, Ch. V, Lem. 5.3])**.** *For each open compact subgroup $\mathcal{H}$ of $\mathcal{U}^{\square}(n)$, there is an open compact subgroup $\mathcal{H}' \subset \mathcal{H}$ (which can be taken to be normal) such that the compatible choice $\Sigma^{(\mathcal{H}')}$ of admissible rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}'}$ defined in Construction 7.3.1.6 remains **smooth** and satisfies the following condition: For each lifting $\Phi_{\mathcal{H}'} = (X, Y, \phi, \varphi_{-2,\mathcal{H}'}, \varphi_{0,\mathcal{H}'})$ of $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ to level $\mathcal{H}'$, and for each vertex ℓ_0 of $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}'}}}$ corresponding to a top-dimensional cone σ_0, we have*
$$\langle \ell_0, x\rangle < \langle \gamma \cdot \ell_0, x\rangle \tag{7.3.1.8}$$
for all $x \in \overline{\sigma}_0 \cap \mathbf{P}^{+}_{\Phi_{\mathcal{H}'}}$ and all $\gamma \in \Gamma_{\Phi_{\mathcal{H}'}}$ such that $\gamma \neq 1$.

Lemma 7.3.1.9 (cf. [42, Ch. V, Lem. 5.5])**.** *Suppose $\sigma \in \Sigma_{\Phi_{\mathcal{H}}}$, and suppose $\sigma_1, \ldots, \sigma_r$ are the one-dimensional faces of σ. For each $1 \leq j \leq r$, consider the unique $y_j \in \sigma_j$ such that $\mathbf{S}^{\vee}_{\Phi_{\mathcal{H}}} \cap \sigma_j = \mathbb{Z}_{\geq 1} \cdot y_j$, so that $K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} \cap \sigma_j = \mathbb{R}_{\geq 1} \cdot (\mathsf{pol}_{\Phi_{\mathcal{H}}}(y_j)^{-1} y_j)$, and let $L_j := \{x \in \mathbf{S}_{\Phi_{\mathcal{H}}} \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \langle x, y_j\rangle = \mathsf{pol}_{\Phi_{\mathcal{H}}}(y_j)\}$. Then each $L_j \cap K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ is a top-dimensional face of $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, whose vertices are in $\mathbf{S}_{\Phi_{\mathcal{H}}} \cap K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ because $y_j \in \mathbf{S}^{\vee}_{\Phi_{\mathcal{H}}}$ and $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ takes integral values on $\mathbf{S}^{\vee}_{\Phi_{\mathcal{H}}}$, and the intersection $\underset{1\leq j\leq r}{\cap}(L_j \cap K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}})$ defines a face of $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ (which we consider dual to σ). Suppose $d \geq 1$ is any integer, and suppose $\ell_0 \in \mathbf{S}_{\Phi_{\mathcal{H}}} \cap d \cdot \big(\underset{1\leq j\leq r}{\cap}(L_j \cap K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}})\big)$ does not lie on any proper face of $d \cdot \big(\underset{1\leq j\leq r}{\cap}(L_j \cap K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}})\big)$. Then there exist $\ell_1, \ldots, \ell_n \in \mathbf{S}_{\Phi_{\mathcal{H}}} \cap K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ (which are not necessarily vertices of $K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$) such that*
$$\mathbb{R}_{\geq 0} \cdot \sigma^{\vee} = \sum_{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} \cap (d \cdot K^{\vee}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}})} \mathbb{R}_{\geq 0} \cdot (\ell - \ell_0) = \sum_{1 \leq i \leq n} \mathbb{R}_{\geq 0} \cdot (d \cdot \ell_i - \ell_0).$$

Remark 7.3.1.10. The integral version of Lemma 7.3.1.9 is not true in general. We cannot replace $\mathbb{R}_{\geq 0}$ with $\mathbb{Z}_{\geq 0}$ in its statements. This difference is immaterial because we are taking normalizations later in the proof of projectivity of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ over S_0. (Indeed, it is the main reason that we have to take normalizations.)

Remark 7.3.1.11. The literal statements of [42, Ch. V, Lem. 5.5], which are stronger than those of Lemma 7.3.1.9, are unfortunately incorrect. For example, if $\mathbf{P}^+_{\Phi_{\mathcal{H}}} = \mathbb{R}_{>0} = \sigma$, then there are no other top-dimensional cones at all, and hence [42, Ch. V, Lem. 5.4] asserts that $\sigma^\vee = \{0\}$—but $\sigma^\vee$ is certainly nonzero. This error was inherited from a similar error in [16, Ch. IV, Sec. 2, p. 330], and was in turn inherited by [42, Ch. V, Lem. 5.5] and the first official version of this work.

7.3.2 Generalities on Normalizations of Blowups

Definition 7.3.2.1. *Let W be any noetherian scheme, and let $\mathcal{I}$ be any coherent sheaf of ideals over W. Then we denote by $\mathrm{Bl}_{\mathcal{I}}(W)$ the **blowup** of W along $\mathcal{I}$, and we denote by $\mathrm{NBl}_{\mathcal{I}}(W)$ the **normalization** of $\mathrm{Bl}_{\mathcal{I}}(W)$. (To take the normalization of a noetherian scheme, we first replace it with its reduced subscheme with the same underlying topological space, next replace it with the disjoint union of its irreducible components, and then take the componentwise normalization.)*

Definition 7.3.2.2. *Let W be any noetherian scheme, let $\mathcal{I}$ be any coherent $\mathscr{O}_W$-ideal, and let $f : \tilde{W} \to W$ be any morphism from a noetherian scheme $\tilde{W}$ such that the image $f^{-1}\mathcal{I} \cdot \mathscr{O}_{\tilde{W}}$ of the canonical morphism $f^*\mathcal{I} \to \mathscr{O}_{\tilde{W}}$ (which we will call the **coherent ideal pullback** of $\mathcal{I}$) is an **invertible** $\mathscr{O}_{\tilde{W}}$-ideal. Then we denote by*

$$\mathrm{Bl}_{\mathcal{I}}(f) : \tilde{W} \to \mathrm{Bl}_{\mathcal{I}}(W)$$

the canonical morphism induced by the universal property of $\mathrm{Bl}_{\mathcal{I}}(W)$ such that f is the composition of $\mathrm{Bl}_{\mathcal{I}}(f)$ with the structural morphism $\mathrm{Bl}_{\mathcal{I}}(W) \to W$.

*If moreover $\tilde{W}$ is a **normal** scheme, and if f induces dominant morphisms from irreducible components of $\tilde{W}$ to irreducible components of W, then we denote by*

$$\mathrm{NBl}_{\mathcal{I}}(f) : \tilde{W} \to \mathrm{NBl}_{\mathcal{I}}(W)$$

the canonical morphism induced by the universal property of $\mathrm{NBl}_{\mathcal{I}}(W)$ such that f is the composition of $\mathrm{NBl}_{\mathcal{I}}(f)$ with the structural morphism $\mathrm{NBl}_{\mathcal{I}}(W) \to W$.

Let us quote the following useful result concerning blowups from [42, Ch. V, §5]:

Proposition 7.3.2.3 ([42, Ch. V, Prop. 5.13]; cf. [16, Ch. IV, §2, p. 327, Lem.])**.** *Suppose we have a commutative diagram*

$$\begin{tikzcd} \tilde{W}' \arrow[r, "\tilde{g}"] \arrow[d, "f'"'] & \tilde{W} \arrow[d, "f"] \\ W' \arrow[r, "g"'] & W \end{tikzcd}$$

of noetherian normal integral schemes such that f and f' are proper, and such that the canonical morphisms $\mathscr{O}_W \to f_*\mathscr{O}_{\tilde{W}}$ and $\mathscr{O}_{W'} \to f'_*\mathscr{O}_{\tilde{W}'}$ are isomorphisms. Suppose that there is a finite group H acting on $\tilde{W}'$ and W', which is compatible with f', and suppose that $\tilde{g}$ and g can be identified with the quotients by H. Suppose that $\imath$ and $\imath'$ are invertible sheaves of ideals over $\tilde{W}$ and $\tilde{W}'$, respectively, such that $\imath' \cong \tilde{g}^*\imath$. For each integer $d \geq 1$, set $\mathcal{I}^{(d)} := f_*\imath^{\otimes d}$ and $\mathcal{I}' := f'_*\imath'$. Then $\mathcal{I}^{(d)}$ and $\mathcal{I}'$ are coherent sheaves of ideals over W and W', respectively.

Suppose the canonical morphism $\imath^{\otimes d} \to (\tilde{g}_*\tilde{g}^*\imath^{\otimes d})^H$ is an isomorphism for every integer $d \geq 1$ (which is automatic when $\tilde{g}$ is flat). Suppose that the canonical morphism $(f')^{-1}\mathcal{I}' \cdot \mathscr{O}_{\tilde{W}'} \to \imath'$ of coherent $\mathscr{O}_{\tilde{W}'}$-ideals is an isomorphism, and that the induced canonical morphism $\mathrm{NBl}_{\mathcal{I}'}(f') : \tilde{W}' \to \mathrm{NBl}_{\mathcal{I}'}(W')$ (see Definition 7.3.2.2) is an isomorphism. Then, for some integer $d_0 \geq 1$, the canonical morphism $f^{-1}\mathcal{I}^{(d_0)} \cdot \mathscr{O}_{\tilde{W}} \to \imath^{\otimes d_0}$ of coherent $\mathscr{O}_{\tilde{W}}$-ideals is also an isomorphism, and the induced canonical morphism $\mathrm{NBl}_{\mathcal{I}^{(d_0)}}(f) : \tilde{W} \to \mathrm{NBl}_{\mathcal{I}^{(d_0)}}(W)$ is an isomorphism.

7.3.3 Main Result on Projectivity of Toroidal Compactifications

Let $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ be any *projective* compatible choice of smooth rational polyhedral cone decomposition data, with a compatible collection $\mathsf{pol} = \{\mathsf{pol}_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ of polarization functions as in Definition 7.3.1.3.

Definition 7.3.3.1. *Let us retain the setting of Σ, pol, and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} = \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ as above. According to 3 of Theorem 6.4.1.1, the complement $\mathsf{D}_{\infty,\mathcal{H}}$ of $\mathsf{M}_{\mathcal{H}}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} = \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (with its reduced structure) is a relative Cartier divisor with normal crossings, each of whose irreducible components is an irreducible component of some $\overline{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ that is the closure of some strata $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ labeled by the equivalence class $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ of some triple $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$ with σ a one-dimensional cone in the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$. Let $\jmath_{\mathcal{H},\mathsf{pol}}$ be the invertible sheaf of ideals over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ supported on $\mathsf{D}_{\infty,\mathcal{H}}$ such that the order of $\jmath_{\mathcal{H},\mathsf{pol}}$ along each irreducible component of $\overline{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is the value of $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ at the $\mathbb{Z}_{>0}$-generator of $\sigma \cap \mathbf{S}^\vee_{\Phi_{\mathcal{H}}}$ for some (and hence every) choice of representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)$. This is well defined because of the compatibility condition for $\mathsf{pol} = \{\mathsf{pol}_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ as in Definition 7.3.1.3.*

Definition 7.3.3.2. *For each integer $d \geq 1$, let $\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}} := \oint_{\mathcal{H},*}(\jmath^{\otimes d}_{\mathcal{H},\mathsf{pol}})$, where $\oint_{\mathcal{H}} : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathsf{M}^{\min}_{\mathcal{H}}$ is the canonical morphism (as described in 3 of Theorem 7.2.4.1). (Then $\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}}$ is a coherent $\mathscr{O}_{\mathsf{M}^{\min}_{\mathcal{H}}}$-ideal because $\oint_{\mathcal{H}}$ is proper and because the canonical morphism $\mathscr{O}_{\mathsf{M}^{\min}_{\mathcal{H}}} \to \oint_{\mathcal{H},*}\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ is an isomorphism.)*

Let us introduce the following condition for $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ and $\mathsf{pol} = \{\mathsf{pol}_{\Phi_{\mathcal{H}}}\}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ (cf. Lemma 7.3.1.7):

Condition 7.3.3.3 (cf. [16, Ch. IV, §2, p. 329] and [42, Ch. V, §5, p. 178])**.** *For each representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ of a cusp label, and for each vertex ℓ_0 of $K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ corresponding to a top-dimensional cone σ_0, we have*

$$\langle \ell_0, x\rangle < \langle \gamma \cdot \ell_0, x\rangle$$

for all $x \in \overline{\sigma}_0 \cap \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ and all $\gamma \in \Gamma_{\Phi_{\mathcal{H}}}$ such that $\gamma \neq 1$.

Let us state the main result of this section. Note that the running neatness assumption on $\mathcal{H}$ (in this section) is indispensable, because we need $\mathsf{M}_{\mathcal{H}}$ to be a scheme before we investigate whether its compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is a scheme for some choice of Σ.

Theorem 7.3.3.4 (cf. [16, Ch. IV, §2.1, Thm.] and [42, Ch. V, Thm. 5.8])**.** *Suppose Σ is projective with a compatible collection pol of polarization functions as in Definition 7.3.1.3, suppose $\jmath_{\mathcal{H},\mathsf{pol}}$ is defined over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} = \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ as in Definition 7.3.3.1, and suppose $\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}}$ is defined over $\mathsf{M}^{\min}_{\mathcal{H}}$ as in Definition 7.3.3.2 for each integer $d \geq 1$. Then there exists an integer $d_0 \geq 1$ such that the following are true:*

1. *The canonical morphism $\oint_{\mathcal{H}}^{-1}\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}} \cdot \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to \jmath^{\otimes d_0}_{\mathcal{H},\mathsf{pol}}$ of coherent $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$-ideals is an isomorphism, which induces a canonical morphism*

$$\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}}) : \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \to \mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\mathsf{M}^{\min}_{\mathcal{H}})$$

by the universal property of the normalization of blowup as in Definition 7.3.2.2.

2. *The canonical morphism* $\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}})$ *above is an isomorphism.*

In particular, $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *is a scheme projective (and smooth) over* S_0. *If Condition 7.3.3.3 is satisfied, then the above two statements are true for every* $d_0 \geq 3$.

The proof can be divided into three rather independent parts. The first is the following reduction step:

Reduction to the case Condition 7.3.3.3 is satisfied. By Lemma 7.3.1.7, there exists a normal open compact subgroup $\mathcal{H}'$ of $\mathcal{H}$ such that Condition 7.3.3.3 is satisfied by the $\Sigma^{(\mathcal{H}')} = \{\Sigma_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ and $\mathsf{pol}^{(\mathcal{H}')} = \{\mathsf{pol}_{\Phi_{\mathcal{H}'}}\}_{[(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})]}$ in Construction 7.3.1.6, and such that $\Sigma^{(\mathcal{H}')}$ is smooth.

Suppose that Theorem 7.3.3.4 is true for $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}'} = \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma^{(\mathcal{H}')}}$, $\jmath^{\otimes d_0'}_{\mathcal{H}',\mathsf{pol}^{(\mathcal{H}')}}$, and $\mathcal{J}^{(d_0')}_{\mathcal{H}',\mathsf{pol}^{(\mathcal{H}')}}$ for some integer $d_0' \geq 1$. In particular, $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}'}$ is projective and smooth over S_0.

By construction, the surjections $\Xi_{\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'}}(\sigma) \twoheadrightarrow \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ are finite flat (with possible ramification along the boundary strata) whenever $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ is induced by $(\Phi_{\mathcal{H}'},\delta_{\mathcal{H}'})$ and σ is a cone in the cone decomposition $\Sigma_{\Phi_{\mathcal{H}'}} = \Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}'}} \cong \mathbf{P}_{\Phi_{\mathcal{H}}}$. Therefore, the canonical surjection $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma^{(\mathcal{H}')}} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (given by Proposition 6.4.2.9) is finite flat. It is the unique finite flat extension of the canonical (finite étale) surjection $\mathsf{M}_{\mathcal{H}'} \twoheadrightarrow \mathsf{M}_{\mathcal{H}}$. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma^{(\mathcal{H}')}}$ is projective and smooth over S_0, the quotient by $\mathcal{H}/\mathcal{H}'$ is also projective and isomorphic to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ over S_0 (by [39, V, 4.1] and by Zariski's main theorem (Proposition 7.2.3.4) as in the proof of Corollary 7.2.5.2). Moreover, we know that $\jmath_{\mathcal{H}',\mathsf{pol}^{(\mathcal{H}')}} \cong (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma^{(\mathcal{H}')}} \twoheadrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma})^*\jmath_{\mathcal{H},\mathsf{pol}}$ by construction. Hence we have verified all the assumptions of Proposition 7.3.2.3, whose application completes the reduction step. □

Now let us prove 1 and 2 of Theorem 7.3.3.4 separately under the assumption that Condition 7.3.3.3 is satisfied.

Proof of 1 of Theorem 7.3.3.4. Assume that Condition 7.3.3.3 holds. To verify that $\oint_{\mathcal{H}}^{-1}\mathcal{J}_{\mathcal{H},\mathsf{pol}} \cdot \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to \jmath_{\mathcal{H},\mathsf{pol}}$ is an isomorphism of coherent $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$-ideals, it suffices to verify the same statement along the completions of strict localizations of $\mathsf{M}^{\min}_{\mathcal{H}}$ at its geometric points. Let $\bar{x}$ be a geometric point along the $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$. According to Corollary 7.2.3.18, we may identify $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]}$ with $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. According to Proposition 7.2.3.16, for each representative $(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})$ of $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}})]$, we have a canonical isomorphism $\mathscr{O}_{(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}} \cong \Big[\prod\limits_{\ell\in\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}} (\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}\Big]^{\Gamma_{\Phi_{\mathcal{H}}}}$ given by (7.2.3.17). As in the proof of Proposition 7.2.3.16, the isomorphism (7.2.3.17) is obtained by taking the *common intersection* of the rings of regular functions over the various completions of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ along the fibers of the structural morphisms $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ over $\bar{x}$. The structural sheaf of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (as an $\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-algebra) can be described symbolically as $\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}} \cong \hat{\bigoplus\limits_{\ell\in\sigma^\vee}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$, and the global sections of its completion along (the fiber over) $\bar{x}$ is isomorphic to $\hat{\bigoplus\limits_{\ell\in\sigma^\vee}} (\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$ (as an $\mathscr{O}_{(\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}})^\wedge_{\bar{x}}}$-algebra).

Let $d \geq 1$ be any integer. Let us first identify the pullback $(\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ of $\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}}$ to $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}$. For each one-dimensional cone $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}} \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ in $\Sigma_{\Phi_{\mathcal{H}}}$, let s_σ be a $\mathbb{Z}_{>0}$-generator of $\sigma \cap \mathbf{S}^\vee_{\Phi_{\mathcal{H}}}$. By definition, the order of $\jmath_{\mathcal{H},\mathsf{pol}}$ along the σ-stratum of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ is given by the value of $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ at s_σ, and

$$\begin{aligned}\sigma_0^\vee &= \{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} : \langle \ell, y\rangle > 0 \ \forall y \in \sigma\}\\ &= \{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} : \langle \ell, y\rangle \geq 1 \ \forall y \in \sigma \cap \mathbf{S}^\vee_{\Phi_{\mathcal{H}}}\}\\ &= \{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} : \langle \ell, s_\sigma\rangle \geq 1\}.\end{aligned}$$

Therefore, in $\hat{\bigoplus\limits_{\ell\in\sigma^\vee}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$, the sheaf of ideals defining the σ-stratum consists of sections whose nonzero terms are supported on those ℓ such that $\langle \ell, s_\sigma\rangle \geq 1$, and hence the pullback of $\jmath^{\otimes d}_{\mathcal{H},\mathsf{pol}}$ consists of sections whose nonzero terms are supported on those ℓ such that $\langle \ell, s_\sigma\rangle \geq d \cdot \mathsf{pol}_{\Phi_{\mathcal{H}}}(s_\sigma)$, or equivalently such that $\langle \ell, t_\sigma\rangle \geq d$ for $t_\sigma := (\mathsf{pol}_{\Phi_{\mathcal{H}}}(s_\sigma))^{-1}s_\sigma$. Note that t_σ is the unique boundary of the half line $\sigma \cap K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ by definition of $K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$, and we have

$$\begin{aligned}K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}} &= \{x \in (\mathbf{S}_{\Phi_{\mathcal{H}}}) \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \langle x, y\rangle \geq 1 \ \forall y \in K_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}\}\\ &= \{x \in (\mathbf{S}_{\Phi_{\mathcal{H}}}) \underset{\mathbb{Z}}{\otimes} \mathbb{R} : \langle x, t_\sigma\rangle \geq 1 \ \forall t_\sigma\},\end{aligned}$$

the first equality being the definition, and the second equality being true because the faces of the boundary of $\overline{K}_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ are spanned by the t_σ's. Therefore, for each particular $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}$, the condition $\langle \ell, t_\sigma\rangle \geq d$ for all t_σ is equivalent to the condition that $\ell \in d \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$. By *taking common intersections* of global sections over the completion along fibers over $\bar{x}$, we see that the sheaf of ideals $(\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}} \subset \mathscr{O}_{(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}}} \cong \Big[\prod\limits_{\ell\in\mathbf{P}^\vee_{\Phi_{\mathcal{H}}}} (\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}\Big]^{\Gamma_{\Phi_{\mathcal{H}}}}$ consists of sections whose nonzero terms are supported on those $\ell \in d \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$.

Now let us investigate the pullback of the canonical morphism $\oint_{\mathcal{H}}^{-1}\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}} \cdot \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to \jmath^{\otimes d}_{\mathcal{H},\mathsf{pol}}$ under $(\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}} \to \mathsf{M}^{\min}_{\mathcal{H}}$. The goal is to show that the pullback is an isomorphism if $d \geq 3$.

Since the strata corresponding to top-dimensional cones meet all the irreducible components of the fiber of $\oint_{\mathcal{H}}$ over $\bar{x}$, it suffices to show that the morphism is an isomorphism after pullback to the completion of each stratum labeled by a top-dimensional cone. Let τ be any top-dimensional cone in $\Sigma_{\Phi_{\mathcal{H}}}$, which corresponds to a vertex ℓ_0 of $K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ (by 4 of Proposition 7.3.1.2). Then $\tau^\perp = \{0\}$, $\tau_0^\vee = \tau^\vee - \{0\}$, and hence the ideal of definition of $\hat{\bigoplus\limits_{\ell\in\tau^\vee}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$, name the sheaf of ideals defining the τ-stratum, consists of sections whose nonzero terms are supported on those nonzero ℓ in $\tau^\vee$. By construction, the (coherent ideal) pullback of $\jmath^{\otimes d}_{\mathcal{H},\mathsf{pol}}$ to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ consists of sections in $\hat{\bigoplus\limits_{\ell\in\tau^\vee}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$ whose nonzero terms are supported on those ℓ lying in $d \cdot \ell_0 + \tau^\vee$, the translation of $\tau^\vee$ by $d \cdot \ell_0$. In other words, it is the sheaf of invertible ideals generated by $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d \cdot \ell_0)$.

Since ℓ_0 is dual to the top-dimensional cone τ in $\mathbf{P}^+_{\Phi_{\mathcal{H}}}$, we claim that the invertible sheaf $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_0)$ over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is relatively ample over $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. To show this, let us reinstate in this paragraph the notation of Sections 6.2.2, 6.2.3, and 6.2.4. Up to choosing a $\mathbb{Z}$-basis $y_1, \ldots, y_r$ of Y, and by completion of squares for quadratic forms, there exists some integer $N \geq 1$ such that $N \cdot \ell_0$ can be represented as a positive definite matrix of the form $ue\,{}^t u$, where e and u are matrices with integer coefficients, and where $e = \mathrm{diag}(e_1, \ldots, e_r)$ is diagonal with positive entries. Consider the *finite morphism* defined by the composition $C_{\Phi_\mathcal{H},\delta_\mathcal{H}} \overset{\text{can.}}{\to} \ddot{C}_{\Phi_1} \overset{\mathrm{pr}_1}{\to} \underline{\mathrm{Hom}}_\mathcal{O}(Y, A) \overset{\text{can.}}{\to} \underline{\mathrm{Hom}}_\mathbb{Z}(Y, A) \overset{u^*}{\to} \underline{\mathrm{Hom}}_\mathbb{Z}(Y, A)$, under which a positive tensor power of $\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell_0)$ is isomorphic to a positive tensor power of the pullback of the line bundle $\underset{1\le i\le r}{\otimes} (\mathrm{pr}_i^*(\mathrm{Id}_A, \lambda_A)^*\mathcal{P}_A)^{\otimes e_i}$ over A. Since λ_A is a polarization (cf. Definition 1.3.2.16), and since all the e_i's are positive, we see that $\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell_0)$ is relatively ample over $\mathsf{M}_\mathcal{H}^{\mathsf{Z}_\mathcal{H}}$, as desired.

Since $d \geq 3$, by Lefschetz's theorem (see, for example, [94, §17, Thm., p. 163]), the invertible sheaf $(\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(d \cdot \ell_0))^\wedge_{\bar{x}} \cong (\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell_0)^{\otimes d})^\wedge_{\bar{x}}$ is generated by its global sections over $(C_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{x}}$, namely, the sections of $(\underline{\mathrm{FJ}}^{(d\cdot\ell_0)}_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{x}} \cong (\mathrm{p}_{\Phi_\mathcal{H},\delta_\mathcal{H}})_*(\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(d\cdot \ell_0))^\wedge_{\bar{x}}$.

Let us write each section f of $(\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ as an infinite sum

$$f = \sum_{\ell \in d\cdot\ell_0+\tau^\vee} f^{(\ell)},$$

where each $f^{(\ell)}$ is a section of $(\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{x}}$. Since f is $\Gamma_{\Phi_\mathcal{H}}$-invariant, we may decompose it into an infinite sum

$$f = \sum_{[\ell]\in(\Gamma_{\Phi_\mathcal{H}}\cdot(d\cdot\ell_0+\tau^\vee))/\Gamma_{\Phi_\mathcal{H}}} f^{[\ell]}$$

of subseries $f^{[\ell]} = \sum\limits_{\ell\in[\ell]} f^{(\ell)}$, where each $[\ell]$ is by definition the $\Gamma_{\Phi_\mathcal{H}}$-orbit of some $\ell \in d\cdot\ell_0 + \tau^\vee$.

Since the ideal of definition of $\mathscr{O}_{\mathfrak{X}_{\Phi_\mathcal{H},\delta_\mathcal{H},\tau}} \cong \hat{\bigoplus\limits_{\ell\in\tau^\vee}} (\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell))^\wedge_{\bar{x}}$ consists of sections whose nonzero terms are supported on those ℓ's in $\tau_0^\vee = \tau^\vee - \{0\}$, we see that $f^{[d\cdot\ell_0]} = \sum\limits_{\gamma\in\Gamma_{\Phi_\mathcal{H}}} f^{(\gamma\cdot(d\cdot\ell_0))}$ is a *leading subseries* of f in the sense that $f - f^{[d\cdot\ell_0]}$ has a higher degree than $f^{[d\cdot\ell_0]}$ in the natural grading defined by the ideal of definition of $\hat{\bigoplus\limits_{\ell\in\tau^\vee}} (\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell))^\wedge_{\bar{x}}$. If Condition 7.3.3.3 is satisfied, then we have the stronger statement that $f^{(d\cdot\ell_0)}$ is a *leading term* of $f^{[d\cdot\ell_0]}$ in the sense that $f^{[d\cdot\ell_0]} - f^{(d\cdot\ell_0)}$ (or equivalently $f - f^{(d\cdot\ell_0)}$) has a higher degree than $f^{(d\cdot\ell_0)}$ in the natural grading defined by the ideal of definition of $\hat{\bigoplus\limits_{\ell\in\tau^\vee}} (\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\ell))^\wedge_{\bar{x}}$. (These are abused terminologies because the leading subseries or terms might be zero.) As a result, if $f_1, \ldots, f_k$ are sections of $(\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ generating $(\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$, then the leading terms $f_1^{(d\cdot\ell_0)}, \ldots, f_k^{(d\cdot\ell_0)}$ of $f_1, \ldots, f_k$, respectively, generate the (coherent ideal) pullback of $(\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ to $(\mathfrak{X}_{\Phi_\mathcal{H},\delta_\mathcal{H},\tau})^\wedge_{\bar{x}}$, and also generate the pullback of $(\Psi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(d\cdot\ell_0))^\wedge_{\bar{x}}$.

Comparing with the above description of the pullback of $\mathcal{I}^{\otimes d}_{\mathcal{H},\mathsf{pol}}$ to $\mathfrak{X}_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma}$, we see that the pullback of $\oint_\mathcal{H}^{-1}\mathcal{J}^{(d)}_{\mathcal{H},\mathsf{pol}} \cdot \mathscr{O}_{\mathsf{M}_\mathcal{H}^{\mathrm{tor}}} \to \mathcal{I}^{\otimes d}_{\mathcal{H},\mathsf{pol}}$ under $(\mathsf{M}_\mathcal{H}^{\min})^\wedge_{\bar{x}} \to \mathsf{M}_\mathcal{H}^{\min}$ is an isomorphism when $d \geq 3$. Since $\bar{x}$ is arbitrary, this proves 1 of Theorem 7.3.3.4 by taking any integer $d_0 \geq 3$. □

Proof of 2 of Theorem 7.3.3.4. Let $d_0 \geq 3$ be any integer such that 1 of Theorem 7.3.3.4 is satisfied. In this case, there is a canonical morphism $\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_\mathcal{H}) : \mathsf{M}_\mathcal{H}^{\mathrm{tor}} \to \mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\mathsf{M}_\mathcal{H}^{\min})$, and our goal is to show that this morphism $\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_\mathcal{H})$ is an isomorphism. By Zariski's main theorem (Proposition 7.2.3.4), it suffices to show that $\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_\mathcal{H})$ is quasi-finite (and hence finite).

As in the proof of 1 of Theorem 7.3.3.4, we may verify this by pulling back to the completions of the strict localizations of $\mathsf{M}_\mathcal{H}^{\min}$ at its geometric points. Let us assume the same setting as in the proof of 1 of Theorem 7.3.3.4, with $(\Phi_\mathcal{H}, \delta_\mathcal{H})$ a representative of a cusp label and $\bar{x}$ a geometric point on the $[(\Phi_\mathcal{H}, \delta_\mathcal{H})]$-stratum of $\mathsf{M}_\mathcal{H}^{\min}$. Consider a cone $\sigma \subset \mathbf{P}^+_{\Phi_\mathcal{H}}$ in $\Sigma_{\Phi_\mathcal{H}}$. Let V be any complete discrete valuation ring with the valuation $\upsilon : \mathrm{Inv}(V) \to \mathbb{Z}$ and with an algebraically closed residue field k, and let $y : \mathrm{Spf}(V) \to \mathsf{M}_\mathcal{H}^{\mathrm{tor}}$ be any morphism centered at a geometric point over $\bar{x}$ that factors through $\mathrm{Spf}(V) \overset{\tilde{y}}{\to} \mathfrak{X}_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma} \overset{\text{can.}}{\to} \mathsf{M}_\mathcal{H}^{\mathrm{tor}}$, such that the induced morphisms $\mathrm{Spec}(V) \to \Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}(\sigma)$ and $\mathrm{Spec}(V) \to \mathsf{M}_\mathcal{H}^{\mathrm{tor}}$ map the generic point of $\mathrm{Spec}(V)$ to $\Xi_{\Phi_\mathcal{H},\delta_\mathcal{H}}$ and $\mathsf{M}_\mathcal{H}$, respectively. Let

$$\bar{y} : \mathrm{Spec}(k) \to \mathsf{M}_\mathcal{H}^{\mathrm{tor}}$$

and

$$\bar{z} := \mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_\mathcal{H}) \circ \bar{y} : \mathrm{Spec}(k) \to \mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\mathsf{M}_\mathcal{H}^{\min})$$

be the induced morphisms. To show that $\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_\mathcal{H})$ is quasi-finite, it suffices to show that there are only finitely many $\bar{y}$ inducing the same $\bar{z}$ (for each σ), because there are only finitely many strata $\mathsf{Z}_{[(\Phi_\mathcal{H},\delta_\mathcal{H},\sigma)]}$ of $\mathsf{M}_\mathcal{H}^{\mathrm{tor}}$ lying above the strata $\mathsf{Z}_{[(\Phi_\mathcal{H},\delta_\mathcal{H})]}$ of $\mathsf{M}_\mathcal{H}^{\min}$.

Consider the composition of canonical morphisms

$$\begin{aligned}&\mathrm{Spf}(V) \overset{\tilde{y}}{\to} (\mathfrak{X}_{\Phi_\mathcal{H},\delta_\mathcal{H},\sigma})^\wedge_{\bar{x}} \overset{\text{can.}}{\to} (\mathsf{M}_\mathcal{H}^{\mathrm{tor}})^\wedge_{\bar{x}} \\ &\overset{\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_\mathcal{H})}{\to} (\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\mathsf{M}_\mathcal{H}^{\min}))^\wedge_{\bar{x}} \overset{\text{can.}}{\to} \mathrm{NBl}_{(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}_\mathcal{H}^{\min})^\wedge_{\bar{x}})\end{aligned} \tag{7.3.3.5}$$

of proper formal algebraic spaces over $(\mathsf{M}_\mathcal{H}^{\min})^\wedge_{\bar{x}}$, where each of the $(\,\cdot\,)^\wedge_{\bar{x}}$ stands for the pullback under $(\mathsf{M}_\mathcal{H}^{\min})^\wedge_{\bar{x}} \to \mathsf{M}_\mathcal{H}^{\min}$, and where the last term is defined in the categories of algebraizable formal schemes in the obvious sense. As we saw in the proof of 1 of Theorem 7.3.3.4, the sheaf of ideals $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}} \subset \mathscr{O}_{(\mathsf{M}_\mathcal{H}^{\min})^\wedge_{\bar{x}}} \cong \Big[\prod\limits_{\ell\in\mathbf{P}^\vee_{\Phi_\mathcal{H}}} (\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{x}}\Big]^{\Gamma_{\Phi_\mathcal{H}}}$ consists of sections whose nonzero terms are supported on those $\ell \in d_0 \cdot K^\vee_{\mathsf{pol}_{\Phi_\mathcal{H}}}$.

By definition of $\mathrm{Bl}_{(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}_\mathcal{H}^{\min})^\wedge_{\bar{x}})$, it has an open covering by affine open formal subschemes $\{\mathfrak{U}_f\}_f$ labeled by nonzero sections f of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$. Over each $\mathfrak{U}_f$, the section f of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ becomes a local generator of the (coherent ideal) pullback of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$. (We allow $\mathfrak{U}_f$ to be empty if f is nowhere a generator.) For each such $\mathfrak{U}_f$, let $\tilde{\mathfrak{U}}_f$ denote the open formal subscheme of $\mathrm{NBl}_{(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}_\mathcal{H}^{\min})^\wedge_{\bar{x}})$ supported on the preimage of $\mathfrak{U}_f$.

Let us fix the choice of a nonzero section f of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ such that (7.3.3.5) factors through $\mathfrak{U}_f$. Then we can evaluate υ on sections of $\Gamma(\mathfrak{U}_f, \mathscr{O}_{\mathfrak{U}_f})$ by pullback under (7.3.3.5). (We adopt the convention that $\upsilon(0) = +\infty$.) By writing $f = \sum\limits_{\ell\in\mathbf{S}_{\Phi_\mathcal{H}}\cap(d_0\cdot K^\vee_{\mathsf{pol}_{\Phi_\mathcal{H}}})} f^{(\ell)}$, where each $f^{(\ell)}$ is a section of $(\underline{\mathrm{FJ}}^{(\ell)}_{\Phi_\mathcal{H},\delta_\mathcal{H}})^\wedge_{\bar{x}}$, we may assume that there is a *leading term* $f^{(\ell_0)} \neq 0$, for some $\ell_0 \in \mathbf{S}_{\Phi_\mathcal{H}} \cap (d_0 \cdot K^\vee_{\mathsf{pol}_{\Phi_\mathcal{H}}})$, such that $v(f^{(\ell_0)}) \leq v(f^{(\ell)})$ for all $\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}} \cap (d_0 \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}})$. In this case, $f^{(\ell_0)}$ is a generator of the pullback of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$, and necessarily also a generator of the pullback of $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_0)$. This forces $v(f^{(\ell_0)}) > 0$ because y is centered on the support of the pullback of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$. Without loss of generality, we may and we shall assume that $f^{(\ell)} \neq 0$ exactly when $\ell \in \Gamma_{\Phi_{\mathcal{H}}} \cdot \ell_0$. Let $\mathfrak{V}_{f^{(\ell_0)}}$ denote the maximal open formal subscheme of $(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$ over which $f^{(\ell_0)}$ is a generator of the pullback of $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_0)$, and let $\mathfrak{W}_{f^{(\ell_0)}}$ denote the preimage of $\mathfrak{V}_{f^{(\ell_0)}}$ under the canonical morphism $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}} \to (C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$. Then the proof of 1 of Theorem 7.3.3.4 shows that $\mathfrak{W}_{f^{(\ell_0)}}$ is the preimage of $\mathfrak{U}_f$ under the canonical morphism $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}} \to \mathrm{Bl}_{(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}}((\mathsf{M}^{\min}_{\mathcal{H}})^\wedge_{\bar{x}})$.

Suppose $\ell_{\mathrm{gen}} \in K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$. Then ℓ_{gen} can be identified with some positive definite pairing over $Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ (as in Section 6.2.5), and hence its stabilizer in $\Gamma_{\Phi_{\mathcal{H}}}$ can be identified with a discrete subgroup of a compact orthogonal subgroup of $\mathrm{GL}_{\mathbb{R}}(Y \underset{\mathbb{Z}}{\otimes} \mathbb{R})$, which must be finite and hence trivial, by the neatness of $\mathcal{H}$. Let $g^{(d_0\cdot\ell_{\mathrm{gen}})}$ be a section of $(\underline{\mathrm{FJ}}^{(d_0\cdot\ell_{\mathrm{gen}})}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}$, namely, a global section of $(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d_0\cdot\ell_{\mathrm{gen}}))^\wedge_{\bar{x}}$. Since the stabilizer of ℓ_{gen} in $\Gamma_{\Phi_{\mathcal{H}}}$ is trivial, we obtain a section $g^{[d_0\cdot\ell_{\mathrm{gen}}]} = \sum\limits_{\gamma\in\Gamma_{\Phi_{\mathcal{H}}}} g^{(\gamma\cdot(d_0\cdot\ell_{\mathrm{gen}}))}$ of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ with $g^{(\gamma\cdot(d_0\cdot\ell_{\mathrm{gen}}))} := \gamma(g^{(d_0\cdot\ell_{\mathrm{gen}})})$ for each $\gamma \in \Gamma_{\Phi_{\mathcal{H}}}$, and the fact that $f^{(\ell_0)}$ is a generator of the (coherent ideal) pullback of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ shows, in particular, that $v(f^{(\ell_0)}) \leq v(g^{(d_0\cdot\ell_{\mathrm{gen}})})$. Since $\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_{\mathrm{gen}})$ is relatively ample over $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$, and since $d_0 \geq 3$ by assumption, the sheaf $(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d_0\cdot\ell_{\mathrm{gen}}))^\wedge_{\bar{x}} \cong (\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_{\mathrm{gen}})^{\otimes d_0})^\wedge_{\bar{x}}$ is generated by its global sections by Lefschetz's theorem (see, for example, [94, §17, Thm., p. 163]). Hence we have arrived at the conclusion that

$$0 < v(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell_0)) \leq v(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(d_0\cdot\ell_{\mathrm{gen}})) \qquad (7.3.3.6)$$

for every $\ell_{\mathrm{gen}} \in K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$.

Suppose ℓ_0 is a point in the *interior* of $d_0 \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$. By convexity of the polarization function, ℓ_0 is a linear combination $\ell_0 = \sum\limits_{1\leq i\leq k} r_i(d_0\cdot\ell_i)$, of elements ℓ_i of $K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ such that $r_i \in \mathbb{Q}_{\geq 0}$ for every $1 \leq i \leq k$ and $\sum\limits_{1\leq i\leq k} r_i > 1$. Let $N \geq 1$ be an integer such that $Nr_i \in \mathbb{Z}_{\geq 0}$ for every $1 \leq i \leq k$. By (7.3.3.6), we have the relation

$$\begin{aligned} 0 < v(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(N\cdot\ell_0)) &< \sum_{1\leq i\leq k} v(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(Nr_i\cdot\ell_0)) \\ &\leq \sum_{1\leq i\leq k} v(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(Nr_i(d_0\cdot\ell_i))) = v(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(N\cdot\ell_0)), \end{aligned}$$

which is impossible. Hence we see that ℓ_0 must be on the boundary of $d_0 \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$.

Suppose ℓ_0 lies on the face of $d_0 \cdot K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ dual to some $\tau \in \Sigma_{\Phi_{\mathcal{H}}}$ as in Lemma 7.3.1.9, so that there exist $\ell_1, \ldots, \ell_n \in \mathbf{S}_{\Phi_{\mathcal{H}}} \cap K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$ (which are not necessarily vertices of $K^\vee_{\mathsf{pol}_{\Phi_{\mathcal{H}}}}$) such that $\mathbb{R}_{\geq 0}\cdot\tau^\vee = \sum\limits_{1\leq i\leq n} \mathbb{R}_{\geq 0}\cdot(d_0\cdot\ell_i - \ell_0)$. By (7.3.3.6) (with $\ell_{\mathrm{gen}} = \ell_i$ there, for each i), we see that $v(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)) \geq 0$ for all $\ell \in \tau^\vee$. Since V and $y : \mathrm{Spf}(V) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ are arbitrary (as long as y factors through $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$, so that $v(\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)) \geq 0$ for all $\ell \in \sigma^\vee$; and as long as the composition (7.3.3.5) factors through $\mathfrak{U}_f$), we see that $\tau^\vee \subset \sigma^\vee$, which happens only when σ is a face of τ, because σ and τ are both in $\Sigma_{\Phi_{\mathcal{H}}}$. Consequently, $\sigma^\perp \cap \tau^\vee$ generates the group $\sigma^\perp$.

Since $\mathfrak{U}_f$ is tautological for the condition that f generates the (coherent ideal) pullback of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ to $\mathfrak{U}_f$, the above argument shows that pullback of sections of $\mathscr{O}_{\mathfrak{U}_f}$ to the open formal subscheme $\mathfrak{W}_{f^{(\ell_0)}}$ of $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma})^\wedge_{\bar{x}}$ form a subset of the sections of $\hat{\bigoplus\limits_{\ell\in\sigma^\vee}} (\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))^\wedge_{\bar{x}}$ over $\mathfrak{V}_{f^{(\ell_0)}}$ whose nonzero terms are supported on those ℓ lying in $\tau^\vee$. Moreover, it shows that the pullback to $\mathfrak{W}_{f^{(\ell_0)}}$ of sections of the (coherent ideal) pullback of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ to $\mathfrak{U}_f$ form a subset of the sections of $\hat{\bigoplus\limits_{\ell\in\sigma^\vee}} (\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))^\wedge_{\bar{x}}$ over $\mathfrak{V}_{f^{(\ell_0)}}$ whose nonzero terms are supported on those ℓ lying in $\ell_0 + \tau^\vee$, the translation of $\tau^\vee$ by ℓ_0. Then the (coherent ideal) pullback of $(\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}})^\wedge_{\bar{x}}$ to $\mathfrak{W}_{f^{(\ell_0)}}$ consists of sections of $\hat{\bigoplus\limits_{\ell\in\sigma^\vee}} (\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))^\wedge_{\bar{x}}$ over $\mathfrak{V}_{f^{(\ell_0)}}$ whose nonzero terms are supported on those ℓ lying in $\ell_0 + \sigma^\vee$, the translation of $\sigma^\vee$ by ℓ_0.

Since $\mathbb{R}_{>0}\cdot(\sigma^\perp \cap \tau^\vee)$ is disjoint from $\mathbb{R}_{>0}\cdot(\ell_0 + \sigma^\vee)$, the morphism $\Gamma(\tilde{\mathfrak{U}}_f, \mathscr{O}_{\tilde{\mathfrak{U}}_f}) \to k$ induced by $\bar{z} : \mathrm{Spec}(k) \to \mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\mathsf{M}^{\min}_{\mathcal{H}})$ determines a compatible collection of morphisms $\{\Gamma(\mathfrak{V}_{f^{(\ell_0)}}, (\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell))^\wedge_{\bar{x}}) \to k\}$ indexed by ℓ's in a finite index semisubgroup of $\sigma^\perp \cap \tau^\vee$. Since $\sigma^\perp \cap \tau^\vee$ generates the group $\sigma^\perp$, this determines a compatible collection of morphisms $\{\Gamma(\mathfrak{V}_{f^{(\ell_0)}}, (\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}) \to k\}$ indexed by ℓ's in a finite index subgroup of $\sigma^\perp$. Since the whole collection $\{\Gamma(\mathfrak{V}_{f^{(\ell_0)}}, (\Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^\wedge_{\bar{x}}) \to k\}_{\ell\in\sigma^\perp}$ determines the morphism $\bar{y} : \mathrm{Spf}(k) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \stackrel{\text{can.}}{\to} \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, this shows that $\mathrm{NBl}_{\mathcal{J}^{(d_0)}_{\mathcal{H},\mathsf{pol}}}(\oint_{\mathcal{H}})$ is quasi-finite, and hence is an isomorphism by Zariski's main theorem (Proposition 7.2.3.4), as desired. □

Appendix A

Algebraic Spaces and Algebraic Stacks

The main purpose of this appendix is to review the general concepts of algebraic spaces and algebraic stacks, which are useful for studying moduli problems. A secondary purpose is to fix our definitions and conventions, because there exist discrepancies among the existing works. (This appendix and the next one are reproduced from the first two chapters and the first two appendices of the author's master's thesis presented to National Taiwan University [78] in the spring of 2001.)

Our main references for algebraic spaces are [7], [8], and [73]. Our main references for algebraic stacks are [36], [11], and [83]. We would like to mention that de Jong has an ongoing online book project on foundation for the theory of stacks and related topics, which can be found on his website.

A.1 Some Category Theory

A.1.1 A Set-Theoretical Remark

In the standard axiomatic set theory (say, Zermelo–Fraenkel), some naive operations of sets are forbidden so that certain logical problems will not arise from these operations. For example, a collection formed by all sets should be called a *class*, but not a *set*.

However, mathematicians seldom need the full generality of axiomatic set theory. To avoid clumsy language, a common solution is to introduce a *universe*, namely, a large *set of sets* (using a naive terminology here) that is closed under all necessary operations, and to consider only those sets in this universe. Then, when we say that we are forming a set of certain sets, we are only forming a set of those corresponding sets in the universe. Hence no logical issue arises in such operations.

More precisely,

Definition A.1.1.1. *A* ***universe*** *is a nonempty set* U *with the following axioms:*

1. *If* $x \in \mathsf{U}$ *and if* $y \in x$*, then* $y \in \mathsf{U}$.
2. *If* $x, y \in \mathsf{U}$*, then* $\{x, y\} \in \mathsf{U}$.
3. *If* $x \in \mathsf{U}$*, then the power set* 2^x*, namely, the set formed by all subsets of* x*, is in* U.
4. *If* $(x_i)_{i \in I \in \mathsf{U}}$ *is a family of elements of* U*, then* $\bigcup\limits_{i \in I} x_i \in \mathsf{U}$.

For a more detailed exposition of the theory of universes, one may consult [12, I, 0] or [12, I, Appendice: Univers (par N. Bourbaki)].

Remark A.1.1.2. In what follows, we shall fix a choice of a universe that is sufficient for our need, and we shall not mention our choice of the universe again.

A.1.2 2-Categories and 2-Functors

Let us summarize some properties of 2-categories and 2-functors from [61] and [49].

Definition A.1.2.1. *A* ***2-category*** C *consists of the following data:*

1. *A set of objects* $\operatorname{Ob} \mathsf{C}$
2. *For each two objects* $X, Y \in \operatorname{Ob} \mathsf{C}$*, a category* $\operatorname{Hom}_{\mathsf{C}}(X, Y)$*, written also as* $\operatorname{Hom}(X, Y)$
3. *For each three objects* $X, Y, Z \in \operatorname{Ob} \mathsf{C}$*, a functor*
$$\mu_{X,Y,Z} : \operatorname{Hom}(X, Y) \times \operatorname{Hom}(Y, Z) \to \operatorname{Hom}(X, Z)$$

such that

(i) *the categories such as* $\operatorname{Hom}(X, Y)$ *are pairwise disjoint;*

(ii) *for each* $X \in \operatorname{Ob} \mathsf{C}$ *there exists the identity morphism* Id_X *in* $\operatorname{Hom}(X, X)$*, which is determined uniquely by the condition that, for each two objects* $X, Y \in \operatorname{Ob} \mathsf{C}$*,*
$$\mu_{X,X,Y}(\operatorname{Id}_X, \cdot\,) = \mu_{X,Y,Y}(\,\cdot\,, \operatorname{Id}_Y) = \operatorname{Id}_{\operatorname{Hom}(X,Y)};$$

(iii) *for each four objects* $X, Y, Z, T \in \operatorname{Ob} \mathsf{C}$*, we have the* ***associative law***
$$\mu_{X,Z,T} \circ (\mu_{X,Y,Z} \times \operatorname{Id}_{\operatorname{Hom}(Z,T)}) = \mu_{X,Y,T} \circ (\operatorname{Id}_{\operatorname{Hom}(X,Y)} \times \mu_{Y,Z,T}).$$

For each two objects X and Y in C, we call an object f in $\operatorname{Hom}(X, Y)$ a *1-morphism* and write it as $f : X \to Y$. Let f and g be two objects of $\operatorname{Hom}(X, Y)$. A morphism $\alpha : f \to g$ is called a *2-morphism*, represented in the form

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & Y \end{tikzcd}.$$

Let X, Y, and Z be three objects in C. For $f \in \operatorname{Ob}\operatorname{Hom}(X, Y)$ and $g \in \operatorname{Ob}\operatorname{Hom}(Y, Z)$ (resp. $\alpha \in \operatorname{Mor}\operatorname{Hom}(X, Y)$ and $\beta \in \operatorname{Mor}\operatorname{Hom}(Y, Z)$), we write $g \circ f$ (resp. $\beta \circ \alpha$) in place of $\mu_{X,Y,Z}(f, g) \in \operatorname{Ob}\operatorname{Hom}(X, Z)$ (resp. $\mu_{X,Y,Z}(\alpha, \beta) \in \operatorname{Mor}\operatorname{Hom}(X, Z)$).

Definition A.1.2.2. *Two objects* X *and* Y *of* C *are* ***equivalent*** *if there exist two 1-morphisms* $u : X \to Y$ *and* $v : Y \to X$ *and two invertible 2-morphisms (or 2-isomorphisms)* $\alpha : v \circ u \to \operatorname{Id}_X$ *and* $\beta : u \circ v \to \operatorname{Id}_Y$.

Definition A.1.2.3. *Consider the diagram*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Z \\ & Y \arrow[ur, "g"'] \arrow[u, phantom, "\Uparrow\alpha"] & \end{tikzcd}$$

of 1-morphisms. If α is a 2-morphism from $g \circ f$ to h, which is a 2-isomorphism, then we say that the diagram is ***commutative****. Diagrams in other forms are defined to be commutative in the same way.*

On the other hand, a diagram of 2-morphisms will be called commutative only if the compositions are ***equal****.*

Now we define the concept of a covariant 2-functor. (A contravariant 2-functor is defined in a similar way.)

Definition A.1.2.4. *Let C and C' be two 2-categories. A* ***2-functor*** *$F : \mathsf{C} \to \mathsf{C}'$ consists of a map*

$$F : \operatorname{Ob}\mathsf{C} \to \operatorname{Ob}\mathsf{C}'$$

and a functor

$$F : \operatorname{Hom}(X, Y) \to \operatorname{Hom}(F(X), F(Y))$$

between each two objects $X, Y \in \operatorname{Ob}\mathsf{C}$ such that

1. *for each $X \in \operatorname{Ob}\mathsf{C}$, $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$;*
2. *for each $X, Y \in \operatorname{Ob}\mathsf{C}$ and $f \in \operatorname{Hom}(X, Y)$, $F(\mathrm{Id}_f) = \mathrm{Id}_{F(f)}$;*
3. *for each diagram*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

in C, there exists a 2-isomorphism $\varepsilon_{g,f} : F(g) \circ F(f) \to F(g \circ f)$ such that

(a) *$\varepsilon_{f,\mathrm{Id}_X} = \varepsilon_{\mathrm{Id}_Y, f} = \mathrm{Id}_{F(f)}$;*

(b) *ε is associative—the diagram*

$$\begin{tikzcd} F(h) \circ F(g) \circ F(f) \arrow[r, "\varepsilon_{h,g} \times \mathrm{Id}_{F(f)}"] \arrow[d, "\mathrm{Id}_{F(h)} \times \varepsilon_{g,f}"'] & F(h \circ g) \circ F(f) \arrow[d, "\varepsilon_{h \circ g, f}"] \\ F(h) \circ F(g \circ f) \arrow[r, "\varepsilon_{h, g \circ f}"] & F(h \circ g \circ f) \end{tikzcd}$$

is commutative;

4. *for each pair of 2-morphisms $\alpha : f \to f'$ and $\beta : g \to g'$ in C, we have $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$;*
5. *for each pair of 2-morphisms $\alpha : f \to f'$ and $\beta : g \to g'$ in C, the diagram*

$$\begin{tikzcd} F(g) \circ F(f) \arrow[r, "F(\beta) \circ F(\alpha)"] \arrow[d, "\varepsilon_{g,f}"'] & F(g') \circ F(f') \arrow[d, "\varepsilon_{g',f'}"] \\ F(g \circ f) \arrow[r, "F(\beta \circ \alpha)"] & F(g' \circ f') \end{tikzcd}$$

is commutative.

By abuse of language, the last condition is usually written as $F(\beta) \circ F(\alpha) = F(\beta \circ \alpha)$. This equality does not make sense literally; but if we fix a choice for all the 2-isomorphisms $\varepsilon_{g,f}$, then there's a unique way to interpret this equality.

Definition A.1.2.5. *Let F and G be two 2-functors from C to C'. A* ***1-morphism*** *(or a* ***natural transformation****) φ from F to G assigns to each object A in C a 1-morphism $\varphi(A) : F(A) \to G(A)$ in C', and to each 1-morphism $f : A \to B$ in C a 2-morphism $\varphi(f) : F(f) \to G(f)$, such that for each 1-morphism $g : A \to B$, the diagram of 1-morphisms*

$$\begin{tikzcd} F(A) \arrow[r, "\varphi(A)"] \arrow[d, "F(g)"'] & G(A) \arrow[d, "G(g)"] \\ F(B) \arrow[r, "\varphi(B)"] & G(B) \end{tikzcd}$$

is commutative (up to 2-isomorphism), and such that for each 2-morphism $\alpha : f \to g$ of 1-morphisms, the diagram of 2-morphisms

$$\begin{tikzcd} F(f) \arrow[r, "\varphi(f)"] \arrow[d, "F(\alpha)"'] & G(f) \arrow[d, "G(\alpha)"] \\ F(g) \arrow[r, "\varphi(g)"] & G(g) \end{tikzcd}$$

is also commutative.

Definition A.1.2.6. *Let φ and ψ be two 1-morphisms of functors from the 2-functor F to the 2-functor G. A 2-morphism θ from φ to ψ assigns to each object A in C a 2-morphism $\theta(A) : \varphi(A) \to \psi(A)$ between 1-morphisms, and to each 1-morphism f in C an identity $\theta(f) : \varphi(f) \overset{=}{\to} \psi(f)$ between 2-morphisms.*

The last statement asserts that a 2-morphism between two 1-morphisms φ and ψ exists only when $\varphi(f) = \psi(f)$ for each 1-morphism f in C.

For each two 2-functors F and G, the 1-morphisms and 2-morphisms from F to G defined above form a category $\mathsf{Hom}(F, G)$ whose objects are 1-morphisms of 2-functors and whose morphisms are 2-morphisms between 1-morphisms.

Remark A.1.2.7. It is not practical to naively define a *(1-)isomorphism* between 2-functors to be a 1-morphism with an inverse. It is more convenient to define it to be a 1-morphism with a *quasi-inverse* (defined in an obvious way analogously to the quasi-inverse of an equivalence of categories) such that their compositions are 2-isomorphic to the identity.

Given any (1-)category C, we may define a 2-category by making the set $\operatorname{Hom}(X, Y)$ into a category whose objects are elements of $\operatorname{Hom}(X, Y)$ and whose only morphisms are identities.

Given a 2-category C, there are two ways of attaching a 1-category. We have to make $\operatorname{Hom}(X, Y)$ into a set. A naive way is just to take the set of objects of $\operatorname{Hom}(X, Y)$, and we obtain the so-called underlying category of C. This has the problem that a 2-functor $F : \mathsf{C} \to \mathsf{C}'$ is not in general a functor of the underlying categories (because we only require the composition of 1-morphisms to be respected up to 2-isomorphism).

A better way of attaching a 1-category to a 2-category is to define the set of morphisms between the objects X and Y as the set of isomorphism classes of objects of $\operatorname{Hom}(X, Y)$: two objects f and g of $\operatorname{Hom}(X, Y)$ are isomorphic if there exists a 2-isomorphism $\alpha : f \overset{\sim}{\to} g$ between them. We call the category obtained in this way the 1-category associated with C. Note that a 2-functor between 2-categories then induces a functor between the associated 1-categories.

A.2 Grothendieck Topologies

The main references for this section are [4] and [114]. Related topics can be found in [39, IV], [12] and [86, III].

Let us begin with an example:

Example A.2.1 (topology in the usual sense). Let X be a topological space, and let T denote the collection of all open subsets of X. The collection T becomes a category if we define

$$\mathrm{Hom}(U,V) = \begin{cases} \emptyset & \text{if } U \not\subset V, \\ \text{inclusion } U \to V & \text{if } U \subset V, \end{cases}$$

for $U, V \in T$. The object X is a final object in the category T. If $\{U_i\}_{i=1}^n$ is a finite subset of T, then the intersection $\bigcap_{i=1}^n U_i$ is equal to the product $\prod_{i=1}^n U_i$ in the category T. If $\{U_i\}_{i\in I}$ is a (possibly infinite) subset of T, then the union $\bigcup_{i\in I} U_i$ is equal to the direct sum $\bigoplus_{i\in I} U_i$ in the category T.

Grothendieck's generalization of the notion of topology consists of replacing the category T of open sets of a topological space X with an arbitrary category, in which (for example) $\mathrm{Hom}(U,V)$ may have more than one element, and of prescribing, in addition for this category, a system $\{U_i \xrightarrow{\phi_i} U\}_{i\in I}$ of *coverings* of its objects.

Definition A.2.2. *A* ***topology*** *(or* ***site****) T consists of a category $\mathrm{cat}(T)$ and a set $\mathrm{cov}(T)$ of* ***coverings****, namely, families $\{U_i \xrightarrow{\phi_i} U\}_{i\in I}$ of morphisms in $\mathrm{cat}(T)$, such that the following properties hold:*

1. *For $\{U_i \to U\}_{i\in I}$ in $\mathrm{cov}(T)$ and a morphism $V \to U$ in $\mathrm{cat}(T)$, all fiber products $U_i \times_U V$ exist, and the induced family $\{U_i \times_U V \to V\}_{i\in I}$ is again in $\mathrm{cov}(T)$.*
2. *Given $\{U_i \to U\}_{i\in I} \in \mathrm{cov}(T)$ and a family $\{V_{i,j} \to U_i\}_{i\in J_i} \in \mathrm{cov}(T)$ for each $i \in I$, the family $\{V_{i,j} \to U\}_{i\in I, j\in J_i}$ obtained by composition of morphisms also belongs to $\mathrm{cov}(T)$.*
3. *If $\phi : U' \to U$ is an isomorphism in $\mathrm{cat}(T)$, then $\{U' \xrightarrow{\phi} U\} \in \mathrm{cov}(T)$.*

Definition A.2.3. *A* ***morphism*** *$f : T \to T'$* ***of topologies*** *is a functor $f : \mathrm{cat}(T) \to \mathrm{cat}(T')$ of the underlying categories with the following two properties:*

1. *$\{U_i \xrightarrow{\phi_i} U\}_{i\in I} \in \mathrm{cov}(T)$ implies $\{f(U_i) \xrightarrow{f(\phi_i)} f(U)\}_{i\in I} \in \mathrm{cov}(T')$.*
2. *For $\{U_i \to U\}_{i\in I} \in \mathrm{cov}(T)$ and a morphism $V \to U$ in $\mathrm{cat}(T)$, the canonical morphisms $f(U_i \times_U V) \to f(U_i) \times_{f(U)} f(V)$ are isomorphisms for all $i \in I$.*

Definition A.2.4. *Let T be a topology, and let C denote a category with products (such as the category of commutative groups or the category of sets).*

A ***presheaf*** *over T with values in C is a contravariant functor $F : T \to \mathsf{C}$ (or, more precisely, a contravariant functor $F : \mathrm{cat}(T) \to \mathsf{C}$).* ***Morphisms of presheaves*** *(with values in C) are defined as morphisms of contravariant functors.*

A presheaf F is a ***sheaf*** *over T if for every covering $\{U_i \to U\}_{i\in I}$ in $\mathrm{cov}(T)$ the diagram*

$$F(U) \to \prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_i \times_U U_j)$$

is exact in C, where the double arrow in the diagram means two morphisms from $\prod_{i\in I} F(U_i)$ to $\prod_{i,j\in I} F(U_i \times_U U_j)$, which are induced by the projections $U_i \times_U U_j \to U_i$ and $U_i \times_U U_j \to U_j$, respectively. The exactness here means that an object $(x_i)_{i\in I} \in \prod_{i\in I} F(U_i)$ is in the image of $F(U) \to \prod_{i\in I} F(U_i)$ if and only if it is mapped by the two morphisms to the same image in $\prod_{i,j\in I} F(U_i \times_U U_j)$. ***Morphisms of sheaves*** *are defined as morphisms of presheaves.*

Let S be a scheme. Let (Sch/S) be the category of *schemes* over S.

Remark A.2.5. For compatibility with our references for algebraic spaces and algebraic stacks later, we shall adopt the convention that *schemes* are always *quasi-separated preschemes* (see [73, II, 1.9], [36, Def. 4.5 and 4.6], [83, 1.4(3)]).

The following lemma quoted from [73, I, 2.26] suggests that we do not lose too much by assuming that schemes are quasi-separated:

Lemma A.2.6. *Let S be a separated noetherian scheme, and let U be a scheme locally of finite type over S. Then U is quasi-separated.*

Consider three topologies on the category (Sch/S): If the coverings are collections of families of morphisms that are *surjective and étale* (resp. *faithfully flat and of finite presentation*, resp. *faithfully flat and quasi-compact*), then we say that the corresponding topology is the *étale* (resp. *fppf*, resp. *fpqc*) topology.

A.3 Properties Stable in the Étale Topology of Schemes

Fix a scheme S. All schemes and morphisms in this section are in (Sch/S).

Definition A.3.1. *We say that a property "P" of schemes is* ***stable in the étale topology*** *if, for each étale covering $X' \to X$ of schemes, the scheme X has property "P" if and only if X' has property "P".*

Proposition A.3.2 (cf. [73, I, 4.9])**.** *The following properties of schemes are stable in the étale topology:* ***locally noetherian****,* ***reduced****,* ***normal****,* ***nonsingular****, and* ***of dimension*** *n* ***over a ground field*** *k.*

Definition A.3.3. *We say that a property "P" of morphisms of schemes is* ***stable in the étale topology*** *if, for each morphism $f : X \to Y$ and each étale covering $U \to Y$, the morphism f has property "P" if and only if the induced morphism $f' : X \times_Y U \to U$ has property "P".*

Proposition A.3.4 (cf. [73, I, 4.10])**.** *The following properties of morphisms of schemes are stable in the étale topology:* ***quasi-compact****,* ***separated****,* ***universally injective****,* ***universally closed****,* ***of finite type****,* ***of finite presentation****,* ***finite****,* ***universally bijective****,* ***quasi-finite****,* ***being an isomorphism****, and* ***quasi-separated****.*

Definition A.3.5. *We say that a property "P" of morphisms of schemes is* ***stable and local on the source in the étale topology*** *if, for each morphism $f : X \to Y$, each étale covering $U \to Y$, and each étale covering $V \to U \times_Y X$, the morphism f has property "P" if and only if the induced morphism $f' : V \to U$ has property "P".*

Proposition A.3.6 (cf. [73, I, 4.11])**.** *The following properties of schemes are stable and local on the source in the étale topology:* ***surjective****,* ***flat****,* ***faithfully flat****,* ***universally open****,* ***étale****,* ***locally of finite presentation****, and* ***locally of finite type****.*

Definition A.3.7. *We say that a property "P" of morphisms of schemes* ***satisfies effective descent in the étale topology*** *(cf. [56, VIII] and [59, IV-2, 2.6 and 2.7]) if a morphism $f : X \to Y$ has property "P" if, for every morphism $Y' \to Y$ from a scheme Y', the induced morphism $f' : X \times_Y Y' \to Y'$ has property "P".*

Proposition A.3.8. *The following properties of schemes satisfy effective descent in the étale topology:* ***open immersions****,* ***affine****,* ***closed immersions****,* ***immersions****,* ***quasi-affine****, and* ***immersions of reduced closed subschemes****.*

A.4 Algebraic Spaces

Fix a scheme S. Consider the category (Sch/S) of schemes over S with the *étale topology*.

Definition A.4.1. *A* ***space*** *over S is a sheaf of sets over the étale site of* (Sch/S)*. We denote by* (Spc/S) *the category of spaces over S, with morphisms the morphisms of sheaves over the étale site of* (Sch/S)*.*

The category (Sch/S) of schemes over S can be identified with a full subcategory of (Spc/S): We associate with each scheme X over S the *functor of points* $U \mapsto X(U) := \mathrm{Hom}_{(\mathrm{Sch}/S)}(U, X)$ over (Sch/S). By [56, VIII, 5.1 and 5.3], this functor is a sheaf over the étale site of (Sch/S), defining a space over S. By Yoneda's lemma, (Sch/S) is a full subcategory of (Spc/S).

By abuse of language, we say that a space over S is a scheme if it comes from a scheme (over S) in this way. In particular, the scheme S over S corresponds to the final object of (Spc/S).

Definition A.4.2. *A morphism $f : X \to Y$ in* (Spc/S) *is called* ***schematic*** *if for all $U \in \mathrm{Ob}(\mathrm{Sch}/S)$ and all $y \in Y(U)$ (viewed as a morphism $y : U \to Y$ in* (Spc/S)*), the fiber product $U \times_{y,Y,f} X$ is a scheme.*

Properties of morphisms of schemes that are stable in the étale topology (see Definition A.3.3), stable and local on the source in the étale topology (see Definition A.3.5), or satisfy effective descent in the étale topology (see Definition A.3.7) extend naturally to schematic morphisms of algebraic spaces. In particular, we can speak of schematic morphisms that are *quasi-compact*, *surjective*, *étale*, and *closed immersions* (see Propositions A.3.4, A.3.6, and A.3.8).

Proposition A.4.3. *Let X be a space over S. The following are equivalent:*

1. *The morphism $\Delta_X : X \to X \times_S X$ is schematic.*
2. *For each scheme U over S, every morphism $U \to X$ is schematic.*
3. *For all schemes U and V over S, and for all morphisms $\phi : U \to X$ and $\psi : V \to X$, the fiber product $U \times_X V$ is a scheme.*

Proof. The implication 2⇔3 follows immediately from the definition. The implication 1⇒3 is valid because the fiber product $U \times_X V$ is the fiber product of the morphisms $\Delta_X : X \to X \times_S X$ and $\phi \times \psi : U \times_S V \to X \times_S X$. For the implication 3⇒1, each morphism $U \to X \times_S X$ induces the following commutative diagram.

$$\begin{tikzcd}
U \underset{X \times_S X}{\times} X \arrow[r] \arrow[d] & U \times_X U \arrow[r] \arrow[d] & X \arrow[d, "\Delta_X"] \\
U \arrow[r, "\Delta_U"'] & U \times_S U \arrow[r] & X \times_S X
\end{tikzcd}$$

Both squares are Cartesian, and $U \times_X U$ is a scheme by hypothesis. Hence $U \underset{X \times_S X}{\times} X$ is also a scheme. □

Definition A.4.4. *An algebraic space over S is a space X over S satisfying:*

1. *(****Quasi-separateness****; cf. Remark A.2.5) The diagonal morphism $\Delta_X : X \to X \times_S X$ is schematic and quasi-compact.*
2. *There exist a scheme X' over S, and a morphism of spaces $X' \xrightarrow{\pi} X$ over S (automatically schematic; cf. Proposition A.4.3) that is étale and surjective.*

Let $(\text{Alg-Spc}/S)$ be the full subcategory of (Spc/S) whose objects are algebraic spaces over S. The subcategory (Sch/S) of schemes over S is naturally contained in $(\text{Alg-Spc}/S)$.

Definition A.4.5. *A morphism $f : X \to Y$ in* (Spc/S) *is called* ***representable*** *if for all $U \in \mathrm{Ob}(\mathrm{Sch}/S)$ and all $y \in Y(U)$, the fiber product $U \times_{y,Y,f} X$ is an* ***algebraic space****.*

Remark A.4.6. For morphisms in (Spc/S), the property of being *schematic* is different from the property of being *representable*.

Unless otherwise specified, let us assume from now on that all schemes, spaces, and morphisms are over S (namely, in (Spc/S)) in the following subsections of Section A.4.

A.4.1 Quotients of Equivalence Relations

Definition A.4.1.1. *An* ***equivalence relation*** *$X_\bullet$ in* (Spc/S) *is given by the data of two spaces X_0 and X_1, and a monomorphism $X_1 \xrightarrow{\delta} X_0 \times_S X_0$ of spaces (namely, a monomorphism of sheaves of sets over the étale site of* (Sch/S)*) such that for each object U in* (Sch/S)*, the graph of the function $X_1(U) \xrightarrow{\delta(U)} X_0(U) \times_S X_0(U)$ is the graph of an equivalence relation in the category of sets.*

For each fiber product $Z_1 \underset{Z}{\times} Z_2$ in a category, we define by pr_i the projection from $Z_1 \underset{Z}{\times} Z_2$ to Z_i. There might be conflicts of notation, but the meaning of pr_i should be clear from the context.

A *quotient of the equivalence relation* $X_\bullet$ is defined by the cokernel of the diagram

$$X_1 \underset{\mathrm{pr}_2 \circ\delta}{\overset{\mathrm{pr}_1 \circ\delta}{\rightrightarrows}} X_0 \;.$$

Such a quotient Q exists in the category of spaces over S, and X_1 can be canonically identified with the fiber product $X_0 \underset{Q}{\times} X_0$. Conversely, for all epimorphisms $X_0 \twoheadrightarrow Q$ in (Spc/S), Q is identified with the quotient of the equivalence relation given by the canonical morphism $X_0 \underset{Q}{\times} X_0 \to X_0 \underset{S}{\times} X_0$ over S.

Proposition A.4.1.2 ([73, II, 1.3])**.** *A space over S is an algebraic space if and only if it is a quotient of an equivalence relation $X_\bullet$ in (Spc/S) such that X_0 and X_1 are schemes, such that $\delta : X_1 \to X_0 \underset{S}{\times} X_0$ is a quasi-compact monomorphism, and such that $\mathrm{pr}_1 \circ \delta$ and $\mathrm{pr}_2 \circ \delta$ are étale morphisms from X_1 to X_0. (Note that the morphisms δ, $\mathrm{pr}_1 \circ \delta$ and $\mathrm{pr}_2 \circ \delta$ are automatically schematic because their sources and targets are schemes.)*

More precisely,

1. *if X is an algebraic space, and if $\pi : X_0 \to X$ is a morphism from a scheme X_0 to X that is surjective and étale, then the projections $\mathrm{pr}_i : X_1 := X_0 \underset{X}{\times} X_0 \to X_0$ are étale for $i = 1, 2$, the morphism $(\mathrm{pr}_1, \mathrm{pr}_2) : X_1 \to X_0 \underset{S}{\times} X_0$ is quasi-compact, and X is the quotient of the equivalence relation defined by this morphism;*
2. *if $X_\bullet$ is an equivalence relation in (Spc/S) with X_0 and X_1 schemes, with $\delta : X_1 \to X_0 \underset{S}{\times} X_0$ quasi-compact, and with $\mathrm{pr}_1 \circ \delta$ and $\mathrm{pr}_2 \circ \delta$ étale, then the quotient of $X_\bullet$ is an algebraic space over S, and the canonical morphism from X_0 to the quotient is schematic, étale, and surjective.*

Remark A.4.1.3. Given a morphism $\pi : X_0 \to X$ from a scheme X_0 to a space X, let $X_1 := X_0 \underset{X}{\times} X_0$. This defines a morphism $\delta : X_1 \to X_0 \underset{S}{\times} X_0$ by $\delta := (\mathrm{pr}_1, \mathrm{pr}_2) : X_1 \to X_0 \underset{S}{\times} X_0$. Then the morphism δ is related to $\Delta_X : X \to X \underset{S}{\times} X$ by the following *Cartesian* diagram.

$$\begin{tikzcd}
X_1 = X_0 \underset{X}{\times} X_0 \arrow[r] \arrow[d, "\delta"'] & X \arrow[d, "\Delta_X"] \\
X_0 \underset{S}{\times} X_0 \arrow[r] & X \underset{S}{\times} X
\end{tikzcd}$$

In particular, if π is étale and surjective, then δ is schematic and quasi-compact if and only if Δ_X is.

Thanks to Proposition A.4.1.2, we can alternatively define the following:

Definition A.4.1.4. *A space X over S is called an algebraic space over S if it is the quotient of an* ***étale equivalence relation*** *$X_\bullet$, namely, an equivalence relation such that X_0 and X_1 are schemes, such that δ is quasi-compact, and such that $\mathrm{pr}_1 \circ\delta$ and $\mathrm{pr}_2 \circ\delta$ are étale.*

Hence an algebraic space can be viewed (noncanonically) as the quotient of a scheme by an étale equivalence relation.

A.4.2 Properties of Algebraic Spaces

A substantial part of the theory of schemes is generalized to the theory of algebraic spaces by Artin [8] and Knutson [73]. Since the theory of algebraic stacks will be based on the theory of algebraic spaces, we summarize here many of the important definitions and results from [73].

Proposition A.4.2.1 ([73, II, 1.4])**.** *Let X_1 and X_2 be algebraic spaces. Let $\pi_1 : X_1' \to X_1$ and $\pi_2 : X_2' \to X_2$ be étale coverings, where X_1' and X_2' are schemes. Then π_1 and π_2 are automatically schematic, and $X_1' \underset{X_1}{\times} X_1'$ and $X_2' \underset{X_2}{\times} X_2'$ are representable by schemes. Let g and h be morphisms in the following diagram*

$$\begin{tikzcd}
X_1' \underset{X_1}{\times} X_1' \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] \arrow[d, "g"] & X_1' \arrow[r, "\pi_1"] \arrow[d, "h"] & X_1 \arrow[d, dotted, "f"] \\
X_2' \underset{X_2}{\times} X_2' \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X_2' \arrow[r, "\pi_2"] & X_2
\end{tikzcd}$$

of solid arrows. Then there is a unique morphism $f : X_1 \to X_2$ with $\pi_2 \circ h = f \circ \pi_1$. Conversely, every morphism $f : X_1 \to X_2$ is induced in this way for some choices of π_1, π_2, g, h.

Hence, in order to study a morphism $f : X_1 \to X_2$ of algebraic spaces, we may choose π_1 and π_2 as in Proposition A.4.2.1 and study instead the corresponding morphisms g and h of schemes.

Proposition A.4.2.2 ([73, II, 1.5])**.** *Disjoint unions and fiber products exist in the category of algebraic spaces.*

For an algebraic space X, the diagonal morphism $X \to X \underset{S}{\times} X$ is schematic. Hence, if a property of schemes is defined by imposing conditions on the diagonal morphisms, and if those conditions make sense for schematic morphisms of algebraic spaces, then we can define the property for algebraic spaces as well.

Definition A.4.2.3. *We define X to be* ***locally separated*** *(resp.* ***separated****) if the schematic morphism $\Delta_X : X \to X \underset{S}{\times} X$ is a quasi-compact (resp. closed) immersion (see Propositions A.3.4 and A.3.8).*

Lemma A.4.2.4 (cf. [73, II, 1.8])**.** *To verify that X is locally separated (resp. separated), it suffices to verify that for some étale surjection $X' \to X$ from a scheme, the morphism $X' \underset{X}{\times} X' \to X' \underset{S}{\times} X'$, defined by the pullback of $\Delta_X : X \to X \underset{S}{\times} X$ to $X' \underset{S}{\times} X' \to X \underset{S}{\times} X$ (see Remark A.4.1.3), is a quasi-compact (resp. closed) immersion.*

Definition A.4.2.5. *A morphism $f : X_1 \to X_2$ between algebraic spaces is* ***étale*** *if there are étale coverings $\pi_1 : X_1' \to X_1$ and $\pi_2 : X_2' \to X_2$ from schemes and an étale morphism $f' : X_1' \to X_2'$ (cf. Proposition A.4.2.1) such that $f \circ \pi_1 = \pi_2 \circ f'$.*

Definition A.4.2.6. *An algebraic space X is **quasi-compact** if X admits a covering $X' \to X$ with X' a quasi-compact scheme. A morphism $f : X \to Y$ of algebraic spaces is **quasi-compact** if for every étale morphism $U \to Y$, with U a quasi-compact scheme, $U \underset{Y}{\times} X$ is quasi-compact.*

These definitions enable us to extend the descent theory of schemes [56, VIII] to algebraic spaces [73, II, 3].

Definition A.4.2.7. *Let "P" be a property of schemes that is **stable in the étale topology** (see Definition A.3.1 and Proposition A.3.2). We say that an algebraic space X has property "P" if and only if there is an étale covering $X' \to X$ from a scheme X such that X' has property "P".*

Definition A.4.2.8. *Let "P" be a property of morphisms of schemes that is **stable and local on the source in the étale topology** (see Definition A.3.5 and Proposition A.3.6). We say that a morphism $f : X \to Y$ of algebraic spaces has property "P" if and only if there is an étale covering $U \to Y$ and an étale covering $V \to U \underset{Y}{\times} X$, where U and V are schemes, such that the induced morphism $f' : V \to U$ has property "P".*

Definition A.4.2.9. *Let $f : X \to Y$ be a morphism of algebraic spaces. We say f is **of finite type** if it is locally of finite type and quasi-compact. We say f is **of finite presentation** if it is locally of finite presentation, quasi-compact, and the induced morphism $X \to X \underset{Y}{\times} X$ is quasi-compact. We say f is **quasi-finite** if it is locally quasi-finite and quasi-compact.*

It is immediate from the local nature of the definition that these properties of morphisms are stable in the étale topology.

Definition A.4.2.10. *Let "P" be a property of morphisms of schemes **satisfying effective descent in the étale topology** (see Definition A.3.7 and Proposition A.3.8). We say that a morphism $f : X \to Y$ of algebraic spaces has property "P" if and only if for every morphism $Y' \to Y$ from a scheme Y' to Y, the fiber product $X \underset{Y}{\times} Y'$ is a scheme and the induced morphism $f' : X \underset{Y}{\times} Y' \to Y'$ has property "P". Therefore, if f is an open immersion, a closed immersion, an immersion, an affine morphism, or a quasi-affine morphism, and if Y is a scheme, then so is X. In particular, a subspace of a scheme is a scheme.*

Definition A.4.2.11. *A morphism $f : X \to Y$ of algebraic spaces is **quasi-separated** (resp. **locally separated**, resp. **separated**) if the induced morphism $X \to X \underset{Y}{\times} X$ is quasi-compact (resp. a quasi-compact immersion, resp. a closed immersion).*

Proposition A.4.2.12 ([73, II, 3.10])**.** *The class of quasi-separated (resp. locally separated, resp. separated) morphisms of algebraic spaces is stable in the étale topology. Moreover, for each morphism $f : X \to Y$ such that X and Y are algebraic spaces and such that Y is separated, X is quasi-separated (resp. locally separated, resp. separated) if and only if the morphism f is quasi-separated (resp. locally separated, resp. separated).*

A.4.3 Quasi-Coherent Sheaves on Algebraic Spaces

As mentioned above, the extension of the definition of étaleness to morphisms between algebraic spaces and the corresponding properties enable us to speak of the étale site of $(\text{Alg-Spc}\,/S)$ in an obvious way. For each sheaf F over the étale site of $(\text{Sch}\,/S)$, we may extend it in a unique way to the étale site of $(\text{Alg-Spc}\,/S)$. Concretely, for each algebraic space X with an étale covering $X' \to X$ from a scheme X', we define

$$F(X) = \ker(F(X') \rightrightarrows F(X' \underset{X}{\times} X')).$$

In particular, we may extend the *structural sheaf of rings* $\mathscr{O}$ over $(\text{Sch}\,/S)$ (which assigns to each $\text{Spec}(R)$ the ring R) to the étale site of algebraic spaces. The *sheaf of units* $\mathscr{O}^\times$ and *sheaf* $\mathscr{N}$ *of nilpotent elements* of $\mathscr{O}$ are extended in a similar way.

For a particular algebraic space X, the restriction of $\mathscr{O}$ to the étale site of $(\text{Alg-Spc}\,/X)$ (defined in the obvious way) is the *structural sheaf* of X, denoted by $\mathscr{O}_X$. For example, for each scheme U with a morphism $U \to X$, the structural sheaf $\mathscr{O}_X$ assigns to U the global sections of the structural sheaf $\mathscr{O}_U$ of U.

Definition A.4.3.1. *Let X be an algebraic space, and let $\mathscr{O}_X$ be its structural sheaf of rings. An $\mathscr{O}_X$-module F is **quasi-coherent** (resp. **coherent**, resp. **locally free of rank** r) if for each morphism $i : U \to X$ from a scheme U, the induced sheaf i^*F (which is defined in a natural way to be the sheaf over U assigning $i^*F(V) = F(V)$ to each $V \to U$) is quasi-coherent (resp. coherent, resp. locally free of rank r) in the usual sense (of sheaves over schemes).*

A.4.4 Points and the Zariski Topology of Algebraic Spaces

Let X be an algebraic space. Consider morphisms of algebraic spaces of the form $i : p = \text{Spec}(k) \to X$ (with k a field) that are monomorphisms in the category $(\text{Spc}\,/S)$. Two such morphisms $i_1 : p_1 \to X$ and $i_2 : p_2 \to X$ are considered equivalent if there is an isomorphism $e : p_1 \to p_2$ with $i_2 \circ e = i_1$.

Definition A.4.4.1. *A **point** of X is defined to be an equivalence class of such morphisms of algebraic spaces. By abuse of language, we say that p is in X and write $p \in X$. The **residue field** $k(p)$ of X at p is defined to be the field k above. A **geometric point** $j : q \to X$ is a morphism of algebraic spaces with $q = \text{Spec}(\bar{k})$ for some algebraically closed field $\bar{k}$. (Note that j need not be a monomorphism. Hence a geometric point is usually not a point.)*

Proposition A.4.4.2 ([73, II, 6.2])**.** *Let $f : q \to X$ be a morphism of algebraic spaces with $q = \text{Spec}(k)$ for some field k. Then there is a point p of X such that f factors through $q \to p \to X$.*

Theorem A.4.4.3 ([73, II, 6.4])**.** *Let X be an algebraic space and $p \to X$ a point of X. Then there is an affine scheme U and an étale morphism $U \to X$ such that $p \to X$ factors through $p \to U \to X$.*

Definition A.4.4.4. *Let X be an algebraic space. The **associated underlying topological space** $|X|$ of X is defined to be the collection of points of X (modulo equivalence relations of points). The set $|X|$ is given a topological structure by taking a subset $U \subset |X|$ to be open if $U = |Y|$ for some open subspace Y of X. This topology on $|X|$ is called the **Zariski topology**.*

Proposition A.4.4.5 ([73, II, 6.10])**.** *$|X|$ is a topological space and there is a one–one correspondence between open subspaces of X and open subsets of $|X|$, and a one–one correspondence between reduced closed subspaces of X and closed subsets of $|X|$. Also, $X \mapsto |X|$ is a functor.*

Definition A.4.4.6. *X is irreducible if and only if $|X|$ is, and the definitions of* ***topologically dense subspace****,* ***surjective morphism****,* ***universally open morphism****,* ***open morphism****,* ***immersion****,* ***closed morphism****, and* ***universally closed*** *can be made in terms of the underlying topological spaces. For example, a morphism is defined to be* ***universally closed*** *if for each algebraic space X' with $X' \to X$, the induced map $|Y \underset{X}{\times} X'| \to |X'|$ is closed. (These are compatible with the previous definitions. See [73, II, 6.11].)*

Finally,

Definition A.4.4.7. *A morphism $f : X \to Y$ of algebraic spaces is* ***proper*** *if it is separated, of finite type, and* ***universally closed****.*

A.5 Categories Fibered in Groupoids

Recall that a *groupoid* is a category whose morphisms are all isomorphisms.

Definition A.5.1. *Let Y be any category. A* ***category fibered in groupoids*** *over Y is a category X with a morphism $p : \mathsf{X} \to \mathsf{Y}$ (called the* ***structural morphism****) such that*

1. *for each morphism $V \xrightarrow{\phi} U$ in Y and each object x of X such that $p(x) = U$, there exists a morphism $y \xrightarrow{f} x$ in X such that $p(f) = \phi$;*
2. *for each diagram $y \xrightarrow{f} x \xleftarrow{g} z$ in X whose image in Y is $V \xrightarrow{\phi} U \xleftarrow{\psi} W$ and for each morphism $V \xrightarrow{\chi} W$ in Y such that $\phi = \psi \circ \chi$, there exists a* ***unique*** *morphism $y \xrightarrow{h} z$ in X such that $f = g \circ h$ and $p(h) = \chi$.*

For each $U \in \operatorname{Ob} \mathsf{Y}$, we denote by X_U or $\mathsf{X}(U)$ the *fiber* (*category*) of X over U, whose objects are those $x \in \operatorname{Ob} \mathsf{X}$ such that $p(x) = U$, and whose morphisms are those $f \in \operatorname{Mor} \mathsf{X}$ such that $p(f) = \mathrm{Id}_U$. By definition, the category X_U is a groupoid.

For each morphism $V \xrightarrow{\phi} U$ in Y and each object $x \in \mathsf{X}_U$, the set of morphisms $y \xrightarrow{f} x$ such that $p(y) = V$ and $p(f) = \phi$ is a torsor under the group $\operatorname{Hom}_V(y, y)$ of automorphisms of y in X_V. Let us specify once for each $V \xrightarrow{\phi} U$ and $x \in \mathsf{X}_U$ a choice of $y \xrightarrow{f} x$, and write it as $\phi^* x \to x$ or $x_V \to x$. Moreover, for each $f \in \operatorname{Mor} \mathsf{X}_U$, we denote by $\phi^* f$ or $f|_V$ the unique morphism g making the diagram

$$\begin{tikzcd} \phi^* x' \arrow[r] & x' \\ \phi^* x \arrow[u, leftarrow, "g"'] \arrow[r] & x \arrow[u, leftarrow, "f"] \end{tikzcd}$$

commutative. Then we have defined a functor $\phi^* : \mathsf{X}_U \to \mathsf{X}_V$ for each morphism $V \xrightarrow{\phi} U$ in Y. We call this functor the *functor of base change* by ϕ and denote it by $\cdot|_V$. For each two morphisms $W \xrightarrow{\psi} V$ and $V \xrightarrow{\phi} U$ in Y, we have a canonical isomorphism (up to 2-isomorphism) between the two functors $\psi^* \circ \phi^*$ and $(\phi \circ \psi)^*$.

The category of groupoids form a 2-category (see Section A.1.2) (Gr) whose objects are groupoids, whose 1-morphisms are functors between groupoids, and whose 2-morphisms are the natural transformations between them. The functors of base change as above allow us to associate with the category X fibered in groupoids a (contravariant) 2-functor F from Y (viewed as a 2-category in the canonical way) to (Gr), by assigning to each U in Y the groupoid $\mathsf{F}(U) = \mathsf{X}_U$ in (Gr), and by assigning to each morphism $\phi : U \to V$ in Y the functor (1-morphism) $\phi^* : \mathsf{X}_V \to \mathsf{X}_U$ between groupoids. Two different choices made above in the construction of the functors of base change may result in two different 2-functors, but the 2-functors thus defined are canonically isomorphic.

Definition A.5.2. *A* ***morphism*** *$f : \mathsf{X} \to \mathsf{X}'$ of categories fibered in groupoids over Y is a functor of categories from X to X' such that $p_{\mathsf{X}'} \circ f = p_{\mathsf{X}}$, where p_{X} (resp. $p_{\mathsf{X}'}$) is the structural morphism of X (resp. Y). We denote by $\operatorname{Hom}_{\mathsf{Y}}(\mathsf{X}, \mathsf{X}')$ the category whose objects are morphisms of categories fibered in groupoids over Y, and whose morphisms are natural transformations.*

The categories fibered in groupoids over a category Y form a 2-category (see Section A.1.2) (Ct-F-Gr $/\mathsf{Y}$) whose objects are categories fibered in groupoids over Y, whose 1-morphisms are morphisms between categories fibered in groupoids over Y, and whose 2-morphisms are the natural transformations between them.

Definition A.5.3. *A morphism $f : \mathsf{X} \to \mathsf{X}'$ in* (Ct-F-Gr $/\mathsf{Y}$) *is a* ***monomorphism*** *(resp. an* ***isomorphism****) if, for all $U \in \operatorname{Ob} \mathsf{Y}$, the functor $f_U : \mathsf{X}_U \to \mathsf{X}'_U$ is fully faithful (resp. an equivalence of categories) (see Remark A.1.2.7). If f is an isomorphism of categories fibered in groupoids, then we say that X and X' are isomorphic (under f).*

Construction A.5.4 (fiber product of categories fibered in groupoids). Suppose we have two 1-morphisms $F : \mathsf{X} \to \mathsf{X}''$ and $F' : \mathsf{X}' \to \mathsf{X}''$ in (Ct-F-Gr $/\mathsf{Y}$). We would like to construct the fiber product $\mathsf{X} \underset{F, \mathsf{X}'', F'}{\times} \mathsf{X}'$: For each $U \in \operatorname{Ob} \mathsf{Y}$, the fiber $(\mathsf{X} \underset{F, \mathsf{X}'', F'}{\times} \mathsf{X}')_U$ consists of objects in the form (x, x', g), where $x \in \operatorname{Ob} \mathsf{X}_U$, where $x' \in \operatorname{Ob} \mathsf{X}'_U$, and where g is a morphism $F(x) \to F'(x')$ in X''_U. A morphism from (x_1, x'_1, g_1) to (x_2, x'_2, g_2) in this fiber is a pair $(f : x_1 \to x_2, f' : x'_1 \to x'_2)$, where $f \in \operatorname{Mor} \mathsf{X}_U$ and $f' \in \operatorname{Mor} \mathsf{X}'_U$, such that $g_2 \circ F(f) = F'(f') \circ g_1$. Finally, for each morphism $V \xrightarrow{\phi} U$ in Y, we have $\phi^*(x, x', g) = (\phi^* x, \phi^* x', \phi^* g)$ and $\phi^*(f, f') = (\phi^* f, \phi^* f')$.

Definition A.5.5. *We call the fiber product $\mathsf{X} \underset{F, \mathsf{X}'', F'}{\times} \mathsf{X}'$ the* ***pullback*** *of $F : \mathsf{X} \to \mathsf{X}''$ to X' (under $F' : \mathsf{X}' \to \mathsf{X}''$). When X' is a full subcategory of X'', we define the pullback of F to X' using the canonical 1-morphism $F' : \mathsf{Y}' \to \mathsf{X}''$.*

Definition A.5.6. *Let S be any scheme. A* ***category fibered in groupoids*** *over S is a category X fibered in groupoids over* (Sch $/S$)*. Morphisms of categories fibered in groupoids over S are morphisms of categories fibered in groupoids over* (Sch $/S$).

Then the categories fibered in groupoids over S form a 2-category (Ct-F-Gr $/S$), namely, the category (Ct-F-Gr $/$(Sch $/S$)).

Each sheaf F over (Sch $/S$) is naturally identified with a category fibered in groupoids over (Sch $/S$), by considering for each scheme U the set $F(U)$ to be a category whose morphisms are all identities. Then $F(U)$ is a groupoid for each U, and the conditions of being a category fibered in groupoids are naturally satisfied. It is immediate that the categories of spaces, algebraic spaces, and schemes are all natural full sub-2-categories of (Ct-F-Gr $/S$).

Recall the following definition due to Grothendieck [59, IV-3, 8.14.2] and Artin [7, 4.4]:

Definition A.5.7. *A contravariant functor $F : (\mathrm{Sch}/S) \to (\mathrm{Sets})$ is **locally of finite presentation** over S if, for each filtering projective system of affine schemes $\mathrm{Spec}(A_i)$ over S, the canonical map*

$$\varinjlim_{i\in I} F(\mathrm{Spec}(A_i)) \to F(\varprojlim_{i\in I} \mathrm{Spec}(A_i)) \tag{A.5.8}$$

is a bijection.

We make the following analogous definition for categories fibered in groupoids:

Definition A.5.9. *A category X fibered in groupoids over S is **locally of finite presentation** if, for each filtering projective system of affine schemes $\mathrm{Spec}(A_i)$ over S, the canonical functor*

$$\varinjlim_{i\in I} \mathsf{X}(\mathrm{Spec}(A_i)) \to \mathsf{X}(\varprojlim_{i\in I} \mathrm{Spec}(A_i)) \tag{A.5.10}$$

defines an equivalence of categories.

A.5.1 Quotients of Groupoid Spaces

Let C be a groupoid, namely, a category whose morphisms are all isomorphisms. Then we obtain two sets $\mathrm{Mor}\,\mathsf{C}$ and $\mathrm{Ob}\,C$, and three maps of sets,

$$\begin{aligned} &(\textit{source}) && s : \mathrm{Mor}\,\mathsf{C} \to \mathrm{Ob}\,\mathsf{C},\\ &(\textit{target}) && t : \mathrm{Mor}\,\mathsf{C} \to \mathrm{Ob}\,\mathsf{C},\\ &(\textit{multiplication}) && m : \mathrm{Mor}\,\mathsf{C} \underset{s,\mathrm{Ob}\,\mathsf{C},t}{\times} \mathrm{Mor}\,\mathsf{C} \to \mathrm{Mor}\,\mathsf{C}, \end{aligned}$$

satisfying the following axioms:

1. (*Associativity*) $m \circ (\mathrm{Id}, m) = m \circ (m, 1)$.
2. (*Existence of identity*) There exists a map $e : \mathrm{Ob}\,\mathsf{C} \to \mathrm{Mor}\,\mathsf{C}$ such that $s \circ e = t \circ e = \mathrm{Id}$ and $m \circ (\mathrm{Id}, e \circ s) = m \circ (e \circ t, \mathrm{Id}) = \mathrm{Id}$.
3. (*Existence of inverse*) There exists a map $[-1] : \mathrm{Mor}\,\mathsf{C} \to \mathrm{Mor}\,\mathsf{C}$ such that $s \circ [-1] = t$, $t \circ [-1] = s$, $m \circ (\mathrm{Id}, [-1]) = e \circ s$, and $m \circ ([-1], \mathrm{Id}) = e \circ t$.

Explicitly, if $f : X \to Y$ and $g : Y \to Z$ are elements in $\mathrm{Mor}\,\mathsf{C}$, then the maps are defined by $s(f) = X$, $t(f) = Y$, $m(g, f) = g \circ f$, $e(X) = \mathrm{Id}_X$, and $[-1](f) = f^{-1}$. (The inverse f^{-1} of f is defined because f is an isomorphism by assumption.) Conversely, suppose we have two sets C_0 and C_1, together with maps $s : C_1 \to C_0$, $t : C_1 \to C_0$, and $m : C_1 \underset{s,C_0,t}{\times} C_1 \to C_1$ satisfying all the axioms, then we can define a groupoid C with $\mathrm{Ob}\,\mathsf{C} = C_0$ and $\mathrm{Mor}\,\mathsf{C} = C_1$. In other words, the datum of a groupoid C is equivalent to the data of the tuple (C_0, C_1, s, t, m).

Remark A.5.1.1. Suppose we have a tuple (C_0, C_1, s, t, m) defining a groupoid C as above. Then s and t define a map of sets $(s, t) : C_1 \to C_0 \times C_0$, and the image of (s, t) is the graph of an equivalence relation in the category of sets. The fiber of (s, t) over an element (X, Y) of C_0 is the category $\mathrm{Hom}(X, Y)$ in C. Suppose all automorphisms in C are identity morphisms. That is, for every $X \in \mathrm{Ob}\,\mathsf{C}$, the set $\mathrm{Hom}(X, X)$ has only one element Id_X. Then (s, t) is a monomorphism, and its graph defines an equivalence relation on C_0. Therefore, the study of equivalence relations can be viewed as a special case of the study of groupoids.

Let S be a scheme and let (Spc/S) be the category of spaces over S.

Definition A.5.1.2. *A **groupoid space** $X_\bullet$ in (Spc/S) is given by data consisting of two spaces X_0 and X_1, and five morphisms*

$$s : X_1 \to X_0, \qquad t : X_1 \to X_0, \qquad m : X_1 \underset{s,X_0,t}{\times} X_1 \to X_1,$$

$$e : X_0 \to X_1, \qquad \text{and} \qquad [-1] : X_1 \to X_1$$

of spaces such that, for each object U in (Sch/S), the tuple

$$(X_0(U), X_1(U), s(U), t(U), m(U))$$

defines a groupoid, with $e(U)$ and $[-1](U)$ giving the maps required by the axioms.

By abuse of notation, we shall often say that a groupoid space $X_\bullet$ in (Spc/S) is given by data consisting of two spaces X_0 and X_1, and a morphism

$$X_1 \xrightarrow{\delta} X_0 \underset{S}{\times} X_0$$

of spaces. The morphisms s and t are given by $s = \mathrm{pr}_1 \circ \delta$ and $t = \mathrm{pr}_2 \circ \delta$. The morphisms m, e, and $[-1]$ are suppressed in the notation, just as they are seldom mentioned when denoting group schemes.

Construction A.5.1.3. The *quotient of the groupoid space $X_\bullet$* is defined (up to isomorphism) in $(\text{Ct-F-Gr}/S)$ as follows. Let $X_\bullet$ be given by the data of two spaces X_0 and X_1, and three morphisms $s : X_1 \to X_0$, $t : X_1 \to X_0$, and $m : X_1 \underset{s,X_0,t}{\times} X_1 \to X_1$ of spaces. We shall denote by X the quotient of $X_\bullet$ in (Ct-F-Gr). First let us describe the fibers of X. For every object U in (Sch/S), the fiber category X_U is the groupoid defined by the tuple $(X_0(U), X_1(U), s(U), t(U), m(U))$. In particular, $\mathrm{Ob}\,\mathsf{X}_U = X_0(U)$ and $\mathrm{Mor}\,\mathsf{X}_U = X_1(U)$. Then $\mathrm{Ob}\,\mathsf{X} = \coprod_{U\in\mathrm{Ob}(\mathrm{Sch}/S)} \mathrm{Ob}\,\mathsf{X}_U$, and we define $p : \mathrm{Ob}\,\mathsf{X} \to \mathrm{Ob}(\mathrm{Sch}/S)$ by setting $p(x) = U$ for $x \in \mathrm{Ob}\,\mathsf{X}_U$.

Suppose $x, y \in \mathrm{Ob}\,\mathsf{X}$. Let $U = p(x)$ and $V = p(y)$. For each $\phi \in \mathrm{Hom}(U, V)$, we define $\mathrm{Hom}_\phi(x, y)$ to be the subset $\mathrm{Hom}_U(x, x) := (s(U), t(U))^{-1}(x, x)$ of $\mathrm{Mor}\,\mathsf{X}_U = X_1(U)$. Then we define the set $\mathrm{Hom}(x, y)$ to be $\coprod_{\phi\in\mathrm{Hom}(U,V)} \mathrm{Hom}_\phi(x, y)$, and define $p : \mathrm{Mor}\,\mathsf{X} \to \mathrm{Mor}\,\mathrm{Ob}(\mathrm{Sch}/S)$ by setting $p(f) = \phi$ for $f \in \mathrm{Hom}_\phi(x, y)$.

Suppose $x, y, z \in \mathrm{Ob}\,\mathsf{X}$, $f \in \mathrm{Hom}(x, y)$, and $g \in \mathrm{Hom}(y, z)$. Let $U = p(x)$, $V = p(y)$, $W = p(z)$, $\phi = p(f)$, and $\psi = p(g)$. Then f is an element of $\mathrm{Hom}_\phi(x, y) = \mathrm{Hom}_U(x, x) \subset X_1(U)$, and g is an element of $\mathrm{Hom}_\psi(y, z) = \mathrm{Hom}_V(y, y) \subset X_1(V)$. The morphism $\phi : U \to V$ defines a pullback $\phi^*(g)$ in $X_1(U)$. This element has both source and target equal to $\phi^*(y) = x$ because s and t are morphisms of spaces. Then we define the element $g \circ f$ in $\mathrm{Hom}_{\psi\circ\phi}(x, z) = \mathrm{Hom}_U(x, x) \subset X_1(U)$ to be $m(\phi^*(g), f)$. This defines a composition law for $\mathrm{Mor}\,\mathsf{X}$ because of the axioms satisfied by m.

By construction, $p : \mathsf{X} \to (\mathrm{Sch}/S)$ is a morphism of categories. To show that it defines an object in $(\text{Ct-F-Gr}/S)$, we need to verify that it satisfies both of the conditions in Definition A.5.1. Condition 1 is satisfied if we take y to be $\phi^*(x)$, and take f to be $\mathrm{Id}_y \in \mathrm{Hom}_V(y, y) = \mathrm{Hom}_\phi(y, x)$. Condition 2 is satisfied if we take $h := m^{-1}([-1](\chi^*(g)), f) \in \mathrm{Hom}_V(y, y) = \mathrm{Hom}_\chi(y, z)$, where $f \in \mathrm{Hom}_V(y, y) = \mathrm{Hom}_\phi(y, x)$ and $g \in \mathrm{Hom}_W(z, z) = \mathrm{Hom}_\psi(z, x)$.

There is a canonical morphism $\pi : X_0 \to \mathsf{X}$ defined by setting $\pi : \mathrm{Ob}\,X_0 \to \mathrm{Ob}\,\mathsf{X}$ to be the identity $\mathrm{Id}_{X_0} : X_0 \to X_0$, and by setting $\pi : \mathrm{Mor}\,X_0 \to \mathrm{Mor}\,\mathsf{X}$ to be the morphism $e : X_0 \to X_1$. The two morphisms $\pi \circ s : X_1 \to \mathsf{X}$ and $\pi \circ t : X_1 \to \mathsf{X}$ define a morphism $(\pi \circ s, \pi \circ t) : X_1 \to X_0 \underset{\mathsf{X}}{\times} X_0$, which is an isomorphism by the explicit description of $X_0 \underset{\mathsf{X}}{\times} X_0$ in Construction A.5.4. (This finishes Construction A.5.1.3.)

Remark A.5.1.4. In Construction A.5.1.3, the quotient X of $X_\bullet$ is constructed by adding morphisms (defined by X_1) to the space X_0 (viewed as an object in (Ct-F-Gr)). When the groupoid space defines an equivalence relation of spaces in (Spc/S) (as in Remark A.5.1.1), the quotient X agrees up to isomorphism in $(\text{Ct-F-Gr}/S)$ with the quotient of equivalence relations in Section A.4.1 defined by cokernel. This is because isomorphisms in $(\text{Ct-F-Gr}/S)$ are defined by equivalences of categories.

A.6 Stacks

From now on (until the end of Appendix A), let us fix a choice of a scheme S. Unless otherwise specified, all objects and morphisms will be defined over S.

Definition A.6.1. *A* ***stack*** F *over* S *is a* ***sheaf of groupoids*** *over the étale site of* (Sch/S)*; that is, a 2-functor (presheaf)* $\mathsf{F}: (\mathrm{Sch}/S) \to (\mathrm{Gr})$*, satisfying the following axioms: Let* $(V_i \to U)_{i\in I}$ *be a covering of* U *in the étale site of* (Sch/S).

1. *If* x *and* y *are two objects of* $\mathsf{F}(U)$*, and if* $\{f_i : x|_{V_i} \to y|_{V_i}\}_{i\in I}$ *is a collection of morphisms such that* $f_i|_{V_{ij}} = f_j|_{V_{ij}}$ *for all* $i,j \in I$*, then there is a morphism* $f : x \to y$ *of* $\mathsf{F}(U)$ *such that* $f|_{V_i} = f_i$ *for all* $i \in I$.
2. *If* x *and* y *are two objects of* $\mathsf{F}(U)$*, and if* $f : x \to y$ *and* $g : x \to y$ *are two morphisms such that* $f|_{V_i} = g|_{V_i}$ *for all* $i \in I$*, then* $f = g$.
3. *If, for each* $i \in I$*,* x_i *is an object of* $\mathsf{F}(V_i)$*, and if* $\{f_{ij} : x_i|_{V_{ij}} \xrightarrow{\sim} x_j|_{V_{ij}}\}_{i,j\in I}$ *is a collection of isomorphisms satisfying the* ***cocycle condition*** $(f_{ik}|_{V_{ijk}}) = (f_{jk}|_{V_{ijk}}) \circ (f_{ij}|_{V_{ijk}})$ *for all* $i,j,k \in I$*, then there is an object* $x \in \mathsf{F}(U)$ *and a collection* $\{f_i : x|_{V_i} \xrightarrow{\sim} x_i\}_{i\in I}$ *of isomorphisms such that* $(f_j|_{V_{ij}}) = f_{ij} \circ (f_i|_{V_{ij}})$ *for all* $i,j \in I$.

A stack over S naturally defines a category fibered in groupoids over (Sch/S). We denote by (St/S) the full sub-2-category of $(\text{Ct-F-Gr}/S)$ whose objects are stacks over S. Then a morphism of stacks is a *monomorphism* (resp. an *isomorphism*) if it is a monomorphism (resp. an isomorphism) of categories fibered in groupoids (see Definition A.5.3).

Lemma A.6.2. *Let* X *be a stack over* S*, and let* U *be a scheme over* S*. The functor* $u : \mathsf{Hom}_S(U, \mathsf{X}) \to \mathsf{X}(U)$ *sending a morphism of stacks* $f : (\mathrm{Sch}/U) \to \mathsf{X}$ *to* $f(\mathrm{Id}_U)$ *is an equivalence of categories.*

Proof. This follows from Yoneda's lemma. □

The sub-2-category (St/S) of $(\text{Ct-F-Gr}/S)$ is stable under taking arbitrary projective limits, and in particular, stable under taking fiber products. However, it is not stable under taking inductive limits; yet we may consider the associated stacks of the prestacks defined in a natural way by taking the inductive limits. (The process of taking inductive limits is stable in $(\text{Ct-F-Gr}/S)$.)

The category (Spc/S) is a full subcategory of (St/S) by viewing 1-functors as 2-functors in the canonical way. Then both $(\text{Alg-Spc}/S)$ and (Sch/S) are subcategories of (St/S).

Definition A.6.3. *A stack* Y *is a* ***substack*** *of* X *if it is a full subcategory of* X*, and if the following are satisfied:*

1. *If an object* x *of* X *is in* Y*, then all objects isomorphic to* x *are also in* Y.
2. *For all morphisms of schemes* $f : V \to U$*, if* x *is in* $\mathsf{Y}(U)$*, then* f^*x *is in* $\mathsf{Y}(V)$.
3. *Let* $\{V_i \to U\}_{i\in I}$ *be a covering of* U *in the étale site of* (Sch/S)*. Then* x *is in* $\mathsf{Y}(U)$ *if and only if* $x|_{V_i}$ *is in* $\mathsf{Y}(V_i)$ *for all* i.

Definition A.6.4. *Let* $f : \mathsf{Z} \to \mathsf{X}$ *be a (1-)morphism of stacks over* S*. We say that* f *is an* ***epimorphism*** *if, for each* $U \in \mathrm{Ob}(\mathrm{Sch}/S)$ *and* $x \in \mathrm{Ob}\,\mathsf{X}_U$*, there is a covering* $U' \to U$ *in the étale site of* (Sch/S) *and* $z' \in \mathrm{Ob}\,\mathsf{Z}_{U'}$ *such that* $f_{U'}(z')$ *is isomorphic to* $x|_{U'}$ *in* $\mathsf{X}_{U'}$.

Definition A.6.5. *A stack* X *is said to be* ***representable by an algebraic space*** *(resp. a* ***scheme****) if there is an algebraic space (resp. a scheme)* X *such that the stack associated with* X *is isomorphic to* X.

Remark A.6.6. Elsewhere in our text, if not particularly stated, an algebraic stack will be called representable if it is representable by an algebraic space.

Definition A.6.7. *A morphism of stacks* $f : \mathsf{X} \to \mathsf{Y}$ *is called* ***representable*** *if for every morphism* $U \to \mathsf{Y}$ *from a scheme* U*, the fiber product stack* $U \underset{\mathsf{Y}}{\times} \mathsf{X}$ *is representable by an algebraic space.*

Suppose "P" is a property of morphisms of algebraic spaces that is local in nature on the target for the étale topology on (Sch/S) *and stable under arbitrary base change. Then we say that* f *has property "P" if for every* $U \to \mathsf{Y}$ *from a scheme* U*, the pullback morphism* $U \underset{\mathsf{Y}}{\times} \mathsf{X} \to U$ *(of algebraic spaces) has property "P".*

Remark A.6.8. Here is a list of such properties summarized from [83, 3.10] (cf. Definitions A.4.2.5, A.4.2.6, A.4.2.8, A.4.2.9, A.4.2.10, A.4.2.11, A.4.4.6, and A.4.4.7):

1. surjective, radicial, and universally bijective [59, I, 3.6.1, 3.6.4, and 3.7.6]
2. universally open, universally closed, separated, quasi-compact, locally of finite type, locally of finite presentation, of finite type, of finite presentation, being an immersion, being an open immersion, being a closed immersion, being an open immersion with dense image, affine, quasi-affine, entire, finite, quasi-finite, and proper [59, IV-2, 2.5.1 and IV-4, 17.7.5]
3. fibers are geometrically connected, geometrically reduced, and geometrically irreducible [59, IV-2, 4.5.6 and 4.6.10]
4. locally of finite type and of relative dimension $\leq d$, and locally of finite type and of pure relative dimension d [59, IV-2, 5.5.1 and 4.1.4]
5. flat, unramified, smooth, and étale [59, IV-2, 2.2.13 and IV-4, 17.7.4]

Lemma A.6.9. *Let* $f : \mathsf{X} \to \mathsf{Y}$ *be a representable morphism of stacks over* S*, and let* $\mathsf{Y}' \to \mathsf{Y}$ *be an arbitrary morphism of stacks over* S*. Then the morphism* $f' : \mathsf{X}' = \mathsf{X} \underset{\mathsf{Y}}{\times} \mathsf{Y}' \to \mathsf{Y}'$ *induced by base change is representable.*

Moreover, let "P" be any property that is local in nature on the target for the topology chosen on the étale site of (Sch/S) *and stable under arbitrary base change. If* f *has property "P" then* f' *has property "P" too.*

Proof. This is immediate from the definitions. □

Lemma A.6.10 ([83, 3.12(a)]). *Let $f : \mathsf{X} \to \mathsf{Y}$ be a representable morphism of stacks over S. If Y is representable by an algebraic space, then so is X.*

Let $\Delta_{\mathsf{X}} : \mathsf{X} \to \mathsf{X} \underset{S}{\times} \mathsf{X}$ be the diagonal morphism. By Lemma A.6.2, two morphisms from a scheme U to $\mathsf{X} \underset{S}{\times} \mathsf{X}$ are equivalent to two objects x and y of X_U, respectively. By taking the fiber product, we obtain the following Cartesian diagram.

$$\begin{tikzcd} \underline{\mathrm{Isom}}_U(x,y) \arrow[r] \arrow[d] & \mathsf{X} \arrow[d, "\Delta_{\mathsf{X}}"] \\ U \arrow[r, "{(x,y)}"] & \mathsf{X} \underset{S}{\times} \mathsf{X} \end{tikzcd}$$

Proposition A.6.11 (cf. Proposition A.4.3). *Let X be a stack over S. The following are equivalent:*

1. *The morphism $\Delta_{\mathsf{X}} : \mathsf{X} \to \mathsf{X} \underset{S}{\times} \mathsf{X}$ is representable.*
2. *The stack $\underline{\mathrm{Isom}}_U(x, y)$ over S is representable for every scheme U over S and for every $x, y \in \mathrm{Ob}\,\mathsf{X}_U$.*
3. *For every algebraic space X over S, every morphism $X \to \mathsf{X}$ is representable.*
4. *For every scheme U over S, every morphism $U \to \mathsf{X}$ is representable.*
5. *For all schemes U and V over S, and for all morphisms $U \to \mathsf{X}$ and $V \to \mathsf{X}$, the fiber product $U \underset{\mathsf{X}}{\times} V$ is representable.*

Proof. The implications 1⇔2 and 3⇒4⇔5 follow immediately from the definitions.

1⇒5: Since the diagram

$$\begin{tikzcd} U \underset{\mathsf{X}}{\times} V \arrow[r] \arrow[d] & \mathsf{X} \arrow[d, "\Delta_{\mathsf{X}}"] \\ U \underset{S}{\times} V \arrow[r] & \mathsf{X} \underset{S}{\times} \mathsf{X} \end{tikzcd}$$

is Cartesian, $U \underset{\mathsf{X}}{\times} V$ is representable if Δ_{X} is.

5⇒1: Consider the following diagram.

$$\begin{tikzcd} U \underset{\mathsf{X} \underset{S}{\times} \mathsf{X}}{\times} \mathsf{X} \arrow[r] \arrow[d] & U \underset{\mathsf{X}}{\times} U \arrow[r] \arrow[d] & \mathsf{X} \arrow[d, "\Delta_{\mathsf{X}}"] \\ U \arrow[r, "\Delta_U"] & U \underset{S}{\times} U \arrow[r] & \mathsf{X} \underset{S}{\times} \mathsf{X} \end{tikzcd}$$

Both squares are Cartesian and $U \underset{\mathsf{X}}{\times} U$ is representable by hypothesis. Hence $U \underset{\mathsf{X} \underset{S}{\times} \mathsf{X}}{\times} \mathsf{X}$ is representable.

5⇒3: For each scheme U over S, and for each morphism $U \to \mathsf{X}$, the induced morphism $X \underset{\mathsf{X}}{\times} U \to X$ is representable by the following diagram.

$$\begin{tikzcd} V \underset{\mathsf{X}}{\times} U \arrow[r] \arrow[d] & X \underset{\mathsf{X}}{\times} U \arrow[r] \arrow[d] & U \arrow[d] \\ V \arrow[r] & X \arrow[r] & \mathsf{X} \end{tikzcd}$$

Hence $X \underset{\mathsf{X}}{\times} U$ is representable by Lemma A.6.10. □

A.7 Algebraic Stacks

There are definitions of algebraic stacks in the literature that are not equivalent to each other (see [36], [11], and [83]; see also de Jong's *Stacks Project* on his website). The following is the definition taken from [83].

Definition A.7.1. *An* ***Artin stack*** *is a stack X such that we have the following:*

1. (***Quasi-separateness***, *cf. Remark A.2.5*) *The diagonal morphism Δ_{X} is representable, quasi-compact, and separated.* (*More precisely, for each $U \in \mathrm{Ob}(\mathrm{Sch}\,/S)$ and $x, y \in \mathrm{Ob}\,\mathsf{X}_U$, we require the presheaf $\underline{\mathrm{Isom}}_U(x, y)$ of isomorphisms from x to y to be representable by an algebraic space quasi-compact and separated over U. The separateness here is automatic if X is representable by a scheme or an algebraic space, because the diagonal morphism Δ_{X} is a monomorphism for a scheme or an algebraic space by definition* (*see Definition A.4.4*)*, and a monomorphism is automatically separated* (*see [59, I, 55.1]*).)
2. *There exist an algebraic space X over S, and a morphism $p : X \to \mathsf{X}$ of stacks over S* (*automatically representable by Proposition A.6.11*)*, called an* ***atlas*** *or a* ***presentation*** *of X, that is surjective* (*see Definition A.6.4*) *and* ***smooth***.

However, for the purpose of studying moduli spaces of abelian schemes with additional structures, it is usually desirable to consider the following narrower definition:

Definition A.7.2. *A* ***Deligne–Mumford stack*** *over S is an algebraic stack over S that admits an* ***étale*** *presentation.* (*That is, in Definition A.7.1, p is an étale morphism.*)

Convention A.7.3. *From now on, unless otherwise specified, our* ***algebraic stacks*** *will be Deligne–Mumford stacks.* (*We will still explicitly specify Deligne–Mumford stacks when necessary.*) *A presentation of a Deligne–Mumford stack X will be understood to be an étale and surjective morphism $p : X \to \mathsf{X}$ from an algebraic space X to the stack X.* (*In [36], the presentation is assumed to be from a* ***scheme****; yet there is no essential difference, because algebraic spaces admit étale surjections from schemes.*)

We denote by $(\text{Alg-St}\,/S)$ the full subcategory of (Deligne–Mumford) algebraic stacks in $(\mathrm{St}\,/S)$. A stack over S associated with an algebraic space (over S) is clearly an algebraic stack. Hence $(\text{Alg-Spc}\,/S)$ is a sub-2-category of $(\text{Alg-St}\,/S)$.

Proposition A.7.4 ([83, 4.4]). *An algebraic stack X over S is representable* (*by an algebraic space; see Definition A.6.5 and Remark A.6.6*) *if and only if the diagonal 1-morphism*

$$\Delta_{\mathsf{X}} : \mathsf{X} \to \mathsf{X} \underset{S}{\times} \mathsf{X} \tag{A.7.5}$$

is a monomorphism.

In the remainder of this section, the reader is encouraged to compare the definitions and properties (for algebraic stacks) with the corresponding ones (for algebraic spaces) in Section A.4.

A.7.1 Quotients of Étale Groupoid Spaces

Proposition A.7.1.1 (cf. [83, 4.3.1 and 4.3.2])**.** *A category X fibered in groupoids over S is an algebraic stack if and only if it is a quotient for a groupoid space $X_\bullet = (X_0, X_1, s, t, m, e, [-1])$ in (Spc/S) such that X_0 and X_1 are algebraic spaces, such that $\delta = (s,t) : X_1 \to X_0 \underset{S}{\times} X_0$ is quasi-compact and separated, and such that s and t are étale morphisms from X_1 to X_0. (The morphisms δ, s, and t are automatically representable because their sources and targets are algebraic spaces.) (See Definition A.5.1.2 and Construction A.5.1.3.) More precisely, we have the following:*

1. *If X is an algebraic stack, and if $\pi : X_0 \to \mathsf{X}$ is a morphism from an algebraic space X_0 to X that is surjective and étale, then the projections $\mathrm{pr}_i : X_1 := X_0 \underset{\mathsf{X}}{\times} X_0 \to X_0$ are étale for $i = 1, 2$, and the morphism $(\mathrm{pr}_1, \mathrm{pr}_2) : X_1 \to X_0 \underset{S}{\times} X_0$ is quasi-compact and separated. Let $s := \mathrm{pr}_1 : X_1 \to X_0$ and $t := \mathrm{pr}_2 : X_1 \to X_0$. Let $m : X_1 \underset{t, X_0, s}{\times} X_1 \cong X_0 \underset{\mathsf{X}}{\times} X_0 \underset{\mathsf{X}}{\times} X_0 \to X_1$ be induced by the projection $X_0 \underset{S}{\times} X_0 \underset{S}{\times} X_0 \to X_0 \underset{S}{\times} X_0$ to the first and third factors. Let $e : X_0 \to X_1 = X_0 \underset{\mathsf{X}}{\times} X_0$ be the morphism induced by the diagonal $\Delta_{X_0} : X_0 \to X_0 \underset{S}{\times} X_0$. Let $[-1] : X_1 \to X_1$ be induced by the morphism $X_0 \underset{S}{\times} X_0 \to X_0 \underset{S}{\times} X_0$ switching the two factors. Then $(X_0, X_1, s, t, m, e, [-1])$ defines a groupoid space $X_\bullet$, and X is isomorphic to the quotient of the groupoid space $X_\bullet$.*
2. *Conversely, if $X_\bullet = (X_0, X_1, s, t, m, e, [-1])$ is a groupoid space in (Spc/S) with X_0 and X_1 algebraic spaces, with $\delta = (s,t) : X_1 \to X_0 \underset{S}{\times} X_0$ quasi-compact and separated, and with s and t étale, then the quotient X of $X_\bullet$ is an algebraic stack over S, and the canonical morphism from X_0 to X is representable, étale, and surjective.*

Hence an algebraic stack can be viewed as the quotient of an algebraic space by an étale groupoid space.

Remark A.7.1.2. Given a morphism $\pi : X_0 \to \mathsf{X}$ from an algebraic space X_0 to a stack X, let $X_1 := X_0 \underset{\mathsf{X}}{\times} X_0$. This defines a morphism $\delta : X_1 \to X_0 \underset{S}{\times} X_0$ by $\delta := (\mathrm{pr}_1, \mathrm{pr}_2) : X_0 \underset{\mathsf{X}}{\times} X_0 \to X_0 \underset{S}{\times} X_0$. Then the morphism δ is related to $\Delta_{\mathsf{X}} : \mathsf{X} \to \mathsf{X} \underset{S}{\times} \mathsf{X}$ by the following *Cartesian* diagram.

$$\begin{tikzcd} X_0 \underset{\mathsf{X}}{\times} X_0 \arrow[r] \arrow[d, "\delta"'] & \mathsf{X} \arrow[d, "\Delta_{\mathsf{X}}"] \\ X_0 \underset{S}{\times} X_0 \arrow[r] & \mathsf{X} \underset{S}{\times} \mathsf{X} \end{tikzcd}$$

In particular, if π is étale and surjective, then δ is representable, quasi-compact, and separated if and only if Δ_{X} is.

Thanks to Proposition A.7.1.1, we can alternatively define the following:

Definition A.7.1.3. *A category X fibered in groupoids over S is called an algebraic stack over S if it is the quotient of an **étale groupoid space** $X_\bullet$, namely, a groupoid space $X_\bullet = (X_0, X_1, s, t, m, e, [-1])$ such that X_0 and X_1 are algebraic spaces, such that $\delta = (s,t)$ is quasi-compact and separated, and such that s and t are étale.*

A.7.2 Properties of Algebraic Stacks

The properties of an algebraic stack are characterized by its (étale) presentations:

Definition A.7.2.1. *Let "P" be a property of algebraic spaces that is stable in the étale topology (cf. Definitions A.3.1 and A.4.2.7). Then we say that an algebraic stack X over S has property "P" if and only if for one (and hence for every) presentation $p : X \to \mathsf{X}$, the algebraic space X has property "P".*

Definition A.7.2.2. *Let "P" be a property of morphisms of algebraic spaces that is stable and local on the source in the étale topology (cf. Definitions A.3.5 and A.4.2.8). Suppose $f : \mathsf{X} \to \mathsf{Y}$ is a morphism of algebraic stacks. Then we say that f has property "P" if and only if for one (and hence for every) commutative diagram of stacks*

$$\begin{tikzcd} X' \arrow[r, "p'"] \arrow[dr, "f'"'] & Y \underset{\mathsf{Y}}{\times} \mathsf{X} \arrow[r] \arrow[d] & \mathsf{X} \arrow[d, "f"] \\ & Y \arrow[r, "p"] & \mathsf{Y} \end{tikzcd}$$

where p (resp. p') is a presentation for Y (resp. $Y \underset{\mathsf{Y}}{\times} \mathsf{X}$), the induced f' has property "P".

Proposition A.7.2.3. *A scheme (resp. an algebraic space) satisfies a property "P" as a scheme (resp. an algebraic stack) if and only if it satisfies the property "P" as an algebraic stack. A morphism of schemes (resp. algebraic spaces) satisfies a property "P" as a morphism of schemes (resp. algebraic stacks) if and only if it satisfies the property "P" as a morphism of algebraic stacks.*

Proposition A.7.2.4 (cf. [59, IV-3, 8.14.2], [7, 4.4], and [83, 4.15])**.** *A scheme (resp. an algebraic space, resp. an algebraic stack) over S is locally of finite presentation as a scheme (resp. an algebraic space, resp. an algebraic stack) if and only if it is locally of finite presentation as a category fibered in groupoids (see Definition A.5.9).*

Definition A.7.2.5. *An algebraic stack is called **quasi-compact** if there exists a presentation $p : X \to \mathsf{X}$ with X quasi-compact. A morphism $f : \mathsf{X} \to \mathsf{Y}$ of algebraic stacks is called **quasi-compact** if for every morphism from a scheme Y into Y, the fiber product $Y \underset{\mathsf{Y}}{\times} \mathsf{X}$ is a quasi-compact algebraic stack over Y.*

Lemma A.7.2.6. *In Definition A.7.2.5 it suffices to require that there exists a surjection $f : Y \to \mathsf{X}$ from a quasi-compact algebraic space Y.*

Proof. Suppose $p : X \to \mathsf{X}$ is any presentation of X, with an étale covering $\{X_\alpha\}_{\alpha \in J}$ of X by affine schemes. Then $\{Y \underset{\mathsf{X}}{\times} X_\alpha\}_{\alpha \in J}$ is an étale covering of $Y \underset{\mathsf{X}}{\times} X$. For each α, the image W_α of $Y \underset{\mathsf{X}}{\times} X_\alpha$ in Y is open because the étale (or smooth) morphism p is open. By assumption, Y is quasi-compact, so there is a finite subset J' of J

such that $\{W_\alpha\}_{\alpha\in J'}$ cover Y. Hence, by surjectivity of f, the images of $\{W_\alpha\}_{\alpha\in J'}$ cover X. Therefore, we may replace X with the finite disjoint union of $\{X_\alpha\}_{\alpha\in J'}$ and obtain a quasi-compact presentation of X, as desired. $\square$

Definition A.7.2.7. *We define a morphism $f : \mathsf{X} \to \mathsf{Y}$ to be* ***of finite type*** *if it is quasi-compact and locally of finite type, and to be* ***of finite presentation*** *if it is quasi-compact and locally of finite presentation. An algebraic stack is* ***noetherian*** *if it is quasi-compact, quasi-separated, and locally noetherian.*

Definition A.7.2.8. *An algebraic stack X is* ***separated*** *if the (representable) diagonal morphism Δ_{X} is universally closed (hence proper, since it is separated and of finite type).*

A morphism $f : \mathsf{X} \to \mathsf{Y}$ of algebraic stacks is ***separated*** *(resp.* ***quasi-separated****) if for every morphism $U \to \mathsf{Y}$ from a separated scheme U, the fiber product $U \underset{\mathsf{Y}}{\times} \mathsf{X}$ is separated (resp. quasi-separated).*

Lemma A.7.2.9. *An algebraic stack X is separated if it is a quotient of an étale groupoid space $X_\bullet = (X_0, X_1, s, t, m, e, [-1])$ such that $\delta = (s,t)$ is quasi-compact, separated, and* ***finite****.*

Proof. This follows from Proposition A.7.1.1 and Remark A.7.1.2. $\square$

Definition A.7.2.10. *A morphism $f : \mathsf{X} \to \mathsf{Y}$ is said to be* ***proper*** *if it is separated, of finite type, and universally closed.*

Let $f : \mathsf{X} \to S$ be a morphism of finite type from an algebraic stack X to a noetherian scheme S. Assume the diagonal morphism $\mathsf{X} \to \mathsf{X} \underset{S}{\times} \mathsf{X}$ is separated and quasi-compact.

Theorem A.7.2.11 (valuative criterion for separateness; see [83, 7.8])**.** *A morphism f as above is separated if and only if, for each commutative diagram*

$$\begin{tikzcd} & \mathsf{X} \arrow[d, "f"] \\ \mathrm{Spec}(R) \arrow[ur, shift left, "g_1"] \arrow[ur, shift right, "g_2"'] \arrow[r] & S \end{tikzcd}$$

in which R is a complete discrete valuation ring R with algebraically closed residue field, each isomorphism between the restrictions of g_1 and g_2 to the generic point of $\mathrm{Spec}(R)$ *extends to an isomorphism between the morphisms g_1 and g_2.*

Theorem A.7.2.12 (valuative criterion for properness; see [83, 7.12])**.** *Suppose f is separated as in Theorem A.7.2.11. Then f is proper if and only if, for each commutative diagram*

$$\begin{tikzcd} & & \mathsf{X} \arrow[d, "f"] \\ \mathrm{Spec}(K) \arrow[urr, "g"] \arrow[r] & \mathrm{Spec}(R) \arrow[r] & S \end{tikzcd}$$

in which R is a discrete valuation ring with fraction field K, there exists a finite extension K' of K such that the restriction of g to $\mathrm{Spec}(K')$ *extends to a morphism from* $\mathrm{Spec}(R')$ *to X, where R' is the integral closure of R in K'.*

Remark A.7.2.13. To prove that a given f is proper, it suffices to verify the above criterion under the additional hypothesis that R is complete and has an algebraically closed residue field. Furthermore, if there is an *open dense subset* U of X (see Definition A.7.4.3), then it is enough to test only those g's that factor through U (see [83, 7.12.4]).

Definition A.7.2.14. *The* ***disjoint union*** $\mathsf{X} = \coprod_{i\in I} \mathsf{X}_i$ *of a family $(\mathsf{X}_i)_{i\in I}$ of stacks is the stack a section of which over a scheme U consists of a decomposition $U = \coprod_{i\in I} U_i$ of U and a section of x_i over U_i for each i.*

The empty stack $\emptyset$ is the algebraic stack represented by the empty scheme. A stack is *connected* if it is nonempty and is not the disjoint union of two nonempty stacks.

Proposition A.7.2.15 ([36, Prop. 4.14]; see [83, 4.9])**.** *A locally noetherian algebraic stack is in one and only one way the disjoint union of a family of connected algebraic stacks (called its* ***connected components****).*

We denote by $\pi_0(\mathsf{X})$ the set of connected components of a locally noetherian algebraic stack X. If $p : X \to \mathsf{X}$ is étale and surjective, then $\pi_0(\mathsf{X})$ is the cokernel of $\pi_0(X \underset{\mathsf{X}}{\times} X) \rightrightarrows \pi_0(X)$, namely, the quotient of the equivalence relation of sets defined by the two projections.

Definition A.7.2.16. *A substack Y of X is called* ***open*** *(resp.* ***closed****, resp.* ***locally closed****) if the inclusion morphism $\mathsf{Y} \to \mathsf{X}$ is representable and is an open immersion (resp. closed immersion, resp. locally closed immersion).*

Definition A.7.2.17. *An algebraic stack X is* ***irreducible*** *if it is nonempty and if, for every two nonempty open substacks Y_1 and Y_2 in X, their intersection $\mathsf{Y}_1 \cap \mathsf{Y}_2$ is nonempty.*

A.7.3 Quasi-Coherent Sheaves on Algebraic Stacks

Definition A.7.3.1. *Let X be an algebraic stack. The* ***étale site*** *$\mathsf{X}_{\text{ét}}$ of X is the category with objects that are the étale morphisms $u : U \to \mathsf{X}$ from schemes, where a morphism from (U,u) to (V,v) is a morphism of schemes $\phi : U \to V$ together with a 2-morphism between the 1-morphisms $u : U \to \mathsf{X}$ and $v \circ \phi : U \to \mathsf{X}$. A collection of morphisms $\phi_i : (U_i, u_i) \to (U,u)$ is a covering family if the underlying family of morphisms of schemes is surjective.*

The site $\mathsf{X}_{\text{ét}}$ is in a natural way ringed (the meaning of this is made clear by the following definitions). When we speak of *sheaves* over X we mean sheaves over $\mathsf{X}_{\text{ét}}$.

Definition A.7.3.2. *A* ***quasi-coherent sheaf*** *$\mathscr{F}$ over the algebraic stack X consists of the following data:*

1. *For each morphism $U \to \mathsf{X}$ where U is a scheme, a quasi-coherent sheaf $\mathscr{F}_U$ over U.*
2. *For each morphism $f : U \to V$ over X, an isomorphism $\phi_f : \mathscr{F}_U \xrightarrow{\sim} f^*\mathscr{F}_V$ satisfying the* ***cocycle condition****; namely, for all morphisms $f : U \to V$ and $g : V \to W$ over X, we have $\phi_{g\circ f} = \phi_f \circ f^*\phi_g$.*

We say that $\mathscr{F}$ is ***coherent*** *(resp.* ***of finite type****, resp.* ***of finite presentation****, resp.* ***locally free****) if $\mathscr{F}_U$ is coherent (resp. of finite type, resp. of finite presentation, resp. locally free) for every U.*

A morphism of quasi-coherent sheaves $h : \mathscr{F} \to \mathscr{F}'$ is a collection of morphisms of sheaves $h_U : \mathscr{F}_U \to \mathscr{F}'_U$ compatible with all the isomorphisms ϕ_f as above.

This definition is compatible with that in Section A.4.3.

Definition A.7.3.3. *Let* X *be an algebraic stack. The* ***structural sheaf*** $\mathscr{O}_\mathsf{X}$ *is defined by taking* $(\mathscr{O}_\mathsf{X})_U = \mathscr{O}_U$ *for each morphism* $U \to X$ *from a scheme* U*.*

Proposition A.7.3.4 ([36]; see [83, 14.2.4])**.** *Let* X *be an algebraic stack. Then the functor that assigns to each algebraic stack* $f : \mathsf{T} \to \mathsf{X}$ *the* $\mathscr{O}_\mathsf{X}$*-algebra* $f_* \mathscr{O}_\mathsf{T}$ *induces an antiequivalence between the following two categories:*

1. *The category of algebraic stacks* ***schematic*** *and affine over* X*. (This is analogous to schematic morphisms of algebraic spaces: A morphism* $f : \mathsf{X} \to \mathsf{Y}$ *is* ***schematic*** *if, for each morphism* $U \to \mathsf{Y}$ *from a scheme* U*, the fiber product* $U \underset{\mathsf{Y}}{\times} \mathsf{X}$ *is representable by a* ***scheme****.)*

2. *The category of quasi-coherent* $\mathscr{O}_\mathsf{X}$*-algebras.*

Definition A.7.3.5. *Let* X *be an algebraic stack, and let* $\mathscr{F}$ *be a quasi-coherent* $\mathscr{O}_\mathsf{X}$*-algebra. For all étale morphisms* $u : U \to \mathsf{X}$ *with* U *an affine scheme, let* $\mathscr{A}_U$ *be the integral closure of* $\Gamma(U, (\mathscr{O}_\mathsf{X})_U) = \Gamma(U, \mathscr{O}_U)$ *in* $\mathscr{F}_U$*. By [59, II, 6.3.4], the* $\mathscr{A}_U$ *for variable* U *are the sections over* U *of a quasi-coherent sheaf* $\mathscr{A}$ *over* X*, which we will call the* ***integral closure*** *of* $\mathscr{O}_\mathsf{X}$ *in* $\mathscr{F}$*.*

Definition A.7.3.6. *Let* $f : \mathsf{T} \to \mathsf{X}$ *be schematic and affine. The algebraic stack associated by Proposition A.7.3.4 with the integral closure of* $\mathscr{O}_\mathsf{X}$ *in* $f^* \mathscr{O}_\mathsf{T}$ *will be called the* ***normalization*** *of* X *with respect to* T*. Its formation is compatible with arbitrary étale base change.*

A.7.4 Points and the Zariski Topology of Algebraic Stacks

Let X be an algebraic stack. Consider the set of pairs (x, k), where k is a field over S and where $x : \operatorname{Spec}(k) \to \mathsf{X}$ is an object in $\mathsf{X}_{\operatorname{Spec}(k)}$. (This is slightly different from the definition of a point of algebraic spaces in Definition A.4.4.1. Here we do not assume that the morphism is a monomorphism.) We define two elements (x, k) and (x', k') of this set to be *equivalent* if there is a third field k'' with coverings $\operatorname{Spec}(k'') \to \operatorname{Spec}(k)$ and $\operatorname{Spec}(k'') \to \operatorname{Spec}(k')$ in the étale site of (Sch/S) such that the induced objects $x|_{\operatorname{Spec}(k)}$ and $x'|_{\operatorname{Spec}(k')}$ are isomorphic in $\mathsf{X}(\operatorname{Spec}(k''))$. It is clear that this is an equivalence relation.

Definition A.7.4.1. *A* ***point*** *of the algebraic stack* X *is an equivalence class in the set defined above. The set of points of* X*, called the* ***associated underlying topological space****, is denoted by* $|\mathsf{X}|$*.*

Suppose $x : \operatorname{Spec}(k) \to \mathsf{X}$ is a representative of a point of X, and $f : \mathsf{X} \to \mathsf{Y}$ is a morphism. Then we have by composition a point of Y, the equivalence class of $f \circ x : \operatorname{Spec}(k) \to \mathsf{Y}$. Hence we have a map $|f| : |\mathsf{X}| \to |\mathsf{Y}|$, which we often denote by f if there's no confusion.

Definition A.7.4.2. *Let* X *be an algebraic stack. The* ***Zariski topology*** *on* $|\mathsf{X}|$ *is defined by taking open sets to be subsets of the form* $U = |\mathsf{Y}|$ *for some open substack* Y*.*

Definition A.7.4.3. *As in the case of algebraic spaces, we may define a morphism* $f : \mathsf{X} \to \mathsf{Y}$ *between algebraic stacks to be open, closed, with dense image, universally closed, etc., using the induced map* $|f| : |\mathsf{X}| \to |\mathsf{Y}|$ *between underlying topological spaces.*

A.7.5 Coarse Moduli Spaces

The results in this section are quoted from [38, I, §8]. The justifications for some of the claims can be found in [68], with explanations supplied by [32].

Let X be an algebraic stack over S.

Definition A.7.5.1. *A* ***coarse moduli space*** *of* X *is an algebraic space* $[\mathsf{X}]$ *over* S*, with a morphism* $\pi : \mathsf{X} \to [\mathsf{X}]$ *over* S *such that we have the following:*

1. *Each morphism from* X *to an algebraic space* Z *over* S *factors through* π *and induces a morphism from* $[\mathsf{X}]$ *to* Z*.*

2. *If* $\bar{s} : \operatorname{Spec}(k) \to S$ *is a geometric point of* S *(where* k *is algebraically closed), then* π *induces a bijection between the set of isomorphism classes of objects in* X *over* $\bar{s}$ *(namely, isomorphism classes of morphisms from* $\bar{s}$ *to* X *over* S*) and* $[\mathsf{X}](\bar{s})$*.*

Remark A.7.5.2. The term *coarse* here is meant to be in contrast to the term *fine* in a *fine moduli space* for X, namely, an algebraic space that *represents* the algebraic stack. The idea is that we can always interpret algebraic stacks as moduli problems.

If S is locally noetherian and X is separated of finite type over S, then we can show that X admits a coarse moduli space $[\mathsf{X}]$. Here are some of its properties:

1. Let $x : \operatorname{Spec}(k) \to \mathsf{X}$ be a geometric point of X, let $\mathscr{O}^h_{\mathsf{X},x}$ be the *strict local ring* of X at x, let $\mathscr{O}^h_{[\mathsf{X}],\pi(x)}$ be the strict local ring of $[\mathsf{X}]$ at $\pi(x)$, and let $\operatorname{Aut}(x)$ be the group of automorphisms of the object x in X over $\operatorname{Spec}(k)$. Then the morphism π induces an isomorphism
$$\operatorname{Spec}(\mathscr{O}^h_{\mathsf{X},x}) / \operatorname{Aut}(x) \xrightarrow{\sim} \operatorname{Spec}(\mathscr{O}^h_{[\mathsf{X}],\pi(x)}).$$
Suppose that $H \subset \operatorname{Aut}(x)$ is the subgroup of automorphisms of the object $x : \operatorname{Spec}(k) \to \mathsf{X}$ that extends to automorphisms of the object $\operatorname{Spec}(\mathscr{O}^h_{\mathsf{X},x}) \to \mathsf{X}$. Then the group $\operatorname{Aut}(x)/H$ acts faithfully on $\mathscr{O}^h_{\mathsf{X},x}$.

2. The formation of a coarse moduli space does not commute with arbitrary base change. However, it commutes with flat base change, and with arbitrary base change when π is étale. Moreover, if $[\mathsf{X}']$ is a coarse moduli scheme of $\mathsf{X}' := \mathsf{X} \underset{S}{\times} S'$, the morphism
$$[\mathsf{X}'] \to [\mathsf{X}] \underset{S}{\times} S'$$
is *radicial* (or *universally injective*); namely, for each field K, the induced morphism on K-valued points is injective; see [59, I, 3.5.4].

3. If $u : U \to \mathsf{X}$ is étale surjective, then $[\mathsf{X}]$ is a geometric and uniform categorical quotient of U by the groupoid space $U \underset{\mathsf{X}}{\times} U$ in the category of algebraic spaces (see [68] for more details).

Appendix B

Deformations and Artin's Criterion

In this appendix we review the basic notions of infinitesimal deformations, which can be generalized to the context of 2-categories, and give a proof of Artin's criterion for algebraic stacks using his theory of algebraization.

Throughout this appendix, the index i is an integer ≥ 0, and all projective limits, unless otherwise specified, run through all integers $i \geq 0$.

B.1 Infinitesimal Deformations

Let Λ be a noetherian complete local ring with residue field k.

Notation B.1.1. *We denote by C_Λ the category of Artinian local Λ-algebras with residue field k and by $\hat{\mathsf{C}}_\Lambda$ the category of noetherian complete local Λ-algebras with residue field k.*

A covariant functor F from C_Λ to (Sets), the category of sets, extends to $\hat{\mathsf{C}}_\Lambda$ by the formula

$$\hat{F}(R) = \varprojlim F(R/\mathfrak{m}^{i+1})$$

for $R \in \hat{\mathsf{C}}_\Lambda$ with maximal ideal $\mathfrak{m}$. Conversely, a covariant functor F from $\hat{\mathsf{C}}_\Lambda$ into (Sets) induces by restriction a functor $F|_{\mathsf{C}_\Lambda} : \mathsf{C}_\Lambda \to$ (Sets).

For each covariant functor F from $\hat{\mathsf{C}}_\Lambda$ into (Sets), there is a canonical map

$$F(R) \to \hat{F}(R) = \varprojlim F(R/\mathfrak{m}^{i+1}). \tag{B.1.2}$$

In general, we do not know whether it is a bijection or not.

For each R in $\hat{\mathsf{C}}_\Lambda$ with maximal ideal m, we define a functor h_R on C_Λ by setting $h_R(A) = \operatorname{Hom}(R, A)$ for each $A \in \mathsf{C}_\Lambda$.

Lemma B.1.3. *If F is any functor on C_Λ, and if R is in $\hat{\mathsf{C}}_\Lambda$, then we have a canonical isomorphism*

$$\hat{F}(R) \xrightarrow{\sim} \operatorname{Hom}(h_R, F). \tag{B.1.4}$$

Proof. Let $\hat{\xi} = \varprojlim \xi_i$ be in $\hat{F}(R)$, where $\{\xi_i \in F(R/\mathfrak{m}^{i+1})\}_{i\geq 0}$ is a compatible system of elements (i.e., ξ_{i+1} induces ξ_i in $F(R/\mathfrak{m}^{i+1})$ for $i \geq 0$). Each $u : R \to A$ factors through $u_{i_0} : R/\mathfrak{m}^{i_0+1} \to A$ for some $i_0 \geq 0$, and we assign to each $u \in h_R(A)$ the element $F(u_{i_0})(\xi_{i_0})$ of $F(A)$. Conversely, for each natural transformation $h_R \to F$, and for each integer $i \geq 0$, let ξ_i be the image of the canonical homomorphism $(R \to R/\mathfrak{m}^{i+1}) \in h_R(R/\mathfrak{m}^{i+1})$ in $F(R/\mathfrak{m}^{i+1})$. Then $\{\xi_i \in F(R/\mathfrak{m}^{i+1})\}_{i\geq 0}$ form a compatible system of elements by the functorial property of the natural transformation $h_R \to F$. Thus we get an element $\hat{\xi} = \varprojlim \xi_i$ in $\hat{F}(R)$. It is clear that this association is an inverse of the previous assignment $\hat{F}(R) \to \operatorname{Hom}(h_R, F)$. Hence we have indeed an isomorphism (B.1.4). □

After the above identification, we are ready to make the following definitions.

Definition B.1.5. *A covariant functor F from C_Λ to* (Sets) *is called* ***prorepresentable*** *if there exist $R \in \operatorname{Ob} \hat{\mathsf{C}}_\Lambda$ and $\hat{\xi} \in \hat{F}(R)$ that induce an isomorphism*

$$\hat{\xi} : h_R(A) = \operatorname{Hom}(R, A) \xrightarrow{\sim} F(A) \tag{B.1.6}$$

of functors over C_Λ. The algebra R is then uniquely determined.

Definition B.1.7. *Let F be a covariant functor from $\hat{\mathsf{C}}_\Lambda$ to* (Sets)*. Suppose $F|_{\mathsf{C}_\Lambda}$ is prorepresentable. Then we say that it is* ***effectively prorepresentable*** *if there exists some $\xi \in F(R)$ (with R as above) that induces an isomorphism as the $\hat{\xi}$ in (B.1.6) via the canonical map (B.1.2). (Note that $F(R)$ and $\hat{F}(R)$ are not the same in general.)*

Let F be a covariant functor from $\hat{\mathsf{C}}_\Lambda$ into (Sets), and consider an element

$$\xi_0 \in F(k). \tag{B.1.8}$$

Definition B.1.9. *By an* ***infinitesimal deformation*** *of ξ_0, we mean an element $\eta \in F(A)$ where $A \in \operatorname{Ob} \mathsf{C}_\Lambda$ is an Artinian local Λ-algebra with residue field k, and η induces $\xi_0 \in F(k)$ by functoriality.*

Definition B.1.10. *A* ***formal deformation*** *of ξ_0 is an element*

$$\hat{\xi} = \varprojlim \xi_i \in \hat{F}(R) = \varprojlim F(R/m^{i+1})$$

where $R \in \operatorname{Ob} \hat{\mathsf{C}}_\Lambda$ is a noetherian complete local Λ-algebra with maximal ideal $\mathfrak{m}$ and residue field k, and where $\{\xi_i \in F(R/\mathfrak{m}^{i+1})\}_{i\geq 0}$ is a compatible system of elements (i.e., ξ_{i+1} induces ξ_i in $F(R/\mathfrak{m}^{i+1})$ for each $i \geq 0$) with the element ξ_0 given above.

Definition B.1.11. *A formal deformation $\hat{\xi} = \varprojlim \xi_i \in \hat{F}(R)$ of ξ_0 is said to be* ***effective*** *if there is an element $\xi \in F(R)$ that induces ξ_i for each i.*

Definition B.1.12. *A formal deformation $\hat{\xi} \in \hat{F}(R)$ is said to be* ***versal*** *(resp.* ***universal****) if it has the following property: Let $A' \to A$ be any surjection of Artinian local Λ-algebras in C_Λ, let $\eta' \in F(A')$ be any infinitesimal deformation of ξ_0, and let $\eta \in F(A)$ be the infinitesimal deformation of ξ_0 induced by η'. Then each homomorphism $R \to A$ that induces $\eta \in F(A)$ by $\hat{\xi} \in \hat{F}(R)$ and (B.1.4) can be* ***lifted*** *(resp.* ***uniquely lifted****) to a morphism $R \to A'$ (whose composition with $A' \to A$ is the given morphism $R \to A$) such that η' is induced by the homomorphism $R \to A'$.*

If $F(k)$ has only one element ξ_0, then the existence of a formal universal deformation of ξ_0 is equivalent to the prorepresentability of F. (This is slightly weaker than the prorepresentability defined in [51], which considers also Artinian local rings whose residue fields are finite extensions of k. Nevertheless, we will deal with finite extensions directly in our context. See Section B.1.1 for more information.)

B.1.1 Structures of Complete Local Rings

In our study of deformation, we may need to consider the case where the field k is a finite field extension of a residue field $k(s)$ of an excellent scheme S, and a functor F will be defined on the category of complete local noetherian $\mathscr{O}_S$-algebras with residue field k. (The excellence of S implies, in particular, that the formal completions of S are noetherian.) Under these assumptions, we would like to find a noetherian complete local $\mathscr{O}_S$-algebra Λ with residue field k such that a complete local noetherian ring with residue field k is a local $\mathscr{O}_{S,s}$-algebra (with compatible structural morphisms to k) if and only if it is a local Λ-algebra. If this is possible, then F defines naturally (by restriction) a functor from $\hat{\mathsf{C}}_\Lambda$ to (Sets), which we again denote by F, and we are justified to study the restricted functor only.

This problem is solved by Cohen's structural theorems on complete local rings. We first consider the case where the characteristic of the complete local ring is equal to its residue field.

Definition B.1.1.1. *A local ring $(R, \mathfrak{m}, k)$ is called **equicharacteristic** if* $\mathrm{char}(R) = \mathrm{char}(k)$.

*For such an equicharacteristic local ring, a subfield $k' \subset R$ is called a **coefficient field** if k' is mapped isomorphically to k under the canonical homomorphism $R \to R/\mathfrak{m} = k$, or equivalently, if $R = k' + \mathfrak{m}$.*

Theorem B.1.1.2. *Let $(R, \mathfrak{m}, k)$ be a complete equicharacteristic local ring. Then there is a coefficient field $k' \subset R$. If the maximal ideal $\mathfrak{m}$ can be generated by n elements, then R is a homomorphic image of the formal power series ring $k[[X_1, \ldots, X_n]]$.*

Proof. The original proof can be found in [27, Thm. 9]. For references that are more accessible, see [88, Thm. 28.3], [41, Thm. 7.7], or [97, Thm. 31.1]. □

On the other hand, if $\mathrm{char}(R) \neq \mathrm{char}(k)$, then necessarily $\mathrm{char}(k) = p$ for some prime number p. In this case, it is not possible to have such a coefficient field, because p times the units of k must be zero in R.

Nevertheless, it is still possible to have a so-called *coefficient ring* $R_0 \subset R$, where R_0 is a complete local ring with maximal ideal pR_0 and $R = R_0 + \mathfrak{m}$. Namely, $k = R/\mathfrak{m} \cong R_0/pR_0$. More precisely, the coefficient ring is a homomorphic image of a *p-ring*.

Definition B.1.1.3. *A p-**ring** is a discrete valuation ring of characteristic zero whose maximal ideal is generated by the prime number p.*

The simplest examples of a p-ring are the rings $\mathbb{Z}_{(p)}$ and $\mathbb{Z}_p$.

Theorem B.1.1.4 (see [88, Thm. 29.1])**.** *Let (A, tA, k) be a discrete valuation ring and k' an extension field of k, where t is the uniformizer of A. Then there exists a discrete valuation ring (A', tA', k') containing (A, tA, k) with the same uniformizer t.*

Corollary B.1.1.5. *For each given field K of characteristic p, there is a p-ring A_K having K as its residue field.*

Proof. This is immediate by applying Theorem B.1.1.4 to the rings $\mathbb{Z}_{(p)}$ or $\mathbb{Z}_p$. □

Theorem B.1.1.6 (see [88, Thm. 29.2] or [27, Thm. 11])**.** *Let $(R, \mathfrak{m}, K)$ be a complete local ring, let (A, tA, k) be a p-ring, and let $\phi_0 : k \to K$ be a homomorphism. Then there exists a local homomorphism $\phi : A \to R$ that induces ϕ_0 on the residue fields.*

Then the following corollary is immediate:

Corollary B.1.1.7. *A complete p-ring is uniquely determined* (*up to isomorphism*) *by its residue field.*

Theorem B.1.1.8 (see [27, Thm. 12] or [97, Thm. 31.1])**.** *Let $(R, \mathfrak{m}, k)$ be a complete local ring with* $\mathrm{char}(k) = p$*. Then there is a coefficient ring $R_0 \subset R$, which is a homomorphic image of a complete p-ring with residue field k. If the maximal ideal $\mathfrak{m}$ has a minimal generator of n elements, then R is a homomorphic image of the formal power series ring $R_0[[X_1, \ldots, X_n]]$. If $p \notin \mathfrak{m}^2$, then it is a homomorphic image of $R_0[[X_1, \ldots, X_{n-1}]]$ with only $n-1$ variables.*

The proof of Theorem B.1.1.4 requires Zorn's lemma. Hence it does not provide us with an explicit construction of the extension. Alternatively, we shall consider the *Witt vectors*, which give an explicit construction of the unique complete p-ring with residue field k when k is *perfect* of characteristic p.

Over each commutative ring A, the construction of the Witt vectors $W(A)$ as a projective limit $\varprojlim_{i \geq 1} W_i(A)$ can be found in, for example, [110, II, 6]. The upshot is the following:

Theorem B.1.1.9 (see [110, II, 6, Thm. 8])**.** *If k is a perfect field of characteristic p, then $W(k)$ is a complete p-ring with residue field k.*

Then it follows from Corollary B.1.1.7 that every complete p-ring with perfect residue field k is isomorphic to $W(k)$.

Example B.1.1.10. $W(\mathbb{F}_p) = \mathbb{Z}_p$ and $W_i(\mathbb{F}_p) = \mathbb{Z}/p^i\mathbb{Z}$.

Now let us return to our problem:

Lemma B.1.1.11. *Let S be an **excellent** scheme, s a point of S, and k a finite field extension of the residue field $k(s)$ of S. Then there is a noetherian complete local $\mathscr{O}_{S,s}$-algebra Λ with residue field k such that a noetherian complete local ring R with residue field k is a local $\mathscr{O}_{S,s}$-algebra* (*with compatible structural morphisms to k*) *if and only if it is a local Λ-algebra. Concretely, Λ is the completion of $\hat{\mathscr{O}}_{S,s} \underset{A_{k(s)}}{\otimes} A_k$ at the maximal ideal given by the kernel of the homomorphism $\hat{\mathscr{O}}_{S,s} \underset{A_{k(s)}}{\otimes} A_k \to k$ induced by* $\mathrm{Spec}(k) \to S$*. If* $\mathrm{char}(k) = 0$*, then $A_{k(s)}$* (*resp. A_k*) *is simply $k(s)$* (*resp. k*)*. If* $\mathrm{char}(k) = p > 0$*, then $A_{k(s)}$* (*resp. A_k*) *is the complete p-ring with residue field $k(s)$* (*resp. k*) (*see Corollary B.1.1.5*).

Alternatively, but less explicitly, one can consider the completion of the Henselization of $\mathscr{O}_{S,s} \to k$ (see [44, p. 16], where k does not have to be $k(s)$).

Proof of Lemma B.1.1.11. By Theorems B.1.1.2 and B.1.1.6, given any complete local ring R with residue field k, there is a local homomorphism $A_k \to R$ making R a local A_k-algebra. Similarly, there is a local homomorphism $A_{k(s)} \to \hat{\mathscr{O}}_{S,s}$ making the ring $\hat{\mathscr{O}}_{S,s}$ a local $A_{k(s)}$-algebra. Since the homomorphism $\mathscr{O}_{S,s} \to R$ factors through $\hat{\mathscr{O}}_{S,s} \to R$, we have an induced homomorphism $\hat{\mathscr{O}}_{S,s} \underset{A_{k(s)}}{\otimes} A_k \to R$. Hence it suffices to take Λ to be the completion of $\hat{\mathscr{O}}_{S,s} \underset{A_{k(s)}}{\otimes} A_k$ at the maximal ideal given by the kernel of the induced homomorphism $\hat{\mathscr{O}}_{S,s} \underset{A_{k(s)}}{\otimes} A_k \to k$. □

The following corollary of Lemma B.1.1.11 is a restatement of the remarks made at the beginning of this section.

Corollary B.1.1.12. *With assumptions as in Lemma B.1.1.11, let F be a contravariant functor $F : (\mathrm{Sch}/S) \to (\mathrm{Sets})$, and let k be a finite field extension of a residue field $k(s)$ of S. Then there is a noetherian complete local $\mathscr{O}_{S,s}$-algebra Λ with residue field k such that a noetherian complete local ring with residue field k is an $\mathscr{O}_{S,s}$-algebra (with compatible structural morphisms to k) if and only if it is a local Λ-algebra. Hence F defines a (covariant) functor $\hat{\mathsf{C}}_\Lambda \to (\mathrm{Sets})$ by restriction (and by taking spectra of objects and morphisms in $\hat{\mathsf{C}}_\Lambda$), again denoted by F by abuse of notation.*

This will be used implicitly in Theorem B.2.1, Theorem B.2.2, Theorem B.2.1.6 (Artin's algebraization theorems) and explicitly in Theorem B.3.9. Here we say that it will be used *implicitly* because the notions of deformation were defined by Artin in a more general way in [6]. Hence Lemma B.1.1.11 and Corollary B.1.1.12 will be used to justify their (natural) compatibility with our more restrictive definitions. Lemma B.1.1.11 and Corollary B.1.1.12 will be used *explicitly* in Theorem B.3.9, where we need to associate deformation functors defined in the sense of Section B.1 with categories fibered in groupoids.

B.2 Existence of Algebraization

The following useful theorems are given by Artin in [6]:

Theorem B.2.1 (existence of algebraization; see [6, Thm. 1.6])**.** *Let S be a scheme or an algebraic space locally of finite type over a field or an excellent Dedekind domain. Consider a contravariant functor $F : (\mathrm{Sch}/S) \to (\mathrm{Sets})$ locally of finite presentation over S (see Definition A.5.7 and Proposition A.7.2.4). Let $s \in S$ be a point whose residue field $k(s)$ is of finite type over $\mathscr{O}_S$, let k' be a finite field extension of $k(s)$, and let $\xi_0 \in F(k')$. Suppose that an effective formal versal deformation (R, ξ) of ξ_0 exists, where R is a noetherian complete local $\mathscr{O}_S$-algebra with residue field k' and $\xi \in F(R)$. (Our language of deformation theory is applicable in this case, by taking Λ as in Lemma B.1.1.11 and Corollary B.1.1.12.) Then there is a scheme X of finite type over S, a closed point $x \in X$ with residue field $k(x) = k'$, and an element $\widetilde{\xi} \in F(X)$, such that the triple $(X, x, \widetilde{\xi})$ is a versal deformation of ξ_0. More precisely, there is an isomorphism $\hat{\mathscr{O}}_{X,x} \cong R$ such that $\widetilde{\xi}$ induces ξ via the composition $\mathrm{Spec}(R) \xrightarrow{\sim} \mathrm{Spec}(\hat{\mathscr{O}}_{X,x}) \to X$. The isomorphism $\hat{\mathscr{O}}_{X,x} \cong R$ is unique if (R, ξ) is universal.*

Theorem B.2.2 (uniqueness of algebraization; see [6, Thm. 1.7])**.** *With notation as in Theorem B.2.1, suppose that the element $\xi \in F(R)$ is uniquely determined by the set $\{\xi_i\}_{i \geq 0}$ of its truncations (namely, images of ξ induced by homomorphisms $R \to R/m^{i+1}$). Then the triple $(X, x, \widetilde{\xi})$ is unique up to local isomorphism for the étale topology, in the following sense: If $(X', x', \widetilde{\xi}')$ is another algebraization, then there is a third one $(X'', x'', \widetilde{\xi}'')$ and a diagram $X \xleftarrow{f} X'' \xrightarrow{f'} X'$, where f and f' are étale morphisms, which sends $x \xleftarrow{f} x'' \xrightarrow{f'} x'$ and $\widetilde{\xi} \xleftarrow{f} \widetilde{\xi}'' \xrightarrow{f'} \widetilde{\xi}'$.*

B.2.1 Generalization from Sets to Groupoids

The above theorems of algebraization consider the case where F is a contravariant functor from (Sch/S) to (Sets). However, in our later application, we will need to consider contravariant 2-functors from (Sch/S) to the 2-category (Gr) of groupoids. With such a contravariant 2-functor $\mathsf{F} : (\mathrm{Sch}/S) \to (\mathrm{Gr})$, we can canonically associate a contravariant functor $\overline{\mathsf{F}} : (\mathrm{Sch}/S) \to (\mathrm{Sets})$ such that, for each scheme U, $\overline{\mathsf{F}}(U)$ is the set of isomorphism classes of $\mathsf{F}(U)$. The functorial properties of $\overline{\mathsf{F}}$ are implied by the 2-functorial properties of F. Furthermore, we have a canonical morphism (of 2-functors) $\mathsf{F} \to \overline{\mathsf{F}}$ defined by sending each object of $\mathsf{F}(U)$ to the isomorphism class containing it in $\overline{\mathsf{F}}(U)$. When $U = \mathrm{Spec}(A)$ is an affine scheme, we shall also denote $\mathsf{F}(U)$ by $\mathsf{F}(A)$, by abuse of notation. We extend such an abuse of notation to morphisms between affine schemes.

To generalize the notions of deformations studied in Section B.1, we would like to consider a functor of the form $\mathsf{F} : \hat{\mathsf{C}}_\Lambda \to (\mathrm{Gr})$. Let $\xi_0 \in \mathrm{Ob}\,\mathsf{F}(k)$ be an object in the groupoid $\mathsf{F}(k)$. Let $R \in \hat{\mathsf{C}}_\Lambda$ be any noetherian complete ring with residue field k.

Definition B.2.1.1. *A **formal deformation** of ξ_0 is an object $\hat{\xi} = \varprojlim \xi_i$ with ξ_0 given above in the projective limit $\hat{\mathsf{F}}(R) = \varprojlim \mathsf{F}(R/\mathfrak{m}^{i+1})$, where $\{\xi_i \in \mathrm{Ob}\,\mathsf{F}(R/\mathfrak{m}^{i+1})\}_{i \geq 0}$ is a projective system compatible up to 2-isomorphism.*

Definition B.2.1.2. *A formal deformation $\hat{\xi} = \varprojlim \xi_i$ is called **effective** if there is an object $\xi \in F(R)$ inducing objects isomorphic to ξ_i in $\mathsf{F}(R/\mathfrak{m}^{i+1})$ for all i.*

Let h_R be the functor assigning to each Artinian local Λ-algebra A the set $h_R(A)$ of homomorphisms $R \to A$. Ideally, in (B.1.4), we would hope to have a canonical *isomorphism* from $\hat{\mathsf{F}}(R)$ to $\mathsf{Hom}(h_R, \mathsf{F})$. However, this is not always possible. In general we can only show the existence of an *equivalence of categories* between the two categories.

Lemma B.2.1.3. *There is an equivalence of categories*

$$\hat{\mathsf{F}}(R) \to \mathsf{Hom}(h_R, \mathsf{F}) \tag{B.2.1.4}$$

between the categories $\hat{\mathsf{F}}(R)$ and $\mathsf{Hom}(h_R, \mathsf{F})$.

Proof. Let $\hat{\xi} = \varprojlim \xi_i$ be an object in $\hat{\mathsf{F}}(R)$. Each morphism $u : R \to A$, where A is an Artinian local ring in $\mathrm{Ob}\,\mathsf{C}_\Lambda$, must factor through $u_i : R/\mathfrak{m}^{i+1} \to A$ for some i, and we assign to u the object $\mathsf{F}(u_i)(\xi_i)$ induced by ξ_i in $\mathsf{F}(A)$.

The above assignment depends on the way we factor the homomorphism. In general, the object we assign in $\mathsf{F}(A)$ is not canonically determined. Therefore, to define a morphism from h_R to F, we must choose once for all possible homomorphisms of the form $R \to A$ the corresponding ways we factor the homomorphisms.

Let us adopt the following rule: For each Artinian local ring A in $\mathrm{Ob}\,\mathsf{C}_\Lambda$, consider the least i_0 such that $\mathfrak{m}^{i_0+1} = 0$ holds for the maximal ideal $\mathfrak{m}$ of A, and such that $R \to A$ factors through $R/\mathfrak{m}^{i_0+1} \to A$. (Each morphism $h_R \to \mathsf{F}$ defined by another choice of factorization is isomorphic to this morphism.) Hence the object-level assignment of the functor (B.2.1.4) is complete.

The morphism-level assignment then follows naturally. Each morphism $f : \hat{\xi} \to \hat{\eta}$ in $\hat{\mathsf{F}}(R)$ is given by a series of morphisms $f_i : \xi_i \to \eta_i$ in $\mathsf{F}(R/\mathfrak{m}^{i+1})$. For each A in $\mathrm{Ob}\,\mathsf{C}_\Lambda$ with the i_0 chosen as above, and for each morphism $u : R \to A$, the morphism from $\mathsf{F}(u_{i_0})(\xi_{i_0})$ to $\mathsf{F}(u_{i_0})(\eta_{i_0})$ is defined by $\mathsf{F}(u_{i_0})(f_{i_0})$.

Conversely, for each natural transformation $h_R \to \mathsf{F}$, define for each i the object $\xi_i \in \mathrm{Ob}\,\mathsf{F}(R/\mathfrak{m}^{i+1})$ to be the image induced by the canonical homomorphism $(R \to R/\mathfrak{m}^{i+1}) \in h_R(R/\mathfrak{m}^{i+1})$. Then we obtain a projective system $\{\xi_i \in \mathrm{Ob}\,\mathsf{F}(R/\mathfrak{m}^{i+1})\}_{i\geq 0}$ compatible up to 2-isomorphism.

One can check that this association gives a quasi-inverse of the previous association. For each morphism from h_R to F induced by an element $\hat{\xi} \in \hat{\mathsf{F}}(R)$ through (B.2.1.4), it is always true that the above converse induces an element in $\hat{\mathsf{F}}(R)$ isomorphic to $\hat{\xi}$. Hence the two categories in (B.2.1.4) are equivalent. □

The notions of *versal* and *universal* formal deformations can be generalized naturally in the following way.

Definition B.2.1.5. *A formal deformation $\hat{\xi}$ is **versal** (resp. **universal**) if it has the following property: Let $A' \to A$ be any surjection of Artinian local Λ-algebras in C_Λ, let $\eta' \in \mathrm{Ob}\,\mathsf{F}(A')$ be any infinitesimal deformation of ξ_0, and let $\eta \in \mathrm{Ob}\,\mathsf{F}(A)$ be the infinitesimal deformation of ξ_0 induced by η'. Then each homomorphism $R \to A$ that induces $\eta \in F(A)$ by $\hat{\xi} \in \hat{\mathsf{F}}(R)$ and (B.2.1.4) can be lifted (resp. uniquely lifted) to a homomorphism $R \to A'$ that induces (by $\hat{\xi}$ and (B.2.1.4)) an object isomorphic to η'.*

Let us turn to the case of a 2-functor associated with a category fibered in groupoids F over (Sch/S), that is, a contravariant 2-functor $\mathsf{F} : (\mathrm{Sch}/S) \to (\mathrm{Gr})$ assigning to each U in (Sch/S) the fiber groupoid F_U. For an object $\xi_0 \in \mathrm{Ob}\,\mathsf{F}(k)$ with k an $\mathscr{O}_S$-field of finite type, the existence of an *effective formal deformation* of ξ_0 is to say that there is an object $\xi \in \mathrm{Ob}\,\mathsf{F}(R)$, where R is a noetherian complete local $\mathscr{O}_S$-algebra with maximal ideal $\mathfrak{m}$ and residue field k, such that ξ induces an object isomorphic to ξ_0 in $\mathrm{Ob}\,\mathsf{F}(k)$. By viewing ξ_0 (resp. ξ) as a morphism $\mathrm{Spec}(k) \to \mathsf{F}$ (resp. $\mathrm{Spec}(R) \to \mathsf{F}$), the above statement amounts to the assertion that the diagram

$$\begin{tikzcd} \mathrm{Spec}(k) \arrow[r] \arrow[dr, "\xi_0"'] & \mathrm{Spec}(R) \arrow[d, "\xi"] \\ & \mathsf{F} \end{tikzcd}$$

is commutative. (Note that such a diagram of groupoids is defined to be commutative only up to 2-isomorphism.) To say that the effective formal deformation is *versal* (resp. *universal*), it amounts to saying that, for each diagram of solid arrows

$$\begin{tikzcd} \mathrm{Spec}(R) \arrow[d, "\xi"'] & \mathrm{Spec}(A) \arrow[l] \arrow[d, hook] \\ \mathsf{F} & \mathrm{Spec}(A') \arrow[l] \arrow[ul, dotted] \end{tikzcd}$$

where $A' \to A$ is a surjection of Artinian local $\mathscr{O}_S$-algebras with residue field k, there is a dotted arrow (resp. a unique dotted arrow) $\mathrm{Spec}(A') \to \mathrm{Spec}(R)$ making the diagram commute.

Theorem B.2.1 can be reformulated in the following form, to be used in the proof of Theorem B.3.7 in the next section.

Theorem B.2.1.6 (modified version of Theorem B.2.1)**.** *Let S be a scheme or an algebraic space locally of finite type over a field or an excellent Dedekind domain. Consider a category F fibered in groupoids over (Sch/S) which is locally of finite presentation (see Definition A.5.9 and Proposition A.7.2.4). Let $s \in S$ be a point whose residue field $k(s)$ is of finite type over $\mathscr{O}_S$, let k' be a finite field extension of $k(s)$, and let $\xi_0 \in \mathrm{Ob}\,\mathsf{F}(k')$. Suppose that an effective formal versal deformation (R, ξ) of ξ_0 exists, where R is a noetherian complete local $\mathscr{O}_S$-algebra with residue field k' and $\xi \in \mathrm{Ob}\,\mathsf{F}(R)$. (Our language of deformation is applicable in this case for the same reason mentioned in Theorem B.2.1.) Then there is a scheme X of finite type over S, a closed point $x \in X$ with residue field $k(x) = k'$, and an element $\widetilde{\xi} \in \mathrm{Ob}\,\mathsf{F}(X)$, such that the triple $(X, x, \widetilde{\xi})$ is a versal deformation of ξ_0. More precisely, there is an isomorphism $\hat{\mathscr{O}}_{X,x} \cong R$ such that $\widetilde{\xi}$ induces an object in $\mathrm{Ob}\,\mathsf{F}(R)$ isomorphic to ξ via the composition $\mathrm{Spec}(R) \xrightarrow{\sim} \mathrm{Spec}(\hat{\mathscr{O}}_{X,x}) \to X$. The isomorphism $\hat{\mathscr{O}}_{X,x} \cong R$ is unique if (R, ξ) is universal.*

Proof. To prove the theorem, we consider the contravariant functor $\overline{\mathsf{F}} : (\mathrm{Sch}/S) \to (\mathrm{Sets})$ associated with the category F fibered in groupoids, with the canonical surjection $\pi : \mathsf{F} \to \overline{\mathsf{F}}$ as before. Consider the effective versal deformation $\eta \in \overline{\mathsf{F}}(R)$ of $\pi(\xi_0) \in \overline{\mathsf{F}}(k')$ given by the image $\pi(\xi)$ of ξ in $\overline{\mathsf{F}}(R)$. It is immediate that $\overline{\mathsf{F}}$ satisfies all the requirements of Theorem B.2.1, and hence there is a triple $(X, x, \widetilde{\eta})$ with $\widetilde{\eta} \in \overline{\mathsf{F}}(X)$, and an isomorphism $\hat{\mathscr{O}}_{X,x} \cong R$ (which is unique if the formal versal deformation is universal), such that $\widetilde{\eta}$ induces η via the composition $\mathrm{Spec}(R) \xrightarrow{\sim} \mathrm{Spec}(\hat{\mathscr{O}}_{X,x}) \to X$. Now the element $\widetilde{\eta} \in \overline{\mathsf{F}}(X)$ is an equivalence class of objects in $\mathsf{F}(X)$. Let us take any object $\widetilde{\xi}$ in $\mathrm{Ob}\,\mathsf{F}(X)$ in the class of $\widetilde{\eta}$. Since $\widetilde{\eta}$ induces η in $\overline{\mathsf{F}}(R)$, the object induced by $\widetilde{\xi}$ in $\mathrm{Ob}\,\mathsf{F}(R)$ must be isomorphic to ξ. Hence the triple $(X, x, \widetilde{\xi})$ satisfies our requirement, as desired. □

B.3 Artin's Criterion for Algebraic Stacks

Artin proved a criterion which is useful for proving that a functor is representable by an algebraic space. The same proof provides the following criterion in [38] for proving that a stack is algebraic. The theorem we will generalize is [6, Thm. 3.4]. The reader might be interested in consulting the other parts of [6], the article [9], and the book [10].

Before starting the proof, we need some preparation. Throughout this section, we assume that S is a scheme or an algebraic space locally of finite type over a field or an excellent Dedekind domain.

Definition B.3.1. *Let X be a category fibered in groupoids over (Sch/S), let X be a scheme over S, and let $\xi : X \to \mathsf{X}$ be a morphism in $(\mathrm{Ct\text{-}F\text{-}Gr}/S)$ (i.e., $\xi \in \mathrm{Ob}\,\mathsf{X}(X)$). Let x be a point of X. We say that ξ is **formally étale** at x if, for every commutative diagram of solid arrows*

$$\begin{tikzcd} X \arrow[d, "\xi"'] & Z_0 \arrow[l, "f_0"'] \arrow[d, hook] \\ \mathsf{X} & Z \arrow[l] \arrow[ul, dotted, "f"'] \end{tikzcd} \tag{B.3.2}$$

where Z is the spectrum of an Artinian local $\mathscr{O}_S$-algebra, where Z_0 is a closed subscheme of Z defined by a nilpotent ideal, and where f_0 is a morphism sending the unique closed point of Z_0 to x, there exists a unique dotted arrow f making the diagram commutative.

Remark B.3.3. By [59, IV-4, 17.14.1 and 17.14.2], the property of being étale and being formally étale are equivalent for morphisms locally of finite presentation. A special case of this fact can also been found in [10, I, Prop. 1.1], and a discussion of one of the two directions of the proof can be found in [91, Rem. 3.22]. This fact is generalized to the case of a morphism locally of finite presentation from a scheme to an algebraic space by [6, Lem. 3.3].

The properties of being étale and being formally étale are equivalent in the following case:

Proposition B.3.4. *Let X be a scheme over S, let X be a category fibered in groupoids over* $(\mathrm{Sch}\,/S)$*, and let $\xi : X \to \mathsf{X}$ be a morphism in* $(\text{Ct-F-Gr}\,/S)$*. Assume that ξ is a representable morphism that is locally of finite representation. Namely, for each scheme U over S, and each morphism $U \to \mathsf{X}$, the fiber product $X \times_{\mathsf{X}} U$ is representable by an algebraic space locally of finite presentation over U. Let $x \in X$. Then ξ is formally étale at x if and only if the following condition holds:*

Let $U \to \mathsf{X}$ be any morphism from a scheme U to X, and let C be the preimage of x in $X \times_{\mathsf{X}} U$. Then the projection $X \times_{\mathsf{X}} U \to U$ is étale at every point of C.

Before the proof of Proposition B.3.4, we need some technical preparation. For the convenience of readers, we quote the following proposition:

Proposition B.3.5 ([60, I, 6.2.6(v)])**.** *If the composition of two morphisms of schemes $f : X \to Y$ and $g : Y \to Z$ is locally of finite presentation, and if g is locally of finite type, then f is locally of finite presentation.*

Then we have the following lemma:

Lemma B.3.6. *Let X be an algebraic space, and let U and V be two schemes. Suppose that we have morphisms $V \to X \to U$ of algebraic spaces such that $V \to X$ is étale surjective and such that $X \to U$ is locally of finite presentation. Then $V \to X$ is locally of finite presentation.*

Proof. Consider any étale covering $W \to X$ from a scheme W. Since $X \to U$ is locally of finite presentation, $W \to U$ and $V \times_X W \to U$ are locally of finite presentation over W (because the property of being locally of finite presentation is stable in the étale topology). Applying Proposition B.3.5 to the composition $V \times_X W \to W \to U$, we see that $V \times_X W \to W$ is locally of finite presentation. As a result, the morphism $V \to X$ is locally of finite presentation by definition. □

Proof of Proposition B.3.4. Suppose the condition in the proposition is satisfied (for every scheme U). We would like to show that $\xi : X \to \mathsf{X}$ is formally étale. Let Z be the spectrum of an Artinian local $\mathscr{O}_S$-algebra, with Z_0 a closed subscheme of Z defined by a nilpotent ideal (cf. Definition B.3.1). Each morphism $f_0 : Z_0 \to X$ with set-theoretic image a closed point $x_0 \in X$ induces a morphism $Z_0 \to X \times_{\mathsf{X}} Z$ with set-theoretic image y for some closed point y of $X \times_{\mathsf{X}} Z$ whose image in X is x_0. Consider an affine scheme V and an étale morphism $V \to X \times_{\mathsf{X}} Z$ as in Theorem A.4.4.3 such that $y \to X \times_{\mathsf{X}} Z$ factors through $y \to V \to X \times_{\mathsf{X}} Z$. Since $V \to X \times_{\mathsf{X}} Z$ is étale and locally of finite presentation (by Lemma B.3.6), and since being étale is equivalent to being formally étale in this case (as explained in Remark B.3.3), the morphism from the closed point of Z_0 to y extends to a morphism from Z_0 to V. Therefore we have a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "\xi"'] & X \times_{\mathsf{X}} Z \arrow[l] \arrow[d] & V \arrow[l] \arrow[d] & Z_0 \arrow[l] \arrow[d, hook] \\
\mathsf{X} & Z \arrow[l] \arrow[r, equal] & Z & Z \arrow[l, "\mathrm{Id}_Z"] \arrow[ul, dotted]
\end{tikzcd}$$

of solid arrows. By hypothesis (with $U = Z$ in the condition of the proposition), $V \to Z$ is étale at the preimage of y and is locally of finite presentation. Hence, by Remark B.3.3 again, we have an induced morphism $Z \to V$ (dotted in the diagram) making the diagram commute. The composition $Z \to V \to X \times_{\mathsf{X}} Z \to X$ then gives the desired morphism f. If f' is another such morphism, then by the universal property of the fiber product $X \times_{\mathsf{X}} Z$ and by the formal étaleness of $V \to X \times_{\mathsf{X}} Z$, the morphism f' extends uniquely to a morphism $Z \to V$ making the diagram commutative. Then f' and f must be the same, by the formal étaleness of $V \to Z$. This shows that ξ is formally étale.

Conversely, suppose $U \to \mathsf{X}$ is any morphism as in the condition of the proposition. Let $y \in C$ be any point whose image in X is x. By Theorem A.4.4.3, there is an affine scheme V and an étale morphism $V \to X \times_{\mathsf{X}} U$ such that $y \to X \times_{\mathsf{X}} U$ factors through $y \to V \to X \times_{\mathsf{X}} U$. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[d, "\xi"'] & X \times_{\mathsf{X}} U \arrow[l] \arrow[d] & V \arrow[l] \arrow[d] & Z_0 \arrow[l, "f_0"'] \arrow[d, hook] \\
\mathsf{X} & U \arrow[l] \arrow[r, equal] & U & Z \arrow[l] \arrow[ul, dotted, "f"'] \arrow[ull, dotted] \arrow[ulll, dotted]
\end{tikzcd}$$

of solid arrows, where f_0 is a morphism sending Z_0 set-theoretically to y. The composition $Z_0 \to V \to X \times_{\mathsf{X}} U \to X$ defines a morphism from Z_0 to x set-theoretically. Hence by formal étaleness of ξ, there is a morphism from $Z \to X$ extending the composition. By the universal property of the fiber product $X \times_{\mathsf{X}} U$, this morphism induces a morphism $Z \to X \times_{\mathsf{X}} U$. The morphism $V \to X \times_{\mathsf{X}} U$ is étale, is locally of finite presentation by Lemma B.3.6, and hence is formally étale by Remark B.3.3. Therefore we have an induced morphism $f : Z \to V$ making the above diagram commute. Since the choices of Z and Z_0 are arbitrary (independent of U and V), we see that $V \to U$ is étale at y by Remark B.3.3. Hence $X \times_{\mathsf{X}} U \to U$ is étale at y by definition. □

Theorem B.3.7 (Artin's criterion)**.** *Let S be a scheme of finite type over a field or an excellent Dedekind domain. Let X be a category fibered in groupoids over* $(\mathrm{Sch}\,/S)$*. Then X is an algebraic stack locally of finite type over S if and only if the following conditions hold:*

1. *X is a stack for the étale topology* (*see Definition A.6.1*).
2. *X is locally of finite presentation* (*see Definition A.5.9*).
3. *Suppose ξ and η are two objects in $\mathsf{X}(U)$, where U is a scheme of finite type over S. Then $\mathrm{Isom}_U(\xi, \eta)$ is an algebraic space locally of finite type over S.*

4. *For each field k_0 of finite type over S with a 1-morphism $i : \operatorname{Spec}(k_0) \to \mathsf{X}$, there exist a noetherian complete local ring R, a morphism j from the spectrum of a finite separable extension k_0' of k_0 to the closed point s of $\operatorname{Spec}(R)$, and a commutative diagram*

$$\begin{tikzcd} \operatorname{Spec}(k_0') \arrow[r] \arrow[d, "j"'] & \operatorname{Spec}(k_0) \arrow[d, "i"] \\ \operatorname{Spec}(R) \arrow[r, "\xi"'] & \mathsf{X} \end{tikzcd} \tag{B.3.8}$$

with ξ formally étale at s.

5. *If ξ is a 1-morphism in $(\text{Ct-F-Gr}/S)$ from a scheme U of finite type over S to X, and if ξ is formally étale at a point u (of U) of finite type over S, then ξ is formally étale in a neighborhood of u (in U).*

Proof. We first prove the necessity. Let X be an algebraic stack locally of finite type over S.

Condition 1 is trivially satisfied by definition, and condition 3 follows from Proposition A.6.11. Condition 2 is also automatic, because X is locally of finite type over S, where S is locally of finite type over a field or an excellent Dedekind domain. Therefore X is locally of finite presentation as an algebraic stack over S, and hence at the same time locally of finite presentation as a category fibered in groupoids over S by Proposition A.7.2.4.

To verify condition 4, let k_0 be any field of finite type over S with a 1-morphism $i : U = \operatorname{Spec}(k_0) \to S$. Let $X \to \mathsf{X}$ be the presentation of X. Consider the algebraic space U' representing the fiber product $X \underset{\mathsf{X}}{\times} U$, and take any point $\operatorname{Spec}(k_0') \to U'$. Since U' is étale and of finite presentation over $U = \operatorname{Spec}(k_0)$, we see that k_0' is a finite separable extension of k_0. By Theorem A.4.4.3, the morphism $\operatorname{Spec}(k_0') \to X$ must factor through an affine scheme V with $V \to X$ an étale morphism, making the diagram

$$\begin{tikzcd} \operatorname{Spec}(k_0') \arrow[r] \arrow[d] & U' = X \underset{\mathsf{X}}{\times} U \arrow[r] \arrow[d] & U = \operatorname{Spec}(k_0) \arrow[d] \\ V \arrow[r] & X \arrow[r] & \mathsf{X} \end{tikzcd}$$

commutative. Since X is locally of finite type over S, V is also locally of finite type over S (cf. Lemma B.3.6, with identical proof), and we may replace V with an affine neighborhood of $\operatorname{Spec}(k_0')$ that is of finite type over S. By the hypothesis on S, V is excellent. By taking the formal completion of $\mathscr{O}_V$ with respect to the image s of $\operatorname{Spec}(k_0')$, we get the desired noetherian complete local ring $R = \hat{\mathscr{O}}_{V,s}$ inducing the diagram (B.3.8).

To verify condition 5, let ξ be a 1-morphism (in $(\text{Ct-F-Gr}/S)$) from a scheme U of finite type over S to X, which is formally étale at a point u of finite type over S. By Proposition B.3.4, the morphism $U' = X \underset{\mathsf{X}}{\times} U \to X$ is étale at every point whose image through $U' \to U$ is u. Since being étale is an open condition, we see that $U' \to X$ is étale at a neighborhood of a point in U' whose image under $U' \to U$ is u. Now since $X \to \mathsf{X}$ is étale, the induced projection $U' \to U$ is étale, and hence open. Then $U \to \mathsf{X}$ is étale at an open neighborhood of u, as desired.

Conversely, suppose that all the conditions are satisfied. We would like to construct a representable morphism $X \to \mathsf{X}$ that is étale and surjective.

Let p be any point of X of finite type over S. This means that there is a field k_0^p of finite type over S with a 1-morphism $i^p : U = \operatorname{Spec}(k_0^p) \to S$. Condition 4 can be interpreted as the existence of an effective universal formal deformation (R^p, ξ^p) of $\xi_0^p \in \operatorname{Ob}\mathsf{X}(k^p)$, where R^p is a noetherian complete local ring with residue field k^p (see Section B.2.1). Here we have the commutative diagram

$$\begin{tikzcd} \operatorname{Spec}(k_0'^p) \arrow[r] \arrow[d] & \operatorname{Spec}(k_0^p) \arrow[d] \\ s = \operatorname{Spec}(k^p) \arrow[r] & S \end{tikzcd}$$

in which $\operatorname{Spec}(k_0'^p) \to \operatorname{Spec}(k_0^p)$ and $\operatorname{Spec}(k_0^p) \to S$ are of finite type. Hence $s = \operatorname{Spec}(k^p) \to S$ must be of finite type. Since X is locally of finite presentation by condition 2, we may apply Theorem B.2.1.6 to deduce that the pair (R^p, ξ^p) is algebraizable, say by $(X^p, x^p, \widetilde{\xi}^p)$, with properties such as being a scheme of finite type over S etc., described in Theorem B.2.1. Moreover, by condition 4, $\widetilde{\xi}^p : X^p \to \mathsf{X}$ is formally étale.

By condition 5, by replacing X^p with an open neighborhood of x^p, we may assume that $\widetilde{\xi}^p : X^p \to \mathsf{X}$ is formally étale at every point. By condition 3, $\widetilde{\xi}^p : X^p \to \mathsf{X}$ is representable and étale. Namely, the pullback $\widetilde{\xi}^p_U : X^p \underset{\mathsf{X}}{\times} U \to U$ of $\widetilde{\xi}^p$ is étale for every scheme $U \to \mathsf{X}$.

Now we claim that, if we take X to be the disjoint union $\coprod_p X^p$ for representatives p in each equivalence class of points of finite type of X, then $X \to \mathsf{X}$ is a presentation of X.

Let U be any scheme of finite type over S with a morphism $U \to \mathsf{X}$. Then the projection $X \underset{\mathsf{X}}{\times} U \to U$ is étale. To show that the projection $X \to \mathsf{X}$ is surjective, it suffices to show that every point u of U of finite type over S is in the image, since such points are dense. Write $u = \operatorname{Spec}(k)$ for some k. Let $p = \operatorname{Spec}(k_0^p)$ be the representative of the equivalence class of points of finite type equivalent to u that is used in forming the disjoint union $X = \coprod_p X^p$. By construction, this point p is equivalent to the point $x^p = \operatorname{Spec}(k^p)$ of X^p. Take a common separable field extension k' of k^p and k. The two morphisms $\operatorname{Spec}(k') \to \operatorname{Spec}(k^p) \to X^p$ and $\operatorname{Spec}(k') \to u \to U$ induce a morphism from $\operatorname{Spec}(k')$ to the algebraic space $X^p \underset{\mathsf{X}}{\times} U$. By Proposition A.4.4.2, this factors through a point of $X^p \underset{\mathsf{X}}{\times} U$ whose image in the scheme U is some point u'. Then the morphism $\operatorname{Spec}(k') \to U$ factors through both points u and u', which means that $u' = u$. Hence $X^p \underset{\mathsf{X}}{\times} U \to U$ covers u as desired. This completes the proof. $\square$

Theorem B.3.9. *If the residue fields of finite type of S are perfect, then, to establish the result of Theorem B.3.7, we may replace condition 4 of Theorem B.3.7 with the following one:*

Let $s \in S$, and let k_0 be a finite field extension of $k(s)$. Suppose that $u : \operatorname{Spec}(k_0) \to S$ is of finite type. Then by hypothesis, $k(s)$ is perfect and $k_0/k(s)$ is a separable extension. By Lemma B.1.1.11, there exists a complete local ring Λ_{k_0} with residue field k_0, together with a morphism $\bar{u} : \operatorname{Spec}(\Lambda_{k_0}) \to S$ that is formally étale and extends to u by composition with $\operatorname{Spec}(k_0) \to \operatorname{Spec}(\Lambda_{k_0})$. (This follows from the construction of Λ_{k_0} in Lemma B.1.1.11, the universal property of complete p-rings, and the separability of the residue field extension.) For $\xi_0 \in \operatorname{Ob}\mathsf{X}(k_0)$, we denote by $\mathsf{D}(\xi_0)$ the following category over the opposite category of $\hat{\mathsf{C}}_{\Lambda_{k_0}}$: For

$A \in \operatorname{Ob} \hat{\mathsf{C}}_{\Lambda_{k_0}}$, an object of $\mathsf{D}(\xi_0)(A)$ is an object ξ of $\mathsf{X}(\operatorname{Spec}(A))$, equipped with an isomorphism

$$(\text{image of } \xi \text{ in } \mathsf{X}(\operatorname{Spec}(k_0))) \xrightarrow{\sim} \xi_0. \tag{B.3.10}$$

Then the condition to replace with is the following one:

4′. For k_0 and ξ_0 as above, the covariant functor $D := \overline{\mathsf{D}}$ from $\hat{\mathsf{C}}_{\Lambda_{k_0}}$ to (Sets) *defined by*

$$A \mapsto \{\text{isomorphism classes in } \mathsf{D}(\xi_0)(A)\}$$

is effectively prorepresentable (by some noetherian complete local ring R, and some $\xi \in \operatorname{Ob} \mathsf{X}(R)$ satisfying the restriction (B.3.10)).

This condition may be verified after replacing the field k_0 with a finite extension.

Proof. Recall that, in our proof of Theorem B.3.7, condition 4 is equivalent to the existence of a certain effective universal deformation of the 2-functor associated with the category X fibered in groupoids, which is needed for applying Theorem B.2.1.6. However, the proof of Theorem B.2.1.6 merely requires the existence of an effective universal deformation of the functor canonically associated with the 2-functor associated with X, which is equivalent to the effective prorepresentability of the deformation functor defined above. Hence the theorem follows. □

Theorem B.3.11. *If S is of finite type over a field or over an excellent Dedekind domain having infinitely many points, and if all possible complete local rings R in Theorem B.3.9 are normal and of the same Krull dimension, then we may suppress condition 5 in Theorem B.3.7.*

Proof. We first remark that, under these hypotheses, for each scheme U of finite type over S, the points of finite type of U are all closed points. Moreover, for each integral scheme U of finite type over S, the Krull dimensions of its local rings at closed points are constant. (This is closely related to the discussion of Jacobson schemes in [59, IV-3, §10].)

Proceeding as in the proof of Theorem B.3.7, suppose we have $(X, x, \widetilde{\xi})$ coming from the algebraization of some (R, ξ). (We have dropped the superscripts here.) The condition 5 was used to replace X with an open neighborhood of x such that $\widetilde{\xi} : X \to \mathsf{X}$ is formally étale at every point. Our aim is to show that this is possible without the condition 5.

Since R is normal by assumption, by replacing X with an open neighborhood of x, we may assume that X is normal [59, IV-2, 7.8.3] and integral, and that $\widetilde{\xi} : X \to \mathsf{X}$ is unramified. We claim that $\widetilde{\xi}$ is then étale. It suffices to show that $\widetilde{\xi}$ is formally étale at every closed point of X. Let y be any closed point with $y \to \mathsf{X}$ induced by $\widetilde{\xi} : X \to \mathsf{X}$, and let $(X', x', \widetilde{\xi}')$ be as in condition 4 of Theorem B.3.7 and in Theorem B.2.1.6, so that $x' \to \mathsf{X}$ is equivalent to $y \to \mathsf{X}$. Note that $\widetilde{\xi}'$ is formally étale at x'. We may replace X' with a neighborhood of x' and assume that it is also normal and integral, and that $\widetilde{\xi}' : X' \to \mathsf{X}$ is unramified. The Krull dimensions of X and X' are both equal to some d, by assumption.

Now let z be the (unique) point of $X' \underset{\mathsf{X}}{\times} X$ lying over (x', y). Since $\widetilde{\xi}'$ is formally étale at x', the morphism $X' \underset{\mathsf{X}}{\times} X \to X$ is étale at z (by Proposition B.3.4). Hence the dimension of $(X' \underset{\mathsf{X}}{\times} X)$ at z is also d. Since $X' \underset{\mathsf{X}}{\times} X \to X'$ is unramified, and since X' is normal and integral, the morphism $X' \underset{\mathsf{X}}{\times} X \to X'$ is automatically étale (by [59, IV-4, 18.10.1 and 18.10.4]). Therefore $\widetilde{\xi}$ is formally étale at y. □

Remark B.3.12. The assumption on Krull dimensions of R in Theorem B.3.11 is not reasonable if S does not satisfy the property that, for each integral scheme U of finite type over S, the Krull dimensions of its local rings at points of finite type over S are constant. This fails, for example, if $S = \operatorname{Spec}(\mathbb{Z}_{(p)})$.

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Index