TOROIDAL COMPACTIFICATIONS OF PEL-TYPE KUGA FAMILIES

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Abstract. We explain how compactifications of Kuga families of abelian varieties over PEL-type Shimura varieties, including for example all those products of universal abelian schemes, can be constructed (up to good isogenies not affecting the relative cohomology) by a uniform method. We also calculate the relative cohomology and explain its various properties crucial for applications to the cohomology of automorphic bundles.

Contents

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Introduction

To study the relations between automorphic forms and Galois representations, it is desirable to understand the cohomology of Shimura varieties with coefficients in algebraic representations of the associated reductive groups (i.e., the so-called *automorphic bundles*).

In the case of PEL-type Shimura varieties, the associated reductive groups are (up to center) twists of products of symplectic, orthogonal, or general linear groups. According to Weyl's construction [53] (see also [17] and [19]), all algebraic representations of a classical group can be realized as summands in the tensor powers of the *standard representation* of the group. In geometry, one is led to consider the cohomology of fiber products of the universal families of abelian varieties over the PEL-type Shimura varieties. Such fiber products are special cases of what we will call *PEL-type Kuga families*, or simply Kuga families. When the PEL-type Shimura variety in question is not compact, the total spaces of such Kuga families are not compact either.

To study cohomology properly, one is often led to the question of the existence of projective smooth compactifications with good properties, such as allowing the Hecke operators to act on their cohomology spaces (but not necessarily the geometric spaces). In what follows, let us simply call such compactifications *good* compactifications. In characteristic zero, such questions can often be handled by the embedded resolution of singularities due to Hironaka [28, 29]. However, more explicit theories exist in our context. The work of Mumford and his collaborators in [4] provides a systematic collection of good compactifications of Shimura varieties with explicit descriptions of local structures, while the work of Pink [48] provides a systematic construction of good compactifications of the Kuga families as well. These compactifications are called *toroidal compactifications.* Their methods are analytic in nature and cannot be truly generalized in mixed characteristics.

Based on the theory of degeneration of polarized abelian varieties initiated by Mumford [44], Faltings and Chai [15, 8, 16] constructed good compactifications over the integers for Siegel moduli spaces defined by the moduli space of principally polarized abelian varieties. In [16], they also constructed good compactifications of fiber products of the universal families by gluing weak relatively complete models along the boundary. We ought to point out that, although most works on compactifications spend most of their pages on the construction of boundary charts, it is only the *gluing* argument that validates the whole construction. (This is not necessarily the case for works using the moduli-theoretic approach, such as [2], [1], or [47]. However, the questions there are not less challenging: What can one say about the boundary structures? Are they equally useful for applications to cohomology?) Thus, even if the construction of toroidal compactifications of Siegel moduli spaces in [16, Ch. IV] has been generalized for all PEL-type Shimura varieties in [38], the gluing of weak relatively complete models has to be carried out separately when one works along the original idea of [16, Ch. VI]. (This is the case in for example [50], in which the assumption that the boundary divisors are regular, i.e. have no crossings, unfortunately rules out all cases where choices of cone decompositions are needed for the Shimura varieties.)

Note that gluing is not just about techniques of descent. Any theory of descent requires an input of some *descent data*. Since a naive generalization of the constructions in [16, Ch. IV] introduces unwanted boundary components, which have to be studied and removed carefully by imposing liftability and pairing conditions as in [38], we have reason to believe that a naive generalization of the construction in [16, Ch. VI, §1] requires delicate modifications, without which even the strongest descent theory cannot be applied.

The aim of this article is to avoid any further argument of gluing, and to treat all PEL-type cases on an equal footing. We shall reduce the construction of toroidal compactifications of PEL-type Kuga families to the construction of toroidal compactifications of Shimura varieties in [38], by systematically realizing the Kuga families as locally closed boundary strata in the toroidal compactifications of (larger) PEL-type Shimura varieties. Partly inspired by Kato's theory of log abelian schemes, we can show that, up to refinements of cone decompositions, the structural morphisms from the Kuga families to the Shimura varieties extend (up to good isogenies not affecting the relative cohomology) to log smooth morphisms with nice properties between the toroidal compactifications. This approach differs fundamentally from the one in [16, Ch. VI]. As Chai pointed out, although no technique can be truly shared between analytic and algebraic constructions, our idea is close in spirit to Pink's in [48]. (See Remark 3.10 below.)

Since we replace Faltings and Chai's construction with a different one, we need to explain that our simpler (but perhaps cruder) construction is not less useful. Thus our second task is to calculate the relative (log) de Rham cohomology of the compactified families. We show that such relative cohomology not only enjoys the same expected properties as in [16, Ch. VI, §1], but also admits natural Hecke actions defined by parabolic subgroups of larger reductive algebraic groups, because our construction uses toroidal boundaries of larger Shimura varieties. This exhibits a large class of endomorphisms on our cohomology spaces, including ones needed in the geometric realization of Weyl's construction (i.e., the realization of automorphic bundles as summands in the relative cohomology of Kuga families).

The outline of this article is as follows. In Section 1, we review some of the results we need from [38]. We consider the investment of this summary worthwhile because, although we do not need to carry out another gluing argument, we do need the full strength of the long work [38]. In Section 2, we define what we mean by PEL-type Kuga families, state our main theorem, and give an outline of the proof. In Section 3, we carry out the construction of toroidal compactifications for these Kuga families that admit log smooth morphisms to the Shimura varieties in question. (This section serves roughly the same purpose as [16, Ch. VI, §1].) In Sections 4 and 5, we show that these toroidal compactifications are indeed *good* by justifying what we mentioned in the previous paragraph. (These two sections serve roughly the same purpose as [16, Ch. VI, §2].) We would like to mention that the use of nerve spectral sequences in Section 4 imitates immediate analogues in [26] and [27] (based on techniques that can be traced back to [36, Ch. I, §3]), while the use of log extensions of polarizations is inspired by Kato's idea of (relative) log Picard groups [32, 3.3]. (See Remark 5.7.) The article ends with Section 6, in which we explain how to define canonical extensions of the so-called principal bundles.

Although used as the main motivation for our construction, applications to cohomology of automorphic bundles will be deferred to some forthcoming papers. There the readers will find the construction of proper smooth integral models useful for studying cohomology with not only rational coefficients, but also integral and torsion coefficients.

We shall follow [38, Notations and Conventions] unless otherwise specified. (Although our references to [38] use the numbering in the original version, the reader is advised to consult the errata and revision (available online) for corrections of typos and minor mistakes, and for improved exposition.)

1. PEL-type moduli problems and their compactifications

In this section, we summarize definitions and main results in [38] that will be needed in this article. We will emphasize definitions such as the ones involved in the description of boundary structures, but will have to be less comprehensive on some fundamental definitions including the ones of level structures.

1A. **Linear algebraic data.**

Let $\mathcal{O}$ be an order in a finite-dimensional semisimple algebra over $\mathbb{Q}$ with a positive involution $\star$. Here an *involution* means an anti-automorphism of order two, and *positivity* of $\star$ means $\mathrm{Tr}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}/\mathbb{R}}(xx^\star) > 0$ for any $x \neq 0$ in $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$. We assume that $\mathcal{O}$ is mapped to itself under $\star$. We shall denote the center of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$ by F.

Let $\mathbb{Z}(1) := \ker(\exp : \mathbb{C} \to \mathbb{C}^\times)$, which is a free $\mathbb{Z}$-module of rank one. Any choice $\sqrt{-1}$ of a square-root of -1 in $\mathbb{C}$ determines an isomorphism $(2\pi\sqrt{-1})^{-1} : \mathbb{Z}(1) \xrightarrow{\sim} \mathbb{Z}$, but there is no canonical isomorphism between $\mathbb{Z}(1)$ and $\mathbb{Z}$. For any commutative $\mathbb{Z}$-algebra R, we denote by $R(1)$ the module $R\underset{\mathbb{Z}}{\otimes}\mathbb{Z}(1)$.

By a PEL-type $\mathcal{O}$-lattice $(L, \langle\cdot,\cdot\rangle, h)$ (as in [38, Def. 1.2.1.3]), we mean the following data:

1. An $\mathcal{O}$-lattice, namely a $\mathbb{Z}$-lattice L with the structure of an $\mathcal{O}$-module.
2. An alternating pairing $\langle\cdot,\cdot\rangle : L \times L \to \mathbb{Z}(1)$ satisfying $\langle bx, y\rangle = \langle x, b^\star y\rangle$ for any $x, y \in L$ and $b \in \mathcal{O}$, together with an $\mathbb{R}$-algebra homomorphism $h : \mathbb{C} \to \mathrm{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})$ satisfying:
   (a) For any $z \in \mathbb{C}$ and $x, y \in L\underset{\mathbb{Z}}{\otimes}\mathbb{R}$, we have $\langle h(z)x, y\rangle = \langle x, h(z^c)y\rangle$, where $\mathbb{C} \to \mathbb{C} : z \mapsto z^c$ is the complex conjugation.

(b) For any choice of $\sqrt{-1}$ in $\mathbb{C}$, the $\mathbb{R}$-bilinear pairing
$$(2\pi\sqrt{-1})^{-1}\langle\,\cdot\,,h(\sqrt{-1})\,\cdot\,\rangle:(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})\times(L\underset{\mathbb{Z}}{\otimes}\mathbb{R})\to\mathbb{R}$$
is symmetric and positive definite. (This last condition forces $\langle\,\cdot\,,\,\cdot\,\rangle$ to be nondegenerate.)

The tuple $(\mathcal{O},^\star,L,\langle\,\cdot\,,\,\cdot\,\rangle,h)$ (over $\mathbb{Z}$) then gives us an integral version of the tuple $(B,^\star,V,\langle\,\cdot\,,\,\cdot\,\rangle,h)$ (over $\mathbb{Q}$) in [37] and related works. (We favor lattices over $\mathbb{Z}$ rather than their analogues over $\mathbb{Q}$ (or over $\mathbb{Z}_{(p)}$ for some p) because we will work with isomorphism classes rather than isogeny classes; cf. Remark 1.7 below.)

Definition 1.1 (cf. [38, Def. 1.2.1.5])**.** *Let a PEL-type $\mathcal{O}$-lattice $(L,\langle\,\cdot\,,\,\cdot\,\rangle,h)$ be given as above. For any $\mathbb{Z}$-algebra R, set*
$$\mathrm{G}(R):=\{(g,r)\in\mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}R}(L\underset{\mathbb{Z}}{\otimes}R)\times\mathbf{G}_{\mathrm{m}}(R):\langle gx,gy\rangle=r\langle x,y\rangle,\forall x,y\in L\underset{\mathbb{Z}}{\otimes}R\}.$$
In other words, $\mathrm{G}(R)$ *is the group of symplectic automorphisms of* $L\underset{\mathbb{Z}}{\otimes}R$ (*respecting the pairing* $\langle\,\cdot\,,\,\cdot\,\rangle$ *up to a scalar multiple; cf.* [38, Def. 1.1.4.11])*. For any $\mathbb{Z}$-algebra homomorphism $R\to R'$, we have by definition a natural homomorphism* $\mathrm{G}(R)\to\mathrm{G}(R')$*, making* G *a group functor* (*or in fact an affine group scheme*) *over $\mathbb{Z}$.*

The projection to the second factor $(g,r)\mapsto r$ defines a morphism $\nu:\mathrm{G}\to\mathbf{G}_{\mathrm{m}}$, which we call the **similitude character***. For simplicity, we shall often denote elements (g,r) in* G *by simply g, and denote by $\nu(g)$ the value of r when we need it.* (*If $L\neq\{0\}$ and R is flat over $\mathbb{Z}$, then the value of r is uniquely determined by g. Hence there is little that we lose when suppressing r from the notation. However, this suppression is indeed an abuse of notation in general. For example, when $L=\{0\}$, we have* $\mathrm{G}=\mathbf{G}_{\mathrm{m}}$.)

Let $\Box$ be any set of rational primes. (It can be either an empty set, a finite set, or an infinite set.) We denote by $\mathbb{Z}_{(\Box)}$ the unique localization of $\mathbb{Z}$ (at the multiplicative subset of $\mathbb{Z}$ generated by nonzero integers prime to $\Box$) having $\Box$ as its set of height one primes, and denote by $\hat{\mathbb{Z}}^{\Box}$ (resp. $\mathbb{A}^{\infty,\Box}$, resp. $\mathbb{A}^{\Box}$) the integral adeles (resp. finite adeles, resp. adeles) away from $\Box$. Then we have definitions for $\mathrm{G}(\mathbb{Q})$, $\mathrm{G}(\mathbb{A}^{\infty,\Box})$, $\mathrm{G}(\mathbb{A}^{\infty})$, $\mathrm{G}(\mathbb{R})$, $\mathrm{G}(\mathbb{A}^{\Box})$, $\mathrm{G}(\mathbb{A})$, $\mathrm{G}(\mathbb{Z})$, $\mathrm{G}(\mathbb{Z}/n\mathbb{Z})$, $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, $\mathrm{G}(\hat{\mathbb{Z}})$, $\mathcal{U}^{\Box}(n):=\ker(\mathrm{G}(\hat{\mathbb{Z}}^{\Box})\to\mathrm{G}(\hat{\mathbb{Z}}^{\Box}/n\hat{\mathbb{Z}}^{\Box})=\mathrm{G}(\mathbb{Z}/n\mathbb{Z}))$ for any n prime to $\Box$, and $\mathcal{U}(n):=\ker(\mathrm{G}(\hat{\mathbb{Z}})\to\mathrm{G}(\hat{\mathbb{Z}}/n\hat{\mathbb{Z}})=\mathrm{G}(\mathbb{Z}/n\mathbb{Z}))$.

Following Pink [48, 0.6], we define the neatness of open compact subgroups $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ as follows: View $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ as a *subgroup* of $\mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}}(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})\times\mathbf{G}_{\mathrm{m}}(\hat{\mathbb{Z}}^{\Box})$. (Or we may use any faithful linear algebraic representation of G.) Then, for each rational prime $p>0$ not in $\Box$, it makes sense to talk about *eigenvalues* of elements g_p in $\mathrm{G}(\mathbb{Z}_p)$, which are elements in $\bar{\mathbb{Q}}_p^\times$. Let $g=(g_p)\in\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, with p running through rational primes such that $\Box\nmid p$. For each such p, let Γ_{g_p} be the subgroup of $\bar{\mathbb{Q}}_p^\times$ generated by eigenvalues of g_p. For any embedding $\bar{\mathbb{Q}}\hookrightarrow\bar{\mathbb{Q}}_p$, consider the subgroup $(\bar{\mathbb{Q}}^\times\cap\Gamma_{g_p})_{\mathrm{tors}}$ of torsion elements of $\bar{\mathbb{Q}}^\times\cap\Gamma_{g_p}$, which is independent of the choice of the embedding $\bar{\mathbb{Q}}\hookrightarrow\bar{\mathbb{Q}}_p$.

Definition 1.2 ([38, Def. 1.4.1.8])**.** *We say that $g=(g_p)$ is* **neat** *if* $\underset{p\notin\Box}{\cap}(\bar{\mathbb{Q}}^\times\cap\Gamma_{g_p})_{\mathrm{tors}}=\{1\}$*. We say that an open compact subgroup $\mathcal{H}$ of* $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$ *is* **neat** *if all its elements are neat.*

Remark 1.3. The usual Serre's lemma that no nontrivial root of unity can be congruent to 1 modulo n if $n \geq 3$ shows that $\mathcal{H}$ is neat if $\mathcal{H} \subset \mathcal{U}^{\Box}(n)$ for some $n \geq 3$ such that $\Box \nmid n$.

Remark 1.4. Definition 1.2 makes no reference to the group $\mathrm{G}(\mathbb{Q})$ of rational elements. For the related notion of neatness for arithmetic groups, see [6, 17.1].

1B. **Definition of moduli problems.** Let us fix a PEL-type $\mathcal{O}$-lattice $(L, \langle \cdot, \cdot \rangle, h)$ as in the previous section. Let F_0 be the *reflex field* of $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h)$ defined as in [37, p. 389] or [38, Def. 1.2.5.4]. We shall denote the ring of integers in F_0 by $\mathcal{O}_{F_0}$, and use similar notations for other number fields. (This is in conflict with the notation of the order $\mathcal{O}$, but the precise interpretation will be clear from the context.)

Let $\mathrm{Disc} = \mathrm{Disc}_{\mathcal{O}/\mathbb{Z}}$ be the discriminant of $\mathcal{O}$ over $\mathbb{Z}$ (as in [38, Def. 1.1.1.6]; see also [38, Prop. 1.1.1.12]). Closely related to Disc is the invariant $\mathrm{I}_{\mathrm{bad}}$ for $\mathcal{O}$ defined in [38, Def. 1.2.1.17], which is either 2 or 1, depending on whether type D factors are involved. Let $L^{\#} := \{x \in L \underset{\mathbb{Z}}{\otimes} \mathbb{Q} : \langle x, y \rangle \in \mathbb{Z}(1), \forall y \in L\}$ denote the dual lattice of L with respect to the pairing $\langle \cdot, \cdot \rangle$.

Definition 1.5. *We say that a prime number p is* **bad** *if $p | \mathrm{I}_{\mathrm{bad}} \mathrm{Disc}[L^{\#} : L]$. We say a prime number p is* **good** *if it is not bad. We say that $\Box$ is a set of good primes if it does not contain any bad primes.*

Let us fix a choice of a set $\Box$ of *good primes*. By abuse of notation, let $\mathcal{O}_{F_0,(\Box)}$ be the localization of $\mathcal{O}_{F_0}$ at the multiplicative set generated by rational prime numbers not in $\Box$. Let $\mathsf{S}_0 := \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ and let $(\mathrm{Sch}/\mathsf{S}_0)$ be the category of schemes over S_0. For any open compact subgroup $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, there is an associated moduli problem $\mathsf{M}_{\mathcal{H}}$ defined as follows:

Definition 1.6 (cf. [38, Def. 1.4.1.4])**.** *The moduli problem $\mathsf{M}_{\mathcal{H}}$ is defined as the category fibred in groupoids over $(\mathrm{Sch}/\mathsf{S}_0)$ whose fiber over each S is the groupoid $\mathsf{M}_{\mathcal{H}}(S)$ described as follows: The objects of $\mathsf{M}_{\mathcal{H}}(S)$ are tuples $(G, \lambda, i, \alpha_{\mathcal{H}})$, where:*

1. *G is an abelian scheme over S.*
2. *$\lambda : G \to G^{\vee}$ is a polarization of degree prime to $\Box$.*
3. *$i : \mathcal{O} \to \mathrm{End}_S(G)$ defines an $\mathcal{O}$-structure of (G, λ) (satisfying the Rosati condition $i(b)^{\vee} \circ \lambda = \lambda \circ i(b^{\star})$ for any $b \in \mathcal{O}$).*
4. *$\underline{\mathrm{Lie}}_{G/S}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-module structure given naturally by i satisfies the determinantal condition in* [38, Def. 1.3.4.2] *given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot, \cdot \rangle, h)$.*
5. *$\alpha_{\mathcal{H}}$ is an (integral) level-$\mathcal{H}$ structure of (G, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}, \langle \cdot, \cdot \rangle)$ as in* [38, Def. 1.3.7.8].

The isomorphisms $(G, \lambda, i, \alpha_{\mathcal{H}}) \sim_{\mathrm{isom.}} (G', \lambda', i', \alpha'_{\mathcal{H}})$ of $\mathsf{M}_{\mathcal{H}}(S)$ are given by (naive) isomorphisms $f : G \xrightarrow{\sim} G'$ such that $\lambda = f^{\vee} \circ \lambda' \circ f$, $f \circ i(b) = i'(b) \circ f$ for all $b \in \mathcal{O}$, and $f \circ \alpha_{\mathcal{H}} = \alpha'_{\mathcal{H}}$ (symbolically).

Remark 1.7. The definition here using isomorphism classes is not as canonical as the ones proposed by Grothendieck and Deligne using quasiisogeny classes (as in [37]). For the relation between their definitions and ours, see [38, §1.4]. We introduce the definition (using isomorphisms) here mainly because this is the definition most concrete for the study of compactifications.

Theorem 1.8 ([38, Thm. 1.4.1.12 and Cor. 7.2.3.10]). *The moduli problem $\mathsf{M}_{\mathcal{H}}$ is a smooth separated algebraic stack of finite type over S_0. It is representable by a quasiprojective scheme if the objects it parameterizes have no nontrivial automorphism, which is in particular the case when $\mathcal{H}$ is* **neat** (*as in Definition* 1.2).

We shall insist *from now on* the following technical condition on PEL-type $\mathcal{O}$-lattices:

Condition 1.9 (cf. [38, Cond. 1.4.3.9]). *The PEL-type $\mathcal{O}$-lattice $(L, \langle\cdot,\cdot\rangle, h)$ is chosen such that the action of $\mathcal{O}$ on L extends to an action of some maximal order $\mathcal{O}'$ in B containing $\mathcal{O}$.*

1C. **Cusp labels.** Although there is no *rational boundary components* in the theory of arithmetic compactifications (in mixed characteristics), we have developed in [38, §5.4] the notion of *cusp labels* that serves a similar purpose. (While $\mathrm{G}(\mathbb{Q})$ plays an important role in the analytic theory over $\mathbb{C}$, it does not play any obvious role in the algebraic theory over $\mathcal{O}_{F_0,(\Box)}$. This is partly due to the so-called failure of Hasse's principle; see for example [37, §8] and [38, Rem. 1.4.3.11].)

Unlike in the analytic theory over $\mathbb{C}$, where boundary components are naturally parameterized by group-theoretic objects, the only algebraic machinery we have is the theory of semiabelian degenerations of abelian varieties with PEL structures. The cusp labels are (by their very design) part of the parameters (which we call the *degeneration data*) for such (semiabelian) degenerations.

Definition 1.10 (cf. [38, §1.2.6]). *Let R be any noetherian $\mathbb{Z}$-algebra. Suppose we have an increasing filtration $\mathtt{F} = \{\mathtt{F}_{-i}\}$ on $L \underset{\mathbb{Z}}{\otimes} R$, indexed by nonpositive integers $-i$, such that $\mathtt{F}_0 = L \underset{\mathbb{Z}}{\otimes} R$.*

1. *We say that $\mathtt{F}$ is* **integrable** *if, for any i, $\mathrm{Gr}^{\mathtt{F}}_{-i} := \mathtt{F}_{-i}/\mathtt{F}_{-i-1}$ is integrable in the sense that $\mathrm{Gr}^{\mathtt{F}}_{-i} \cong M_i \underset{\mathbb{Z}}{\otimes} R$ (as R-modules) for some $\mathcal{O}$-lattice M_i.*
2. *We say that $\mathtt{F}$ is* **split** *if there exists (noncanonically) some isomorphism $\mathrm{Gr}^{\mathtt{F}} := \underset{i}{\oplus} \mathrm{Gr}^{\mathtt{F}}_{-i} \overset{\sim}{\to} \mathtt{F}_0$ of R-modules.*
3. *We say that $\mathtt{F}$ is* **admissible** *if it is both integrable and split.*
4. *Let m be an integer. We say that $\mathtt{F}$ is* **m-symplectic with respect to** *$(L, \langle\cdot,\cdot\rangle)$ if, for any i, $\mathtt{F}_{-m+i}$ and $\mathtt{F}_{-i}$ are annihilators of each other under the pairing $\langle\cdot,\cdot\rangle$ on $\mathtt{F}_0$.*

We shall only work with $m = 3$, and we shall suppress m in what follows. The fact that $\hat{\mathbb{Z}}^\Box$ involves *bad primes* (cf. Definition 1.5) is the main reason that we may have to allow nonprojective filtrations.

Definition 1.11 ([38, Def. 5.2.7.1]). *We say that a symplectic admissible filtration $\mathtt{Z}$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$ is* **fully symplectic with respect to** *$(L, \langle\cdot,\cdot\rangle)$ if there is a symplectic admissible filtration $\mathtt{Z}_{\mathbb{A}^\Box} = \{\mathtt{Z}_{-i,\mathbb{A}^\Box}\}$ on $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\Box$ that* **extends** *$\mathtt{Z}$ in the sense that $\mathtt{Z}_{-i,\mathbb{A}^\Box} \cap (L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box) = \mathtt{Z}_{-i}$ in $L \underset{\mathbb{Z}}{\otimes} \mathbb{A}^\Box$ for all i.*

Definition 1.12 ([38, Def. 5.2.7.3]). *A symplectic-liftable admissible filtration $\mathtt{Z}_n$ on L/nL is called* **fully symplectic-liftable** *with respect to $(L, \langle\cdot,\cdot\rangle)$ if it is the reduction modulo n of some admissible filtration $\mathtt{Z}$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$ that is fully symplectic with respect to $(L, \langle\cdot,\cdot\rangle)$ as in Definition 1.11.*

Degenerations into semiabelian schemes induce filtrations on Tate modules and on Lie algebras of the generic fibers. While the symplectic-liftable admissible filtrations represent (certain orbits of) filtrations on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ induced by filtrations on Tate modules via the level structures, the fully symplectic-liftable ones are equipped with (certain orbits of) filtrations on $L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ induced by the filtrations on Lie algebras via the Lie algebra condition (4) in Definition 1.6. (One may interpret the Lie algebra condition as the "de Rham" (or rather "Hodge") component of a certain "complete level structure", the direct product of whose "ℓ-adic" components being a level structure in the usual sense.) Such (orbits of) filtrations are the crudest invariants of degenerations we consider.

Definition 1.13 (cf. [38, Def. 5.4.1.3])**.** *Given a fully symplectic admissible filtration* $\mathtt{Z}$ *on* $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ *with respect to* $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ *as in Definition* 1.11*, a* **torus argument** Φ *for* $\mathtt{Z}$ *is a tuple* $\Phi := (X, Y, \phi, \varphi_{-2}, \varphi_0)$*, where:*

(1) *X and Y are $\mathcal{O}$-lattices of the same $\mathcal{O}$-multirank (see* [38, Def. 5.2.2.5]*), and $\phi : Y \hookrightarrow X$ is an $\mathcal{O}$-equivariant embedding.*
(2) *$\varphi_{-2} : \mathrm{Gr}^{\mathtt{Z}}_{-2} \overset{\sim}{\to} \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1))$ and $\varphi_0 : \mathrm{Gr}^{\mathtt{Z}}_0 \overset{\sim}{\to} Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$ are isomorphisms (of $\hat{\mathbb{Z}}^\square$-modules) such that the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{20} : \mathrm{Gr}^{\mathtt{Z}}_{-2} \times \mathrm{Gr}^{\mathtt{Z}}_0 \to \hat{\mathbb{Z}}^\square(1)$ defined by $\mathtt{Z}$ is the pullback of the pairing*

$$\langle\,\cdot\,,\,\cdot\,\rangle_\phi : \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square) \to \hat{\mathbb{Z}}^\square(1)$$

defined by the composition

$$\mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1)) \times (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square) \overset{\mathrm{Id} \times \phi}{\to} \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1)) \times (X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square) \to \hat{\mathbb{Z}}^\square(1),$$

with the sign convention that $\langle\,\cdot\,,\,\cdot\,\rangle_\phi(x, y) = x(\phi(y)) = (\phi(y))(x)$ for any $x \in \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1))$ and any $y \in Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$.

Definition 1.14 (cf. [38, Def. 5.4.1.4 and 5.4.1.5])**.** *Given a fully symplectic-liftable admissible filtration* $\mathtt{Z}_n$ *on* L/nL *with respect to* $(L, \langle\,\cdot\,,\,\cdot\,\rangle)$ *as in Definition* 1.12*, a* **torus argument** Φ_n **at level** n *for* $\mathtt{Z}_n$ *is a tuple* $\Phi_n := (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$*, where:*

(1) *X and Y are $\mathcal{O}$-lattices of the same $\mathcal{O}$-multirank, and $\phi : Y \hookrightarrow X$ is an $\mathcal{O}$-equivariant embedding.*
(2) *$\varphi_{-2,n} : \mathrm{Gr}^{\mathtt{Z}}_{-2,n} \overset{\sim}{\to} \mathrm{Hom}(X/nX, (\mathbb{Z}/n\mathbb{Z})(1))$ (resp. $\varphi_{0,n} : \mathrm{Gr}^{\mathtt{Z}}_{0,n} \overset{\sim}{\to} Y/nY$) is an isomorphism that is the reduction modulo n of some isomorphism $\varphi_{-2} : \mathrm{Gr}^{\mathtt{Z}}_{-2} \overset{\sim}{\to} \mathrm{Hom}_{\hat{\mathbb{Z}}^\square}(X \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \hat{\mathbb{Z}}^\square(1))$ (resp. $\varphi_0 : \mathrm{Gr}^{\mathtt{Z}}_0 \overset{\sim}{\to} (Y \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square)$), such that $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$ form a torus argument as in Definition* 1.13*.*

We say in this case that Φ_n is the reduction modulo n of Φ.

Two torus arguments $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ and $\Phi'_n = (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n})$ at level n are **equivalent** *if and only if there exists a pair of isomorphisms*

$$(\gamma_X : X' \overset{\sim}{\to} X, \gamma_Y : Y \overset{\sim}{\to} Y')$$

(of $\mathcal{O}$-lattices) such that $\phi = \gamma_X \phi' \gamma_Y$, $\varphi'_{-2,n} = {}^t\gamma_X \varphi_{-2,n}$, *and* $\varphi'_{0,n} = \gamma_Y \varphi_{0,n}$. *In this case, we say that* Φ_n *and* Φ'_n *are equivalent under the pair of isomorphisms* $\gamma = (\gamma_X, \gamma_Y)$, *which we denote as* $\gamma = (\gamma_X, \gamma_Y) : \Phi_n \xrightarrow{\sim} \Phi'_n$.

The torus arguments record the isomorphism classes of the torus parts of degenerations of abelian schemes with PEL structures. These are the second crudest invariants of degenerations we consider.

Definition 1.15 ([38, Def. 5.4.1.9]). *A* (**principal**) **cusp label at level** n *for a PEL-type* $\mathcal{O}$*-lattice* $(L, \langle \cdot\,,\,\cdot \rangle, h)$, *or a cusp label of the moduli problem* M_n, *is an equivalence class* $[(\mathsf{Z}_n, \Phi_n, \delta_n)]$ *of triples* $(\mathsf{Z}_n, \Phi_n, \delta_n)$, *where:*

1. Z_n *is an admissible filtration on* L/nL *that is fully symplectic-liftable in the sense of Definition* 1.12.
2. Φ_n *is a torus argument at level* n *for* Z_n.
3. $\delta_n : \mathrm{Gr}_n^{\mathsf{Z}} \xrightarrow{\sim} L/nL$ *is a liftable splitting.*

Two triples $(\mathsf{Z}_n, \Phi_n, \delta_n)$ *and* $(\mathsf{Z}'_n, \Phi'_n, \delta'_n)$ *are equivalent if* Z_n *and* Z'_n *are identical, and if* Φ_n *and* Φ'_n *are equivalent as in Definition* 1.14.

The liftable splitting δ_n in any triple $(\mathsf{Z}_n, \Phi_n, \delta_n)$ is noncanonical and auxiliary in nature. Such splittings are needed for analyzing the "degeneration of pairings" in general PEL cases (unlike in the special case in Faltings–Chai [16, Ch. IV, §6]).

To proceed from principal cusp labels at level n to general cusp labels at level $\mathcal{H}$, where $\mathcal{H}$ is an open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^\square)$, we form *étale orbits* of the objects we have thus defined. The precise definitions are complicated (see [38, Def. 5.4.2.1, 5.4.2.2, and 5.4.2.4]) but the idea is simple: For any $\mathcal{H}$ as above, consider those $n \geq 1$ sufficiently divisible such that $\square \nmid n$ and $\mathcal{U}^\square(n) \subset \mathcal{H}$. Then we have a compatible system of finite groups $H_n = \mathcal{H}/\mathcal{U}^\square(n)$, and an object at level $\mathcal{H}$ is simply defined to be a compatible system of étale H_n-orbits of objects at running levels n as above. Then we arrive at the notions of *torus arguments* $\Phi_\mathcal{H} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ *at level* $\mathcal{H}$, and of *representatives* $(\mathsf{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})$ *of cusp labels* $[(\mathsf{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})]$ *at level* $\mathcal{H}$. (The liftability condition is implicit in such a definition, as in the definition of level structures we omitted.) By abuse of language, we call these $\mathcal{H}$-orbits of $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_0)$, $(\mathsf{Z}, \Phi, \delta)$, and $[(\mathsf{Z}, \Phi, \delta)]$, respectively.

For simplicity, we shall often omit $\mathsf{Z}_\mathcal{H}$ from the notation.

Lemma 1.16 (cf. [38, Lem. 5.2.7.5 in the revision]). *Let* Z_n *be an admissible filtration on* L/nL *that is fully symplectic-liftable with respect to* $(L, \langle \cdot\,,\,\cdot \rangle)$. *Let* $(\mathrm{Gr}_{-1}^{\mathsf{Z}}, \langle \cdot\,,\,\cdot \rangle_{11})$ *be induced by some fully symplectic lifting* Z *of* Z_n, *and let* $(\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{Z}}, \langle \cdot\,,\,\cdot \rangle_{11,\mathbb{R}}, h_{-1})$ *be determined by* [38, Prop. 5.1.2.2 in the revision] *by any extension* $\mathsf{Z}_{\mathbb{A}^\square}$ *in Definition* 1.11 (*which has the same reflex field* F_0 *as* $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot\,,\,\cdot \rangle, h)$ *does*). *Then there is associated* (*noncanonically*) *a PEL-type* $\mathcal{O}$*-lattice* $(L^{\mathsf{Z}_n}, \langle \cdot\,,\,\cdot \rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ *satisfying Condition* 1.9 *such that:*

1. $[(L^{\mathsf{Z}_n})^\# : L^{\mathsf{Z}_n}]$ *is prime to* $\square$.
2. *There exist* (*noncanonical*) $\mathcal{O}$*-equivariant isomorphisms*
$$(\mathrm{Gr}_{-1}^{\mathsf{Z}}, \langle \cdot\,,\,\cdot \rangle_{11}) \xrightarrow{\sim} (L^{\mathsf{Z}_n} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square, \langle \cdot\,,\,\cdot \rangle^{\mathsf{Z}_n})$$
and
$$(\mathrm{Gr}_{-1,\mathbb{R}}^{\mathsf{Z}}, \langle \cdot\,,\,\cdot \rangle_{11,\mathbb{R}}, h_{-1}) \xrightarrow{\sim} (L^{\mathsf{Z}_n} \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot\,,\,\cdot \rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n}).$$

(3) *The moduli problem* $\mathsf{M}_n^{\mathsf{Z}_n}$ *defined by the noncanonical* $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ *as in Definition* 1.6 *is canonical in the sense that it depends* (*up to isomorphism*) *only on* Z_n, *but not on the choice of* $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$.

Definition 1.17 (cf. [38, Def. 5.4.2.6 and the errata])**.** *The PEL-type* $\mathcal{O}$*-lattice*

$$(L^{\mathsf{Z}_\mathcal{H}}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_\mathcal{H}}, h^{\mathsf{Z}_\mathcal{H}})$$

is a fixed (*noncanonical*) *choice of any of the PEL-type* $\mathcal{O}$*-lattice* $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ *in Lemma* 1.16 *for any element* Z_n *in any* Z_{H_n} (*in* $\mathsf{Z}_\mathcal{H} = \{\mathsf{Z}_{H_n}\}$, *a compatible collection of étale orbits* Z_{H_n} *at various levels* n *such that* $\Box \nmid n$ *and* $\mathcal{U}^\Box(n) \subset \mathcal{H}$). *The elements of* H_n *leaving* Z_n *invariant induce a subgroup of* $\mathrm{G}_{(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})}(\mathbb{Z}/n\mathbb{Z})$. *Let* $\mathcal{H}_h$ *be the preimage of this subgroup under*

$$\mathrm{G}_{(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})}(\hat{\mathbb{Z}}^\Box) \twoheadrightarrow \mathrm{G}_{(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})}(\mathbb{Z}/n\mathbb{Z}).$$

Then we define $\mathsf{M}_{\mathcal{H}_h}$ *to be the moduli problem defined by* $(L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$ *with level-*$\mathcal{H}_h$ *structures as in Lemma* 1.16. (*The isomorphism class of this final moduli problem is independent of the choice of* $(L^{\mathsf{Z}_\mathcal{H}}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_\mathcal{H}}, h^{\mathsf{Z}_\mathcal{H}}) = (L^{\mathsf{Z}_n}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathsf{Z}_n}, h^{\mathsf{Z}_n})$.) *We define* $\mathsf{M}_\mathcal{H}^{\Phi_\mathcal{H}}$ *to be the quotient of* $\coprod \mathsf{M}_n^{\mathsf{Z}_n}$ *by* H_n, *where the disjoint union is over representatives* $(\mathsf{Z}_n, \Phi_n, \delta_n)$ (*with the same* (X, Y, ϕ)) *in* $(\mathsf{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})$, *which is finite étale over* $\mathsf{M}_{\mathcal{H}_h}$ *by construction.* (*The isomorphism class of* $\mathsf{M}_\mathcal{H}^{\Phi_\mathcal{H}}$ *is independent of the choice of* n *and the representatives* $(\mathsf{Z}_n, \Phi_n, \delta_n)$ *we use.*) *We then* (*abusively*) *define* $\mathsf{M}_\mathcal{H}^{\mathsf{Z}_\mathcal{H}}$ *to be the quotient of* $\mathsf{M}_\mathcal{H}^{\Phi_\mathcal{H}}$ *by the subgroup of* Γ_ϕ *stabilizing* $\Phi_\mathcal{H}$ (*whose action factors through a finite quotient group*), *which depends only on the cusp label* $[(\mathsf{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})]$, *but not on the choice of the representative* $(\mathsf{Z}_\mathcal{H}, \Phi_\mathcal{H}, \delta_\mathcal{H})$. *By construction, we have finite étale morphisms* $\mathsf{M}_\mathcal{H}^{\Phi_\mathcal{H}} \to \mathsf{M}_\mathcal{H}^{\mathsf{Z}_\mathcal{H}} \to \mathsf{M}_{\mathcal{H}_h}$ (*which can be identified with* $\mathsf{M}_{\mathcal{H}'_h} \to \mathsf{M}_{\mathcal{H}''_h} \to \mathsf{M}_{\mathcal{H}_h}$ *for some canonically determined open compact subgroups* $\mathcal{H}'_h \subset \mathcal{H}''_h \subset \mathcal{H}_h$).

Such *boundary moduli problems* $\mathsf{M}_\mathcal{H}^{\mathsf{Z}_\mathcal{H}}$ are the fundamental building blocks in the construction of toroidal boundary charts for $\mathsf{M}_\mathcal{H}$. (They actually appear in the boundary of the minimal compactification of $\mathsf{M}_\mathcal{H}$, which we call *cusps*. They are parameterized by the cusp labels of $\mathsf{M}_\mathcal{H}$.)

It is important to study the relations among cusp labels of different multiranks.

Definition 1.18 (cf. [38, Def. 5.4.1.15])**.** *A* **surjection**

$$(\mathsf{Z}_n, \Phi_n, \delta_n) \twoheadrightarrow (\mathsf{Z}'_n, \Phi'_n, \delta'_n)$$

between representatives of cusp labels at level n, *where* $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ *and* $\Phi'_n = (X', Y', \phi', \varphi'_{-2,n}, \varphi'_{0,n})$, *is a pair* (*of surjections*) $(s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y')$ (*of* $\mathcal{O}$*-lattices*) *such that:*

(1) *Both* s_X *and* s_Y *are admissible surjections* (*i.e., with kernels defining filtrations that are admissible in the sense of Definition* 1.10), *and they are compatible with* ϕ *and* ϕ' *in the sense that* $s_X\phi = \phi' s_Y$.

(2) $\mathsf{Z}'_{-2,n}$ *is an admissible submodule of* $\mathsf{Z}_{-2,n}$, *and the natural embedding* $\mathrm{Gr}^{\mathsf{Z}'}_{-2,n} \hookrightarrow \mathrm{Gr}^{\mathsf{Z}}_{-2,n}$ *satisfies* $\varphi_{-2,n} \circ (\mathrm{Gr}^{\mathsf{Z}'}_{-2,n} \hookrightarrow \mathrm{Gr}^{\mathsf{Z}}_{-2,n}) = s_X^* \circ \varphi'_{-2,n}$.

(3) $\mathsf{Z}_{-1,n}$ *is an admissible submodule of* $\mathsf{Z}'_{-1,n}$, *and the natural surjection* $\mathrm{Gr}^{\mathsf{Z}}_{0,n} \twoheadrightarrow \mathrm{Gr}^{\mathsf{Z}'}_{0,n}$ *satisfies* $s_Y \circ \varphi_{0,n} = \varphi'_{0,n} \circ (\mathrm{Gr}^{\mathsf{Z}}_{0,n} \twoheadrightarrow \mathrm{Gr}^{\mathsf{Z}'}_{0,n})$.

In this case, we write $s = (s_X, s_Y) : (\mathsf{Z}_n, \Phi_n, \delta_n) \twoheadrightarrow (\mathsf{Z}'_n, \Phi'_n, \delta'_n)$

By taking orbits as before, there is a corresponding notion for general cusp labels:

Definition 1.19 (cf. [38, Def. 5.4.2.12]). *A* **surjection** $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *between representatives of cusp labels at level $\mathcal{H}$, where $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ and $\Phi'_{\mathcal{H}} = (X', Y', \phi', \varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$, is a pair (of surjections) $s = (s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y')$ (of $\mathcal{O}$-lattices) such that:*

1. *Both s_X and s_Y are admissible surjections, and they are compatible with ϕ and ϕ' in the sense that $s_X \phi = \phi' s_Y$.*
2. *$\mathsf{Z}'_{\mathcal{H}}$ and $(\varphi'_{-2,\mathcal{H}}, \varphi'_{0,\mathcal{H}})$ are assigned to $\mathsf{Z}_{\mathcal{H}}$ and $(\varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ respectively under $s = (s_X, s_Y)$ as in* [38, Lem. 5.4.2.11].

In this case, we write $s = (s_X, s_Y) : (\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$.

Definition 1.20 (cf. [38, Def. 5.4.2.13]). *We say that there is a* **surjection** *from a cusp label at level $\mathcal{H}$ represented by some $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ to a cusp label at level $\mathcal{H}$ represented by some $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ if there is a surjection (s_X, s_Y) from $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ to $(\mathsf{Z}'_{\mathcal{H}}, \Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$.*

This is well defined by [38, Lem. 5.4.1.16].

The surjection among cusp labels can be naturally seen when we have the so-called *two-step degenerations* (see [16, Ch. III, §10] and [38, §4.5.6 in the revision]). This notion will be further developed in Definitions 1.32, 1.37, and 1.38 below.

1D. **Cone decompositions.** For any torus argument $\Phi_n = (X, Y, \phi, \varphi_{-2,n}, \varphi_{0,n})$ at level n, consider the finitely generated commutative group (i.e., $\mathbb{Z}$-module)

$$\ddot{\mathbf{S}}_{\Phi_n} := ((\tfrac{1}{n}Y) \underset{\mathbb{Z}}{\otimes} X) / \begin{pmatrix} y \otimes \phi(y') - y' \otimes \phi(y) \\ (b\frac{1}{n}y) \otimes \chi - (\frac{1}{n}y) \otimes (b^\star \chi) \end{pmatrix}_{\substack{y, y' \in Y \\ \chi \in X, b \in \mathcal{O}}} \tag{1.21}$$

and set $\mathbf{S}_{\Phi_n} := \ddot{\mathbf{S}}_{\Phi_n, \mathrm{free}}$, the free quotient of $\ddot{\mathbf{S}}_{\Phi_n}$. (See [38, (6.2.3.5) and Conv. 6.2.3.26].) Then, for a general torus argument $\Phi_{\mathcal{H}} = (X, Y, \phi, \varphi_{-2,\mathcal{H}}, \varphi_{0,\mathcal{H}})$ at level $\mathcal{H}$, there is a recipe [38, Lem. 6.2.4.4] that gives a corresponding free commutative group $\mathbf{S}_{\Phi_{\mathcal{H}}}$ (which can be identified with a finite index subgroup of some $\mathbf{S}_{\Phi_n}$).

The group $\mathbf{S}_{\Phi_{\mathcal{H}}}$ provides indices for certain "Laurent series expansions" near the boundary strata. In the modular curve case, it is canonically isomorphic to $\mathbb{Z}$, which means there is a canonical parameter q near the boundary — i.e., the *cusps*. The expansion of modular forms with respect to this parameter then gives the familiar q-expansion along the cusps. The compactification of the modular curves can be described locally near each of the cusps by $\mathrm{Spec}(R[q^i]_{i \in \mathbb{Z}}) \hookrightarrow \mathrm{Spec}(R[q^i]_{i \in \mathbb{Z}_{\geq 0}})$ for some suitable base ring R. For $\mathsf{M}_{\mathcal{H}}$, we would like to have an analogous theory in which the torus with the character group $\mathbf{S}_{\Phi_{\mathcal{H}}}$ can be partially compactified by adding normal crossings divisors in a smooth scheme. This is best achieved by the theory of *toroidal embeddings* developed in [36]. Many terminologies in such a theory will naturally show up in our description of the toroidal boundary charts, and we will review them in what follows.

Let $\mathbf{S}^\vee_{\Phi_{\mathcal{H}}} := \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_{\mathcal{H}}}, \mathbb{Z})$ be the $\mathbb{Z}$-dual of $\mathbf{S}_{\Phi_{\mathcal{H}}}$, and let $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}} := \mathbf{S}^\vee_{\Phi_{\mathcal{H}}} \underset{\mathbb{Z}}{\otimes} \mathbb{R} = \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\Phi_{\mathcal{H}}}, \mathbb{R})$. By construction of $\mathbf{S}_{\Phi_{\mathcal{H}}}$, the $\mathbb{R}$-vector space $(\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ is isomorphic to the space of Hermitian pairings $(\!| \cdot, \cdot |\!) : (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R} = B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$, by sending a Hermitian pairing $(\!| \cdot, \cdot |\!)$ to the function $y \otimes \phi(y') \mapsto \mathrm{Tr}_{B/\mathbb{Q}} (\!| y, y' |\!)$ in $\mathrm{Hom}_{\mathbb{R}}((Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}), \mathbb{R}) \cong (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$. (See [38, Lem. 1.1.4.6].)

Definition 1.22 (cf. [38, beginning of §6.1.1]). (1) *A subset of* $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$ *is called a* **cone** *if it is invariant under the natural multiplication action of* $\mathbb{R}^\times_{>0}$ *on the* $\mathbb{R}$*-vector space* $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$.

(2) *A cone in* $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$ *is* **nondegenerate** *if its closure does not contain any nonzero* $\mathbb{R}$*-vector subspace of* $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$.

(3) *A* **rational polyhedral cone** *in* $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$ *is a cone in* $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$ *of the form* $\sigma = \mathbb{R}_{>0}v_1 + \ldots + \mathbb{R}_{>0}v_n$ *with* $v_1, \ldots, v_n \in (\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{Q} = \mathbf{S}^\vee_{\Phi_\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

(4) *A* **supporting hyperplane** *of* σ *is a hyperplane* P *in* $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$ *such that* σ *does not overlap with both sides of* P.

(5) *A* **face** *of* σ *is a rational polyhedral cone* τ *such that* $\overline{\tau} = \overline{\sigma} \cap P$ *for some supporting hyperplane* P *of* σ. (*Here an overline on a cone means its closure in the ambient space* $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$.)

Let $\mathbf{P}_{\Phi_\mathcal{H}}$ be the subset of $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$ corresponding to positive semidefinite Hermitian pairings $(\!|\cdot,\cdot|\!) : (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to B \underset{\mathbb{Q}}{\otimes} \mathbb{R}$, with radical (namely the annihilator of the whole space) *admissible* in the sense that it is the $\mathbb{R}$-span of some *admissible submodule* Y' of Y. (We say a submodule Y' of Y is admissible if $Y' \subset Y$ defines an admissible filtration on Y; cf. Definition 1.10. In particular, the quotient Y/Y' is also an $\mathcal{O}$-lattice.)

Definition 1.23 ([38, Def. 6.2.4.1 and 5.4.1.6]). *The group* $\Gamma_{\Phi_\mathcal{H}}$ *is the subgroup of elements* $\gamma = (\gamma_X, \gamma_Y)$ *in* $\mathrm{GL}_\mathcal{O}(X) \times \mathrm{GL}_\mathcal{O}(Y)$ *satisfying* $\phi = \gamma_X \phi \gamma_Y$, $\varphi_{-2,\mathcal{H}} = {}^t\gamma_X \varphi_{-2,\mathcal{H}}$, *and* $\varphi_{0,\mathcal{H}} = \gamma_Y \varphi_{0,\mathcal{H}}$ (*if we view the latter two as collections of orbits*).

The group $\Gamma_{\Phi_\mathcal{H}}$ acts on $\mathbf{S}_{\Phi_\mathcal{H}}$, and its induced action preserves the subset $\mathbf{P}_{\Phi_\mathcal{H}}$ of $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$. (The group $\Gamma_{\Phi_\mathcal{H}}$ is the automorphism group of the torus argument $\Phi_\mathcal{H}$. Such automorphism groups show up naturally because torus arguments are only determined up to isomorphism.)

Definition 1.24 (cf. [38, Def. 6.1.1.12]). *A* $\Gamma_{\Phi_\mathcal{H}}$**-admissible rational polyhedral cone decomposition** *of* $\mathbf{P}_{\Phi_\mathcal{H}}$ *is a collection* $\Sigma = \{\sigma_j\}_{j\in J}$ *with some indexing set* J *such that:*

(1) *Every* σ_j *is a nondegenerate rational polyhedral cone.*

(2) $\mathbf{P}_{\Phi_\mathcal{H}}$ *is the disjoint union of all the* σ_j*'s in* Σ. *For each* $j \in J$, *the closure of* σ_j *in* $\mathbf{P}_{\Phi_\mathcal{H}}$ *is a disjoint union of* σ_k*'s with* $k \in J$. *In other words,* $\mathbf{P}_{\Phi_\mathcal{H}} = \coprod_{j\in J} \sigma_j$ *is a stratification of* $\mathbf{P}_{\Phi_\mathcal{H}}$. *Moreover, each* σ_k *appearing in the closure of* σ_j *as above is a face of* σ_j.

(3) Σ *is invariant under the action of* $\Gamma_{\Phi_\mathcal{H}}$ *on* $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$, *in the sense that* $\Gamma_{\Phi_\mathcal{H}}$ *permutes the cones in* Σ. *Under this action, the set* $\Sigma/\Gamma_{\Phi_\mathcal{H}}$ *of* $\Gamma_{\Phi_\mathcal{H}}$*-orbits is finite.*

Definition 1.25 (cf. [38, Def. 6.1.1.13]). *A rational polyhedral cone* σ *in* $(\mathbf{S}_{\Phi_\mathcal{H}})^\vee_\mathbb{R}$ *is* **smooth** *with respect to the integral structure given by* $\mathbf{S}^\vee_{\Phi_\mathcal{H}}$ *if we have* $\sigma = \mathbb{R}_{>0}v_1 + \ldots + \mathbb{R}_{>0}v_n$ *with* $v_1, \ldots, v_n$ *part of a* $\mathbb{Z}$*-basis of* $\mathbf{S}^\vee_{\Phi_\mathcal{H}}$.

Definition 1.26 (cf. [38, Def. 6.1.1.14]). *A* $\Gamma_{\Phi_\mathcal{H}}$**-admissible smooth rational polyhedral cone decomposition** *of* $\mathbf{P}_{\Phi_\mathcal{H}}$ *is a* $\Gamma_{\Phi_\mathcal{H}}$*-admissible rational polyhedral cone decomposition* $\{\sigma_j\}_{j\in J}$ *of* $\mathbf{P}_{\Phi_\mathcal{H}}$ *in which every* σ_j *is* **smooth**.

Definition 1.27 (cf. [38, Def. 7.3.1.1]). *Let* $\Sigma_{\Phi_\mathcal{H}} = \{\sigma_j\}_{j\in J}$ *be any* $\Gamma_{\Phi_\mathcal{H}}$*-admissible rational polyhedral cone decomposition of* $\mathbf{P}_{\Phi_\mathcal{H}}$. *An* (**invariant**) **polarization**

function *on* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *for the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *is a* $\Gamma_{\Phi_{\mathcal{H}}}$*-invariant continuous piecewise linear function* $\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}$ *such that:*

(1) $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ *is linear (i.e., coincides with a linear function) on each cone* σ_j *in* $\Sigma_{\Phi_{\mathcal{H}}}$. *(In particular,* $\mathsf{pol}_{\Phi_{\mathcal{H}}}(tx) = t\mathsf{pol}_{\Phi_{\mathcal{H}}}(x)$ *for any* $x \in \mathbf{P}_{\Phi_{\mathcal{H}}}$ *and* $t \in \mathbb{R}_{\geq 0}$.*)*

(2) $\mathsf{pol}_{\Phi_{\mathcal{H}}}((\mathbf{P}_{\Phi_{\mathcal{H}}} \cap \mathbf{S}^\vee_{\Phi_{\mathcal{H}}}) - \{0\}) \subset \mathbb{Z}_{>0}$. *(In particular,* $\mathsf{pol}_{\Phi_{\mathcal{H}}}(x) > 0$ *for any nonzero* x *in* $\mathbf{P}_{\Phi_{\mathcal{H}}}$.*)*

(3) $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ *is linear (in the above sense) on a rational polyhedral cone* σ *in* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *if and only if* σ *is contained in some cone* σ_j *in* $\Sigma_{\Phi_{\mathcal{H}}}$.

(4) *For any* $x, y \in \mathbf{P}_{\Phi_{\mathcal{H}}}$, *we have* $\mathsf{pol}_{\Phi_{\mathcal{H}}}(x+y) \geq \mathsf{pol}_{\Phi_{\mathcal{H}}}(x) + \mathsf{pol}_{\Phi_{\mathcal{H}}}(y)$. *This is called the* **convexity** *of* $\mathsf{pol}_{\Phi_{\mathcal{H}}}$.

If such a polarization function exists, then we say that the $\Gamma_{\Phi_{\mathcal{H}}}$*-admissible rational polyhedral cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *is* **projective**.

Definition 1.28. *An* **admissible boundary component** *of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *is the image of* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ *under the embedding* $(\mathbf{S}_{\Phi'_{\mathcal{H}}})^\vee_{\mathbb{R}} \hookrightarrow (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ *defined by some surjection* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$. *(See Definition* 1.19.*)*

We shall always assume that the following technical condition is satisfied:

Condition 1.29 (cf. [16, Ch. IV, Rem. 5.8(a)]; see also [38, Cond. 6.2.5.25 in the revision])**.** *The cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}} = \{\sigma_j\}_{j \in J}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *is chosen such that, for any* $j \in J$, *if* $\gamma\overline{\sigma}_j \cap \overline{\sigma}_j \neq \{0\}$ *for some* $\gamma \in \Gamma_{\Phi_{\mathcal{H}}}$, *then a power of* γ *acts as the identity on the smallest admissible boundary component of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *containing* $\gamma\overline{\sigma}_j \cap \overline{\sigma}_j$.

This condition is used to ensure that there are no self-intersections of toroidal boundary strata when the level $\mathcal{H}$ is neat.

To describe the toroidal boundary of $\mathsf{M}_{\mathcal{H}}$, we will need not only cusp labels but also the cones:

Definition 1.30 (cf. [38, Def. 6.2.6.1])**.** *Let* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *be two representatives of cusp labels at level* $\mathcal{H}$, *let* $\sigma \subset (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$, *and let* $\sigma' \subset (\mathbf{S}_{\Phi'_{\mathcal{H}}})^\vee_{\mathbb{R}}$. *We say that the two triples* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *are* **equivalent** *if there exists a pair of isomorphisms* $\gamma = (\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$ *(of* $\mathcal{O}$*-lattices) such that:*

(1) *The two representatives* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *are equivalent under* γ *(as in* [38, Def. 5.4.2.4]*, the general level analogue of Definition* 1.15*).*

(2) *The isomorphism* $(\mathbf{S}_{\Phi'_{\mathcal{H}}})^\vee_{\mathbb{R}} \xrightarrow{\sim} (\mathbf{S}_{\Phi_{\mathcal{H}}})^\vee_{\mathbb{R}}$ *induced by* γ *sends* σ' *to* σ.

In this case, we say that the two triples $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *are equivalent under the pair of isomorphisms* $\gamma = (\gamma_X, \gamma_Y)$.

Definition 1.31 (cf. [38, Def. 6.2.6.2])**.** *Let* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *be two representatives of cusp labels at level* $\mathcal{H}$, *and let* $\Sigma_{\Phi_{\mathcal{H}}}$ *(resp.* $\Sigma_{\Phi'_{\mathcal{H}}}$*) be a* $\Gamma_{\Phi_{\mathcal{H}}}$*-admissible (resp.* $\Gamma_{\Phi'_{\mathcal{H}}}$*-admissible) smooth rational polyhedral cone decomposition of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *(resp.* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$*). We say that the two triples* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ *are* **equivalent** *if* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *are equivalent under some pair of isomorphisms* $\gamma = (\gamma_X : X' \xrightarrow{\sim} X, \gamma_Y : Y \xrightarrow{\sim} Y')$, *and if under one (and hence every) such* γ *the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi_{\mathcal{H}}}$ *is identified with the cone decomposition* $\Sigma_{\Phi'_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi'_{\mathcal{H}}}$. *In this case we say that the two triples* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ *and* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ *are equivalent under the pair of isomorphisms* $\gamma = (\gamma_X, \gamma_Y)$.

The compatibility among cone decompositions over different cusp labels is described as follows:

Definition 1.32 (cf. [38, Def. 6.2.6.4]). *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, and let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). A* **surjection** *$(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ is given by a surjection $s = (s_X : X \twoheadrightarrow X', s_Y : Y \twoheadrightarrow Y') : (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ (see Definition 1.19) that induces an embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ such that the restriction $\Sigma_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\Phi'_{\mathcal{H}}}}$ of the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ to $\mathbf{P}_{\Phi'_{\mathcal{H}}}$ is the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$.*

This allows us to define:

Definition 1.33 (cf. [38, Cond. 6.3.3.1 and Def. 6.3.3.2]). *A* **compatible choice of admissible smooth rational polyhedral cone decomposition data** *for $\mathsf{M}_{\mathcal{H}}$ is a complete set $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ of* **compatible choices** *of $\Sigma_{\Phi_{\mathcal{H}}}$ (satisfying Condition 1.29) such that, for every surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ of representatives of cusp labels, the cone decompositions $\Sigma_{\Phi_{\mathcal{H}}}$ and $\Sigma_{\Phi'_{\mathcal{H}}}$ define a surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ as in Definition 1.32.*

Definition 1.34 ([38, Def. 7.3.1.3]). *We say that a compatible choice $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$ (see Definition 1.33) is* **projective** *if it satisfies the following condition: There is a collection $\mathsf{pol} = \{\mathsf{pol}_{\Phi_{\mathcal{H}}} : \mathbf{P}_{\Phi_{\mathcal{H}}} \to \mathbb{R}_{\geq 0}\}$ of polarization functions labeled by representatives $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of cusp labels, each $\mathsf{pol}_{\Phi_{\mathcal{H}}}$ being a polarization function of the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ in Σ (see Definition 1.27), which are* **compatible** *in the following sense: For any surjection $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ of representatives of cusp labels (see Definition 1.19) inducing an embedding $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$, we have $\mathsf{pol}_{\Phi_{\mathcal{H}}}|_{\mathbf{P}_{\Phi'_{\mathcal{H}}}} = \mathsf{pol}_{\Phi'_{\mathcal{H}}}$.*

The most important relations among cone decompositions and among compatible choices of them are the so-called *refinements*:

Definition 1.35 (cf. [38, Def. 6.2.6.3]). *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ be two representatives of cusp labels at level $\mathcal{H}$, and let $\Sigma_{\Phi_{\mathcal{H}}}$ (resp. $\Sigma_{\Phi'_{\mathcal{H}}}$) be a $\Gamma_{\Phi_{\mathcal{H}}}$-admissible (resp. $\Gamma_{\Phi'_{\mathcal{H}}}$-admissible) smooth rational polyhedral cone decomposition of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ (resp. $\mathbf{P}_{\Phi'_{\mathcal{H}}}$). We say that the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ is a* **refinement** *of the triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ if $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ and $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ are equivalent under some pair of isomorphisms $\gamma = (\gamma_X, \gamma_Y)$, and if under one (and hence every) such γ the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi_{\mathcal{H}}}$ is identified with a refinement of the cone decomposition $\Sigma_{\Phi'_{\mathcal{H}}}$ of $\mathbf{P}_{\Phi'_{\mathcal{H}}}$. In this case we say that the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ is a refinement of the triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \Sigma_{\Phi'_{\mathcal{H}}})$ under the pair of isomorphisms $\gamma = (\gamma_X, \gamma_Y)$.*

Definition 1.36 (cf. [38, Def. 6.4.2.2]). *Let $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ and $\Sigma' = \{\Sigma'_{\Phi_{\mathcal{H}}}\}$ be two compatible choices of admissible smooth rational polyhedral cone decomposition data for $\mathsf{M}_{\mathcal{H}}$. We say that Σ* **refines** *Σ' if the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma_{\Phi_{\mathcal{H}}})$ is a refinement of the triple $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \Sigma'_{\Phi_{\mathcal{H}}})$, as in Definition 1.35, for $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ running through all representatives of cusp labels.*

Finally, we would like to describe the relations among the equivalence classes $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$, which will describe the "incidence relations" among (closures of) the toroidal boundary strata.

Definition 1.37 (cf. [38, Def. 6.3.2.14]). *Let $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ be a representative of a cusp label at level $\mathcal{H}$, and let $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ be a nondegenerate smooth rational polyhedral cone. We say that a triple $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ is a* **face** *of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ if:*

(1) $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *is the representative of some cusp label at level* $\mathcal{H}$*, such that there exists a surjection* $s = (s_X, s_Y) : (\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}) \twoheadrightarrow (\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}})$ *as in Definition* 1.19.

(2) $\sigma' \subset \mathbf{P}^+_{\Phi'_{\mathcal{H}}}$ *is a nondegenerate smooth rational polyhedral cone, such that for one (and hence every) surjection* $s = (s_X, s_Y)$ *as above, the image of* σ' *under the induced embedding* $\mathbf{P}_{\Phi'_{\mathcal{H}}} \hookrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ *is contained in the* $\Gamma_{\Phi_{\mathcal{H}}}$*-orbit of a face of* σ*.*

Note that this definition is insensitive to the choices of representatives in the classes $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ and $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$. This justifies the following:

Definition 1.38 (cf. [38, Def. 6.3.2.15])**.** *We say that the equivalence class* $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$ *of* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *is a* **face** *of the equivalence class* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ *of* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *if some triple equivalent to* $(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')$ *is a face of some triple equivalent to* $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$*.*

1E. Arithmetic toroidal compactifications.

Definition 1.39 (cf. [38, Def. 5.3.2.1])**.** *Let* S *be a normal locally noetherian algebraic stack. A tuple* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *over* S *is called a* **degenerating family of type** $\mathsf{M}_{\mathcal{H}}$*, or simply a* **degenerating family** *when the context is clear, if there exists a dense subalgebraic stack* S_1 *of* S*, such that* S_1 *is defined over* $\operatorname{Spec}(\mathcal{O}_{F_0,(\Box)})$*, and such that:*

(1) *By viewing group schemes as relative schemes (cf.* [23]*),* G *is a semiabelian scheme over* S *whose restriction* G_{S_1} *to* S_1 *is an abelian scheme. In this case, the dual semiabelian scheme* $G^\vee$ *exists (up to unique isomorphism), whose restriction* $G^\vee_{S_1}$ *to* S_1 *is the dual abelian scheme of* G_{S_1}*.*

(2) $\lambda : G \to G^\vee$ *is a group homomorphism that induces by restriction a prime-to-*$\Box$ *polarization* λ_{S_1} *of* G_{S_1}*.*

(3) $i : \mathcal{O} \to \operatorname{End}_S(G)$ *is a homomorphism that defines by restriction an* $\mathcal{O}$*-structure* $i_{S_1} : \mathcal{O} \to \operatorname{End}_{S_1}(G_{S_1})$ *of* (G_{S_1}, λ_{S_1})*.*

(4) $(G_{S_1}, \lambda_{S_1}, i_{S_1}, \alpha_{\mathcal{H}}) \to S_1$ *defines a tuple parameterized by the moduli problem* $\mathsf{M}_{\mathcal{H}}$*.*

We will only talk about (semiabelian) degenerations (of abelian varieties with PEL structures) of this form.

Definition 1.40 (cf. [38, Def. 6.3.1])**.** *Let* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *be a degenerating family of type* $\mathsf{M}_{\mathcal{H}}$ *over* S *(as in Definition* 1.39*) over* $\mathsf{S}_0 = \operatorname{Spec}(\mathcal{O}_{F_0,(\Box)})$*. Let* $\underline{\mathrm{Lie}}^\vee_{G/S} := e^*_G \Omega^1_{G/S}$ *be the dual of* $\underline{\mathrm{Lie}}_{G/S}$*, and let* $\underline{\mathrm{Lie}}^\vee_{G^\vee/S} := e^*_G \Omega^1_{G^\vee/S}$ *be the dual of* $\underline{\mathrm{Lie}}_{G^\vee/S}$*. Note that* $\lambda : G \to G^\vee$ *induces an* $\mathcal{O}$*-equivariant morphism* $\lambda^* : \underline{\mathrm{Lie}}^\vee_{G^\vee/S} \to \underline{\mathrm{Lie}}^\vee_{G/S}$*. (Here the* $\mathcal{O}$*-action on* $\underline{\mathrm{Lie}}^\vee_{G/S}$ *is a left action after twisting by the involution* $\star$*.) Then we define the sheaf* $\underline{\mathrm{KS}} = \underline{\mathrm{KS}}_{(G,\lambda)/S} = \underline{\mathrm{KS}}_{(G,\lambda,i,\alpha_{\mathcal{H}})/S}$ *by setting*

$$\underline{\mathrm{KS}} := (\underline{\mathrm{Lie}}^\vee_{G/S} \underset{\mathcal{O}_S}{\otimes} \underline{\mathrm{Lie}}^\vee_{G^\vee/S}) / \begin{pmatrix} \lambda^*(y) \otimes z - \lambda^*(z) \otimes y \\ (b^\star x) \otimes y - x \otimes (by) \end{pmatrix}_{\substack{x \in \underline{\mathrm{Lie}}^\vee_{G/S} \\ y,z \in \underline{\mathrm{Lie}}^\vee_{G^\vee/S} \\ b \in \mathcal{O}}}.$$

Analogues of the sheaf $\underline{\mathrm{KS}}$ appear naturally in the deformation theory of abelian varieties with PEL structures (without degenerations). The point of Definition 1.40 is that it extends the conventional definition (for abelian schemes with PEL structures) to the context of (semiabelian) degenerating families (see Definition 1.39).

Theorem 1.41 (cf. [38, Thm. 6.4.1.1 and 7.3.3.4]). *To each compatible choice $\Sigma = \{\Sigma_{\Phi_{\mathcal{H}}}\}$ of admissible smooth rational polyhedral cone decomposition data as in Definition* 1.33*, there is associated a proper smooth algebraic stack $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, which is an algebraic space when $\mathcal{H}$ is neat (as in Definition* 1.2*), containing $\mathsf{M}_{\mathcal{H}}$ as an open dense subalgebraic stack, together with a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ (as in Definition* 1.39*) such that:*

(1) *The restriction $(G_{\mathsf{M}_{\mathcal{H}}}, \lambda_{\mathsf{M}_{\mathcal{H}}}, i_{\mathsf{M}_{\mathcal{H}}}, \alpha_{\mathcal{H}})$ of the degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ to $\mathsf{M}_{\mathcal{H}}$ is the tautological (i.e., universal) tuple over $\mathsf{M}_{\mathcal{H}}$.*

(2) *$\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ has a stratification by locally closed subalgebraic stacks*

$$\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma} = \coprod_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]} \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]},$$

with $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ running through a complete set of equivalence classes of $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ (as in Definition 1.30*) with $\sigma \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$ and $\sigma \in \Sigma_{\Phi_{\mathcal{H}}} \in \Sigma$. (Here $\mathsf{Z}_{\mathcal{H}}$ is suppressed in the notation by our convention.)*

In this stratification, the $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$-stratum $\mathsf{Z}_{[(\Phi'_{\mathcal{H}},\delta'_{\mathcal{H}},\sigma')]}$ lies in the closure of the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ if and only if $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$ is a face of $[(\Phi'_{\mathcal{H}}, \delta'_{\mathcal{H}}, \sigma')]$ as in Definition 1.38*.*

The $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is smooth and isomorphic to the support of the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ for any representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ of $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$, where the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$ (before quotient by $\Gamma_{\Phi_{\mathcal{H}},\sigma}$, the subgroup of $\Gamma_{\Phi_{\mathcal{H}}}$ formed by elements mapping σ to itself) admits a canonical structure as the completion of an affine toroidal embedding $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)$ (along its σ-stratum $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}$) of a torus torsor $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over an abelian scheme torsor $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ over a finite étale cover $\mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}$ of the smooth algebraic stack $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ in Definition 1.17*. (Note that $\mathsf{Z}_{\mathcal{H}}$ and the isomorphism class of $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ depend only on the class $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$, but not on the choice of the representative $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$.)*

In particular, $\mathsf{M}_{\mathcal{H}}$ is an open dense stratum in this stratification.

(3) *The complement of $\mathsf{M}_{\mathcal{H}}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ (with its reduced structure) is a relative Cartier divisor $\mathsf{D}_{\infty,\mathcal{H}}$ with normal crossings, such that each connected component of a stratum of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} - \mathsf{M}_{\mathcal{H}}$ is open dense in an intersection of irreducible components of $\mathsf{D}_{\infty,\mathcal{H}}$ (including possible self-intersections). When $\mathcal{H}$ is neat, the irreducible components of $\mathsf{D}_{\infty,\mathcal{H}}$ have no self-intersections (cf. Condition* 1.29*,* [38, Rem. 6.2.5.26 in the revision]*, and* [16, Ch. IV, Rem. 5.8(a)]*).*

(4) *The extended Kodaira–Spencer morphism* [38, Def. 4.6.3.32] *for $G \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ induces an isomorphism*

$$\mathrm{KS}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} : \underline{\mathrm{KS}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \xrightarrow{\sim} \Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty]$$

(see Definition 1.40*). Here the sheaf $\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty]$ is the sheaf of modules of log 1-differentials on $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ over S_0, with respect to the relative Cartier divisor $\mathsf{D}_{\infty,\mathcal{H}}$ with normal crossings.*

(5) *The formal completion*

$$(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$$

of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ along the $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$-stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is canonically isomorphic to the formal algebraic stack $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ for any representative

$(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)$ *of* $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \sigma)]$. (*To form the formal completion along a given locally closed stratum, we first remove the other strata appearing in the closure of this stratum from the total space, and then form the formal completion of the remaining space along this stratum.*)

This isomorphism respects stratifications in the sense that, given any étale (*i.e., formally étale and of finite type; see* [21, I, 10.13.3]) *morphism* $\mathrm{Spf}(R, I) \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ *inducing a morphism* $\mathrm{Spec}(R) \to \Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$, *the stratification of* $\mathrm{Spec}(R)$ (*inherited from* $\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\sigma)/\Gamma_{\Phi_{\mathcal{H}},\sigma}$; *see* [38, Prop. 6.3.1.6 and Def. 6.3.2.16 in the revision]) *makes the induced morphism* $\mathrm{Spec}(R) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *a strata-preserving morphism.*

The pullback to $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ *of the degenerating family* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *over* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *is the Mumford family*

$$({}^{\heartsuit}G,\ {}^{\heartsuit}\lambda,\ {}^{\heartsuit}i,\ {}^{\heartsuit}\alpha_{\mathcal{H}})$$

over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}/\Gamma_{\Phi_{\mathcal{H}},\sigma}$ (*see* [38, §6.2.5]) *after we identify the bases using the isomorphism.* (*Here both the pullback of* $(G, \lambda, i, \alpha_{\mathcal{H}})$ *and the Mumford family* $({}^{\heartsuit}G,\ {}^{\heartsuit}\lambda,\ {}^{\heartsuit}i,\ {}^{\heartsuit}\alpha_{\mathcal{H}})$ *are considered as relative schemes with additional structures; cf.* [23].)

(6) *Let* S *be an irreducible noetherian normal scheme over* S_0. *Suppose we have a degenerating family* $(G^{\dagger}, \lambda^{\dagger}, i^{\dagger}, \alpha^{\dagger}_{\mathcal{H}})$ *of type* $\mathsf{M}_{\mathcal{H}}$ *over* S *as in Definition* 1.39. *Then* $(G^{\dagger}, \lambda^{\dagger}, i^{\dagger}, \alpha^{\dagger}_{\mathcal{H}}) \to S$ *is the pullback of* $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *via a* (*necessarily unique*) *morphism* $S \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ (*over* S_0) *if and only if the following condition is satisfied:*

Consider any dominant morphism $\mathrm{Spec}(V) \to S$ *centered at a geometric point* $\bar{s}$ *of* S, *where* V *is a complete discrete valuation ring with quotient field* K, *algebraically closed residue field* k, *and discrete valuation* υ. *Let*

$$(G^{\ddagger}, \lambda^{\ddagger}, i^{\ddagger}, \alpha^{\ddagger}_{\mathcal{H}}) \to \mathrm{Spec}(V)$$

be the pullback of $(G^{\dagger}, \lambda^{\dagger}, i^{\dagger}, \alpha^{\dagger}_{\mathcal{H}}) \to S$. *This pullback family defines an object of* $\mathrm{DEG}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ *over* $\mathrm{Spec}(V)$, *which corresponds to a tuple*

$$(A^{\ddagger}, \lambda^{\ddagger}_A, i^{\ddagger}_A, \underline{X}^{\ddagger}, \underline{Y}^{\ddagger}, \phi^{\ddagger}, c^{\ddagger}, (c^{\vee})^{\ddagger}, \tau^{\ddagger}, [(\alpha^{\natural}_{\mathcal{H}})^{\ddagger}])$$

in $\mathrm{DD}_{\mathrm{PEL},\mathsf{M}_{\mathcal{H}}}$ *under* [38, Thm. 5.3.1.17]. *Then we have a fully symplectic-liftable admissible filtration* $\mathsf{Z}^{\ddagger}_{\mathcal{H}}$ *determined by* $[(\alpha^{\natural}_{\mathcal{H}})^{\ddagger}]$. *Moreover, the étale sheaves* $\underline{X}^{\ddagger}$ *and* $\underline{Y}^{\ddagger}$ *are necessarily constant, because the base ring* V *is strict local. Hence it makes sense to say we also have a uniquely determined torus argument* $\Phi^{\ddagger}_{\mathcal{H}}$ *at level* $\mathcal{H}$ *for* $\mathsf{Z}^{\ddagger}_{\mathcal{H}}$.

On the other hand, we have objects $\underline{\Phi}_{\mathcal{H}}(G^{\ddagger})$, $\underline{\mathbf{S}}_{\underline{\Phi}_{\mathcal{H}}(G^{\ddagger})}$, *and* $\underline{B}(G^{\ddagger})$ (*see* [38, Constr. 6.3.1.1]), *which define objects* $\Phi^{\ddagger}_{\mathcal{H}}$, $\mathbf{S}_{\Phi^{\ddagger}_{\mathcal{H}}}$ *and in particular* $B^{\ddagger} : \mathbf{S}_{\Phi^{\ddagger}_{\mathcal{H}}} \to \mathrm{Inv}(V)$ *over the special fiber. Then* $\upsilon \circ B^{\ddagger} : \mathbf{S}_{\Phi^{\ddagger}_{\mathcal{H}}} \to \mathbb{Z}$ *defines an element of* $\mathbf{S}^{\vee}_{\Phi^{\ddagger}_{\mathcal{H}}}$, *where* $\upsilon : \mathrm{Inv}(V) \to \mathbb{Z}$ *is the homomorphism induced by the discrete valuation of* V.

Then the condition is that, for any $\mathrm{Spec}(V) \to S$ *as above, and for any choice of* $\delta^{\ddagger}_{\mathcal{H}}$ (*which is immaterial, because this choice will not be used*), *there is a cone* $\sigma^{\ddagger}$ *in the cone decomposition* $\Sigma_{\Phi^{\ddagger}_{\mathcal{H}}}$ *of* $\mathbf{P}_{\Phi^{\ddagger}_{\mathcal{H}}}$ (*given by the choice of*

Σ; cf. Definition 1.33) *such that $\overline{\sigma}^\ddagger$ contains all the $\upsilon \circ B^\ddagger$ obtained in this way.*

(7) *If $\mathcal{H}$ is neat and Σ is projective* (*see Definition* 1.34), *then $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is projective* (*and hence a scheme*) *over S_0.*

Statement (1) means the tautological tuple over $\mathsf{M}_{\mathcal{H}}$ extends to a degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}})$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. (Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is normal, this extension is unique by a result of Raynaud; see [49, IX, 1.4] or [16, Ch. I, Prop. 2.7].) Statements (2), (3), (4), (5), and (7) are self-explanatory. Statement (6) can be interpreted as a "universal property" for the degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ among degenerating families over normal locally noetherian bases, as in Definition 1.39, satisfying moreover some conditions describing the "degenerating patterns" over pullbacks to complete discrete valuation rings with algebraically closed residue fields. This "universal property" will be crucial in the main construction of this article (in Section 3A below).

2. Kuga families and their compactifications

Let $\mathcal{O}$, $\star$, $(L, \langle \cdot , \cdot \rangle)$, h, and $\Box$ be as in the previous section. Then we have a moduli problem $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ for each open compact $\mathcal{H}$ of $\mathrm{G}(\hat{\mathbb{Z}}^{\Box})$, with a toroidal compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ for each choice of Σ.

For simplicity, let us maintain the following:

Convention 2.1. *All morphisms between schemes or algebraic stacks over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ will be defined over S_0, unless otherwise specified.*

2A. **PEL-type Kuga families.** Let Q be any $\mathcal{O}$-lattice. Consider the abelian scheme $G_{\mathsf{M}_{\mathcal{H}}}$ over $\mathsf{M}_{\mathcal{H}}$ in (1) of Theorem 1.41. By [38, Prop. 5.2.3.8], the group functor $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})$ over $\mathsf{M}_{\mathcal{H}}$ is representable by a proper smooth group scheme which is an extension of a finite étale group scheme, whose rank has no prime factors other than those of Disc, by an abelian scheme $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ$, which we call the *fiberwise geometric identity component* of $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})$.

Example 2.2. If $Q \cong \mathcal{O}^{\oplus s}$ for some integer $s \geq 0$, then $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ = \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}}) \cong G^{\times s}_{\mathsf{M}_{\mathcal{H}}}$ is the s-fold fiber product of $G_{\mathsf{M}_{\mathcal{H}}}$ over $\mathsf{M}_{\mathcal{H}}$.

Example 2.3. If $\mathcal{O} \cong \mathrm{M}_k(\mathcal{O}_F)$ and Q is of finite index in $\mathcal{O}_F^{\oplus k}$ for some integer $k \geq 1$, then the relative dimension of $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ$ over $\mathsf{M}_{\mathcal{H}}$ is $1/k$ of the relative dimension of $G_{\mathsf{M}_{\mathcal{H}}}$ over $\mathsf{M}_{\mathcal{H}}$.

Definition 2.4. *A* **PEL-type Kuga family** *over $\mathsf{M}_{\mathcal{H}}$ is an abelian scheme $\mathsf{N} \to \mathsf{M}_{\mathcal{H}}$ that is $\mathbb{Z}^\times_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ$ for some $\mathcal{O}$-lattice Q.*

Consider $\mathrm{Diff}^{-1} = \mathrm{Diff}^{-1}_{\mathcal{O}/\mathbb{Z}}$, the inverse different of $\mathcal{O}$ over $\mathbb{Z}$ [38, Def. 1.1.1.11] with its canonical left $\mathcal{O}$-module structure. Since the trace pairing $\mathrm{Diff}^{-1} \times \mathcal{O} \to \mathbb{Z} : (y, x) \mapsto \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(yx)$ is perfect by definition, for each $\mathcal{O}$-lattice Q, we may identify $Q^\vee := \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{Z})$ with $\mathrm{Hom}_{\mathcal{O}}(Q, \mathrm{Diff}^{-1})$. By composition with the involution $\star : \mathcal{O} \xrightarrow{\sim} \mathcal{O}^{\mathrm{op}}$, the natural right action of $\mathcal{O}$ on Diff^{-1} induced a left action of $\mathcal{O}$ on Diff^{-1}, which commutes with the natural left action of $\mathcal{O}$ on Diff^{-1}. Accordingly, the $\mathbb{Z}$-module $Q^\vee$ is torsion-free and has a canonical left $\mathcal{O}$-structure induced by the right action of $\mathcal{O}^{\mathrm{op}}$ on Diff^{-1} (and $\star : \mathcal{O} \xrightarrow{\sim} \mathcal{O}^{\mathrm{op}}$). In other words, $Q^\vee$ is an

$\mathcal{O}$-lattice. Then the trace pairing induces a perfect pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_Q : Q^\vee \times Q \to \mathbb{Z} : (f,x) \mapsto \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(f(x)).$$

For any $b \in \mathcal{O}$, $f \in Q^\vee$, and $x \in Q$, we have

$$\langle bf, x\rangle_Q = \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(f(x)b^\star) = \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(b^\star f(x)) = \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(f(b^\star x)) = \langle f, b^\star x\rangle.$$

Lemma 2.5. *There exists an embedding $j_Q : Q^\vee \hookrightarrow Q$ of $\mathcal{O}$-lattices inducing an isomorphism $j_Q : Q^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Q \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$ of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$-modules such that the pairing*

$$\langle j_Q^{-1}(\,\cdot\,),\,\cdot\,\rangle_Q : (Q \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Q \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}$$

is positive definite.

Proof. By the explicit classification [38, (1.2.1.10), Prop. 1.2.1.13, and Lem. 1.2.1.23], there exists an isomorphism $j_{Q,0} : Q^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{R} \xrightarrow{\sim} Q \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$-modules such that the induced pairing $\langle j_{Q,0}^{-1}(\,\cdot\,),\,\cdot\,\rangle_Q : (Q \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Q \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}$ is positive definite. If $\Box$ is the set of all rational prime numbers, then necessarily $\mathcal{O} = \mathbb{Z}$, and the lemma is clear. Otherwise, we know that $\mathrm{Isom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}}(Q^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}, Q \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)})$ is dense in $\mathrm{Isom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(Q^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{R}, Q \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ (with the topology induced by $\mathbb{R}$). Hence there exists an element $j_{Q,1} : Q^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Q \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$ close to $j_{Q,0}$ in $\mathrm{Isom}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(Q^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{R}, Q \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ such that the induced pairing $\langle j_{Q,1}^{-1}(\,\cdot\,),\,\cdot\,\rangle_Q : (Q \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Q \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}$ is still positive definite. By multiplying $j_{Q,1}$ by a positive element in $\mathbb{Z}_{(\Box)}^\times$, we may assume that it maps $Q^\vee$ to a submodule of Q. Then the induced morphism $j_Q : Q^\vee \to Q$ satisfies the requirement of the lemma. $\Box$

Lemma 2.6. *The abelian scheme $\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}})$ is isomorphic to the dual abelian scheme of $\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}})$.*

Proof. Let s be the common rank of Q and $Q^\vee$ as free $\mathbb{Z}$-modules. Let $\{e_1, \ldots, e_s\}$ be a $\mathbb{Z}$-basis of Q, and let $\{e_1^\vee, \ldots, e_s^\vee\}$ be the dual $\mathbb{Z}$-basis of $Q^\vee$, such that $e_i^\vee(e_j) = \delta_{ij}$ for any $1 \leq i, j \leq s$. Then the choices of bases define canonical isomorphisms

$$\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}}) \cong G_{\mathsf{M}_{\mathcal{H}}}^{\times s} \tag{2.7}$$

and

$$\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}}) \cong (G^\vee_{\mathsf{M}_{\mathcal{H}}})^{\times s}. \tag{2.8}$$

As a result, $\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}}) \cong G_{\mathsf{M}_{\mathcal{H}}}^{\times s}$ is isomorphic to the dual abelian scheme of $\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}}) \cong (G^\vee_{\mathsf{M}_{\mathcal{H}}})^{\times s}$. $\Box$

Lemma 2.9. *Let $j_Q : Q^\vee \hookrightarrow Q$ be as in Lemma 2.5. Then the isogeny*

$$\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q, \mathbb{Z}} : \underline{\mathrm{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}}) \to \underline{\mathrm{Hom}}_{\mathbb{Z}}(Q^\vee, G^\vee_{\mathsf{M}_{\mathcal{H}}})$$

induced canonically by j_Q and $\lambda_{\mathsf{M}_{\mathcal{H}}} : G_{\mathsf{M}_{\mathcal{H}}} \to G^\vee_{\mathsf{M}_{\mathcal{H}}}$, which is of degree prime to $\Box$ because both $[Q : j_Q(Q^\vee)]$ and $\deg(\lambda_{\mathsf{M}_{\mathcal{H}}})$ are prime to $\Box$, is a polarization.

Proof. We need to show that the invertible sheaf

$$(\mathrm{Id}_{\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}})}, \lambda_{\mathsf{M}_{\mathcal{H}}, j_Q, \mathbb{Z}})^* \mathcal{P}_{\underline{\mathrm{Hom}}_{\mathbb{Z}}(Q, G_{\mathsf{M}_{\mathcal{H}}})}$$

is relative ample over $\mathsf{M}_{\mathcal{H}}$. Using the choice of basis $\{e_1, \ldots, e_s\}$ (resp. $\{e_1^\vee, \ldots, e_s^\vee\}$) of Q (resp. $Q^\vee$) as in the proof of Lemma 2.6, the morphism j_Q can be represented by $e_i^\vee \mapsto \sum_{1\leq j\leq s} a_{ij}e_j$ for some integers a_{ij}, for each $1 \leq i \leq s$. These integers form a positive definite matrix $a = (a_{ij})$, because the induced pairing $\langle j_Q^{-1}(\,\cdot\,), \,\cdot\,\rangle_Q : (Q \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Q \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}$ is positive definite. By completion of squares for quadratic forms, we know that there exist an integer $m \geq 1$ such that $ma = ud\,{}^t u$ for some matrices d and u with integral coefficients, where $d = \mathrm{diag}(d_1, \ldots, d_s)$ is diagonal with positive entries. As a result, the morphism $m\lambda_{\mathsf{M}_\mathcal{H},j_Q,\mathbb{Z}}$ factors as a composition

$$m\lambda_{\mathsf{M}_\mathcal{H},j_Q,\mathbb{Z}} = [\,{}^t u]^* \circ \lambda_{\mathsf{M}_\mathcal{H},d,\mathbb{Z}} \circ [u]^*$$

of morphisms

$$[u]^* : \underline{\mathrm{Hom}}_\mathbb{Z}(Q, G_{\mathsf{M}_\mathcal{H}}) \to \underline{\mathrm{Hom}}_\mathbb{Z}(Q, G_{\mathsf{M}_\mathcal{H}}),$$
$$\lambda_{\mathsf{M}_\mathcal{H},d,\mathbb{Z}} : \underline{\mathrm{Hom}}_\mathbb{Z}(Q, G_{\mathsf{M}_\mathcal{H}}) \to \underline{\mathrm{Hom}}_\mathbb{Z}(Q^\vee, G^\vee_{\mathsf{M}_\mathcal{H}}),$$
$$[\,{}^t u]^* : \underline{\mathrm{Hom}}_\mathbb{Z}(Q^\vee, G^\vee_{\mathsf{M}_\mathcal{H}}) \to \underline{\mathrm{Hom}}_\mathbb{Z}(Q^\vee, G^\vee_{\mathsf{M}_\mathcal{H}}).$$

If we identify $\underline{\mathrm{Hom}}_\mathbb{Z}(Q, G_{\mathsf{M}_\mathcal{H}})$ and $\underline{\mathrm{Hom}}_\mathbb{Z}(Q^\vee, G^\vee_{\mathsf{M}_\mathcal{H}})$ as dual abelian schemes of each other using the canonical isomorphisms (2.7) and (2.8) defined by the dual bases $\{e_1, \ldots, e_s\}$ and $\{e_1^\vee, \ldots, e_s^\vee\}$, then $[\,{}^t u]^* = ([u]^*)^\vee$, and $\lambda_{\mathsf{M}_\mathcal{H},d,\mathbb{Z}} = (d_1\lambda_{\mathsf{M}_\mathcal{H}}) \times (d_2\lambda_{\mathsf{M}_\mathcal{H}}) \times \ldots \times (d_s\lambda_{\mathsf{M}_\mathcal{H}}) : G^{\times s}_{\mathsf{M}_\mathcal{H}} \to (G^\vee_{\mathsf{M}_\mathcal{H}})^{\times s}$ is a polarization. Since $[u]^*$ is finite, this implies that $\lambda_{\mathsf{M}_\mathcal{H},j_Q,\mathbb{Z}}$ is also a polarization, as desired. $\square$

Proposition 2.10. *The abelian scheme $\underline{\mathrm{Hom}}_\mathcal{O}(Q^\vee, G^\vee_{\mathsf{M}_\mathcal{H}})^\circ$ is $\mathbb{Z}^\times_{(\Box)}$-isogenous to the dual abelian scheme of $\underline{\mathrm{Hom}}_\mathcal{O}(Q, G_{\mathsf{M}_\mathcal{H}})^\circ$.*

Proof. Since $\lambda_{\mathsf{M}_\mathcal{H},j_Q,\mathbb{Z}}$ is a polarization by Lemma 2.9, the induced morphism

$$\begin{aligned}\lambda_{\mathsf{M}_\mathcal{H},j_Q} :\ & \underline{\mathrm{Hom}}_\mathcal{O}(Q, G_{\mathsf{M}_\mathcal{H}})^\circ \hookrightarrow \underline{\mathrm{Hom}}_\mathbb{Z}(Q, G_{\mathsf{M}_\mathcal{H}}) \\ & \overset{\lambda_{\mathsf{M}_\mathcal{H},j_Q,\mathbb{Z}}}{\to} \underline{\mathrm{Hom}}_\mathbb{Z}(Q^\vee, G^\vee_{\mathsf{M}_\mathcal{H}}) \twoheadrightarrow (\underline{\mathrm{Hom}}_\mathcal{O}(Q, G_{\mathsf{M}_\mathcal{H}})^\circ)^\vee\end{aligned} \tag{2.11}$$

is also a polarization. (Since the condition of being a polarization can be checked fiber by fiber [14, 1.2, 1.3, 1.4], it suffices to note that the restriction of an ample invertible sheaf to a closed subscheme is again ample.) Since $\lambda_{\mathsf{M}_\mathcal{H},j_Q,\mathbb{Z}}$ maps $\underline{\mathrm{Hom}}_\mathcal{O}(Q, G_{\mathsf{M}_\mathcal{H}})^\circ$ onto the subscheme $\underline{\mathrm{Hom}}_\mathcal{O}(Q^\vee, G^\vee_{\mathsf{M}_\mathcal{H}})^\circ$ of $\underline{\mathrm{Hom}}_\mathbb{Z}(Q^\vee, G^\vee_{\mathsf{M}_\mathcal{H}})$, we obtain an isogeny

$$\underline{\mathrm{Hom}}_\mathcal{O}(Q^\vee, G^\vee_{\mathsf{M}_\mathcal{H}})^\circ \to (\underline{\mathrm{Hom}}_\mathcal{O}(Q, G_{\mathsf{M}_\mathcal{H}})^\circ)^\vee.$$

The degree of this isogeny is prime to $\Box$ because $\lambda_{\mathsf{M}_\mathcal{H},j_Q,\mathbb{Z}}$ is. $\square$

Corollary 2.12 (of the proof of Proposition 2.10)**.** *Let $j_Q : Q^\vee \hookrightarrow Q$ be as in Lemma 2.5. Then the canonical morphism*

$$\lambda_{\mathsf{M}_\mathcal{H},j_Q} : \underline{\mathrm{Hom}}_\mathcal{O}(Q, G_{\mathsf{M}_\mathcal{H}})^\circ \to (\underline{\mathrm{Hom}}_\mathcal{O}(Q, G_{\mathsf{M}_\mathcal{H}})^\circ)^\vee$$

induced by j_Q and $\lambda_{\mathsf{M}_\mathcal{H}} : G_{\mathsf{M}_\mathcal{H}} \to G^\vee_{\mathsf{M}_\mathcal{H}}$ (as in (2.11)) is a polarization of degree prime to $\Box$.

Corollary 2.13. *If a Kuga family $\mathsf{N} \to \mathsf{M}_\mathcal{H}$ is $\mathbb{Z}^\times_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_\mathcal{O}(Q, G_{\mathsf{M}_\mathcal{H}})^\circ$ for some $\mathcal{O}$-lattice Q, then we have canonical isomorphisms over $\mathsf{M}_\mathcal{H}$:*

$$\underline{\mathrm{Lie}}_{\mathsf{N}/\mathsf{M}_\mathcal{H}} \cong \underline{\mathrm{Hom}}_\mathcal{O}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}), \qquad \underline{\mathrm{Lie}}_{\mathsf{N}^\vee/\mathsf{M}_\mathcal{H}} \cong \underline{\mathrm{Hom}}_\mathcal{O}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}),$$
$$\underline{\mathrm{Lie}}^\vee_{\mathsf{N}/\mathsf{M}_\mathcal{H}} \cong \underline{\mathrm{Hom}}_\mathcal{O}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}), \qquad \underline{\mathrm{Lie}}^\vee_{\mathsf{N}^\vee/\mathsf{M}_\mathcal{H}} \cong \underline{\mathrm{Hom}}_\mathcal{O}(Q, \underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}).$$

Remark 2.14. We do not need to choose a polarization $\mathsf{N} \to \mathsf{N}^\vee$ in the isomorphisms in Corollary 2.13. The sheaves on the right-hand sides of the isomorphisms are locally free because the order $\mathcal{O}$ is maximal at any good prime (see Definition 1.5 and [38, Prop. 1.1.1.17]), and because lattices over maximal orders are projective modules (see [38, Prop. 1.1.1.20]).

2B. **Main theorem.** (Convention 2.1 will persist until the end of this article.)

Theorem 2.15. *Let Q be any $\mathcal{O}$-lattice. Suppose that $\mathcal{H}$ is* **neat** (*as in Definition* 1.2)*, so that the moduli problem $\mathsf{M}_{\mathcal{H}}$ it defines is representable by a quasiprojective scheme, and so that $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} = \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ is a proper smooth algebraic space over S_0. Then there is a set $\mathbf{K}_{Q,\mathcal{H},\Sigma}$, equipped with a reflexive and transitive binary relation $\succ$, parameterizing the following data:*

(1) *For each $\kappa \in \mathbf{K}_{Q,\mathcal{H},\Sigma}$, there is a $\mathbb{Z}^\times_{(\Box)}$-isogeny $\kappa^{\mathrm{isog}} : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ \to \mathsf{N}_\kappa$ over $\mathsf{M}_{\mathcal{H}}$, together with an open immersion $\kappa^{\mathrm{tor}} : \mathsf{N}_\kappa \hookrightarrow \mathsf{N}^{\mathrm{tor}}_\kappa$ of schemes over S_0, such that the scheme $\mathsf{N}^{\mathrm{tor}}_\kappa$ is* **projective** *and* **smooth** *over S_0, and that the complement of N_κ in $\mathsf{N}^{\mathrm{tor}}_\kappa$* (*with its reduced structure*) *is a relative Cartier divisor $\mathsf{E}_{\infty,\kappa}$ with simple normal crossings.*

   For each relation $\kappa' \succ \kappa$ in $\mathbf{K}_{Q,\mathcal{H},\Sigma}$, there is a proper log étale morphism $f^{\mathrm{tor}}_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa'} \to \mathsf{N}^{\mathrm{tor}}_\kappa$ extending the canonical $\mathbb{Z}^\times_{(\Box)}$-isogeny

$$f_{\kappa',\kappa} := \kappa^{\mathrm{isog}} \circ ((\kappa')^{\mathrm{isog}})^{-1} : \mathsf{N}_{\kappa'} \to \mathsf{N}_\kappa$$

   such that $R^i(f^{\mathrm{tor}}_{\kappa',\kappa})_* \mathscr{O}_{\mathsf{N}^{\mathrm{tor}}_{\kappa'}} = 0$ for $i > 0$.

(2) *For each $\kappa \in \mathbf{K}_{Q,\mathcal{H},\Sigma}$, the structural morphism $f_\kappa : \mathsf{N}_\kappa \to \mathsf{M}_{\mathcal{H}}$ extends* (*necessarily uniquely*) *to a morphism $f^{\mathrm{tor}}_\kappa : \mathsf{N}^{\mathrm{tor}}_\kappa \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, which is* **proper** *and* **log smooth** (*as in* [33, 3.3] *and* [32, 1.6]) *if we equip $\mathsf{N}^{\mathrm{tor}}_\kappa$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ with the canonical* (*fine*) *log structures given respectively by the relative Cartier divisors with* (*simple*) *normal crossings $\mathsf{E}_{\infty,\kappa}$ and $\mathsf{D}_{\infty,\mathcal{H}}$* (*see* (1) *above and* (3) *of Theorem* 1.41)*. Then we have the following commutative diagram:*

$$\begin{tikzcd}
\mathsf{N}_\kappa \arrow[r, hook, "+NCD"] \arrow[d, "f_\kappa", "\substack{proper\\ smooth}"'] & \mathsf{N}^{\mathrm{tor}}_\kappa \arrow[d, "f^{\mathrm{tor}}_\kappa", "\substack{proper\\ log\ smooth}"'] \arrow[dr, "\substack{projective\\ smooth}"] & \\
\mathsf{M}_{\mathcal{H}} \arrow[r, hook, "+NCD"'] & \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \arrow[r, "\substack{proper\\ smooth}"'] & \mathsf{S}_0
\end{tikzcd}$$

   If $\kappa' \succ \kappa$, then we have the compatibility $f^{\mathrm{tor}}_{\kappa'} = f^{\mathrm{tor}}_\kappa \circ f^{\mathrm{tor}}_{\kappa',\kappa}$.

(3) *Let us fix a choice of $\kappa \in \mathbf{K}_{Q,\mathcal{H},\Sigma}$ and suppress the subscript κ from the notation.* (*All canonical isomorphisms will be required to be compatible with the canonical isomorphisms defined by pullback under $f^{\mathrm{tor}}_{\kappa',\kappa}$ for each relation $\kappa' \succ \kappa$.*) *Then the following are true:*
   (a) *Let $\Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}[d\log\infty]$ and $\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty]$ denote the sheaves of modules of log 1-differentials over S_0 given by the* (*respective*) *canonical log structures defined in* (2)*. Let*

$$\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} := (\Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}[d\log\infty])/((f^{\mathrm{tor}})^*(\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty])).$$

   Then there is a canonical isomorphism

$$(f^{\mathrm{tor}})^*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})) \cong \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \tag{2.16}$$

between locally free sheaves over $\mathsf{N}^{\mathrm{tor}}$*, extending the composition of canonical isomorphisms*

$$f^*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})) \cong f^*\underline{\mathrm{Lie}}^\vee_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \cong \Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \tag{2.17}$$

over N.

(b) *For any integer* $b \geq 0$*, there exists a canonical isomorphism*

$$\begin{aligned} R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \\ \cong (\wedge^b(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} (\wedge^a(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))). \end{aligned} \tag{2.18}$$

of locally free sheaves over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$*, compatible with cup products and exterior products, extending the canonical isomorphism over* $\mathsf{M}_{\mathcal{H}}$ *induced by the composition of canonical isomorphisms*

$$R^b f_*(\mathscr{O}_{\mathsf{N}}) \cong \wedge^b \underline{\mathrm{Lie}}_{\mathsf{N}^\vee/\mathsf{M}_{\mathcal{H}}} \cong \wedge^b(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})). \tag{2.19}$$

(c) *Let* $\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} := \wedge^\bullet \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ *be the log de Rham complex associated with* $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ (*with differentials inherited from* $\Omega^\bullet_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}$). *Let the* (*relative*) **log de Rham cohomology** *be defined by*

$$\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) := R^i f^{\mathrm{tor}}_*(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}).$$

Then the (*relative*) **Hodge spectral sequence**

$$E_1^{a,b} := R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \Rightarrow \underline{H}^{a+b}_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \tag{2.20}$$

degenerates at E_1 *terms, and defines a* **Hodge filtration** *on* $\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$ *with locally free graded pieces given by* $R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ *for integers* $a + b = i$*, extending the canonical Hodge filtration on* $\underline{H}^i_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}})$.
As a result, for any integer $i \geq 0$*, there is a canonical isomorphism*

$$\wedge^i \underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \xrightarrow{\sim} \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}),$$

compatible with the Hodge filtrations defined by (2.20)*, extending the canonical isomorphism* $\wedge^i \underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \xrightarrow{\sim} \underline{H}^i_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}})$ *over* $\mathsf{M}_{\mathcal{H}}$ (*defined by cup product*).

(d) *For any* $j_Q : Q^\vee \hookrightarrow Q$ *as in Lemma* 2.5*, the* $\mathbb{Z}^\times_{(\Box)}$*-polarization* $\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q} : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ \to (\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ)^\vee$ *in Corollary* 2.12 *defines canonically* (*as in* [14, 1.5]) *a perfect pairing*

$$\langle \cdot, \cdot \rangle_{\lambda_{\mathsf{M}_{\mathcal{H}}, j_Q}} : \underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1).$$

Then $\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$ *is the unique subsheaf of*

$$(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*(\underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}))$$

satisfying the following conditions:

(i) $\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$ *is locally free of finite rank over* $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$.

(ii) *The sheaf* $f^{\mathrm{tor}}_*(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ *can be identified as the subsheaf of*

$$(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*(f_*(\Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}))$$

formed (locally) by sections that are also sections of $\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}})$. *(Here we view all sheaves canonically as subsheaves of* $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\text{tor}}_{\mathcal{H}})_*(\underline{H}^1_{\text{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}))$.)

(iii) $\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}})$ *is self-dual under the push-forward*

$$(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\text{tor}}_{\mathcal{H}})_*\langle\,\cdot\,,\,\cdot\,\rangle_{\lambda_{\mathsf{M}_{\mathcal{H}}},j_Q}.$$

(e) *The Gauss–Manin connection*

$$\nabla : \underline{H}^\bullet_{\text{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \to \underline{H}^\bullet_{\text{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0} \tag{2.21}$$

extends to an integrable connection

$$\nabla : \underline{H}^\bullet_{\text{log-dR}}(\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}}) \to \underline{H}^\bullet_{\text{log-dR}}(\mathsf{N}^{\text{tor}}/\mathsf{M}^{\text{tor}}_{\mathcal{H}}) \underset{\mathscr{O}_{\mathsf{M}^{\text{tor}}_{\mathcal{H}}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\text{tor}}_{\mathcal{H}}/\mathsf{S}_0} \tag{2.22}$$

with log poles along $\mathsf{D}_{\infty,\mathcal{H}}$*, called the* **extended Gauss–Manin connection***, satisfying the usual Griffiths transversality with the Hodge filtration defined by* (2.20).

(4) *(Hecke actions.) Suppose we have an element* $g_h \in \mathrm{G}(\mathbb{A}^{\infty,\Box})$*, and suppose we have a (neat) open compact subgroup* $\mathcal{H}'$ *of* $\mathrm{G}(\hat{\mathbb{Z}}^\Box)$ *such that* $g_h^{-1}\mathcal{H}'g_h \subset \mathcal{H}$. *Suppose* $\Sigma' = \{\Sigma'_{\Phi'_{\mathcal{H}'}}\}$ *is a compatible choice of admissible smooth rational polyhedral cone decomposition data for* $\mathsf{M}_{\mathcal{H}'}$*, which* g_h*-refines* Σ *as in* [38, Def. 6.4.3.3]. *(The notion was called "dominance" in the original version, but changed to the more common "refinement" in the revision.) Then there is also a set* $\mathbf{K}_{Q,\mathcal{H}',\Sigma'}$*, equipped with a reflexive and transitive binary relation* $\succ$ *as* $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ *is, parameterizing (for* $\kappa' \in \mathbf{K}_{Q,\mathcal{H}',\Sigma'}$*)* $\mathbb{Z}^\times_{(\Box)}$*-isogenies* $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}'}})^\circ \to \mathsf{N}'_{\kappa'}$ *over* $\mathsf{M}_{\mathcal{H}'}$*, together with open immersions* $\mathsf{N}'_{\kappa'} \hookrightarrow (\mathsf{N}'_{\kappa'})^{\text{tor}}$ *of schemes over* S_0*, satisfying analogues of properties* (1), (2)*, and* (3) *above. The constructions of* $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ *and* $\mathbf{K}_{Q,\mathcal{H}',\Sigma'}$ *(and the objects they parameterize) satisfy the compatibility with* g_h *in the sense that, for each* $\kappa \in \mathbf{K}_{Q,\mathcal{H},\Sigma}$*, there is an element* $\kappa' \in \mathbf{K}_{Q,\mathcal{H}',\Sigma'}$ *such that the following are true:*

(a) *There exists a (necessarily unique) finite étale morphism* $[g_h]_{\kappa',\kappa} : \mathsf{N}'_{\kappa'} \to \mathsf{N}_\kappa$ *covering the morphism* $[g_h] : \mathsf{M}_{\mathcal{H}'} \to \mathsf{M}_{\mathcal{H}}$ *given by* [38, Prop. 6.4.3.4]*, inducing a prime-to-*$\Box$ *isogeny* $\mathsf{N}'_{\kappa'} \to \mathsf{N}_\kappa \underset{\mathsf{M}_{\mathcal{H}}}{\times} \mathsf{M}_{\mathcal{H}'}$*, which agrees with the* $\mathbb{Z}^\times_{(\Box)}$*-isogeny induced by* $(\kappa')^{\text{isog}}$*,* κ^{isog}*, and the* $\mathbb{Z}^\times_{(\Box)}$*-isogeny* $G_{\mathsf{M}_{\mathcal{H}'}} \to G_{\mathsf{M}_{\mathcal{H}}} \underset{\mathsf{M}_{\mathcal{H}}}{\times} \mathsf{M}_{\mathcal{H}'}$ *realizing* $G_{\mathsf{M}_{\mathcal{H}}} \underset{\mathsf{M}_{\mathcal{H}}}{\times} \mathsf{M}_{\mathcal{H}'}$ *as a Hecke twist of* $G_{\mathsf{M}_{\mathcal{H}'}}$ *by* g_h. *(Here all the base changes from* $\mathsf{M}_{\mathcal{H}}$ *to* $\mathsf{M}_{\mathcal{H}'}$ *use the morphism* $[g_h]$.)

(b) *There exists a (necessarily unique) proper log étale morphism*

$$[g_h]^{\text{tor}}_{\kappa',\kappa} : (\mathsf{N}'_{\kappa'})^{\text{tor}} \to \mathsf{N}^{\text{tor}}_\kappa \tag{2.23}$$

extending the morphism $[g_h]_{\kappa',\kappa}$ *and covering the morphism* $[g_h]^{\text{tor}} : \mathsf{M}^{\text{tor}}_{\mathcal{H}',\Sigma'} \to \mathsf{M}^{\text{tor}}_{\mathcal{H},\Sigma}$ *given by* [38, Prop. 6.4.3.4]*, such that*

$$R^i([g_h]^{\text{tor}}_{\kappa',\kappa})_*\mathscr{O}_{(\mathsf{N}'_{\kappa'})^{\text{tor}}} = 0 \tag{2.24}$$

for any $i > 0$.

(c) *There is a canonical isomorphism*

$$([g_h]^{\mathrm{tor}})^* \underline{H}^{a+b}_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}_\kappa/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}) \xrightarrow{\sim} \underline{H}^{a+b}_{\text{log-dR}}((\mathsf{N}'_{\kappa'})^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma'})$$

respecting the Hodge filtrations and compatible with the canonical isomorphisms

$$([g_h]^{\mathrm{tor}}_{\kappa',\kappa})^* \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}_\kappa/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \xrightarrow{\sim} \overline{\Omega}^1_{(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma'}},$$
$$([g_h]^{\mathrm{tor}})^* \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \xrightarrow{\sim} \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma'}},$$
$$([g_h]^{\mathrm{tor}})^* \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}} \xrightarrow{\sim} \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}',\Sigma'}},$$

and the canonical isomorphisms in (3) *for* $\mathsf{N}^{\mathrm{tor}}_\kappa$ *and* $(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}$.

(5) ($\mathbb{Z}^\times_{(\Box)}$*-isogenies.*) *Let* g_l *be an element of* $\mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{A}^{\infty,\Box}}(Q \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box})$. *Then the submodule* $g_l(Q \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box)$ *in* $Q \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}$ *determines a unique* $\mathcal{O}$*-lattice* Q' (*up to isomorphism*), *together with a unique choice of an isomorphism* $[g_l]_Q : Q \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)} \xrightarrow{\sim} Q' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$, *inducing an isomorphism* $Q \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box} \xrightarrow{\sim} Q' \underset{\mathbb{Z}}{\otimes} \mathbb{A}^{\infty,\Box}$ *matching* $g_l(Q \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box)$ *with* $Q' \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$, *and inducing a canonical* $\mathbb{Z}^\times_{(\Box)}$*-isogeny*

$$[g_l]^*_Q : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q', G_{\mathsf{M}_\mathcal{H}})^\circ \to \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_\mathcal{H}})^\circ$$

defined by $[g_l]_Q$. *For* $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q', G_{\mathsf{M}_\mathcal{H}})^\circ$, *there is also a set* $\mathbf{K}_{Q',\mathcal{H},\Sigma}$, *equipped with a reflexive and transitive binary relation* $\succ$ *as* $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ *is, parameterizing* (*for* $\kappa' \in \mathbf{K}_{Q',\mathcal{H},\Sigma}$) $\mathbb{Z}^\times_{(\Box)}$*-isogenies*

$$\underline{\mathrm{Hom}}_{\mathcal{O}}(Q', G_{\mathsf{M}_\mathcal{H}})^\circ \to \mathsf{N}'_{\kappa'}$$

over $\mathsf{M}_\mathcal{H}$, *together with open immersions* $\mathsf{N}'_{\kappa'} \hookrightarrow (\mathsf{N}'_{\kappa'})^{\mathrm{tor}}$ *of schemes over* S_0, *satisfying analogues of properties* (1), (2), *and* (3) *above. The constructions of* $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ *and* $\mathbf{K}_{Q',\mathcal{H},\Sigma}$ (*and the objects they parameterize*) *satisfy the compatibility with* g_l *in the sense that, for each* $\kappa \in \mathbf{K}_{Q,\mathcal{H},\Sigma}$, *there is an element* $\kappa' \in \mathbf{K}_{Q',\mathcal{H},\Sigma}$ *such that the following are true:*

(a) *The* $\mathbb{Z}^\times_{(\Box)}$*-isogeny* $[g_l]^*_{\kappa',\kappa} := \kappa^{\mathrm{isog}} \circ [g_l]^*_Q \circ ((\kappa')^{\mathrm{isog}})^{-1} : \mathsf{N}'_{\kappa'} \to \mathsf{N}_\kappa$ *is an* **isogeny** (*not just a quasiisogeny*), *and hence defines a finite étale morphism.*

(b) *There exists a* (*necessarily unique*) *proper log étale morphism*

$$([g_l]^*_{\kappa',\kappa})^{\mathrm{tor}} : (\mathsf{N}'_{\kappa'})^{\mathrm{tor}} \to \mathsf{N}^{\mathrm{tor}}_\kappa \tag{2.25}$$

extending the morphism $[g_l]^*_{\kappa',\kappa}$ *over* $\mathsf{M}_\mathcal{H}$, *such that*

$$R^i([g_l]^*_{\kappa',\kappa})^{\mathrm{tor}}_* \mathscr{O}_{(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}} = 0 \tag{2.26}$$

for any $i > 0$.

(c) *For any integer* $i \geq 0$, *there is a canonical isomorphism*

$$(([g_l]^*_{\kappa',\kappa})^{\mathrm{tor}})^* : \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}_\kappa/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}) \xrightarrow{\sim} \underline{H}^i_{\text{log-dR}}((\mathsf{N}'_{\kappa'})^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma})$$

extending the canonical isomorphism

$$([g_l]^*_{\kappa',\kappa})^* : \underline{H}^i_{\mathrm{dR}}(\mathsf{N}_\kappa/\mathsf{M}_\mathcal{H}) \xrightarrow{\sim} \underline{H}^i_{\mathrm{dR}}(\mathsf{N}'_{\kappa'}/\mathsf{M}_\mathcal{H})$$

induced by $[g_l]_Q$*, respecting the Hodge filtrations and inducing canonical isomorphisms*

$$(([g_l]^*_{\kappa',\kappa})^{\mathrm{tor}})^* : R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}_\kappa/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \xrightarrow{\sim} R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$$

(*for integers* $a+b=i$) *compatible* (*under the canonical isomorphisms in* (3) *for* $\mathsf{N}^{\mathrm{tor}}_\kappa$ *and* $(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}$) *with the canonical isomorphisms*

$$([g_l]^*_Q)^* : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}}((Q')^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$$

and

$$([g_l]^*_Q)^* : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}}((Q')^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}).$$

2C. **Outline of the proof.** The proof of Theorem 2.15 consists of the following steps:

(1) Find a PEL-type $\mathcal{O}$-lattice $(\widetilde{L}, \langle \cdot, \cdot \rangle^\sim, \widetilde{h})$, a fully symplectic admissible filtration $\widetilde{\mathsf{Z}}$ on $\widetilde{L} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\square$, a torus argument $\widetilde{\Phi}$, and a splitting $\widetilde{\delta}$ for $\widetilde{\mathsf{Z}}$, such that, for some choices of $\widetilde{\mathcal{H}}$, $\widetilde{\Sigma}$, and $\widetilde{\sigma}$, the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}$ of the toroidal compactification $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}} = \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ has a canonical structure of an abelian scheme over $\mathsf{M}_{\mathcal{H}}$, and such that there exists a canonical $\mathbb{Z}^\times_{(\square)}$-isogeny

$$\kappa^{\mathrm{isog}} : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ \to \widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}.$$

Then we take N_κ to be this $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}$.

Take $\mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$ to be the set of all such triples $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$, with the binary relation

$$\kappa' = (\widetilde{\mathcal{H}}', \widetilde{\Sigma}', \widetilde{\sigma}') \succ \kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$$

defined when $\widetilde{\mathcal{H}}' \subset \widetilde{\mathcal{H}}$ and $\widetilde{\Sigma}'$ refines $\widetilde{\Sigma}$ as in [38, Def. 6.4.2.8], and when the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}'}, \widetilde{\delta}_{\widetilde{\mathcal{H}}'}, \widetilde{\sigma}')]$-stratum of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'}$ is mapped (surjectively) to the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}} = \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ under the canonical morphism $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ given by [38, Prop. 6.4.2.9].

For $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$, take $\mathsf{N}^{\mathrm{tor}}_\kappa$ to be the closure of the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$. For $\kappa' = (\widetilde{\mathcal{H}}', \widetilde{\Sigma}', \widetilde{\sigma}') \succ \kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$, the morphism $f^{\mathrm{tor}}_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa'} \to \mathsf{N}^{\mathrm{tor}}_\kappa$ is just the morphism induced by the canonical proper morphism $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ given by [38, Prop. 6.4.2.9].

(2) Show that $\mathsf{N}^{\mathrm{tor}}_\kappa$ is projective and smooth over S_0 for $\kappa \in \mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$.

(3) Find a condition on κ that guarantees the existence of a morphism $f^{\mathrm{tor}}_\kappa : \mathsf{N}^{\mathrm{tor}}_\kappa \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ extending the structural morphism $f_\kappa : \mathsf{N}_\kappa \to \mathsf{M}_{\mathcal{H}}$.

(4) Take $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ to be the subset of $\mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$ consisting of elements κ satisfying the condition we have found. Show that this subset is nonempty and has an induced binary relation $\succ$; note that the conditions we need can always be achieved after suitable refinements of cone decompositions. This verifies (1) and (2) of Theorem 2.15.

(5) For each $\kappa \in \mathbf{K}_{Q,\mathcal{H},\Sigma}$, verify that the morphism $f^{\mathrm{tor}}_\kappa : \mathsf{N}^{\mathrm{tor}}_\kappa \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ extending $\mathsf{N}_\kappa \to \mathsf{M}_{\mathcal{H}}$ is log smooth, and verify (3a) of Theorem 2.15.

(6) Assuming (3b) and (3c), verify (4) and (5) of Theorem 2.15 using the Hecke actions on the double tower $\{\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}\}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$.

(7) Verify (3b), (3c), and (3d) of Theorem 2.15 using explicit descriptions of the formal fibers of f_κ^{tor} along (locally closed) strata of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$. (A crucial step for (3b) requires the notion of log extensions of polarizations we mentioned in the introduction.)

We will carry out these steps in Sections 3–5. We will make frequent references to results cited in Section 1, and also to the original statements in [38].

2D. **System of notation.** Although the underlying ideas are simple, the notation can be quite heavy. (This seems unavoidable in general works on compactifications.) We decided to keep the notation informative (and hence complicated), because we believe it is more difficult to keep track of three sets of cusp labels and cone decompositions with simplified notation. We understand that the heaviness of notation will inevitably be an enormous burden on the readers, and hence we would like to provide some guidance by explaining the key features in the system of notation, as follows:

- The superscript $^{\mathrm{tor}}$ stands for *toroidal compactifications* (or objects related to them). For morphisms this typically means extensions to morphisms between toroidal compactifications.
- Depending on the context, the overlines can have different meanings:
  - For geometric objects they almost always mean *closures*.
  - For sheaves of differentials (or related objects) they mean the *log versions*.
  - Notable exceptions (to the above two) are in Sections 3B–3C below, where overlines can also stand for *quotients* of group schemes or sheaves.
- Objects for the "given" moduli problem $\mathsf{M}_{\mathcal{H}}$ and its compactifications are denoted as in Section 1.
- Objects for the "larger" moduli problem $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ (mentioned in step 1 above) will be denoted with either $\sim$ (tilde) or $\breve{}$ (breve) on top of the symbols in Section 1. The difference is the following:
  - Symbols with $\sim$ will be used for defining $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ and its compactifications $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$, and for realizing the Kuga families we would like to compactify as boundary strata $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$ of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$.
  - Symbols with $\breve{}$ will be used for the boundary strata of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ appearing in the closure of the realizations $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$. (These strata are parameterized by faces $[(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})]$ of $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]$.) In other words, they parameterize the boundary strata of the toroidal compactification of the Kuga families we consider.
- In the local descriptions of toroidal boundary structures, we will encounter notations of the forms $(\,\cdot\,)(\sigma)$ and $(\,\cdot\,)_\sigma$.
  - When the object $(\,\cdot\,)$ being modified is a scheme with action by some torus, $(\,\cdot\,)(\sigma)$ will stand for the affine toroidal embedding adding the σ-stratum (which then also adds all the strata for nontrivial faces of σ),

while $(\,\cdot\,)_\sigma$ will stand for the closed σ-stratum (without the nontrivial face strata).

– The formal version of $(\,\cdot\,)_\sigma$ (often denoted in Fraktur) will mean the formal completion of $(\,\cdot\,)(\sigma)$ along $(\,\cdot\,)_\sigma$.

The notation will be most heavy in Sections 4–5, where the calculation of relative cohomology is carried out in detail. For readers only interested in applications to cohomology of Shimura varieties, the statements of Theorem 2.15, the two propositions in Section 3D, and the applications in Section 6 are all they need.

3. Constructions of compactifications and morphisms

3A. **Kuga families as toroidal boundary strata.** The goal of this subsection is to carry out steps (1) and (2) of Section 2C.

Let Q be an $\mathcal{O}$-lattice as in Theorem 2.15. Identify $Q^\vee$ with $\operatorname{Hom}_{\mathcal{O}}(Q, \mathrm{Diff}^{-1})$ and give it an $\mathcal{O}$-lattice structure as in Section 2A. The (surjective) trace map $\mathrm{Tr}_{\mathcal{O}/\mathbb{Z}} : \mathrm{Diff}^{-1} \to \mathbb{Z}$ induces a *perfect* pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_Q : Q^\vee \times Q \to \mathbb{Z} : (f, x) \mapsto \mathrm{Tr}_{\mathcal{O}/\mathbb{Z}}(f(x)).$$

By extension of scalars, the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_Q$ induces a perfect pairing between $Q^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ and $Q \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. By Condition 1.9, the action of $\mathcal{O}$ on L extends to an action of some maximal order $\mathcal{O}'$ in B containing $\mathcal{O}$. Let us fix the choice of such a maximal order $\mathcal{O}'$. By [38, Prop. 1.1.1.17], $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)} \neq \mathcal{O}' \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}$ for a prime number $p > 0$ only when $p |\,\mathrm{Disc}$. Let $Q_0 := \mathcal{O}' \cdot Q \subset Q \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ and $Q_{-2} := \operatorname{Hom}_{\mathcal{O}}(Q, \mathrm{Diff}^{-1}_{\mathcal{O}'/\mathbb{Z}})(1) \subset Q^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{Q}(1)$. Then the induced pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle_Q : Q_{-2} \times Q_0 \to \mathbb{Q}(1)$$

has values in $\mathbb{Z}(1)$. The localizations of this pairing at primes of $\mathbb{Z}$ are perfect except at those dividing Disc.

Let $(\widetilde{L}, \langle\,\cdot\,,\,\cdot\,\rangle^\sim, \widetilde{h})$ be the symplectic $\mathcal{O}$-lattice given by the following data:

(1) An $\mathcal{O}$-lattice $\widetilde{L} := Q_{-2} \oplus L \oplus Q_0$, where Q_{-2} and Q_0 are defined as above. (Note that $\widetilde{L}$ satisfies Condition 1.9 by construction.)

(2) A symplectic $\mathcal{O}$-pairing $\langle\,\cdot\,,\,\cdot\,\rangle^\sim : \widetilde{L} \times \widetilde{L} \to \mathbb{Z}(1)$ defined (symbolically) by the matrix

$$\langle x, y\rangle^\sim := {}^t\begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \end{pmatrix} \begin{pmatrix} & & \langle\,\cdot\,,\,\cdot\,\rangle_Q \\ & \langle\,\cdot\,,\,\cdot\,\rangle & \\ -\,{}^t\langle\,\cdot\,,\,\cdot\,\rangle_Q & & \end{pmatrix} \begin{pmatrix} y_{-2} \\ y_{-1} \\ y_0 \end{pmatrix},$$

namely by

$$\langle x, y\rangle^\sim := \langle x_{-2}, y_0\rangle_Q + \langle x_{-1}, y_{-1}\rangle - \langle y_{-2}, x_0\rangle_Q,$$

where $x = \begin{pmatrix} x_{-2} \\ x_{-1} \\ x_0 \end{pmatrix}$ and $y = \begin{pmatrix} y_{-2} \\ y_{-1} \\ y_0 \end{pmatrix}$ are elements of $\widetilde{L} = Q_{-2} \oplus L \oplus Q_0$ expressed (vertically) in terms of components in the direct summands.

Let $j_Q : Q^\vee \hookrightarrow Q$ be an embedding of $\mathcal{O}$-lattices given by Lemma 2.5, so that the pairing $\langle j_Q^{-1}(\,\cdot\,), \,\cdot\,\rangle_Q : (Q \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Q \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}$ is positive definite. Consider the

$\mathbb{R}$-algebra homomorphism $\widetilde{h}:\mathbb{C}\to\operatorname{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(\widetilde{L}\underset{\mathbb{Z}}{\otimes}\mathbb{R})$ defined by

$$z = z_1+\sqrt{-1}\,z_2$$
$$\mapsto \widetilde{h}(z) := \begin{pmatrix} z_1\operatorname{Id}_{Q_{-2}\underset{\mathbb{Z}}{\otimes}\mathbb{R}} & & -z_2((2\pi\sqrt{-1})\circ j_Q^{-1}) \\ & h(z) & \\ z_2(j_Q\circ(2\pi\sqrt{-1})^{-1}) & & z_1\operatorname{Id}_{Q_0\underset{\mathbb{Z}}{\otimes}\mathbb{R}} \end{pmatrix},$$

where $2\pi\sqrt{-1}:\mathbb{Z}\xrightarrow{\sim}\mathbb{Z}(1)$ and $(2\pi\sqrt{-1})^{-1}:\mathbb{Z}(1)\xrightarrow{\sim}\mathbb{Z}$ stand for the isomorphisms defined by the choice of $\sqrt{-1}$ in $\mathbb{C}$, and where the matrix acts (symbolically) on elements $x=\begin{pmatrix}x_{-2}\\x_{-1}\\x_0\end{pmatrix}$ of $\widetilde{L}\underset{\mathbb{Z}}{\otimes}\mathbb{R}$ by left multiplication. In other words,

$$\widetilde{h}(z)\begin{pmatrix}x_{-2}\\x_{-1}\\x_0\end{pmatrix} = \begin{pmatrix} z_1x_{-2}-z_2((2\pi\sqrt{-1})\circ j_Q^{-1})(x_0) \\ h(z)x_{-1} \\ z_2(j_Q\circ(2\pi\sqrt{-1})^{-1})(x_{-2})+z_1x_0 \end{pmatrix}.$$

Then $\widetilde{h}$ is a polarization of $(\widetilde{L},\langle\cdot,\cdot\rangle^{\sim})$ making $(\widetilde{L},\langle\cdot,\cdot\rangle^{\sim},\widetilde{h})$ a PEL-type $\mathcal{O}$-lattice. Note that the reflex field of $(\widetilde{L}\underset{\mathbb{Z}}{\otimes}\mathbb{R},\langle\cdot,\cdot\rangle^{\sim},\widetilde{h})$ is also F_0.

By construction of $(\widetilde{L},\langle\cdot,\cdot\rangle^{\sim})$, there is a fully symplectic admissible filtration on $\widetilde{L}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ induced by

$$0\subset Q_{-2}\subset Q_{-2}\oplus L\subset Q_{-2}\oplus L\oplus Q_0=\widetilde{L}.$$

More precisely, we have

$$\begin{aligned}
\widetilde{\mathrm{Z}}_{-3} &:= 0,\\
\widetilde{\mathrm{Z}}_{-2} &:= Q_{-2}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box},\\
\widetilde{\mathrm{Z}}_{-1} &:= (Q_{-2}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})\oplus(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}),\\
\widetilde{\mathrm{Z}}_{0} &:= (Q_{-2}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})\oplus(L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})\oplus(Q_0\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box})=\widetilde{L}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box},
\end{aligned}$$

so that there are canonical isomorphisms

$$\operatorname{Gr}^{\widetilde{\mathrm{Z}}}_{-2}\cong Q_{-2}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box},\quad \operatorname{Gr}^{\widetilde{\mathrm{Z}}}_{-1}\cong L\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box},\quad \operatorname{Gr}^{\widetilde{\mathrm{Z}}}_{0}\cong Q_0\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$$

matching the pairings $\operatorname{Gr}^{\widetilde{\mathrm{Z}}}_{-2}\times\operatorname{Gr}^{\widetilde{\mathrm{Z}}}_{0}\to\hat{\mathbb{Z}}^{\Box}(1)$ and $\operatorname{Gr}^{\widetilde{\mathrm{Z}}}_{-1}\times\operatorname{Gr}^{\widetilde{\mathrm{Z}}}_{-1}\to\hat{\mathbb{Z}}^{\Box}(1)$ induced by $\langle\cdot,\cdot\rangle^{\sim}$ with $\langle\cdot,\cdot\rangle_Q$ and $\langle\cdot,\cdot\rangle$, respectively.

Let $\widetilde{X}:=\operatorname{Hom}_{\mathcal{O}}(Q_{-2},\operatorname{Diff}^{-1}(1))$ and $\widetilde{Y}:=Q_0$. The pairing $\langle\cdot,\cdot\rangle_Q : Q_{-2}\times Q_0\to\mathbb{Z}(1)$ induces a canonical embedding $\widetilde{\phi}:\widetilde{Y}\hookrightarrow\widetilde{X}$ and there are canonical isomorphisms $\widetilde{\varphi}_{-2}:\operatorname{Gr}^{\widetilde{\mathrm{Z}}}_{-2}\xrightarrow{\sim}\operatorname{Hom}_{\hat{\mathbb{Z}}^{\Box}}(\widetilde{X}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box},\hat{\mathbb{Z}}^{\Box}(1))$ and $\widetilde{\varphi}_0:\operatorname{Gr}^{\widetilde{\mathrm{Z}}}_{0}\xrightarrow{\sim}\widetilde{Y}\underset{\mathbb{Z}}{\otimes}\hat{\mathbb{Z}}^{\Box}$ (of $\hat{\mathbb{Z}}^{\Box}$-modules). These data define a torus argument $\widetilde{\Phi}:=(\widetilde{X},\widetilde{Y},\widetilde{\phi},\widetilde{\varphi}_{-2},\widetilde{\varphi}_0)$ for $\widetilde{\mathrm{Z}}$ as in Definition 1.13.

Let $\widetilde{\delta}$ be the obvious splitting of $\widetilde{\mathrm{Z}}$ induced by the equality $Q_{-2}\oplus L\oplus Q_0=\widetilde{L}$.

Let $\widetilde{\mathrm{G}}$ be the group functor defined by $(\widetilde{L},\langle\cdot,\cdot\rangle^{\sim})$ as in Definition 1.1. For any $\hat{\mathbb{Z}}^{\Box}$-algebra R, let $\widetilde{\mathrm{P}}_{\widetilde{\mathrm{Z}}}(R)$ denote the subgroup of $\widetilde{\mathrm{G}}(R)$ consisting of elements g such that $g(\widetilde{\mathsf{Z}}_{-2} \underset{\hat{\mathbb{Z}}^\Box}{\otimes} R) = \widetilde{\mathsf{Z}}_{-2} \underset{\hat{\mathbb{Z}}^\Box}{\otimes} R$ and $g(\widetilde{\mathsf{Z}}_{-1} \underset{\hat{\mathbb{Z}}^\Box}{\otimes} R) = \widetilde{\mathsf{Z}}_{-1} \underset{\hat{\mathbb{Z}}^\Box}{\otimes} R$. Any element g in $\widetilde{\mathrm{P}}_{\widetilde{\mathsf{Z}}}(R)$ defines an isomorphism

$$\mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-1}(g) : \mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-1} \underset{\hat{\mathbb{Z}}^\Box}{\otimes} R \xrightarrow{\sim} \mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-1} \underset{\hat{\mathbb{Z}}^\Box}{\otimes} R,$$

which corresponds under the canonical isomorphism $\mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-1} \underset{\hat{\mathbb{Z}}^\Box}{\otimes} R \cong L \underset{\mathbb{Z}}{\otimes} R$ above to an element of $\mathrm{G}(R)$. This defines in particular a homomorphism

$$\mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-1} : \widetilde{\mathrm{P}}_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^\Box) \to \mathrm{G}(\hat{\mathbb{Z}}^\Box).$$

Let us also define $\widetilde{\mathrm{P}}'_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^\Box)$ to be the kernel of $\mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-2} \times \mathrm{Gr}^{\widetilde{\mathsf{Z}}}_0$, where $\mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-2}$ and $\mathrm{Gr}^{\widetilde{\mathsf{Z}}}_0$ are defined analogously.

Let $\widetilde{\mathcal{H}}$ be any *neat* open compact subgroup of $\widetilde{\mathrm{G}}(\hat{\mathbb{Z}}^\Box)$ satisfying the following conditions:

(1) $\mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-1}(\widetilde{\mathcal{H}} \cap \widetilde{\mathrm{P}}'_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^\Box)) = \mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-1}(\widetilde{\mathcal{H}} \cap \widetilde{\mathrm{P}}_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^\Box)) = \mathcal{H}$. (Both equalities are conditions. Then $\mathcal{H}$ is a direct factor of $\mathrm{Gr}^{\widetilde{\mathsf{Z}}}(\widetilde{\mathcal{H}} \cap \widetilde{\mathrm{P}}_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^\Box))$.)

(2) The splitting $\widetilde{\delta}$ defines a (group-theoretic) splitting of the surjection $\widetilde{\mathcal{H}} \cap \widetilde{\mathrm{P}}'_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^\Box) \twoheadrightarrow \mathcal{H}$ induced by $\mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-1}$.

(Such an $\widetilde{\mathcal{H}}$ exists because the pairing $\langle\,\cdot\,,\,\cdot\,\rangle^{\sim}$ is the direct sum of the pairings on $Q_{-2} \oplus Q_0$ and on L.) The data of $\mathcal{O}$, $(\widetilde{L}, \langle\,\cdot\,,\,\cdot\,\rangle^{\sim}, \widetilde{h})$, $\Box$, and $\widetilde{\mathcal{H}} \subset \widetilde{\mathrm{G}}(\hat{\mathbb{Z}}^\Box)$ define a moduli problem $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ as in Definition 1.6.

Take any compatible choice $\widetilde{\Sigma}$ of admissible smooth rational polyhedral cone decomposition data for $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ that is *projective* (see Definitions 1.33 and 1.34). Since $\widetilde{\mathcal{H}}$ is neat, any such $\widetilde{\Sigma}$ defines a toroidal compactification $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}} = \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ which is projective and smooth over S_0 by (7) of Theorem 1.41.

Let $(\widetilde{\mathsf{Z}}, \widetilde{\Phi}, \widetilde{\delta})$ be as above, and let $(\widetilde{\mathsf{Z}}_{\widetilde{\mathcal{H}}}, \widetilde{\Phi}_{\widetilde{\mathcal{H}}} = (\widetilde{X}, \widetilde{Y}, \widetilde{\phi}, \widetilde{\varphi}_{-2,\widetilde{\mathcal{H}}}, \widetilde{\varphi}_{0,\widetilde{\mathcal{H}}}), \widetilde{\delta}_{\widetilde{\mathcal{H}}})$ be the induced triple at level $\widetilde{\mathcal{H}}$, inducing a cusp label $[(\widetilde{\mathsf{Z}}_{\widetilde{\mathcal{H}}}, \widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}})]$ at level $\widetilde{\mathcal{H}}$.

Let $\widetilde{\sigma} \subset \mathbf{P}^+_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ be any *top-dimensional* nondegenerate rational polyhedral cone in the cone decomposition $\widetilde{\Sigma}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ in $\widetilde{\Sigma}$. Then, by (2) of Theorem 1.41, we have a stratum $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}$ of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$.

Since $\widetilde{\sigma}$ is a *top-dimensional* cone in $\widetilde{\Sigma}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$, the locally closed stratum $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}$ (not its closure) is a zero-dimensional torus bundle over the abelian scheme $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}}$ over $\mathsf{M}_{\mathcal{H}}$. (We have canonical isomorphisms $\widetilde{\mathsf{M}}^{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}} \cong \widetilde{\mathsf{M}}^{\widetilde{\mathsf{Z}}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}} \cong \mathsf{M}_{\mathcal{H}}$ because of the first condition above on the choice of $\widetilde{\mathcal{H}}$. The abelian scheme torsor $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}} \to \widetilde{\mathsf{M}}^{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}}$ is an abelian scheme because of the second condition above on the choice of $\widetilde{\mathcal{H}}$.) In other words, $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}$ is canonically *isomorphic* to $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}}$. By the construction of $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}}$ in [38, §§6.2.3–6.2.4], it is canonically $\mathbb{Z}^\times_{(\Box)}$-isogenous to the abelian scheme $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ$. Let us define N_κ to be this stratum $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}$, and denote the canonical morphism $\mathsf{N}_\kappa \to \mathsf{M}_{\mathcal{H}}$ by f_κ. This gives the $\mathbb{Z}^\times_{(\Box)}$-isogeny κ^{isog} :

$\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^\circ \to \mathsf{N}_\kappa$. Note that $\mathsf{N}_\kappa = \widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}$ is canonically isomorphic to $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}}$ for every $\widetilde{\Sigma}$ and every top-dimensional cone $\widetilde{\sigma}$ in $\widetilde{\Sigma}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$.

As planned in step (1) of Section 2C, let us take $\mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$ to be the set of all possible such triples $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$, with the binary relation $\kappa' = (\widetilde{\mathcal{H}}', \widetilde{\Sigma}', \widetilde{\sigma}') \succ \kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$ defined when $\widetilde{\mathcal{H}}' \subset \widetilde{\mathcal{H}}$, when $\widetilde{\Sigma}'$ refines $\widetilde{\Sigma}$ as in [38, Def. 6.4.2.8], and when $(\widetilde{\Phi}_{\widetilde{\mathcal{H}}'}, \widetilde{\delta}_{\widetilde{\mathcal{H}}'}, \widetilde{\sigma}')$ refines $(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})$ as in [38, Def. 6.4.2.6]. In this case, the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}'}, \widetilde{\delta}_{\widetilde{\mathcal{H}}'}, \widetilde{\sigma}')]$-stratum of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'}$ is mapped to the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ by the canonical morphism $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ given by [38, Prop. 6.4.2.9]. Note that the induced morphism $f_{\kappa',\kappa} : \mathsf{N}_{\kappa'} \to \mathsf{N}_\kappa$, which is $\kappa^{\mathrm{isog}} \circ ((\kappa')^{\mathrm{isog}})^{-1}$ by definition, can be identified with the canonical $\mathbb{Z}^\times_{(\Box)}$-isogeny $\widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}'}, \widetilde{\delta}_{\widetilde{\mathcal{H}}'}} \to \widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}}$. In particular, it is surjective and is an isogeny of degree prime to $\Box$.

For $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$, take $\mathsf{N}^{\mathrm{tor}}_\kappa$ to be the closure of $\widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}$ in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$. Then we obtain the canonical immersion $\kappa^{\mathrm{tor}} : \mathsf{N}_\kappa \hookrightarrow \mathsf{N}^{\mathrm{tor}}_\kappa$.

When $\kappa' = (\widetilde{\mathcal{H}}', \widetilde{\Sigma}', \widetilde{\sigma}') \succ \kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$, the morphism $f^{\mathrm{tor}}_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa'} \to \mathsf{N}^{\mathrm{tor}}_\kappa$ is simply the morphism induced by the canonical proper morphisms $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ given by [38, Prop. 6.4.2.9]. Note that the latter morphism is étale locally given by equivariant morphisms between toric schemes, and the same is true for the induced morphism $f^{\mathrm{tor}}_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa'} \to \mathsf{N}^{\mathrm{tor}}_\kappa$. Therefore, both the morphism $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ and the induced morphism $f^{\mathrm{tor}}_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa'} \to \mathsf{N}^{\mathrm{tor}}_\kappa$ are log étale essentially by definition (see [33, Thm. 3.5]). Moreover, as in [16, Ch. V, Rem. 1.2(b)] and in the proof of [38, Lem. 7.1.1.3], we have $R^i(f^{\mathrm{tor}}_{\kappa',\kappa})_* \mathscr{O}_{\mathsf{N}^{\mathrm{tor}}_{\kappa'}} = 0$ for $i > 0$ by [36, Ch. I, §3].

Lemma 3.1. *Under the assumption that $\widetilde{\mathcal{H}}$ is neat, the closure of every stratum in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ has no self-intersection.*

Proof. According to Definitions 1.33 and 1.34, the collection $\widetilde{\Sigma}$ of cone decompositions for $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ satisfies Condition 1.29. Hence [38, Lem. 6.2.5.27 in the revision] implies that the closure of any stratum does not intersect itself. (See also [16, Ch. IV, Rem. 5.8(a)].) □

Corollary 3.2. *For any $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma}) \in \mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$, the closure $\mathsf{N}^{\mathrm{tor}}_\kappa$ of $\mathsf{N}_\kappa = \widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]}$ in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$ is projective and smooth over S_0, and the complement of N_κ in $\mathsf{N}^{\mathrm{tor}}_\kappa$ (with its reduced structure) is a relative Cartier divisor with simple normal crossings. Thus the collection of open embeddings $\kappa^{\mathrm{tor}} : \mathsf{N}_\kappa \hookrightarrow \mathsf{N}^{\mathrm{tor}}_\kappa$, with $\kappa \in \mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$, satisfies (1) of Theorem 2.15.*

Proof. Combine Lemma 3.1 with (3) and (7) of Theorem 1.41. □

From now on, let us fix a choice of $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma}) \in \mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$, and suppress κ and $\widetilde{\Sigma}$ from the notation. The compatibility of various objects under compositions with or pullbacks by $f^{\mathrm{tor}}_{\kappa',\kappa} : \mathsf{N}^{\mathrm{tor}}_{\kappa'} \to \mathsf{N}^{\mathrm{tor}}_\kappa$ (for $\kappa' \succ \kappa$ in $\mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$) will be obvious from the constructions.

3B. **Extendability of structural morphisms.** The goal of this subsection is to carry out steps (3) and (4) of Section 2C.

Let $(\widetilde{G}, \widetilde{\lambda}, \widetilde{i}, \widetilde{\alpha}_{\widetilde{\mathcal{H}}})$ be the degenerating family of type $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ over $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$. By construction of N as a boundary stratum of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$, the restriction $\widetilde{G}_{\mathsf{N}}$ of $\widetilde{G}$ to N is an extension of the pullback of the abelian scheme $G_{\mathsf{M}_{\mathcal{H}}}$ over $\mathsf{M}_{\mathcal{H}}$ to N by $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$, by the split torus $\widetilde{T}_{\mathsf{N}}$ over N with character group $\widetilde{X}$. The data of $\widetilde{\lambda}$, $\widetilde{i}$, and $\widetilde{\alpha}_{\widetilde{\mathcal{H}}}$ induce respectively a polarization, an $\mathcal{O}$-endomorphism structure, and a level $\mathcal{H}$-structure on the abelian part of $\widetilde{G}_{\mathsf{N}}$, which agree with the pullbacks of the data λ, i, and $\alpha_{\mathcal{H}}$ over $\mathsf{M}_{\mathcal{H}}$ to N by $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$. By normality of (the closure) $\mathsf{N}^{\mathrm{tor}}$ (of N in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$), and by a result of Raynaud (see [49, IX, 2.4] or [16, Ch. I, Prop. 2.9]), the embedding $\widetilde{T}_{\mathsf{N}} \hookrightarrow \widetilde{G}_{\mathsf{N}}$ of group schemes extends (uniquely) to an embedding $\widetilde{T}_{\mathsf{N}^{\mathrm{tor}}} \hookrightarrow \widetilde{G}_{\mathsf{N}^{\mathrm{tor}}}$ of group schemes, and the quotient

$$\overline{G} := \widetilde{G}_{\mathsf{N}^{\mathrm{tor}}} / \widetilde{T}_{\mathsf{N}^{\mathrm{tor}}}$$

is a semiabelian scheme whose restriction to N can be identified with the pullback of G from $\mathsf{M}_{\mathcal{H}}$ to N. Similarly, we obtain $\overline{G}^{\vee} := \widetilde{G}^{\vee}_{\mathsf{N}^{\mathrm{tor}}} / \widetilde{T}^{\vee}_{\mathsf{N}^{\mathrm{tor}}}$. By another result of Raynaud (see [49, IX, 1.4] or [16, Ch. I, Prop. 2.7]), the semiabelian $\overline{G}$ carries (unique) additional structures $\overline{\lambda} : \overline{G} \to \overline{G}^{\vee}$, $\overline{i}$, and $\overline{\alpha}_{\mathcal{H}}$ such that the restriction of $(\overline{G}, \overline{\lambda}, \overline{i}, \overline{\alpha}_{\mathcal{H}})$ to N is the pullback of the tautological tuple over $\mathsf{M}_{\mathcal{H}}$ by $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$, and so that $(\overline{G}, \overline{\lambda}, \overline{i}, \overline{\alpha}_{\mathcal{H}})$ defines a degenerating family of type $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{N}^{\mathrm{tor}}$.

Now the question is whether the structural morphism $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$ extends (necessarily uniquely) to a (proper) morphism $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ between the compactifications. By (6) of Theorem 1.41, this extendability can be verified after pullback to complete discrete valuation rings (with algebraically closed residue fields).

The stratification of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ induces a stratification of $\mathsf{N}^{\mathrm{tor}}$. By (2) of Theorem 1.41, the strata of $\mathsf{N}^{\mathrm{tor}}$ are parameterized by equivalence classes $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$ having $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$ as a face (as in Definition 1.38). Concretely, they are $\widetilde{\mathcal{H}}$-orbits of data of the following form:

(1) A fully symplectic admissible filtration $\breve{\mathsf{Z}} = \{\breve{\mathsf{Z}}_{-i}\}$ on $\widetilde{L} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ satisfying

$$\widetilde{\mathsf{Z}}_{-2} \subset \breve{\mathsf{Z}}_{-2} \subset \breve{\mathsf{Z}}_{-1} \subset \widetilde{\mathsf{Z}}_{-1}. \tag{3.3}$$

Any such filtration $\breve{\mathsf{Z}}$ induces a fully symplectic admissible filtration $\mathsf{Z} = \{\mathsf{Z}_{-i}\}$ on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ by $\mathsf{Z}_{-2} := \breve{\mathsf{Z}}_{-2}/\widetilde{\mathsf{Z}}_{-2}$ and $\mathsf{Z}_{-1} := \breve{\mathsf{Z}}_{-1}/\widetilde{\mathsf{Z}}_{-2}$, so that there is a canonical isomorphism

$$\mathsf{Z}_{0}/\mathsf{Z}_{-1} \cong \widetilde{\mathsf{Z}}_{-1}/\breve{\mathsf{Z}}_{-1}. \tag{3.4}$$

Conversely, any fully symplectic admissible filtration Z on $L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ induces a fully symplectic admissible filtration $\breve{\mathsf{Z}}$ on $\widetilde{L} \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^{\Box}$ satisfying (3.3) and (3.4).

(2) A torus argument $\breve{\Phi} = (\breve{X}, \breve{Y}, \breve{\phi}, \breve{\varphi}_{-2}, \breve{\varphi}_{0})$ for $\breve{\mathsf{Z}}$ (as in Definition 1.13), together with admissible surjections $s_{\breve{X}} : \breve{X} \twoheadrightarrow \widetilde{X}$ and $s_{\breve{Y}} : \breve{Y} \twoheadrightarrow \widetilde{Y}$ satisfying $s_{\breve{X}}\breve{\phi} = \widetilde{\phi} s_{\breve{Y}}$ and other natural compatibilities with $\breve{\varphi}_{-2}$, $\breve{\varphi}_{0}$, $\widetilde{\varphi}_{-2}$, and $\widetilde{\varphi}_{0}$. (See Definitions 1.18, 1.19, and 1.20.)

Any $\breve{\Phi}$, $s_{\breve{X}}$, and $s_{\breve{Y}}$ determine a torus argument $\Phi = (X, Y, \phi, \varphi_{-2}, \varphi_{0})$ for Z by $X := \ker(s_{\breve{X}})$, $Y := \ker(s_{\breve{Y}})$, and $\phi := \breve{\phi}|_{Y}$, so that there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, "\phi"] & \breve{Y} \arrow[r, "s_{\breve{Y}}"] \arrow[d, "\breve{\phi}"] & \widetilde{Y} \arrow[r] \arrow[d, "\widetilde{\phi}"] & 0 \\ 0 \arrow[r] & X \arrow[r] & \breve{X} \arrow[r, "s_{\breve{X}}"'] & \widetilde{X} \arrow[r] & 0 \end{tikzcd} \tag{3.5}$$

whose horizontal rows are exact sequences.

(3) The existence of some splitting of $\breve{\mathsf{Z}}$, inducing some liftable splitting $\breve{\delta}_{\widetilde{\mathcal{H}}}$ defining the cusp label $(\breve{\mathsf{Z}}_{\widetilde{\mathcal{H}}}, \breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}})$ at level $\widetilde{\mathcal{H}}$.

Given the liftable splitting $\widetilde{\delta}_{\widetilde{\mathcal{H}}}$, the existence of the liftable splitting $\breve{\delta}_{\widetilde{\mathcal{H}}}$ is equivalent to the existence of some liftable splitting $\delta_{\mathcal{H}}$ of $\mathsf{Z}_{\mathcal{H}}$. Then we see that there is a canonical bijection between cusp labels $[(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ at level $\mathcal{H}$ and cusp labels $[(\breve{\mathsf{Z}}_{\widetilde{\mathcal{H}}}, \breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}})]$ at level $\widetilde{\mathcal{H}}$ admitting a surjection to $[(\widetilde{\mathsf{Z}}_{\widetilde{\mathcal{H}}}, \widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}})]$.

(4) Let $\Phi_{\mathcal{H}}$ (resp. $\breve{\Phi}_{\widetilde{\mathcal{H}}}$) be the torus argument for $\mathsf{Z}_{\mathcal{H}}$ (resp. $\widetilde{\mathsf{Z}}_{\widetilde{\mathcal{H}}}$) at level $\mathcal{H}$ (resp. $\widetilde{\mathcal{H}}$) induced by Φ (resp. $\breve{\Phi}$). Then (3.5) induces morphisms

$$\mathbf{S}_{\Phi_{\mathcal{H}}} \hookrightarrow \mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \twoheadrightarrow \mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}, \tag{3.6}$$

where the first morphism is canonical, and where the second morphism is defined by $s_{\breve{X}}$ and $s_{\breve{Y}}$, whose composition is zero. (In general, the morphisms in (3.6) *do not* form an exact sequence.)

The dual of (3.6) defines morphisms

$$\mathbf{P}^{+}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}} \hookrightarrow \mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \twoheadrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}, \tag{3.7}$$

where the first morphism is defined by $s_{\breve{X}}$ and $s_{\breve{Y}}$, and where the second morphism is canonical, whose composition is zero.

Then $\breve{\tau} \subset \mathbf{P}^{+}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ is a cone in the cone decomposition $\widetilde{\Sigma}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ having a face $\breve{\sigma}$ that is a $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$-translation (see Definition 1.23) of the image of $\widetilde{\sigma} \subset \mathbf{P}^{+}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ under the first morphism in (3.7).

By (5) of Theorem 1.41, the formal completion

$$(\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]}}$$

is isomorphic to the formal scheme $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau}} = \widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau}} / \Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\tau}}$ for any representative $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$ of $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$. Here $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\tau}}$ is trivial by [38, Lem. 6.2.5.27 in the revision], and $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau}}$ is the formal completion of $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ along its $\breve{\tau}$-stratum $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\tau}}$.

Let us describe the structure of the scheme $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ in more detail:

(1) By construction, $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ is a scheme over $\widetilde{\mathsf{M}}^{\breve{\mathsf{Z}}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}}$, the latter of which is isomorphic to $\mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ because of (3.3) and (3.4). By the two conditions satisfied by $\widetilde{\mathcal{H}}$ above, we have $\widetilde{\mathsf{M}}^{\breve{\Phi}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}} \cong \mathsf{M}^{\Phi_{\mathcal{H}}}_{\mathcal{H}}$ as finite étale covers of $\widetilde{\mathsf{M}}^{\breve{\mathsf{Z}}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}} \cong \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$. (Note that $\widetilde{\mathsf{M}}^{\breve{\mathsf{Z}}_{\widetilde{\mathcal{H}}}}_{\widetilde{\mathcal{H}}} \cong \mathsf{M}^{\mathsf{Z}_{\mathcal{H}}}_{\mathcal{H}}$ is a scheme by [38, Cor. 7.2.3.10].)

By abuse of notation, we shall simply denote the push-forward

$$(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau}) \twoheadrightarrow \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_* \mathscr{O}_{\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau})}$$

by $\mathscr{O}_{\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau})}$, and view $\mathscr{O}_{\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau})}$ as an $\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebra when there is no confusion. We shall adopt a similar convention for other affine morphisms.

(2) Let $(A, \lambda_A, i_A, \varphi_{-1,\mathcal{H}})$ be the tautological object over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. Then $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$ is the abelian scheme torsor over the finite étale cover $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}^{\check{\Phi}_{\widetilde{\mathcal{H}}}} \cong \mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$ of $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}^{\check{\mathsf{Z}}_{\widetilde{\mathcal{H}}}} \cong \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ parameterizing liftings (to level $\widetilde{\mathcal{H}}$) of data of the form $(\check{c} : \check{X} \to A^\vee, \check{c}^\vee : \check{Y} \to A)$, compatible with $\check{\phi} : \check{Y} \hookrightarrow \check{X}$ and satisfying certain liftability and pairing conditions (coming from the so-called symplectic-liftability on the level structures). By construction, $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \to \mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$ is a torsor under an abelian scheme $\mathbb{Z}_{(\Box)}^\times$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(\check{Y}, A)^\circ$.

(3) The scheme $\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$ is a torsor over $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$ under (the pullback of) the split torus $E_{\check{\Phi}_{\widetilde{\mathcal{H}}}} = \underline{\mathrm{Hom}}(\mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}}, \mathbf{G}_{\mathrm{m}})$, which can be identified with the relative spectrum

$$\underline{\mathrm{Spec}}_{\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}} \Big(\bigoplus_{\check{\ell} \in \mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}}} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}) \Big),$$

where $\widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ is the subsheaf of $\mathscr{O}_{\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$ (considered as an $\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebra by our convention) on which $E_{\check{\Phi}_{\widetilde{\mathcal{H}}}}$ acts by the character $\check{\ell}$. In the case when $\check{\ell} = [\check{y} \otimes \check{\chi}]$, where $\check{y} \in \check{Y}$ and $\check{\chi} \in \check{X}$, there is a canonical identification between $\widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ and the pullback of $(\check{c}^\vee(\check{y}), \check{c}(\check{\chi}))^* \mathcal{P}_A$ over $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$. (See [38, Conv. 6.2.3.26 and end of §6.2.4].)

(4) Consider the subsemigroups of $\mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}}$ (see [38, Def. 6.1.1.9 and 6.1.2.5]):

$$\check{\tau}^\vee = \{\check{\ell} \in \mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}} : \langle \check{\ell}, y \rangle \geq 0, \forall y \in \check{\tau}\},$$
$$\check{\tau}_0^\vee = \{\check{\ell} \in \mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}} : \langle \check{\ell}, y \rangle > 0, \forall y \in \check{\tau}\},$$
$$\check{\tau}^\perp = \{\check{\ell} \in \mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}} : \langle \check{\ell}, y \rangle = 0, \forall y \in \check{\tau}\} \cong \check{\tau}^\vee / \check{\tau}_0^\vee.$$

The scheme $\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau})$ is constructed as an *affine toroidal embedding*

$$\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \hookrightarrow \widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\tau})$$

along $\check{\tau}$ over the abelian scheme $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$, which can be identified with the relative spectrum

$$\underline{\mathrm{Spec}}_{\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}} \Big(\bigoplus_{\check{\ell} \in \check{\tau}^\vee} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}) \Big).$$

(5) Finally, the sheaf of ideals

$$\widetilde{\mathscr{I}}_{\check{\tau}} = \bigoplus_{\check{\ell} \in \check{\tau}_0^\vee} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$$

(see [38, Lem. 6.1.2.6]) defines the $\breve{\tau}$-stratum $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\tau}}$, which can be identified with the relative spectrum

$$\underline{\mathrm{Spec}}_{\mathscr{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\bigoplus_{\breve{\ell}\in\breve{\tau}^{\perp}}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big).$$

Here $\widetilde{\mathscr{I}_{\breve{\tau}}}$ is an $\mathscr{O}_{\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})}$-ideal represented as an $\mathscr{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}$-submodule of $\mathscr{O}_{\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})}$ (the latter being viewed as an $\mathscr{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}$-algebra by our convention).

Suppose $\breve{\sigma}$ is the face of $\breve{\tau}$ that is a $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$-translation of the image of $\widetilde{\sigma}\subset\mathbf{P}^{+}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ under the first morphism in (3.7). Similar to the definition of $\breve{\tau}^{\vee}$, $\breve{\tau}_0^{\vee}$, and $\breve{\tau}^{\perp}$ above, consider the following subsemigroups of $\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$:

$$\begin{aligned}\breve{\sigma}^{\vee}&=\{\breve{\ell}\in\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}:\langle\breve{\ell},y\rangle\geq 0,\forall y\in\breve{\sigma}\},\\ \breve{\sigma}_0^{\vee}&=\{\breve{\ell}\in\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}:\langle\breve{\ell},y\rangle> 0,\forall y\in\breve{\sigma}\},\\ \breve{\sigma}^{\perp}&=\{\breve{\ell}\in\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}:\langle\breve{\ell},y\rangle= 0,\forall y\in\breve{\sigma}\}\cong\breve{\sigma}^{\vee}/\breve{\sigma}_0^{\vee}.\end{aligned}$$

Note that $\breve{\tau}^{\vee}\subset\breve{\sigma}^{\vee}$ and $\breve{\tau}^{\perp}\subset\breve{\sigma}^{\perp}$, but $\breve{\tau}_0^{\vee}\not\subset\breve{\sigma}_0^{\vee}$ in general. The closure $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$ of the $\breve{\sigma}$-stratum on $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})\cong\underline{\mathrm{Spec}}_{\mathscr{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\bigoplus_{\breve{\ell}\in\breve{\tau}^{\vee}}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big)$ is defined by the sheaf of ideals $\bigoplus_{\breve{\ell}\in\breve{\sigma}_0^{\vee}\cap\breve{\tau}^{\vee}}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$. Then we have a canonical isomorphism

$$(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})\cong\underline{\mathrm{Spec}}_{\mathscr{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\bigoplus_{\breve{\ell}\in\breve{\sigma}^{\perp}\cap\breve{\tau}^{\vee}}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big),$$

with the $\breve{\tau}$-stratum

$$(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\tau}}\cong\underline{\mathrm{Spec}}_{\mathscr{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\bigoplus_{\breve{\ell}\in\breve{\tau}^{\perp}}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\Big)$$

(as a closed subscheme of $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$) defined by the sheaf of ideals

$$\widetilde{\mathscr{I}_{\breve{\sigma},\breve{\tau}}}:=\bigoplus_{\breve{\ell}\in\breve{\sigma}^{\perp}\cap\breve{\tau}_0^{\vee}}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}).$$

Let $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\breve{\tau}}$ denote the formal completion of $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$ along $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\tau}}$, which can be canonically identified as a closed formal subscheme of $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau}}$, inducing the closures of the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]$-strata on any good formal $(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})$-model. (See [38, Def. 6.3.1.11] for the definition of good formal models, and see [38, Def. 6.3.2.16 in the revision] for the labeling of the strata by equivalence classes of triples of the form $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]$.) By (5) of Theorem 1.41, the *strata-preserving* canonical isomorphism $(\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})]}}\cong\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau}}$ then induces a canonical isomorphism

$$(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})]}}\cong\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\breve{\tau}}.$$

(Alternatively, one may refer directly to the gluing construction of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ in [38, §6.3.3], based on the crucial [38, Prop. 6.3.2.13].)

By the theory of two-step constructions (see [16, Ch. III, Thm. 10.2] and [38, §4.5.6 in the revision]), the degeneration data of the pullback of $(\overline{G}, \overline{\lambda}, \overline{i}, \overline{\alpha}_{\mathcal{H}})$ to affine open formal subschemes of $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \breve{\tau}}$ can be obtained from the degeneration data of pullback of $(\widetilde{G}, \widetilde{\lambda}, \widetilde{i}, \widetilde{\alpha}_{\widetilde{\mathcal{H}}})$ to affine open formal subschemes of $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau}}$ by restricting objects defined on $\breve{X}$ and $\breve{Y}$ to the subgroups X and Y. Therefore, in order to verify (6) of Theorem 1.41, it suffices to verify the following:

Condition 3.8 (cf. [16, Ch. VI, Def. 1.3])**.** *For each $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$ as above, the image of $\breve{\tau}$ in $\mathbf{P}_{\Phi_{\mathcal{H}}}$ under the* (*canonical*) *second morphism in* (3.7) *is contained in some cone $\tau \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ in the cone decomposition $\Sigma_{\Phi_{\mathcal{H}}}$.*

If Condition 3.8 is satisfied (for $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$), then the structural morphism $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$ extends to a (unique) morphism $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, which is étale locally given by morphisms between toric schemes equivariant under (surjective) morphisms between tori. By construction, we have a commutative diagram

$$\begin{tikzcd} \mathsf{N}^{\mathrm{tor}} \arrow[d, "f^{\mathrm{tor}}"'] & \widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \breve{\tau}} \arrow[l] \arrow[r] \arrow[d] & \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}} \arrow[d] \\ \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} & \mathfrak{X}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau} \arrow[l] \arrow[r] & C_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}} \end{tikzcd} \tag{3.9}$$

of canonical morphisms whenever the image of $\breve{\tau}$ under the (canonical) second morphism in (3.7) is contained in τ.

Remark 3.10. Condition 3.8 is analogous to the condition in [48, 6.25(b)], used in for example [26, Lem. 1.6.5] and related works based on [4]. Unfortunately, we must point out that, apart from some pleasant (and often suggestive) analogies, there is no logical implication between the analytic theory in [4] and [48], and the algebraic theory in [16] and [38]. (One cannot even use $\mathrm{G}(\mathbb{Q})$ in the algebraic theory.) The applicability of Condition 3.8 in our work cannot be proved using [48, 6.25(b)].

As planned in step (4) of Section 2C, let us take $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ to be the subset of $\mathbf{K}^{\mathrm{pre}}_{Q,\mathcal{H},\Sigma}$ consisting of elements κ satisfying Condition 3.8. Since Condition 3.8 can be achieved by replacing any given $\widetilde{\Sigma}$ with a refinement, we see that $\mathbf{K}_{Q,\mathcal{H},\Sigma}$ is nonempty and has an induced binary relation which we still denote by $\succ$.

From now on, assume that our fixed choice $\kappa = (\widetilde{\mathcal{H}}, \widetilde{\Sigma}, \widetilde{\sigma})$ lies in $\mathbf{K}_{Q,\mathcal{H},\Sigma}$.

3C. **Logarithmic smoothness of f^{tor}.** The aim of this subsection is to carry out step (5) of Section 2C.

We need to show that the morphism f^{tor} is log smooth (as in [33, 3.3] and [32, 1.6]) if we equip $\mathsf{N}^{\mathrm{tor}}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ with the canonical fine log structures given respectively by the relative Cartier divisors with simple normal crossings given by the complements $\mathsf{N}^{\mathrm{tor}} - \mathsf{N}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} - \mathsf{M}_{\mathcal{H}}$ with their reduced structures. According to [33, 3.12], we have the following:

Lemma 3.11. *To show that the morphism f^{tor} is log smooth, it suffices to show that the first morphism in the canonical exact sequence*

$$(f^{\mathrm{tor}})^*(\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty]) \to \Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}[d\log\infty] \to \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to 0 \tag{3.12}$$

is **injective***, and that $\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ is* **locally free** *of finite rank.*

By (4) of Theorem 1.41, the extended Kodaira–Spencer morphism [38, Def. 4.6.3.32] for $G \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ induces an isomorphism

$$\mathrm{KS}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} : \underline{\mathrm{KS}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \xrightarrow{\sim} \Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty]$$

over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, while the extended Kodaira–Spencer morphism for $\widetilde{G} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ induces an isomorphism

$$\mathrm{KS}_{\widetilde{G}/\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}/\mathsf{S}_0} : \underline{\mathrm{KS}}_{\widetilde{G}/\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}} \xrightarrow{\sim} \Omega^1_{\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}/\mathsf{S}_0}[d\log\infty]$$

over $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$. Over $\mathsf{N}^{\mathrm{tor}}$, we have canonical extensions $0 \to \widetilde{T}_{\mathsf{N}^{\mathrm{tor}}} \to \widetilde{G}_{\mathsf{N}^{\mathrm{tor}}} \to \overline{G} \to 0$ and $0 \to \widetilde{T}^\vee_{\mathsf{N}^{\mathrm{tor}}} \to \widetilde{G}^\vee_{\mathsf{N}^{\mathrm{tor}}} \to \overline{G}^\vee \to 0$ of group schemes, inducing exact sequences

$$0 \to \mathrm{Lie}^\vee_{\overline{G}/\mathsf{N}^{\mathrm{tor}}} \to \mathrm{Lie}^\vee_{\widetilde{G}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \to \mathrm{Lie}^\vee_{\widetilde{T}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \to 0$$

and

$$0 \to \mathrm{Lie}^\vee_{\overline{G}^\vee/\mathsf{N}^{\mathrm{tor}}} \to \mathrm{Lie}^\vee_{\widetilde{G}^\vee_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \to \mathrm{Lie}^\vee_{\widetilde{T}^\vee_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \to 0.$$

Therefore, there is a canonical surjection

$$\underline{\mathrm{KS}}_{\widetilde{G}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \twoheadrightarrow \underline{\mathrm{KS}}_{\widetilde{T}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}}, \tag{3.13}$$

where $\underline{\mathrm{KS}}_{\widetilde{T}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}}$ is the pullback of the sheaf

$$\underline{\mathrm{KS}}_{\widetilde{T}_{\mathsf{S}_0}/\mathsf{S}_0} := (\underline{\mathrm{Lie}}^\vee_{\widetilde{T}_{\mathsf{S}_0}/\mathsf{S}_0} \underset{\mathscr{O}_{\mathsf{S}_0}}{\otimes} \underline{\mathrm{Lie}}^\vee_{\widetilde{T}^\vee_{\mathsf{S}_0}/\mathsf{S}_0}) / \begin{pmatrix} \lambda^*_{\widetilde{T}_{\mathsf{S}_0}}(y) \otimes z - \lambda^*_{\widetilde{T}_{\mathsf{S}_0}}(z) \otimes y \\ (b^\star x) \otimes y - x \otimes (by) \end{pmatrix}_{\substack{x \in \underline{\mathrm{Lie}}^\vee_{\widetilde{T}_{\mathsf{S}_0}/\mathsf{S}_0} \\ y,z \in \underline{\mathrm{Lie}}^\vee_{\widetilde{T}^\vee_{\mathsf{S}_0}/\mathsf{S}_0} \\ b \in \mathcal{O}}}$$

defined (as for degenerating families in Definition 1.40) by the split tori $\widetilde{T}$ and $\widetilde{T}^\vee$ over S_0 with respective character groups $\widetilde{X}$ and $\widetilde{Y}$. The kernel

$$\underline{\mathrm{K}} := \ker(\underline{\mathrm{KS}}_{\widetilde{G}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \twoheadrightarrow \underline{\mathrm{KS}}_{\widetilde{T}_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}})$$

contains $\underline{\mathrm{KS}}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}$ as a natural subsheaf, and the quotient of $\underline{\mathrm{K}}$ by $\underline{\mathrm{KS}}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}$ is isomorphic to

$$\begin{aligned}
&(\underline{\mathrm{Lie}}^\vee_{\overline{G}/\mathsf{N}^{\mathrm{tor}}} \underset{\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{\widetilde{T}^\vee_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}}) / ((b^\star x) \otimes y - x \otimes (by))_{\substack{x \in \underline{\mathrm{Lie}}^\vee_{\overline{G}/\mathsf{N}^{\mathrm{tor}}} \\ y \in \underline{\mathrm{Lie}}^\vee_{\widetilde{T}^\vee_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}} \\ b \in \mathcal{O}}} \\
&\cong \underline{\mathrm{Hom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}}(\underline{\mathrm{Lie}}_{\widetilde{T}^\vee_{\mathsf{N}^{\mathrm{tor}}}/\mathsf{N}^{\mathrm{tor}}}, \underline{\mathrm{Lie}}^\vee_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}) \\
&\cong \underline{\mathrm{Hom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}}(\underline{\mathrm{Hom}}_{\mathbb{Z}}(\widetilde{Y}, \mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}), \underline{\mathrm{Lie}}^\vee_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}) \\
&\cong \underline{\mathrm{Hom}}_{\mathcal{O}}(\widetilde{Y}^\vee, \underline{\mathrm{Lie}}^\vee_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}) \\
&\cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{\overline{G}/\mathsf{N}^{\mathrm{tor}}})
\end{aligned}$$

Since the pullback of $(G, \lambda, i, \alpha_{\mathcal{H}})$ under $\mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is isomorphic to $(\overline{G}, \overline{\lambda}, \overline{i}, \overline{\alpha}_{\mathcal{H}})$, we have canonical isomorphisms

$$(f^{\mathrm{tor}})^* \underline{\mathrm{KS}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \cong \underline{\mathrm{KS}}_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}$$

and

$$(f^{\mathrm{tor}})^*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})) \cong \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{\overline{G}/\mathsf{N}^{\mathrm{tor}}}).$$

Since the étale local structure of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ along the $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$-stratum is the same as $\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$, the calculation in the proof of [38, Prop. 6.2.5.14] shows that the isomorphism $\mathrm{KS}_{\widetilde{G}/\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}/\mathsf{S}_0}$ induces by restriction (to the closure $\mathsf{N}^{\mathrm{tor}}$ of the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum) an isomorphism

$$\underline{\mathrm{K}} \xrightarrow{\sim} \Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}[d\log\infty] \tag{3.14}$$

making the diagram

$$\begin{tikzcd}
(f^{\mathrm{tor}})^*\underline{\mathrm{KS}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \arrow[r, hook] \arrow[d, "\mathrm{KS}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}"', "\wr"] & \underline{\mathrm{K}} \arrow[d, "\wr"', "(3.14)"] \\
(f^{\mathrm{tor}})^*(\Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty]) \arrow[r] & \Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}[d\log\infty]
\end{tikzcd}$$

commutative. In particular, the bottom arrow (which is the first morphism in (3.12)) is *injective*, and the isomorphism (3.14) induces a canonical isomorphism

$$(f^{\mathrm{tor}})^*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})) \xrightarrow{\sim} \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \tag{3.15}$$

of coherent sheaves over $\mathsf{N}^{\mathrm{tor}}$. (The restriction of (3.15) to N is compatible with the composition of isomorphisms (2.17) because of the same calculation in the proof of [38, Prop. 6.2.5.14].)

Thus the desired isomorphism (2.16) is a consequence of (3.15). Moreover, since $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ (see Remark 2.14) is locally free of finite rank over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, the isomorphism (3.15) shows that the sheaf $\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ is also locally free of finite rank over $\mathsf{N}^{\mathrm{tor}}$. By Lemma 3.11, this shows that f^{tor} is log smooth, and completes the proof of (2) and (3a) of Theorem 2.15.

3D. **Equidimensionality of f^{tor}.** Let us take a closer look at the diagram (3.9). By construction of f^{tor}, given any stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, the preimage

$$\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} := (f^{\mathrm{tor}})^{-1}(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]})$$

has a stratification formed by $\widetilde{\mathsf{Z}}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]}$, where $\breve{\tau}$ runs through cones in $\widetilde{\Sigma}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ satisfying the following conditions:

(1) $\breve{\tau} \subset \mathbf{P}^+_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$.
(2) $\breve{\tau}$ has a face $\breve{\sigma}$ that is a $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$-translation of the image of $\widetilde{\sigma} \subset \mathbf{P}^+_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ under the first morphism in (3.7).
(3) The image of $\breve{\tau}$ under the (canonical) second morphism in (3.7) is contained in $\tau \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$.

The formal completion $(\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$ admits a canonical morphism

$$(\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}},$$

whose precomposition with the canonical morphism

$$(\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]}} \to (\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}},$$

for any stratum $\widetilde{\mathsf{Z}}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]}$ of $\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$, coincides with the composition of canonical morphisms $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \breve{\tau}} \to \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ by its very construction.

Since f^{tor} is étale locally given by morphisms between toric schemes equivariant under (surjective) morphisms between tori, to determine if f^{tor} is equidimensional (cf. [16, Ch. VI, Def. 1.3 and Rem. 1.4]), it suffices to determine if the relative dimension of each of the induced (smooth) morphism $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]} \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ between strata is at most $\dim_{\mathsf{M}_{\mathcal{H}}}(\mathsf{N})$, the relative dimension of $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$.

By abuse of language, we define the $\mathbb{R}$-dimension of a cone to be the $\mathbb{R}$-dimension of its $\mathbb{R}$-span. Then the codimension of $\mathsf{N} = \widetilde{\mathsf{Z}}_{[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$ in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ is $\dim_{\mathbb{R}}(\widetilde{\sigma}) = \dim_{\mathbb{R}}((\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}})$ because $\widetilde{\sigma}$ is top-dimensional. The codimension of

$$\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]} \cong (\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\check{\tau}}$$

in $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ is equal to $\dim_{\mathbb{R}}(\check{\tau})$. Therefore, the codimension of $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]}$ in $\mathsf{N}^{\mathrm{tor}}$ is equal to $\dim_{\mathbb{R}}(\check{\tau}) - \dim_{\mathbb{R}}(\widetilde{\sigma}) = \dim_{\mathbb{R}}(\check{\tau}) - \dim_{\mathbb{R}}((\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}})$. On the other hand, the codimension of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} \cong (\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_{\tau}$ in $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is $\dim_{\mathbb{R}}(\tau)$. Hence we have

$$\begin{aligned} (3.16) \quad & \dim_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}(\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]}) \\ & = \dim_{\mathsf{M}_{\mathcal{H}}}(\mathsf{N}) - (\dim_{\mathbb{R}}(\check{\tau}) - \dim_{\mathbb{R}}((\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}})) + \dim_{\mathbb{R}}(\tau). \end{aligned}$$

Let τ' denote the image of $\check{\tau}$ in $(\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$. By assumption on $\check{\tau}$, we have $\tau' \subset \tau$. If $\tau' = \tau$, then

$$\dim_{\mathbb{R}}(\tau) = \dim_{\mathbb{R}}(\tau') \leq \dim_{\mathbb{R}}(\check{\tau}) - \dim_{\mathbb{R}}((\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}}),$$

and hence (3.16) implies

$$\dim_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}(\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]}) \leq \dim_{\mathsf{M}_{\mathcal{H}}}(\mathsf{N}).$$

(If this is true for all $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]}$, then f^{tor} is equidimensional.) On the other hand, suppose $\tau' \subsetneq \tau$. Then there exists a face of τ'' of τ' such that $\tau'' \subset \tau$ and $\dim_{\mathbb{R}}(\tau'') < \dim_{\mathbb{R}}(\tau)$. Note that τ'' is the image of at least one face of $\check{\tau}$ satisfying the three conditions in the first paragraph of this section. By dropping redundant basis vectors, we may assume moreover that this face $\check{\tau}''$ of $\check{\tau}$ satisfies $\dim_{\mathbb{R}}(\tau'') = \dim_{\mathbb{R}}(\check{\tau}'') - \dim_{\mathbb{R}}((\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}})$. Then we have

$$\dim_{\mathbb{R}}(\tau) > \dim_{\mathbb{R}}(\tau'') = \dim_{\mathbb{R}}(\check{\tau}'') - \dim_{\mathbb{R}}((\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}}),$$

and hence (3.16) implies

$$\dim_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}(\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau}'')]}) > \dim_{\mathsf{M}_{\mathcal{H}}}(\mathsf{N}),$$

which means f^{tor} cannot be equidimensional.

This motivates the following strengthening of Condition 3.8:

Condition 3.17 (cf. [16, Ch. VI, Def. 1.3])**.** *For each* $(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})$ *such that* $\mathsf{Z}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\tau})]}$ *is* (*a stratum*) *in* $\mathsf{N}^{\mathrm{tor}}$, *the image of* $\check{\tau} \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ *under the* (*canonical*) *second morphism in* (3.7) *is* **exactly** *some cone* $\tau \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$ *in the cone decomposition* $\Sigma_{\Phi_{\mathcal{H}}}$.

Proposition 3.18. *The morphism* $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *is equidimensional* (*with relative dimension equal to the one of* $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$), *and hence flat, if and only if Condition* 3.17 *is satisfied, if and only if* f^{tor} *is log integral* (*see* [33, Def. 4.3]).

Proof. The equivalence between Condition 3.17 and equidimensionality has been explained above. Since both $\mathsf{N}^{\mathrm{tor}}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ are regular (because they are smooth over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$), the equidimensionality and flatness of f^{tor} are equivalent by [21, IV-3, 15.4.2 b)⇔e')]. By [33, Prop. 4.1(2)], the log integrality of f^{tor} is equivalent to the flatness of each of the canonical morphisms $\mathbb{Z}[\tau^\vee] \hookrightarrow \mathbb{Z}[\breve{\tau}^\vee]$ (defined when $\mathsf{Z}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})]}$ is mapped to $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$), which is equivalent to the equidimensionality of any such morphism (by the smoothness of $\mathbb{Z}[\tau^\vee]$ and $\mathbb{Z}[\breve{\tau}^\vee]$ over $\mathbb{Z}$, and by [21, IV-3, 15.4.2 b)⇔e')] again), which is equivalent to Condition 3.17 by the same (dimension comparison) argument. □

Proposition 3.19 (cf. [16, Ch. VI, Rem. 1.4])**.** *Condition* 3.17 *can be achieved by replacing both the cone decompositions* $\widetilde{\Sigma}$ *and* Σ *with some refinements.*

Proof. Instead of taking refinements of $\widetilde{\Sigma}$ and Σ separately, we consider the morphism $\mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \twoheadrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ in (3.7) and consider the *graph* of $\widetilde{\Sigma}$. More precisely, using the canonical morphisms $X \hookrightarrow \breve{X}$ and $Y \hookrightarrow \breve{Y}$ compatible with ϕ and $\breve{\phi}$, we obtain canonical morphisms $X' := \breve{X} \oplus X \to \breve{X}$ and $Y' := \breve{Y} \oplus Y \to \breve{Y}$ compatible with $\phi' := \breve{\phi} \oplus \phi$ and $\breve{\phi}$, inducing morphisms $\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \oplus \mathbf{S}_{\Phi_{\mathcal{H}}} \twoheadrightarrow \mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ and $\mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \hookrightarrow \mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \oplus \mathbf{P}_{\Phi_{\mathcal{H}}}$. The image of this latter morphism is the *graph* of $\mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \twoheadrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$. Let us define $\ddot{\mathbf{S}}'$ by X', Y', and ϕ' as in (1.21), and let $\mathbf{S}'$ be its free quotient. Define $\mathbf{P}'$ accordingly as the subset of $(\mathbf{S}')^\vee_{\mathbb{R}}$ consisting of positive semidefinite pairings with admissible radicals, containing the graph of $\mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \twoheadrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ canonically as an admissible boundary component (cf. Definition 1.28). The cone decomposition $\widetilde{\Sigma}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ defines a cone decomposition on this graph, which might fail to be projective or smooth with respect to the structure of the ambient space. But we can find a projective smooth cone decomposition of $\mathbf{P}'$, admissible with respect to the actions of all elements in $\mathrm{GL}_{\mathcal{O}}(X') \times \mathrm{GL}_{\mathcal{O}}(Y')$ respecting ϕ', such that its restriction to the graph refine the cone decomposition defined by $\widetilde{\Sigma}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$. Thus we obtain a simultaneous smooth projective refinement of $\widetilde{\Sigma}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ and $\Sigma_{\Phi_{\mathcal{H}}}$, such that image of cones in $\widetilde{\Sigma}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ under $\mathbf{P}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}} \twoheadrightarrow \mathbf{P}_{\Phi_{\mathcal{H}}}$ are cones in $\Sigma_{\Phi_{\mathcal{H}}}$. Since this construction is compatible with surjections between different choices of $\breve{\Phi}_{\widetilde{\mathcal{H}}}$ and $\Phi_{\mathcal{H}}$, we can conclude by induction on magnitude of cusp labels $(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ as in the proofs of [38, Prop. 6.3.3.3 and 7.3.1.5]. □

Remark 3.20. We will not need Propositions 3.18 and 3.19 in what follows. We supply them here because knowing flatness or log integrality of f^{tor} is useful in many applications.

3E. **Hecke actions.** The aim of this subsection is to explain the proof of statements (4) and (5) of Theorem 2.15, with (4c) and (5c) conditional on (3b) and (3c) of Theorem 2.15. These statements might seem elaborate, but they are self-explanatory and based on the following simple idea: Since N and $\mathsf{N}^{\mathrm{tor}}$ are constructed using the toroidal compactifications of $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$, we can use the Hecke actions on $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ and their (compatible) extensions to toroidal compactifications provided by [38, Prop. 6.4.3.4 in the revision].

Let g_h, $\mathcal{H}'$, Σ', g_l, and Q' be as in (4) and (5) of Theorem 2.15. (For proving (4) and (5) of Theorem 2.15, we may assume in what follows either $g_h = 1$ or $g_l = 1$, although the theory works in a more general context.) Using the splitting $\widetilde{\delta}$ of $\widetilde{\mathsf{Z}}$, we obtain an element $\widetilde{g}$ in $\widetilde{\mathrm{P}}_{\widetilde{\mathsf{Z}}}(\mathbb{A}^{\infty,\Box})$ such that $\mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-1}(\widetilde{g}) = g_h$, and such that $\mathrm{Gr}^{\widetilde{\mathsf{Z}}}_0(\widetilde{g})$ is identified with g_l^{-1} under $\widetilde{\varphi}_0 : \mathrm{Gr}^{\widetilde{\mathsf{Z}}}_0 \xrightarrow{\sim} Q_0 \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box \cong Q \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^\Box$. (See Section 3A.) Let $\widetilde{\mathcal{H}}'$ be a (necessarily neat) subgroup of $\widetilde{\mathrm{G}}(\hat{\mathbb{Z}}^\Box)$ such that $\widetilde{g}^{-1}\widetilde{\mathcal{H}}'\widetilde{g} \subset \widetilde{\mathcal{H}}$, and such that $\mathcal{H}' = \mathrm{Gr}^{\widetilde{\mathsf{Z}}}_{-1}(\widetilde{\mathcal{H}}' \cap \mathrm{P}_{\widetilde{\mathsf{Z}}}(\hat{\mathbb{Z}}^\Box))$. By [38, Prop. 6.4.3.4 in the revision], there exist some choices of $\widetilde{\Sigma}'$ such that the canonical morphism $[\widetilde{g}] : \widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}'} \to \widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ extends canonically to $[\widetilde{g}]^{\mathrm{tor}} : \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}',\widetilde{\Sigma}'} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}},\widetilde{\Sigma}}$. By replacing $\widetilde{\Sigma}'$ with a refinement such that it satisfies Condition 3.8 (with Σ' and) with some choice of $\widetilde{\sigma}'$, and such that the morphism $[\widetilde{g}]^{\mathrm{tor}}$ sends the stratum $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}'},\check{\delta}_{\widetilde{\mathcal{H}}'},\widetilde{\sigma}')]}$ to $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$, we see that the induced morphism from the closure of $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}'},\check{\delta}_{\widetilde{\mathcal{H}}'},\widetilde{\sigma}')]}$ to the closure of $\widetilde{\mathsf{Z}}_{[(\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]}$ gives the existences of the morphisms $[g_h]_{\kappa',\kappa}$, $[g_h]^{\mathrm{tor}}_{\kappa',\kappa}$, $[g_l]^*_{\kappa',\kappa}$, and $([g_l]^*_{\kappa',\kappa})^{\mathrm{tor}}$ as in (4a), (4b), (5a), and (5b) of Theorem 2.15, where $\kappa' = (\widetilde{\mathcal{H}}', \widetilde{\Sigma}', \widetilde{\sigma}')$ lies in $\mathbf{K}_{Q',\mathcal{H}',\Sigma'}$, except that (2.24) and (2.26) still have to be explained.

As in the case of showing $R^i(f^{\mathrm{tor}}_{\kappa',\kappa})_*\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}_{\kappa'}} = 0$ for $i > 0$ in Section 3A, since the morphisms $[g_h]^{\mathrm{tor}}_{\kappa',\kappa}$ and $([g_l]^*_{\kappa',\kappa})^{\mathrm{tor}}$ are étale locally given by equivariant morphisms between toric schemes, we have (by [36, Ch. I, §3]) $R^i([g_h]^{\mathrm{tor}}_{\kappa',\kappa})_*(\mathscr{O}_{(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}}) = 0$ and $R^i([g_l]^*_{\kappa',\kappa})^{\mathrm{tor}}_*(\mathscr{O}_{(\mathsf{N}'_{\kappa'})^{\mathrm{tor}}}) = 0$ for $i > 0$, which are (2.24) and (2.26) of Theorem 2.15.

The remaining statements in (4c) and (5c) of Theorem 2.15 now follow if we assume statements (3b) and (3c) of Theorem 2.15. (See the end of Section 5 below.)

4. Calculation of formal cohomology

Throughout this section, unless otherwise specified, we fix the choice of an arbitrary (locally closed) stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. The aim of this section is to calculate the relative cohomology of the pullback of the structural morphism f^{tor} to the formal completion $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$. (See (5) of Theorem 1.41 for a description of this formal completion. See also the first paragraph of Section 3D for a description of the formal completion $(\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$ of $\mathsf{N}^{\mathrm{tor}}$ along $\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} = (f^{\mathrm{tor}})^{-1}(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]})$.)

4A. **Formal fibers of f^{tor}.** Let $\Gamma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\tau}$ be the subgroup of elements in $\Gamma_{\check{\Phi}_{\widetilde{\mathcal{H}}}}$ stabilizing (both) X and Y and inducing an element in $\Gamma_{\Phi_{\mathcal{H}},\tau}$ (the subgroup of $\Gamma_{\Phi_{\mathcal{H}}}$ formed by elements mapping τ to itself). Since we have tacitly assumed that $\Gamma_{\Phi_{\mathcal{H}},\tau}$ is trivial by Conditions 1.29 and [38, Lem. 6.2.5.27 in the revision], $\Gamma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\tau}$ is also the subgroup of elements in $\Gamma_{\check{\Phi}_{\widetilde{\mathcal{H}}}}$ fixing (both) X and Y. Let $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ be the subgroup of $\Gamma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\tau}$ inducing trivial actions on $\widetilde{X}$ and $\widetilde{Y}$ under the two surjections

$$s_{\check{X}} : \check{X} \twoheadrightarrow \widetilde{X} \quad \text{and} \quad s_{\check{Y}} : \check{Y} \twoheadrightarrow \widetilde{Y},$$

which can be identified as a subgroup of $\mathrm{Hom}_{\mathcal{O}}(\widetilde{X}, X)$, with index prime to $\Box$, sending $\widetilde{\phi}(\widetilde{Y})$ to $\phi(Y)$. (Note that $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ depends not just on $\widetilde{\Phi}_{\widetilde{\mathcal{H}}}$ and $\Phi_{\mathcal{H}}$ but also on $\check{\Phi}_{\widetilde{\mathcal{H}}}$.)

Since $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ does not modify $s_{\check{X}}$ and $s_{\check{Y}}$, it does not modify the first morphism in (3.7). Therefore, if we denote the image of $\widetilde{\sigma}$ in $\mathbf{P}_{\check{\Phi}_{\widetilde{\mathcal{H}}}}$ by $\check{\sigma}$, then $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ maps $\check{\sigma}$

to itself. On the other hand, by Condition 1.29 (and Lemma 3.1), if a cone $\breve{\tau} \subset \mathbf{P}^+_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ in $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ has a face that is a $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\tau}$-translation of $\breve{\sigma}$, then it cannot have a different face that is also a $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\tau}$-translation of $\breve{\sigma}$. Let us denote by $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ the subset of $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ consisting of cones $\breve{\tau}$ satisfying the following conditions (cf. similar conditions in the first paragraph of Section 3D):

(1) $\breve{\tau} \subset \mathbf{P}^+_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$.
(2) $\breve{\tau}$ has $\breve{\sigma}$ as a face.
(3) The image of $\breve{\tau}$ under the (canonical) second morphism in (3.7) is contained in $\tau \subset \mathbf{P}^+_{\Phi_{\mathcal{H}}}$.

Then, to obtain a complete list of representatives of the equivalence classes $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$ parameterizing the strata of $\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$, it suffices to take representatives of $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ modulo the action of $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. (That is, we do not have to consider $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$-translations of $\breve{\sigma}$.)

Let $\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\tau)$ denote the toroidal embedding of $\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$ formed by gluing the affine toroidal embeddings $\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ over $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$, where $\breve{\tau}$ runs through cones in $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$. To minimize confusion, we shall distinguish $\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau}_1)$ and $\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau}_2)$ even when $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau}_1)] = [(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau}_2)]$. For each $\breve{\tau}$ as above (having $\breve{\sigma}$ as a face), recall that we have denoted the closure of the $\breve{\sigma}$-stratum of $\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ by $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$. Let $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\tau)$ denote the union of all such $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$, let $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\tau}$ denote the union of all such $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\tau}}$, and let $\widetilde{\widetilde{\mathfrak{X}}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ denote the formal completion of $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\tau)$ along $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\tau}$.

For each $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, consider the open subscheme $U_{\breve{\tau}}$ of $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\tau}$ formed by the union of all (locally closed) strata of $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\tau}$ that contains the stratum $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\tau}}$ in its closure, and consider the open formal subscheme $\mathfrak{U}_{\breve{\tau}}$ of $\widetilde{\widetilde{\mathfrak{X}}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ supported on $U_{\breve{\tau}}$. The subscheme $U_{\breve{\tau}}$ of $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\tau}$ is the closed subscheme of $\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ given by the intersection of $\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ and $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\tau}$ in $\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\tau)$, and the formal subscheme $\mathfrak{U}_{\breve{\tau}}$ of $\widetilde{\widetilde{\mathfrak{X}}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ is the formal completion of $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$ along $U_{\breve{\tau}}$. The collection $\{U_{\breve{\tau}}\}_{\breve{\tau}\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}$ forms an open covering of $(\widetilde{\widetilde{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\tau}$. We can interpret $\widetilde{\widetilde{\mathfrak{X}}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ as constructed by gluing the collection $\{\mathfrak{U}_{\breve{\tau}}\}_{\breve{\tau}\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}$ of formal schemes along their intersections (of supports).

Explicitly, let us denote by $\breve{\tau}^\vee_{\breve{\sigma}}$ the intersection of $(\breve{\tau}')^\vee_0$ for $\breve{\tau}'$ running through faces of $\breve{\tau}$ in $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ (including $\breve{\tau}$ itself). Then we have the canonical isomorphism

$$U_{\breve{\tau}} \cong \underline{\mathrm{Spec}}_{\mathscr{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}}\Big(\big(\mathop{\oplus}_{\breve{\ell}\in\breve{\tau}^\vee} \widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\big)/\big(\mathop{\oplus}_{\breve{\ell}\in\breve{\tau}^\vee_{\breve{\sigma}}} \widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\big)\Big)$$

of schemes affine over $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$. As $\mathscr{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}$-modules, we have a canonical isomorphism

$$\big(\mathop{\oplus}_{\breve{\ell}\in\breve{\tau}^\vee} \widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\big)/\big(\mathop{\oplus}_{\breve{\ell}\in\breve{\tau}^\vee_{\breve{\sigma}}} \widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\big) \cong \mathop{\oplus}_{\breve{\ell}\in\breve{\tau}^\vee-\breve{\tau}^\vee_{\breve{\sigma}}} \widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}).$$

If we equip $\check{\tau}^\vee - \check{\tau}^\vee_{\check{\sigma}}$ with the semigroup structure induced by the canonical bijection $(\check{\tau}^\vee - \check{\tau}^\vee_{\check{\sigma}}) \to \check{\tau}^\vee/\check{\tau}^\vee_{\check{\sigma}}$, then we may interpret $\underset{\check{\ell}\in\check{\tau}^\vee-\check{\tau}^\vee_{\check{\sigma}}}{\oplus} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ as an $\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebra, with algebra structure given by canonical isomorphisms

$$\widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}) \underset{\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}}{\otimes} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}') \xrightarrow{\sim} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}+\check{\ell}')$$

(inherited from those of $\mathscr{O}_{\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}} \cong \underset{\check{\ell}\in\mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}}}{\oplus} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$) if $\check{\ell}+\check{\ell}' \in \check{\tau}^\vee - \check{\tau}^\vee_{\check{\sigma}}$ and by

$$\widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}) \underset{\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}}{\otimes} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}') \to 0$$

otherwise. Then we have a canonical isomorphism

$$U_{\check{\tau}} \cong \underline{\mathrm{Spec}}_{\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}} \Big(\underset{\check{\ell}\in\check{\tau}^\vee-\check{\tau}^\vee_{\check{\sigma}}}{\oplus} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}) \Big).$$

By definition, we have

$$\check{\tau}^\vee - \check{\tau}^\vee_{\check{\sigma}} = \Big(\underset{\substack{\check{\tau}' \text{ face of } \check{\tau} \\ \text{in } \Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}}}{\cup} \big((\check{\tau}')^\perp \cap \check{\tau}^\vee \big) \Big) \subset \check{\sigma}^\perp \cap \check{\tau}^\vee.$$

The formal scheme $\mathfrak{U}_{\check{\tau}}$, being the formal completion of

$$(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\check{\sigma}}(\check{\tau}) \cong \underline{\mathrm{Spec}}_{\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}} \Big(\underset{\check{\ell}\in\check{\sigma}^\perp\cap\check{\tau}^\vee}{\oplus} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}) \Big)$$

along $U_{\check{\tau}}$, can be canonically identified with the relative formal spectrum of the $\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebra $\underset{\check{\ell}\in\check{\sigma}^\perp\cap\check{\tau}^\vee}{\hat{\oplus}} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ over $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$, where $\hat{\oplus}$ denotes the completion of the sum with respect to the $\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-ideal $\underset{\check{\ell}\in\check{\sigma}^\perp\cap\check{\tau}^\vee_{\check{\sigma}}}{\oplus} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$. Note that all the above canonical isomorphisms correspond to canonical morphisms of $\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebras formed by sums of sheaves of the form $\widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ (with $\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebra structures inherited from that of $\mathscr{O}_{\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$).

The above descriptions imply the following simple but important facts:

Lemma 4.1. *Suppose $\check{\tau}$ and $\check{\tau}'$ are two cones in $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$ such that $\check{\tau}'$ is a face of $\check{\tau}$. Then:*

1. *We have a canonical open immersion $\mathfrak{U}_{\check{\tau}'} \hookrightarrow \mathfrak{U}_{\check{\tau}}$ (resp. $U_{\check{\tau}'} \hookrightarrow U_{\check{\tau}}$) of formal subschemes of $\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$.*
2. *The canonical restriction morphism from $\mathfrak{U}_{\check{\tau}}$ to $\mathfrak{U}_{\check{\tau}'}$ corresponds to the canonical morphism*

$$\underset{\check{\ell}\in\check{\sigma}^\perp\cap\check{\tau}^\vee}{\hat{\oplus}} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}) \to \underset{\check{\ell}\in\check{\sigma}^\perp\cap(\check{\tau}')^\vee}{\hat{\oplus}} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$$

of $\mathscr{O}_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}}$-algebras, where the two instances of $\hat{\oplus}$ denote completions of the sums with respect to the sheaves of ideals $\underset{\check{\ell}\in\check{\sigma}^\perp\cap\check{\tau}^\vee_{\check{\sigma}}}{\oplus} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$ and $\underset{\check{\ell}\in\check{\sigma}^\perp\cap(\check{\tau}')^\vee_{\check{\sigma}}}{\oplus} \widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell})$, respectively.

(3) *The canonical restriction morphism from* $U_{\breve{\tau}}$ *to* $U_{\breve{\tau}'}$ *corresponds to the canonical morphism*

$$\underset{\breve{\ell}\in\breve{\tau}^\vee-\breve{\tau}_{\breve{\sigma}}^\vee}{\oplus}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})\to\underset{\breve{\ell}\in(\breve{\tau}')^\vee-(\breve{\tau}')_{\breve{\sigma}}^\vee}{\oplus}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$$

of $\mathscr{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}}$*-algebras, which maps* $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$ *to* $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$ *when*

$$\breve{\ell}\in(\breve{\tau}^\vee-(\breve{\tau}')_{\breve{\sigma}}^\vee)=(\breve{\tau}^\vee-\breve{\tau}_{\breve{\sigma}}^\vee)\cap((\breve{\tau}')^\vee-(\breve{\tau}')_{\breve{\sigma}}^\vee),$$

and to zero otherwise.

(4) *The correspondences in* (2) *and* (3) *above are canonically compatible with each other.*

By Condition 1.29 (and Lemma 3.1), the action of $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ induces only the trivial action on each stratum it stabilizes. Therefore, the quotient morphism

$$\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}\to\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}\tag{4.2}$$

of formal schemes over S_0 is a *local isomorphism.* The morphism (4.2) is not defined over $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$ when the action of $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ on $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$ is nontrivial. Nevertheless, since $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ acts trivially on $\Phi_{\mathcal{H}}$, it acts trivially on $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, and hence (4.2) is defined over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.

Proposition 4.3. *There is a canonical isomorphism*

$$(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}\cong\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}\tag{4.4}$$

of formal schemes over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$*, characterized by the identifications*

$$(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})]}}\cong\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\breve{\tau}}$$

of formal schemes over $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$ (*compatible with the canonical morphisms*

$$(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})]}}\to(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$$

and $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}\to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$). (*The formation of the formal completion here is similar to the one in* (5) *of Theorem* 1.41.)

Proof. Let $\breve{\tau}\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$. Let $\widetilde{\mathfrak{U}}_{\breve{\tau}}$ denote the completion of $\widetilde{\overline{\Xi}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\tau})$ along $U_{\breve{\tau}}$, which contains $\mathfrak{U}_{\breve{\tau}}$ as a closed formal subscheme (with the same support $U_{\breve{\tau}}$).

Since $U_{\breve{\tau}}$ is the union of $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\tau}'}$ with $\breve{\tau}'$ running through faces of $\breve{\tau}$ in $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, which are cones in $\mathbf{P}^+_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$, the tautological degeneration data over $\widetilde{\mathfrak{U}}_{\breve{\tau}}$ satisfies the positivity condition (with respect to the ideal defining $U_{\breve{\tau}}$), and we obtain by Mumford's construction a degenerating family $({}^\heartsuit\widetilde{G},\,{}^\heartsuit\widetilde{\lambda},\,{}^\heartsuit\widetilde{i},\,{}^\heartsuit\widetilde{\alpha}_{\widetilde{\mathcal{H}}})\to\widetilde{\mathfrak{U}}_{\breve{\tau}}$ as in [38, §6.2.5; especially the paragraph preceding Def. 6.2.5.17], called a *Mumford family.* Note that a Mumford family is defined in the sense of relative schemes, namely as a functorial assignment to each affine open formal subscheme $\mathrm{Spf}(R)$ of $\widetilde{\mathfrak{U}}_{\breve{\tau}}$ a degenerating family over $\mathrm{Spec}(R)$. Therefore (6) of Theorem 1.41 applies, and implies the existence of a canonical (strata-preserving) morphism $\widetilde{\mathfrak{U}}_{\breve{\tau}}\to\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ under which $({}^\heartsuit\widetilde{G},\,{}^\heartsuit\widetilde{\lambda},\,{}^\heartsuit\widetilde{i},\,{}^\heartsuit\widetilde{\alpha}_{\widetilde{\mathcal{H}}})\to\widetilde{\mathfrak{U}}_{\breve{\tau}}$ is the pullback of $(\widetilde{G},\widetilde{\lambda},\widetilde{i},\widetilde{\alpha}_{\widetilde{\mathcal{H}}})\to\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$. Moreover, if

$\breve{\tau}' \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, then the morphisms from $\widetilde{\mathfrak{U}}_{\breve{\tau}}$ and from $\widetilde{\mathfrak{U}}_{\breve{\tau}'}$ to $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ agree over the intersection $\widetilde{\mathfrak{U}}_{\breve{\tau}} \cap \widetilde{\mathfrak{U}}_{\breve{\tau}'}$.

By taking the closures of the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-strata (not as closed subschemes of the supports, but as closed formal subschemes, as in the second last paragraph preceding Condition 3.8), we obtain canonical morphisms $\mathfrak{U}_{\breve{\tau}} \to \mathsf{N}^{\mathrm{tor}}$ for all $\breve{\tau}$ in $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, which patch together, cover all strata above $[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau)]$, and define (4.4) as desired. □

By (5) of Theorem 1.41, we have a canonical isomorphism

$$(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \cong \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}. \tag{4.5}$$

By the very constructions, we may and we shall identify the pullback of f^{tor} to $(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$ with the canonical morphism $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$. By abuse of notation, we shall also denote this pullback by

$$f^{\mathrm{tor}} : \widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}.$$

For each $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, let $\mathfrak{U}_{[\breve{\tau}]}$ denote the image of $\mathfrak{U}_{\breve{\tau}}$ under (4.2), which is isomorphic to $\mathfrak{U}_{\breve{\tau}}$ as a formal scheme over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. By admissibility of $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$, we know that the set $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ is *finite*. Then $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ can be constructed by gluing the finite collection $\{\mathfrak{U}_{[\breve{\tau}]}\}_{[\breve{\tau}]\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}}$ of formal schemes over their intersections. Let us denote by

$$f^{\mathrm{tor}}_{[\breve{\tau}]} : \mathfrak{U}_{[\breve{\tau}]} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$$

the restriction of f^{tor} to $\mathfrak{U}_{[\breve{\tau}]}$. If we choose a representative $\breve{\tau}$ of $[\breve{\tau}]$, then we can identify $f^{\mathrm{tor}}_{[\breve{\tau}]} : \mathfrak{U}_{[\breve{\tau}]} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ with the canonical morphism $f^{\mathrm{tor}}_{\breve{\tau}} : \mathfrak{U}_{\breve{\tau}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ induced by the canonical morphism $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$. Let us denote by

$$g_{\breve{\tau}} : \mathfrak{U}_{\breve{\tau}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}},$$

$$h : \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}} \to C_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}}},$$

and

$$h_{\tau} : \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$$

the canonical morphisms. Then we have a canonical identification $f^{\mathrm{tor}}_{\breve{\tau}} = h_{\tau} \circ g_{\breve{\tau}}$. (Note that $g_{\breve{\tau}}$ is a morphism between affine formal schemes over $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$, and that h_{τ} is the pullback of h to the affine formal scheme $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$.)

For simplicity, let us view $\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$ and $\mathscr{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$ as sheaves over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, and suppress $(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_*$ and $(\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_*$ from the notation. For push-forwards (to $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$) of sheaves over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$, we shall use the notation $\hat{\oplus}$ to denote the completion with respect to (the push-forward of) the ideal of definition of $\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$.

Based on Lemma 4.1, we have the following important facts:

Lemma 4.6. (1) *For any $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, and any integer $d \geq 0$, we have the canonical isomorphisms*

$$R^d(f^{\mathrm{tor}}_{\breve{\tau}})_*(\mathscr{O}_{\mathfrak{U}_{\breve{\tau}}}) \cong \underset{\breve{\ell}\in\breve{\sigma}^{\perp}\cap\breve{\tau}^{\vee}}{\hat{\oplus}} R^d(h_{\tau})_*(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})) \tag{4.7}$$

and

$$R^d(f^{\mathrm{tor}}_{\breve{\tau}})_*(\mathscr{O}_{U_{\breve{\tau}}}) \cong \bigoplus_{\breve{\ell}\in\breve{\tau}^\vee-\breve{\tau}^\vee_{\breve{\sigma}}} R^d(h_\tau)_*(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})) \tag{4.8}$$

over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$.

(2) *For any* $\gamma\in\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$*, we have a commutative diagram*

$$\begin{tikzcd} \mathfrak{U}_{\breve{\tau}} \arrow[r, "\gamma"] \arrow[d, "g_{\breve{\tau}}"'] & \mathfrak{U}_{\gamma\breve{\tau}} \arrow[d, "g_{\gamma\breve{\tau}}"] \\ \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}} \arrow[r, "\gamma"] \arrow[d, "h_\tau"'] & \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}} \arrow[d, "h_\tau"] \\ \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \arrow[r, equal] & \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \end{tikzcd}$$

of formal schemes, (naturally) compatible with the commutative diagram

$$\begin{tikzcd} U_{\breve{\tau}} \arrow[r, "\gamma"] \arrow[d, "g_{\breve{\tau}}"'] & U_{\gamma\breve{\tau}} \arrow[d, "g_{\gamma\breve{\tau}}"] \\ (\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_\tau \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}} \arrow[r, "\gamma"] \arrow[d, "h_\tau"'] & (\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_\tau \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}} \arrow[d, "h_\tau"] \\ (\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_\tau \arrow[r, equal] & (\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_\tau \end{tikzcd}$$

of their supports. Then (4.7) *and* (4.8) *are compatible with the canonical isomorphisms* $\gamma^*\mathscr{O}_{\mathfrak{U}_{\gamma\breve{\tau}}}\to\mathscr{O}_{\mathfrak{U}_{\breve{\tau}}}$ *induced by the canonical isomorphisms* $\gamma^*\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\gamma\breve{\ell})\xrightarrow{\sim}\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$ *over* $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$.

(3) *For any integer* $d\geq 0$*, if* $\breve{\tau}'$ *is a face of* $\breve{\tau}$*, then the canonical morphism* $R^d(f^{\mathrm{tor}}_{\breve{\tau}})_*\mathscr{O}_{\mathfrak{U}_{\breve{\tau}}}\to R^d(f^{\mathrm{tor}}_{\breve{\tau}'})_*\mathscr{O}_{\mathfrak{U}_{\breve{\tau}'}}$ *induced by restriction from* $\mathfrak{U}_{\breve{\tau}}$ *to* $\mathfrak{U}_{\breve{\tau}'}$ *corresponds to the morphism*

$$\widehat{\bigoplus_{\breve{\ell}\in\breve{\sigma}^\perp\cap\breve{\tau}^\vee}} R^d(h_\tau)_*(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}))\to \widehat{\bigoplus_{\breve{\ell}\in\breve{\sigma}^\perp\cap(\breve{\tau}')^\vee}} R^d(h_\tau)_*(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}))$$

over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$*, and the canonical morphism* $R^d(f^{\mathrm{tor}}_{\breve{\tau}})_*\mathscr{O}_{U_{\breve{\tau}}}\to R^d(f^{\mathrm{tor}}_{\breve{\tau}'})_*\mathscr{O}_{U_{\breve{\tau}'}}$ *induced by restriction from* $U_{\breve{\tau}}$ *to* $U_{\breve{\tau}'}$ *corresponds to the morphism*

$$\bigoplus_{\breve{\ell}\in\breve{\tau}^\vee-\breve{\tau}^\vee_{\breve{\sigma}}} R^d(h_\tau)_*(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}))\to \bigoplus_{\breve{\ell}\in(\breve{\tau}')^\vee-(\breve{\tau}')^\vee_{\breve{\sigma}}} R^d(h_\tau)_*(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}))$$

over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$*. Both of these morphisms send* $R^d(h_\tau)_*(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}))$ *(identically) to* $R^d(h_\tau)_*(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}))$ *when it is defined on both sides, and to zero otherwise.*

4B. **Relative cohomology of structural sheaves.** By (4.5), we shall identify $(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}$ with $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$, and identify $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ with $(\Xi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_{\tau}$. For simplicity of notation, we shall use $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ and $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ more often than their counterparts.

Recall that $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is an abelian scheme torsor over the finite étale cover $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$ of $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ (see Lemma 1.17). Let $(A, \lambda_A, i_A, \alpha_{\mathcal{H}_h})$ be the tautological tuple over $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$. Let T (resp. $T^{\vee}$) be the split torus with character group X (resp. Y). For simplicity of notation, we shall denote the pullbacks of A, $A^{\vee}$, T, and $T^{\vee}$, respectively, by the same symbols. The pullback of G (resp. $G^{\vee}$) to $\mathfrak{X}_{\Phi_{\mathcal{H}},\mathsf{Z}_{\mathcal{H}},\tau}$ is an extension of A (resp. $A^{\vee}$) by T (resp. $T^{\vee}$), and this extension is a pullback of the tautological extension $G^{\natural}$ (resp. $G^{\vee,\natural}$) over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$. For simplicity, we shall also denote the pullbacks of $G^{\natural}$ and $G^{\vee,\natural}$, respectively, by the same symbols.

Lemma 4.9. *The morphism $h : \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is proper and smooth, and is a torsor under the pullback to $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ of an abelian scheme $\mathbb{Z}^{\times}_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(\widetilde{X}, A)^{\circ} \to \mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}}$.*

Proof. By forming equivariant quotients, we may (and we shall) replace $\widetilde{\mathcal{H}}$ and $\mathcal{H}$ with principal level subgroups of some level n, so that $\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}} = \widetilde{C}_{\check{\Phi}_n,\check{\delta}_n}$ and $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} = C_{\Phi_n,\delta_n}$ are abelian schemes over $\mathsf{M}_{\mathcal{H}}^{\Phi_{\mathcal{H}}} = \mathsf{M}_n^{\mathsf{Z}_n}$. For simplicity, let us denote the kernel of $\widetilde{C}_{\check{\Phi}_n,\check{\delta}_n} \to C_{\Phi_n,\delta_n}$ by C, viewed as a scheme over $\mathsf{M}_n^{\mathsf{Z}_n}$.

While the abelian scheme torsor $\widetilde{C}_{\check{\Phi}_n,\check{\delta}_n} \to \mathsf{M}_n^{\mathsf{Z}_n}$ parameterizes liftings (to level n) of pairs of the form $(\check{c} : \check{X} \to A^{\vee}, \check{c}^{\vee} : \check{Y} \to A)$ satisfying the compatibility $\check{c}\check{\phi} = \lambda_A \check{c}^{\vee}$ and the liftability and pairing conditions, and while the abelian scheme torsor $C_{\Phi_n,\delta_n} \to \mathsf{M}_n^{\mathsf{Z}_n}$ parameterizes liftings (to level n) of pairs of the form $(c : X \to A^{\vee}, c^{\vee} : Y \to A)$ satisfying the compatibility $c\phi = \lambda_A c^{\vee}$ and the liftability and pairing conditions, the scheme $C \to \mathsf{M}_n^{\mathsf{Z}_n}$ parameterizes liftings of pairs of the form $(\widetilde{c} : \widetilde{X} \to A^{\vee}, \widetilde{c}^{\vee} : \widetilde{Y} \to A)$ satisfying the compatibility $\widetilde{c}\widetilde{\phi} = \lambda_A \widetilde{c}^{\vee}$ and the liftability and pairing conditions induced by the ones of the pairs over $\widetilde{C}_{\check{\Phi}_n,\check{\delta}_n} \to \mathsf{M}_n^{\mathsf{Z}_n}$. Therefore, the same (component annihilating) argument in [38, §§6.2.3–6.2.4] shows that the kernel C of h is an abelian scheme $\mathbb{Z}^{\times}_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(\widetilde{X}, A)^{\circ}$.

Consequently, all geometric fibers of h are smooth and have the same dimension (as the relative dimension of $C \to \mathsf{M}_n^{\mathsf{Z}_n}$). Since both $\widetilde{C}_{\check{\Phi}_n,\check{\delta}_n}$ and C_{Φ_n,δ_n} are smooth over S_0, the morphism h is smooth by [21, IV-3, 15.4.2 e')⇒b), and IV-4, 17.5.1 b)⇒a)]. By [7, §2.2, Prop. 14], smooth morphisms between schemes have sections étale locally. This shows that h is a torsor under the pullback of C to C_{Φ_n,δ_n}. (Regardless of this argument, the morphism h is proper because the morphism $\widetilde{C}_{\check{\Phi}_n,\check{\delta}_n} \to \mathsf{M}_n^{\mathsf{Z}_n}$ is.) □

Consider the union $\widetilde{\mathfrak{N}}_{\check{\sigma},\tau}$ of the cones $\check{\tau}$ in $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, which has a *closed covering* by the closures $\check{\tau}^{\mathrm{cl}}$ (in $\widetilde{\mathfrak{N}}_{\check{\sigma},\tau}$) of the cones $\check{\tau}$ in $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$ (with natural incidence relations among their closures inherited from their realizations as locally closed subsets of $(\mathbf{S}_{\check{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}}$). By definition, the nerve of the open covering $\{\mathfrak{U}_{\check{\tau}}\}_{\check{\tau}\in\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}}$ of $\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, or equivalently the open covering $\{U_{\check{\tau}}\}_{\check{\tau}\in\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}}$ of $(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\check{\sigma}}(\tau)$ (by the supports of the formal schemes $\{\mathfrak{U}_{\check{\tau}}\}_{\check{\tau}\in\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}}$), is naturally identified with the nerve of the (locally finite) closed covering $\{\check{\tau}^{\mathrm{cl}}\}_{\check{\tau}\in\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}}$ of $\widetilde{\mathfrak{N}}_{\check{\sigma},\tau}$. Then the nerve of the open covering

$$\{\mathfrak{U}_{[\check{\tau}]}\}_{[\check{\tau}]\in\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}} \quad \text{of} \quad (\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \cong \widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},$$

or equivalently the open covering

$$\{U_{\check{\tau}}\}_{[\check{\tau}]\in\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}} \quad \text{of} \quad \widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]} \cong (\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$$

of the supports of formal schemes, is naturally identified with the nerve of the (finite) closed covering

$$\{[\check{\tau}]^{\mathrm{cl}}\}_{[\check{\tau}]\in\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}} \quad \text{of} \quad \mathfrak{N}_{\check{\sigma},\tau} := \widetilde{\mathfrak{N}}_{\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},$$

where $[\check{\tau}]^{\mathrm{cl}}$ denotes the closure of $[\check{\tau}]$ in $\mathfrak{N}_{\check{\sigma},\tau}$.

For any sheaf $\mathscr{M}$ on $(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ (such as $\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}} \cong \mathscr{O}_{\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}}$), let us define for any integer $d \geq 0$ the constructible sheaf $\mathscr{H}^d(\mathscr{M})$ on $\mathfrak{N}_{\check{\sigma},\tau}$ which associates with each $[\check{\tau}]$ in $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ locally constant coefficients in

$$\mathscr{H}^d(\mathscr{M})([\check{\tau}]^{\mathrm{cl}}) := H^d(U_{[\check{\tau}]}, \mathscr{M}|_{U_{[\check{\tau}]}}).$$

Then, by [18, II, 5.4.1], there is a spectral sequence

$$E_2^{c,d} := H^c(\mathfrak{N}_{\check{\sigma},\tau}, \mathscr{H}^d(\mathscr{M})) \Rightarrow H^{c+d}((\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, \mathscr{M}). \tag{4.10}$$

The construction of $\mathfrak{N}_{\check{\sigma},\tau}$ depends only on the cone decomposition $\Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, while the constructions of both $\mathscr{H}^d(\mathscr{M})$ and the spectral sequence (4.10) are compatible with restrictions to affine open subschemes of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$. Therefore, we can define the sheaves $\underline{\mathscr{H}}^d(\mathscr{M})$ (of constructible sheaves on $\mathfrak{N}_{\check{\sigma},\tau}$) over $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$, and obtain a spectral sequence

$$E_2^{c,d} := H^c(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{M})) \Rightarrow R^{c+d} f_*^{\mathrm{tor}}(\mathscr{M}). \tag{4.11}$$

Here $H^c(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{M}))$ is interpreted as a sheaf on $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$, and the formation of (4.11) is compatible with morphisms in $\mathscr{M}$. In particular, we have compatible spectral sequences

$$E_2^{c,d} := H^c(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \Rightarrow R^{c+d} f_*^{\mathrm{tor}}(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}) \tag{4.12}$$

and

$$E_2^{c,d} := H^c(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})) \Rightarrow R^{c+d} f_*^{\mathrm{tor}}(\mathscr{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}). \tag{4.13}$$

To calculate the left-hand sides of (4.12) and (4.13), we define the sheaves $\underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}})$ and $\underline{\mathscr{H}}^d(\mathscr{O}_{(\widetilde{\Xi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_{\tau}})$ (of constructible sheaves) on $\widetilde{\mathfrak{N}}_{\check{\sigma},\tau}$ (in the obvious way), which, by Lemma 4.6, carry canonical equivariant actions of the group $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$, and descend to the sheaves $\underline{\mathscr{H}}^d(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})$ and $\underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})$

on $\mathfrak{N}_{\breve{\sigma},\tau}$, respectively. Hence we obtain compatible spectral sequences

$$E_2^{c-e,e} := H^{c-e}(\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, H^e(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}))) \Rightarrow H^c(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \tag{4.14}$$

and

$$E_2^{c-e,e} := H^{c-e}(\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, H^e(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{O}_{(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau}))) \Rightarrow H^c(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})). \tag{4.15}$$

Lemma 4.16. *For any $d \geq 0$, the canonical morphisms*

$$R^d(h_\tau)_*(\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}) \to H^0(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}})) \tag{4.17}$$

and

$$R^d(h_\tau)_*(\mathscr{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}) \to H^0(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{O}_{(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau})) \tag{4.18}$$

are isomorphisms compatible with each other. Moreover, for any integer $e > 0$, we have

$$H^e(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}})) = 0 \tag{4.19}$$

and

$$H^e(\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{O}_{(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_\tau})) = 0. \tag{4.20}$$

Proof. By (4.7), we have

$$\underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}})(\breve{\tau}^{\mathrm{cl}}) \cong R^d(f^{\mathrm{tor}}_{\breve{\tau}})_*(\mathscr{O}_{\mathfrak{U}_{\breve{\tau}}}) \cong \underset{\breve{\ell}\in\breve{\sigma}^\perp\cap\breve{\tau}^\vee}{\hat{\oplus}} R^d(h_\tau)_*(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})),$$

and for any face $\breve{\tau}'$ of $\breve{\tau}$, the canonical morphism

$$\underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}})(\breve{\tau}^{\mathrm{cl}}) \to \underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}})((\breve{\tau}')^{\mathrm{cl}})$$

sends the subsheaf $R^d(h_\tau)_*(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}))$ either (identically) to $R^d(h_\tau)_*(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}))$ when $\breve{\ell} \in \breve{\sigma}^\perp \cap (\breve{\tau}')^\vee$, or to zero otherwise. Since $\underset{\breve{\ell}\notin\breve{\tau}^\vee}{\cup} \breve{\tau}^{\mathrm{cl}} = \underset{\breve{\ell}\notin\breve{\tau}^\vee}{\cup} \breve{\tau}$ is a contractible or empty subset of $\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}$ for any given $\breve{\ell} \in \breve{\sigma}^\perp$ (because it is a deformation retract, defined compatibly over the polyhedral cones overlapping with the boundary, of the convex subset of $\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}$ over which $\breve{\ell}$ is negative), this shows (4.19) for $e > 0$ as usual (by the argument in [36, Ch. I, §3]). On the other hand, since $\underset{\breve{\tau}\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}{\cap} (\breve{\sigma}^\perp \cap \breve{\tau}^\vee) = \tau^\vee$, we see that (4.17) is an isomorphism. The proofs for (4.20) and (4.18) are similar. (Since the nerves involve infinitely many cones, let us briefly explain why we can work weight-by-weight as in [36, Ch. I, §3]. This is because, up to replacing the cone decompositions with locally finite refinements not necessarily carrying $\Gamma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$-actions, which is harmless for proving this lemma, we can compute the cohomology as a limit using unions of finite cone decompositions on expanding convex polyhedral subcones, by proving inductively that the cohomology of one degree lower has the desired properties, using [52, Thm. 3.5.8]; then we can consider the associated graded pieces defined by the completions, and work weight-by-weight, because taking cohomology commutes with taking infinite direct sums for Čech complexes defined by finite coverings, as desired.) □

Lemma 4.21. *The topological space $\mathfrak{N}_{\breve{\sigma},\tau}$ is homotopic to the real torus*

$$\mathbf{T}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} := (\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},$$

whose cohomology groups (by contractibility of $(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$) are

$$H^j(\mathbf{T}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},\mathbb{Z}) \cong H^j(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},\mathbb{Z}) \cong \wedge^j(\mathrm{Hom}_{\mathbb{Z}}(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},\mathbb{Z}))$$

for any $j \geq 0$. Over $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$, we have a canonical isomorphism

$$H^j(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},\mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \cong \wedge^j(\mathrm{Hom}_{\mathcal{O}}(Q^{\vee},\underline{\mathrm{Lie}}_{T^{\vee}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}})). \tag{4.22}$$

Proof. Since $\widetilde{\sigma}$ is a top-dimensional cone in $\mathbf{P}^{+}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$, any $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ (which has $\breve{\sigma}$ as a face) is generated by $\breve{\sigma}$ and some rational basis vectors not contained in the image of the first morphism in (3.7). Moreover, the image of $\breve{\tau}$ under the second morphism in (3.7) is contained in $\tau \subset \mathbf{P}^{+}_{\Phi_{\mathcal{H}}}$. By choosing some (noncanonical) splitting of $s_{\breve{X}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} : \breve{X} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \twoheadrightarrow \widetilde{X} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, we can decompose the real vector space $(\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}}$ (noncanonically) as a direct sum $(\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}} \oplus (\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}} \oplus (\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$, on which the action of $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ is realized by its canonical translation action on the second factor. Along the directions of $(\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}})^{\vee}_{\mathbb{R}}$ and $(\mathbf{S}_{\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}$, we can contract $\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}$ (say, towards some arbitrarily chosen points in the convex sets $\widetilde{\sigma}$ and τ) in a way compatible with the actions of $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. Therefore, $\mathfrak{N}_{\breve{\sigma},\tau} = \widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ is homotopic to the real torus $\mathbf{T}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} = (\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}})^{\vee}_{\mathbb{R}}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$.

The canonical isomorphism (4.22) then follows from the composition of the following canonical isomorphisms:

$$\begin{aligned}
H^j(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},\mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} &\cong (\wedge^j(\mathrm{Hom}_{\mathbb{Z}}(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},\mathbb{Z}))) \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \\
&\cong (\wedge^j(\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{\mathcal{O}}(\widetilde{X},X),\mathbb{Z}_{(\Box)}))) \underset{\mathbb{Z}_{(\Box)}}{\otimes} \mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}} \\
&\cong \wedge^j(\mathrm{Hom}_{\mathcal{O}}(Q^{\vee},\mathrm{Hom}_{\mathbb{Z}}(Y,\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}))) \\
&\cong \wedge^j(\mathrm{Hom}_{\mathcal{O}}(Q^{\vee},\underline{\mathrm{Lie}}_{T^{\vee}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}})).
\end{aligned}$$
□

Lemma 4.23. *There are compatible canonical isomorphisms*

$$R^d(h_{\tau})_*(\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}) \cong \wedge^d(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^{\vee},\underline{\mathrm{Lie}}_{A^{\vee}/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}})$$

and

$$R^d(h_{\tau})_*(\mathscr{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}) \cong \wedge^d(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^{\vee},\underline{\mathrm{Lie}}_{A^{\vee}/\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}))$$

for any integer $d \geq 0$.

Proof. By Lemma 4.9, the morphism $h : \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ is a torsor under an abelian scheme $\mathbb{Z}^{\times}_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q,A)^{\circ}$ (and hence has a section étale locally). Since the cohomology of abelian schemes (with coefficients in the structural sheaves) are free and are compatible with arbitrary base changes (see [5, Prop. 2.5.2] and [43, §5]), we obtain compatible canonical isomorphisms

$$\begin{aligned} R^d(h_\tau)_*(\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}) &\cong \wedge^d(\underline{\mathrm{Lie}}_{(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q,A)^\circ)^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}) \\ &\cong \wedge^d(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee,\underline{\mathrm{Lie}}_{A^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}})) \end{aligned}$$

and

$$\begin{aligned} R^d(h_\tau)_*(\mathscr{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}) &\cong \wedge^d(\underline{\mathrm{Lie}}_{(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q,A)^\circ)^\vee/\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}) \\ &\cong \wedge^d(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee,\underline{\mathrm{Lie}}_{A^\vee/\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})) \end{aligned}$$

for any integer $d \geq 0$. $\square$

Proposition 4.24. *There are compatible canonical isomorphisms*

$$\begin{aligned} (4.25)\quad & H^c(\mathfrak{N}_{\check{\sigma},\tau},\underline{\mathscr{H}}^d(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \\ &\cong (\wedge^c(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee,\underline{\mathrm{Lie}}_{T^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}))) \\ &\qquad \underset{\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}}{\otimes} (\wedge^d(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee,\underline{\mathrm{Lie}}_{A^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}))) \end{aligned}$$

and

$$\begin{aligned} (4.26)\quad & H^c(\mathfrak{N}_{\check{\sigma},\tau},\underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})) \\ &\cong (\wedge^c(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee,\underline{\mathrm{Lie}}_{T^\vee/\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}))) \\ &\qquad \underset{\mathscr{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}{\otimes} (\wedge^d(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee,\underline{\mathrm{Lie}}_{A^\vee/\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}))) \end{aligned}$$

for any integers $c, d \geq 0$.

Proof. By Lemma 4.16, the spectral sequences (4.14) and (4.15) degenerate and show that for any integers c and d we have compatible canonical isomorphisms

$$\begin{aligned} (4.27)\quad & H^c(\mathfrak{N}_{\check{\sigma},\tau},\underline{\mathscr{H}}^d(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \\ &\cong H^c(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, H^0(\widetilde{\mathfrak{N}}_{\check{\sigma},\tau},\underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}}))) \\ &\cong H^c(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},\mathbb{Z}) \underset{\mathbb{Z}}{\otimes} R^d(h_\tau)_*(\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}) \end{aligned}$$

and

$$\begin{aligned} (4.28)\quad & H^c(\mathfrak{N}_{\check{\sigma},\tau},\underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})) \\ &\cong H^c(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}, H^0(\widetilde{\mathfrak{N}}_{\check{\sigma},\tau},\underline{\mathscr{H}}^d(\mathscr{O}_{(\Xi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})_\tau}))) \\ &\cong H^c(\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}},\mathbb{Z}) \underset{\mathbb{Z}}{\otimes} R^d(h_\tau)_*(\mathscr{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}). \end{aligned}$$

Now combine (4.27) and (4.28) with Lemmas 4.21 and 4.23. $\square$

Lemma 4.29. *The spectral sequence* (4.12) *degenerates at E_2 terms. Consequently, since the choice of the stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ is arbitrary, by Grothendieck's fundamental theorem* [21, III-1, 4.1.5] (*and by fpqc descent for the property of local freeness* [20, VIII, 1.11]), *the sheaf $R^b f^{\mathrm{tor}}_*(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})$ is locally free of the same rank as $\wedge^b(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$.*

If, for every **maximal point** *s of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$* (*see* [22, 0, 2.1.2]), *we have*

$$\begin{aligned}(4.30)\quad &\dim_{k(s)}((R^b f^{\mathrm{tor}}_*(\mathscr{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})) \underset{\mathscr{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}{\otimes} k(s)) \\ &\qquad \geq \dim_{k(s)}((R^b f^{\mathrm{tor}}_*(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \underset{\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}}{\otimes} k(s)),\end{aligned}$$

then the spectral sequence (4.13) *degenerates at E_2 terms as well, and there is a canonical isomorphism*

$$(4.31)\qquad R^b f^{\mathrm{tor}}_*(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \mathscr{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \xrightarrow{\sim} R^b f^{\mathrm{tor}}_*(\mathscr{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}).$$

Proof. Let $\mathrm{Spf}(R, I)$ be any connected affine open formal subscheme of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$, with the ideal of definition I satisfying $\mathrm{rad}(I) = I$ for simplicity. Since $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ is smooth and of finite type over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, the ring R is a noetherian domain. (See [39, 33.I and 34.A].) Since $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ is a smooth subscheme of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, the quotient R/I is also a noetherian domain. Let $K := \mathrm{Frac}(R)$ and $k := \mathrm{Frac}(R/I)$ be the fraction fields. By abuse of notation, we shall denote pullbacks of schemes to $\mathrm{Spec}(K)$ (resp. $\mathrm{Spec}(k)$) by the subscript K (resp. k).

Since we have an exact sequence

$$0 \to \underline{\mathrm{Lie}}_{T^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \to \underline{\mathrm{Lie}}_{G^{\vee,\natural}/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \to \underline{\mathrm{Lie}}_{A^\vee/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \to 0$$

of locally free sheaves, we have an equality

$$\begin{aligned}(4.32)\quad &\sum_{c+d=b} \dim_K(\wedge^c(\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{A^\vee_K}))) \underset{K}{\otimes} (\wedge^d(\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{T^\vee_K}))) \\ &\qquad = \dim_K(\wedge^b(\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{G^{\vee,\natural}_K}))) \\ &\qquad\qquad = \dim_K(\wedge^b(\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{G^\vee_K}))),\end{aligned}$$

and an analogous equality with K replaced with k.

By construction of the spectral sequences (4.12) and (4.13), by the canonical isomorphisms (4.25) and (4.26), and by the equality (4.32), we have

$$\begin{aligned}(4.33)\quad &\sum_{c+d=b} \dim_K(H^c(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \underset{\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}}{\otimes} K) \\ &\qquad = \dim_K(\wedge^b(\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{G^\vee_K}))) \\ &\qquad\qquad \geq \dim_K((R^b f^{\mathrm{tor}}_*(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \underset{\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}}{\otimes} K)\end{aligned}$$

and

$$(4.34)\qquad \sum_{c+d=b} \dim_k(H^c(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^d(\mathscr{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})) \underset{\mathscr{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}{\otimes} k)$$
$$= \dim_k(\wedge^b(\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{G_k^\vee})))$$
$$\geq \dim_k(R^b f_*^{\mathrm{tor}}(\mathscr{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}) \underset{\mathscr{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}{\otimes} k).$$

Since the pullback of f^{tor} to the open dense subscheme $\mathsf{M}_{\mathcal{H}}$ of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is simply the abelian scheme $f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$, we have

$$(R^b f_*^{\mathrm{tor}}(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}} \cong R^b f_*(\mathscr{O}_{\mathsf{N}})$$
$$\cong \wedge^b \underline{\mathrm{Lie}}_{\mathsf{N}^\vee/\mathsf{M}_{\mathcal{H}}} \cong \wedge^b(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})).$$

Since the canonical morphism $\mathrm{Spec}(K) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ factors through some maximal point of $\mathsf{M}_{\mathcal{H}}$, this implies that the inequality in (4.33) is an equality, and hence that the spectral sequence (4.12) degenerates at E_2 terms after pullback to K. Since all E_2 terms of this spectral sequence are locally free sheaves, this shows that (4.12) degenerates at E_2 terms after pullback to R. Since the choice of R is arbitrary, this shows that (4.12) degenerates over the whole $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$, and hence $R^b f_*^{\mathrm{tor}}(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})$ is locally free of the same rank as $\wedge^b(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}))$ over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$. (Nevertheless, since f^{tor} is not necessarily flat, this does not imply that the formation of $R^b f_*^{\mathrm{tor}}(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})$ is compatible with arbitrary base change.)

Since the canonical morphism $\mathrm{Spec}(k) \to \mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$ factors through some maximal point of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$, the inequality (4.30) implies that

$$\dim_k(R^b f_*^{\mathrm{tor}}(\mathscr{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}) \underset{\mathscr{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}{\otimes} k)$$
$$\geq \dim_k((R^b f_*^{\mathrm{tor}}(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \underset{\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}}{\otimes} k)$$
$$= \dim_K((R^b f_*^{\mathrm{tor}}(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \underset{\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}}{\otimes} K),$$

and hence the *equality* in (4.33) implies the equality in (4.34), because

$$\dim_k(\wedge^b(\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{G_k^\vee}))) = \dim_K(\wedge^b(\mathrm{Hom}_{\mathcal{O}}(Q^\vee, \mathrm{Lie}_{G_K^\vee}))).$$

Therefore, by the same reasoning as in the case of (4.12) above, the spectral sequence (4.13) also degenerates at E_2 terms. Since the spectral sequences (4.12) and (4.13) are compatible with each other (by their very construction), their degeneracy implies that the canonical morphism

$$R^b f_*^{\mathrm{tor}}(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}) \underset{\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}}{\otimes} \mathscr{O}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}} \to R^b f_*^{\mathrm{tor}}(\mathscr{O}_{\widetilde{\mathsf{Z}}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}})$$

is an isomorphism (by comparing graded pieces) and induces (4.31). □

Remark 4.35. By upper semicontinuity for proper flat morphisms (see [43, §5, Cor. (a)]), the assumption (4.30) is satisfied when f^{tor} is flat, or equivalently when Condition 3.17 is satisfied (by Proposition 3.18), which can be achieved by refining both $\widetilde{\Sigma}$ and Σ (by Proposition 3.19).

Corollary 4.36. *For any integer $b \geq 0$, the canonical (cup product) morphism $\wedge^b(R^1 f^{\mathrm{tor}}_*(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})) \to R^b f^{\mathrm{tor}}_*(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})$ is an isomorphism.*

Proof. As in Lemma 4.29, by properness of f^{tor}, this is true if and only if it is true over the formal completion along each stratum $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}$, which is the case because the canonical morphism induces isomorphisms on all graded pieces defined by spectral sequences such as (4.12), which are compatible with cup products by the very construction (see [18, II, §§5–6]). $\square$

4C. **Degeneracy of the (relative) Hodge spectral sequence.** As in (3c) of Theorem 2.15, let $\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) := R^i f^{\mathrm{tor}}_* \overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ be the (relative) log de Rham cohomology. By the definition of $\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$ as the "relative hypercohomology", the natural (Hodge) filtration on the complex $\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ defines the (relative) Hodge spectral sequence (2.20):

$$E_1^{a,b} := R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \Rightarrow \underline{H}^{a+b}_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}).$$

By (3a) of Theorem 2.15 (which we have proved in Section 3C), there is a canonical isomorphism

$$\begin{aligned}\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} &\cong \wedge^a\Big[(f^{\mathrm{tor}})^*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))\Big]\\ &\cong (f^{\mathrm{tor}})^*\Big[\wedge^a(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))\Big]\end{aligned}$$

of locally free sheaves over $\mathsf{N}^{\mathrm{tor}}$. Then (by the projection formula [21, 0_{I}, 5.4.10.1]) we have canonical isomorphisms

$$R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \cong (R^b f^{\mathrm{tor}}_*(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} (\wedge^a(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}^\vee_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))). \tag{4.37}$$

Lemma 4.38. *If $R^b f^{\mathrm{tor}}_*(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})$ is locally free for every integer $b \geq 0$, then the spectral sequence (2.20) degenerates at the E_1 terms.*

Proof. By (4.37), if $R^b f^{\mathrm{tor}}_*(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})$ is locally free for every integer $b \geq 0$, then all the E_1 terms $R^b f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ of the spectral sequence (2.20) are locally free. Therefore, to show that (2.20) degenerates at E_1 terms, it suffices to show that it degenerates at E_1 terms over the open dense subscheme $\mathsf{M}_{\mathcal{H}}$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, which is true because $f^{\mathrm{tor}}|_{\mathsf{N}} = f : \mathsf{N} \to \mathsf{M}_{\mathcal{H}}$ is an abelian scheme. (See for example [5, Prop. 2.5.2].) $\square$

This proves (3c) of Theorem 2.15, because the local freeness of $R^b f^{\mathrm{tor}}_*(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})$ has been established in Section 4B for every integer $b \geq 0$.

4D. **Gauss–Manin connections with log poles.** In Section 3C, we proved the log smoothness of $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ by verifying Lemma 3.11. For simplicity, let us set

$$\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} := \Omega^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}[d\log\infty] \quad \text{and} \quad \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0} := \Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}[d\log\infty].$$

Then (3.12) can be rewritten as the exact sequence

$$0 \to (f^{\mathrm{tor}})^*(\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}) \to \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0} \to \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to 0, \tag{4.39}$$

which induces the *Koszul filtration* [35, 1.2, 1.3]

$$\mathsf{K}^a(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}) := \mathrm{image}(\overline{\Omega}^{\bullet-a}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0} \underset{\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}}{\otimes} (f^{\mathrm{tor}})^*(\overline{\Omega}^a_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}) \to \overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0})$$

on $\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}$, with graded pieces $\mathrm{Gr}^a_{\mathtt{K}}(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}) \cong \overline{\Omega}^{\bullet-a}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}}{\otimes} (f^{\mathrm{tor}})^*(\overline{\Omega}^a_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0})$.

On the other hand, we have the Hodge filtration

$$\mathtt{F}^a(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}) := \overline{\Omega}^{\bullet\geq a}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}$$

on $\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}$, giving the Hodge filtration

$$\mathtt{F}^a(\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})) := \mathrm{image}(R^i f^{\mathrm{tor}}_*(\mathtt{F}^a(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0})) \to R^i f^{\mathrm{tor}}_*(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}))$$

on $\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$. By applying $R^\bullet f^{\mathrm{tor}}_*$ to the short exact sequence

$$0 \to \overline{\Omega}^{\bullet-1}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}}{\otimes} (f^{\mathrm{tor}})^*(\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}) \to \mathtt{K}^2/\mathtt{K}^0 \to \overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0} \to 0, \tag{4.40}$$

we obtain in the long exact sequence the connecting homomorphisms

$$\begin{aligned} &\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) = R^i f^{\mathrm{tor}}_*(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \\ &\xrightarrow{\nabla} R^{i+1} f^{\mathrm{tor}}_*(\overline{\Omega}^{\bullet-1}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}) \cong \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}. \end{aligned} \tag{4.41}$$

As explained in [35, 1.4], the pullback of ∇ in (4.41) to $\mathsf{M}_{\mathcal{H}}$ is nothing but the usual Gauss–Manin connection on $\underline{H}^i_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}})$. Since the sheaves involved in (4.41) are all locally free,

$$\nabla : \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \to \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$$

satisfies the necessary conditions for being an integrable connection with log poles (because its restriction to the dense subscheme $\mathsf{M}_{\mathcal{H}}$ does). If we take the $\mathtt{F}$-filtration on (4.40), we obtain

$$0 \to (\mathtt{F}^{a-1}(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \underset{\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}}{\otimes} (f^{\mathrm{tor}})^*(\overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}))[-1] \to \mathtt{F}^a(\mathtt{K}^2/\mathtt{K}^0) \to \mathtt{F}^a(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}/\mathsf{S}_0}) \to 0$$

and hence the *Griffiths transversality*

$$\nabla(\mathtt{F}^a(\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}))) \subset \mathtt{F}^{a-1}(\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$$

(as in [35, Prop. 1.4.1.6]). This proves (3e) of Theorem 2.15.

Remark 4.42. By (3c) of Theorem 2.15, the (relative) Hodge spectral sequence

$$E_1^{a,i-a} := R^{i-a} f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \Rightarrow \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$$

degenerates. Then we have $\mathrm{Gr}^a_{\mathtt{F}}(\underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})) \cong R^{i-a} f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$, and we can conclude (as in [35, Prop. 1.4.1.7]) that the induced morphism $\nabla : \mathrm{Gr}^a_{\mathtt{F}} \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \to \mathrm{Gr}^{a-1}_{\mathtt{F}} \underline{H}^i_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$ agrees with the morphism

$$R^{i-a} f^{\mathrm{tor}}_*(\overline{\Omega}^a_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \to R^{i-a+1} f^{\mathrm{tor}}_*(\overline{\Omega}^{a-1}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$$

defined by cup product with the Kodaira–Spencer class defined by the extension class of (4.39). We will revisit a special case of this in Section 6B.

5. Polarizations

The aim of this section is to prove (3b) and (3d) of Theorem 2.15, by studying the log extension of polarizations on the relative de Rham cohomology.

5A. Identification of $R^b f_*^{\mathrm{tor}}(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})$.

By Corollary 2.12, any morphism $j_Q : Q^\vee \hookrightarrow Q$ in Lemma 2.5 (together with the tautological polarization $\lambda_{\mathsf{M}_\mathcal{H}} : G_{\mathsf{M}_\mathcal{H}} \to G^\vee_{\mathsf{M}_\mathcal{H}}$ over $\mathsf{M}_\mathcal{H}$) induces canonically a polarization

$$\lambda_{\mathsf{M}_\mathcal{H},j_Q} : \underline{\mathrm{Hom}}_\mathcal{O}(Q, G_{\mathsf{M}_\mathcal{H}})^\circ \to (\underline{\mathrm{Hom}}_\mathcal{O}(Q, G_{\mathsf{M}_\mathcal{H}})^\circ)^\vee$$

of degree prime to $\Box$, and hence an isomorphism

$$d\lambda_{\mathsf{M}_\mathcal{H},j_Q} : \underline{\mathrm{Hom}}_\mathcal{O}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_\mathcal{O}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}).$$

Therefore, it induces canonically a $\mathbb{Z}^\times_{(\Box)}$-polarization $\lambda_{\mathsf{M}_\mathcal{H},j_Q} : \mathsf{N} \to \mathsf{N}^\vee$, and hence an isomorphism $d\lambda_{\mathsf{M}_\mathcal{H},j_Q} : \underline{\mathrm{Lie}}_{\mathsf{N}/\mathsf{M}_\mathcal{H}} \to \underline{\mathrm{Lie}}_{\mathsf{N}^\vee/\mathsf{M}_\mathcal{H}}$. Over $\mathsf{M}^{\mathrm{tor}}_\mathcal{H}$, the morphisms $j_Q : Q^\vee \hookrightarrow Q$ and $d\lambda : \underline{\mathrm{Lie}}_{G/\mathsf{M}^{\mathrm{tor}}_\mathcal{H}} \to \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_\mathcal{H}}$ induce canonically an isomorphism $d\lambda_{j_Q} : \underline{\mathrm{Hom}}_\mathcal{O}(Q, \underline{\mathrm{Lie}}_{G/\mathsf{M}^{\mathrm{tor}}_\mathcal{H}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_\mathcal{O}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_\mathcal{H}})$ extending $d\lambda_{\mathsf{M}_\mathcal{H},j_Q} : \underline{\mathrm{Hom}}_\mathcal{O}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_\mathcal{O}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}})$.

Let us define $\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_\mathcal{H}} := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_\mathcal{H}}, \mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})$. Its restriction to $\mathsf{M}_\mathcal{H}$ can be canonically identified with $\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_\mathcal{H}} := \underline{\mathrm{Hom}}_{\mathscr{O}_\mathsf{N}}(\Omega^1_{\mathsf{N}/\mathsf{M}_\mathcal{H}}, \mathscr{O}_\mathsf{N})$.

Let us denote by $\jmath : \mathsf{M}_\mathcal{H} \to \mathsf{M}^{\mathrm{tor}}_\mathcal{H}$ the canonical open immersion. Then we have the commutative diagram

$$\begin{tikzcd}
f_*^{\mathrm{tor}}(\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_\mathcal{H}}) \arrow[r, equal, "\sim"', "\text{can.}"] \arrow[d, hook, "\text{res.}"'] & \underline{\mathrm{Hom}}_\mathcal{O}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}) \arrow[d, hook, "\text{res.}"] \arrow[ddd, bend left, "d\lambda_{j_Q}"] \\
\jmath_*(f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_\mathcal{H}})) \arrow[r, equal, "\sim", "\text{can.}"'] \arrow[d, dotted] & \jmath_*(\underline{\mathrm{Hom}}_\mathcal{O}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}})) \arrow[d, "\jmath_*(d\lambda_{\mathsf{M}_\mathcal{H},j_Q})"] \\
\jmath_*(R^1 f_*(\mathscr{O}_\mathsf{N})) \arrow[r, equal, "\sim", "\text{can.}"'] & \jmath_*(\underline{\mathrm{Hom}}_\mathcal{O}(Q, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}})) \\
R^1 f_*^{\mathrm{tor}}(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}) \arrow[u, hook, "\text{res.}"] & \underline{\mathrm{Hom}}_\mathcal{O}(Q, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_\mathcal{H}}) \arrow[u, hook, "\text{res.}"']
\end{tikzcd} \tag{5.1}$$

of sheaves over $\mathsf{M}^{\mathrm{tor}}_\mathcal{H}$, with the dotted arrow induced by $\jmath_*(d\lambda_{\mathsf{M}_\mathcal{H},j_Q})$. By abuse of notation, let us denote the dotted arrow also by $\jmath_*(d\lambda_{\mathsf{M}_\mathcal{H},j_Q})$. We have the following simple observation:

Lemma 5.2. *If $\jmath_*(d\lambda_{\mathsf{M}_\mathcal{H},j_Q})$ maps the image of the canonical injection*

$$f_*^{\mathrm{tor}}(\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_\mathcal{H}}) \hookrightarrow \jmath_*(f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_\mathcal{H}}))$$

isomorphically *to the image of the canonical injection*

$$R^1 f_*^{\mathrm{tor}}(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}) \hookrightarrow \jmath_*(R^1 f_*(\mathscr{O}_\mathsf{N})),$$

then (5.1) *induces the desired canonical isomorphism*

$$R^1 f_*^{\mathrm{tor}}(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}) \cong \underline{\mathrm{Hom}}_\mathcal{O}(Q, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}^{\mathrm{tor}}_\mathcal{H}}) \tag{5.3}$$

extending the canonical isomorphism $R^1 f_*(\mathscr{O}_\mathsf{N}) \cong \underline{\mathrm{Hom}}_\mathcal{O}(Q, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}})$ over $\mathsf{M}_\mathcal{H}$.

Remark 5.4. The question is whether the assumption of Lemma 5.2 can be satisfied. Since this is a question about morphisms between locally free sheaves over the normal base scheme $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, it suffices to verify the statement after localizations at points of codimension one. Therefore, since the statement is tautologically true over $\mathsf{M}_{\mathcal{H}}$, it suffices to verify it over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

5B. **Logarithmic extension of polarizations.** By construction (see Section 3A), $\widetilde{X}^\vee(1) \cong \mathrm{Hom}_{\mathcal{O}}(\widetilde{X}, \mathrm{Diff}^{-1}_{\mathcal{O}'/\mathbb{Z}}(1))$ is the submodule Q_{-2} of $Q^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}(1)$, and $\widetilde{Y}$ is the submodule Q_0 of $Q \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$. Therefore, the embedding $j_Q : Q^\vee \hookrightarrow Q$ corresponds to an element $\widetilde{\ell}_{j_Q}$ of $\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(\Box)}$. The positive definiteness of the induced pairing $\langle j_Q^{-1}(\,\cdot\,), \cdot\,\rangle_Q$ then translates to the strong positivity condition that $\langle \widetilde{\ell}_{j_Q}, y\rangle > 0$ for any $y \in \mathbf{P}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}} - \{0\}$. By replacing j_Q with a multiple by a positive integer prime to $\Box$, we may and we shall assume that $\widetilde{\ell}_{j_Q} \in \mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ (without altering the above strong positivity condition). Then we obtain an invertible sheaf $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})$ over the abelian scheme $\mathsf{N} \to \mathsf{M}_{\mathcal{H}}$. Note that $\widetilde{\ell}_{j_Q} \in \widetilde{\sigma}_0^\vee$.

Lemma 5.5. *The invertible sheaf $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})$ is relatively ample over $\mathsf{M}_{\mathcal{H}}$, and induces twice of a $\mathbb{Z}^\times_{(\Box)}$-polarization $\lambda_{\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})} : \mathsf{N} \to \mathsf{N}^\vee$ (namely a $\mathbb{Z}^\times_{(\Box)}$-isogeny whose sufficiently divisible positive multiple is a polarization). Under the canonical isomorphisms in Corollary* 2.13*, the induced morphism*

$$d\lambda_{\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})} : \underline{\mathrm{Lie}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \to \underline{\mathrm{Lie}}_{\mathsf{N}^\vee/\mathsf{M}_{\mathcal{H}}}$$

is twice a positive $\mathbb{Z}^\times_{(\Box)}$-multiple of

$$d\lambda_{\mathsf{M}_{\mathcal{H}},j_Q} : \underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}).$$

In particular, $d\lambda_{\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})}$ is an isomorphism over $\mathsf{M}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

Proof. Just note that the morphism $\lambda_{\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})}$ is twice a positive $\mathbb{Z}^\times_{(\Box)}$-multiple of the $\mathbb{Z}^\times_{(\Box)}$-polarization $\lambda_{\mathsf{M}_{\mathcal{H}},j_Q}$ in Corollary 2.12. $\Box$

The invertible sheaf $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})$ over N defines a global section of $R^1 f_*(\mathscr{O}^\times_{\mathsf{N}})$, and the morphism

$$d\log : \mathscr{O}^\times_{\mathsf{N}} \to \Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} : a \mapsto a^{-1}da$$

induces a global section $D_{\widetilde{\ell}_{j_Q}} = d\log(\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q}))$ of $R^1 f_*(\Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}})$. Then it is standard (cf. [38, Prop. 2.1.5.14]) that the cup product with $D_{\widetilde{\ell}_{j_Q}}$ induces a composition of morphisms

$$f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}) \overset{\cup D_{\widetilde{\ell}_{j_Q}}}{\to} R^1 f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{N}}}{\otimes} \Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}) \overset{\text{can.}}{\to} R^1 f_*(\mathscr{O}_{\mathsf{N}}),$$

and that this morphism $f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}) \to R^1 f_*(\mathscr{O}_{\mathsf{N}})$ can be identified with the morphism $d\lambda_{\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})}$ under the canonical isomorphisms

$$f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}) \cong \underline{\mathrm{Lie}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}} \quad \text{and} \quad R^1 f_*(\mathscr{O}_{\mathsf{N}}) \cong \underline{\mathrm{Lie}}_{\mathsf{N}^\vee/\mathsf{M}_{\mathcal{H}}}.$$

The first question is whether we can extend the morphism $f_*(\underline{\mathrm{Der}}_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}) \to R^1 f_*(\mathscr{O}_{\mathsf{N}})$ to $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$; and the second question is whether the extended morphism is an isomorphism, at least in codimension one.

A naive approach is to extend the invertible sheaf $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})$ to $\mathsf{N}^{\mathrm{tor}}$. Since $\mathsf{N}^{\mathrm{tor}}$ is projective and smooth over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$, it is locally noetherian and locally factorial. Then [21, IV-4, 21.6.11] implies that the canonical restriction morphism $\mathrm{Pic}(\mathsf{N}^{\mathrm{tor}}) \to \mathrm{Pic}(\mathsf{N})$ is *surjective*.

However, since $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is not smooth, we have little control on the canonical restriction morphism $R^1 f_*^{\mathrm{tor}}(\Omega^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}) \overset{\mathrm{res.}}{\to} \jmath_*(R^1 f_*(\Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}}))$, and there is no obvious reason that the image of the class defined by any extension of $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})$ should induce an *isomorphism* extending $d\lambda_{\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})}$ (at least) in codimension one. (This is mentioned in [16, Ch. VI, end of §2], but with no details.)

An alternative approach is to consider the canonical restriction morphism

$$R^1 f_*^{\mathrm{tor}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}) \overset{\mathrm{res.}}{\to} \jmath_*(R^1 f_*(\Omega^1_{\mathsf{N}/\mathsf{M}_{\mathcal{H}}})). \tag{5.6}$$

By Lemma 4.29, and by (3a) of Theorem 2.15, $R^1 f_*^{\mathrm{tor}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})$ is locally free over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$. Therefore, the morphism (5.6) is injective.

Remark 5.7. The use of $R^1 f_*^{\mathrm{tor}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}})$ is inspired by Kato's idea of (relative) *log Picard groups* mentioned in [32, 3.3]. An application of this idea has been carried out in [46].

So far we have refrained from introducing the log structures (because they had not been necessary), but they are needed (at least formally) here. We shall adopt a notation slightly different from those of [33] and [32]. Let $\breve{\jmath} : \mathsf{N} \to \mathsf{N}^{\mathrm{tor}}$ denote the canonical open immersion. Then the canonical (fine) log structure on $\mathsf{N}^{\mathrm{tor}}$ (which we have been using so far) given by $\mathsf{N}^{\mathrm{tor}} - \mathsf{N}$ (with its reduced structure) can be defined explicitly as the sheaf of monoids $\overline{\mathscr{O}}^{\times}_{\mathsf{N}^{\mathrm{tor}}} := \mathscr{O}_{\mathsf{N}^{\mathrm{tor}}} \cap \breve{\jmath}_* \mathscr{O}^{\times}_{\mathsf{N}}$ (sheafification of the obvious presheaf), with associated sheaf of groups $\overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}$. Clearly, the restriction of $\overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}$ to N is canonically isomorphic to $\mathscr{O}^{\times}_{\mathsf{N}}$.

Definition 5.8. *A relative log invertible sheaf over* $f^{\mathrm{tor}} : \mathsf{N}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *is a global section of* $R^1 f_*^{\mathrm{tor}}(\overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}})$.

Since we do not assume that f^{tor} is flat (or log integral), the appropriate interpretation of relative log invertible sheaves can be quite delicate (and beyond this article).

Lemma 5.9. *To define a global section of* $R^1 f_*^{\mathrm{tor}}(\overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}})$, *it suffices to have the following data:*

(1) *A collection of schemes* $\overline{U}_\alpha$ *over* $\mathsf{N}^{\mathrm{tor}}$ *forming an étale covering. We shall denote the fiber product* $\overline{U}_\alpha \underset{\mathsf{N}^{\mathrm{tor}}}{\times} \overline{U}_\beta$ *(i.e., "intersection" in the étale topology) by* $\overline{U}_{\alpha\beta}$, *denote* $\overline{U}_{\alpha\beta}|_{\mathsf{N}} := \overline{U}_{\alpha\beta} \underset{\mathsf{N}^{\mathrm{tor}}}{\times} \mathsf{N}$ *by* $U_{\alpha\beta}$, *and use similar notations for higher fiber products.*
(2) *A usual invertible sheaf* $\mathcal{L}_\alpha$ *over each* $\overline{U}_\alpha$.
(3) *A comparison isomorphism* $\mathcal{L}_\alpha|_{U_{\alpha\beta}} \cong \mathcal{L}_\beta|_{U_{\alpha\beta}}$ *over each* $U_{\alpha\beta}$, *satisfying the usual cocycle condition over triple fiber products* $U_{\alpha\beta\gamma}$.

Proof. Since the restriction morphism $\overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}(\overline{U}_{\alpha\beta}) \to \overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}(U_{\alpha\beta}) \cong \mathscr{O}^{\times}_{\mathsf{N}}(U_{\alpha\beta})$ is a bijection when the image of $\overline{U}_{\alpha\beta}$ in $\mathsf{N}^{\mathrm{tor}}$ is sufficiently small, the data above define a section of $H^1(\mathsf{N}^{\mathrm{tor}}, \overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}})$, which then defines a section of $H^0(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}, R^1 f^{\mathrm{tor}}_*(\overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}))$ by the Leray spectral sequence in low degrees. (See [18, I, 4.5.1].) $\square$

In the construction of toroidal compactifications in [38, §6.3.3] (following [16, Ch. IV, §5]), there is a strata-preserving étale covering $\widetilde{\mathsf{U}} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ (serving as an étale presentation for the algebraic stack $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$), where $\widetilde{\mathsf{U}}$ is a finite union of the so-called *good algebraic models* of $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$. (See [38, Def. 6.3.2.5].) By taking the closures of the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-strata, we obtain a strata-preserving étale covering $\breve{\mathsf{U}} \to \mathsf{N}^{\mathrm{tor}}$, with strata labeled by triples $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$ having $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$ as a face.

Each connected component $\overline{U}_\alpha$ of $\breve{\mathsf{U}}$ is given by the closure of the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum in a so-called good algebraic $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$-model $\widetilde{U}_\alpha = \mathrm{Spec}(\widetilde{R}_\alpha) \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$, where $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$ is a representative of some $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$ having $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$ as a face (cf. second property in [38, Def. 6.3.2.5]), which we may assume to satisfy $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \tau}$. (See Section 4A. There are usually many α for each $[(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})]$.) Then we also have a strata-preserving étale morphism $\overline{U}_\alpha \to (\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$, which we shall call a *good algebraic* $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$*-model* of $\mathsf{N}^{\mathrm{tor}}$. The (open) $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}, \widetilde{\sigma})]$-stratum in $\overline{U}_\alpha$ is exactly the open subscheme $U_\alpha := \overline{U}_\alpha \underset{\mathsf{N}^{\mathrm{tor}}}{\times} \mathsf{N}$ of $\overline{U}_\alpha$.

Lemma 5.10. *Suppose that, for each $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \tau}$, we have chosen an element $\breve{\ell}_{j_Q, \breve{\tau}}$ in $\breve{\tau}_0^\vee$ that is mapped to $\widetilde{\ell}_{j_Q}$ in $\widetilde{\sigma}_0^\vee$ under the second morphism in* (3.6)*, and that $\breve{\ell}_{j_Q, \gamma\breve{\tau}} = \gamma \breve{\ell}_{j_Q, \breve{\tau}}$ for any $\gamma \in \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \Phi_{\mathcal{H}}}$. (Note that the choice of $\breve{\ell}_{j_Q, \breve{\tau}}$ is unique only up to translation by $\breve{\sigma}^\perp$.) Let $\breve{\mathsf{U}} \to \mathsf{N}^{\mathrm{tor}}$ be any strata-preserving étale covering formed by a finite union of good algebraic models. Then the choices of $\{\breve{\ell}_{j_Q, \breve{\tau}}\}_{\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \tau}}$ and $\breve{\mathsf{U}}$ determine a relative log invertible sheaf $\overline{\mathcal{L}}$ over $\mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ extending the rigidified invertible sheaf $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})$ over N, in the following sense: For each good algebraic $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$-model $\overline{U}_\alpha$ of $\mathsf{N}^{\mathrm{tor}}$, with $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \tau}$, let $\mathcal{L}_\alpha$ denote the pullback of $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q, \breve{\tau}})$ under the composition $\overline{U}_\alpha \to (\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau}) \to \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}}$. Then $\mathcal{L}_\alpha|_{U_\alpha}$ is canonically isomorphic to the pullback of $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{j_Q})$ (from $\mathsf{N} \cong \widetilde{C}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \widetilde{\delta}_{\widetilde{\mathcal{H}}}}$) to U_α. Furthermore, the collection $\{(U_\alpha, \mathcal{L}_\alpha)\}$ satisfies the requirements in Lemma* 5.9*, and defines a log invertible sheaf as in Definition* 5.8*.*

Proof. Let $(\widetilde{G}, \widetilde{\lambda}, \widetilde{i}, \widetilde{\alpha}_{\widetilde{\mathcal{H}}})$ be the degenerating family of type $\widetilde{\mathsf{M}}_{\widetilde{\mathcal{H}}}$ over $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$. Let $\underline{B}(\widetilde{G}) : \underline{\mathbf{S}}_{\underline{\widetilde{\Phi}}_{\widetilde{\mathcal{H}}}(\widetilde{G})} \to \underline{\mathrm{Inv}}(\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}})$ be constructed as in [38, Constr. 6.3.1.1]. If $\widetilde{U}_\alpha$ is a good algebraic $(\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\tau})$-model, then for any $\breve{\ell} \in \mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$, the invertible sheaf $\underline{B}(\widetilde{G})(\widetilde{U}_\alpha)(\breve{\ell})$ over $\widetilde{U}_\alpha$ is canonically isomorphic to the pullback of $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$ under the composition $\widetilde{U}_\alpha \to (\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau}) \to \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}}$ (cf. third property in [38, Def. 6.3.2.5]).

Given that $\underline{B}(\widetilde{G})$ is defined over $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ and functorial with respect to pullback morphisms $\widetilde{U}_{\alpha\beta} \to \widetilde{U}_{\alpha}$, the restriction of the pullback of $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{jQ,\breve{\tau}})$ to the $[(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})]$-stratum of $\widetilde{U}_{\alpha}$ is isomorphic to the pullback of $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{jQ})$ when $(\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}},\widetilde{\sigma})$ is a face of $[(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})]$. In other words, $\mathcal{L}_{\alpha}|_{U_{\alpha}}$ is isomorphic to the pullback of $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{jQ})$ over each U_{α}. Since the isomorphisms $\mathcal{L}_{\alpha}|_{U_{\alpha\beta}} \cong \mathcal{L}_{\beta}|_{U_{\alpha\beta}}$ induced by such identifications satisfy the cocycle condition because $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{jQ})$ is defined on N, the claim follows, as desired. $\square$

Remark 5.11. Any (usual) invertible sheaf over $\mathsf{N}^{\mathrm{tor}}$ extending $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{jQ})$ satisfies the requirements in Lemma 5.9 trivially. The point of Lemma 5.10 is that it provides an explicit extension of $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{jQ})$ (useful for our later argument) over an étale covering of $\mathsf{N}^{\mathrm{tor}}$. (We do not have such an explicit description of a global invertible sheaf extension over $\mathsf{N}^{\mathrm{tor}}$.)

Definition 5.12. *To any relative log invertible sheaf* $\overline{\mathcal{L}}$ *over* $\mathsf{N}^{\mathrm{tor}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ *defined by a global section of* $R^1 f^{\mathrm{tor}}_*(\overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}})$, *we define* $d\log(\overline{\mathcal{L}})$ *to be the image of* $\overline{\mathcal{L}}$ *under the canonical morphism* $R^1 f^{\mathrm{tor}}_*(\overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}) \to R^1 f^{\mathrm{tor}}_*(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ *induced by the canonical morphism* $d\log : \overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}} \to \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$.

Corollary 5.13. *There exists a (unique) global section* $D^{\mathrm{tor}}_{\widetilde{\ell}_{jQ}}$ *of* $R^1 f^{\mathrm{tor}}_*(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ *whose image under the canonical injection* (5.6) *is* $\jmath_*(D_{\widetilde{\ell}_{jQ}})$, *which satisfies* $D^{\mathrm{tor}}_{\widetilde{\ell}_{jQ}} = d\log(\overline{\mathcal{L}})$ *for any* $\overline{\mathcal{L}}$ *constructed in Lemma* 5.10 (*with any choices of* $\breve{\ell}_{jQ,\breve{\tau}}$*'s*).

Proof. The existence of $D^{\mathrm{tor}}_{\widetilde{\ell}_{jQ}}$ is clear because there is always some (usual) invertible sheaf over $\mathsf{N}^{\mathrm{tor}}$ extending $\widetilde{\Psi}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\widetilde{\delta}_{\widetilde{\mathcal{H}}}}(\widetilde{\ell}_{jQ})$ (by [21, IV-4, 21.6.11], since $\mathsf{N}^{\mathrm{tor}}$ is locally noetherian and locally factorial, as mentioned above). The uniqueness of $D^{\mathrm{tor}}_{\widetilde{\ell}_{jQ}}$ is clear because (5.6) is injective. Once we know the unique existence of $D^{\mathrm{tor}}_{\widetilde{\ell}_{jQ}}$, it has to agree with $d\log(\overline{\mathcal{L}})$ for any $\overline{\mathcal{L}}$ constructed in Lemma 5.10. $\square$

Thus we are led to state the following:

Proposition 5.14. *Cup product with the global section* $D^{\mathrm{tor}}_{\widetilde{\ell}_{jQ}}$ *of* $R^1 f^{\mathrm{tor}}_*(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ *in Corollary* 5.13 *induces a composition of morphisms*

$$(5.15)\quad f^{\mathrm{tor}}_*(\underline{\mathrm{Der}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \xrightarrow{\cup D^{\mathrm{tor}}_{\widetilde{\ell}_{jQ}}} R^1 f^{\mathrm{tor}}_*(\underline{\mathrm{Der}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}}{\otimes} \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \xrightarrow{\mathrm{can.}} R^1 f^{\mathrm{tor}}_*(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}).$$

This composition is an isomorphism over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. (*By Lemma* 5.2 *and Remark* 5.4, *this implies the existence of the canonical isomorphism* (5.3).)

We will carry out the proof of Proposition 5.14 in the next subsection.

5C. **Induced morphisms over formal fibers.** We fix the choices of $\{\breve{\ell}_{j_Q,\breve{\tau}}\}_{\breve{\tau}\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}$ and $\breve{\mathsf{U}}$, so that $\overline{\mathcal{L}}$ is constructed as in Lemma 5.10, and so that $D^{\mathrm{tor}}_{\widetilde{\ell}_{j_Q}} = d\log(\overline{\mathcal{L}})$ as in Corollary 5.13.

Since f^{tor} is proper and the sheaves involved are all coherent, by Grothendieck's fundamental theorem [21, III-1, 4.1.5], Proposition 5.14 can be verified by pulling back to formal completions along strata of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$. Let us fix the choice of a cusp label $[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]$ of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$, and consider the canonical morphism

$$\imath : \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \cong (\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}.$$

By abuse of notation, we shall also denote by $\imath^*(\,\cdot\,)$ the pullbacks of objects under pullbacks of the morphism $\imath$. We would like to show that the morphism $\imath^* f^{\mathrm{tor}}_*(\underline{\mathrm{Der}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \to \imath^* R^1 f^{\mathrm{tor}}_*(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})$ defined by cup product with $\imath^*(D^{\mathrm{tor}}_{\widetilde{\ell}_{j_Q}})$ is an *isomorphism* over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

As said in Section 4A, the pullback of f^{tor} to $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ can be identified with the canonical morphism $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$, and $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ has a finite open covering by the collection $\{\mathfrak{U}_{[\breve{\tau}]}\}_{[\breve{\tau}]\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}}$ of open formal subschemes. Let $\breve{\tau}\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ be a representative of $[\breve{\tau}]\in\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. For each such $\breve{\tau}$, recall that the formal scheme $\mathfrak{U}_{\breve{\tau}}$ is the completion of $(\widetilde{\Xi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})_{\breve{\sigma}}(\breve{\tau})$ along $U_{\breve{\tau}}$. By abuse of notation, let us denote the pullback of $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q,\breve{\tau}})$ over $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$ to $\mathfrak{U}_{\breve{\tau}}$ by the same notation. For any $\gamma\in\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$, since $\breve{\ell}_{j_Q,\gamma\breve{\tau}} = \gamma\breve{\ell}_{j_Q,\breve{\tau}}$ (see Lemma 5.10), we have a canonical isomorphism $\gamma^*\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q,\gamma\breve{\tau}}) \xrightarrow{\sim} \widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q,\breve{\tau}})$, where $\gamma : \mathfrak{U}_{\breve{\tau}} \xrightarrow{\sim} \mathfrak{U}_{\gamma\breve{\tau}}$ is the canonical isomorphism (see Lemma 4.6). Hence $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q,\breve{\tau}})$ descends to an unambiguous invertible sheaf $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q,[\breve{\tau}]})$ on $\mathfrak{U}_{[\breve{\tau}]}$.

The étale covering $\breve{\mathsf{U}} \to \mathsf{N}^{\mathrm{tor}}$ induces (by taking formal completion along the pullback of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$) an étale (i.e., formally étale and of finite type; see [21, I, 10.13.3]) covering of $(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$. If $\overline{U}_\alpha$ is a good algebraic $(\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\tau})$-model of $\mathsf{N}^{\mathrm{tor}}$, then the formal completion $(\overline{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ of $\overline{U}_\alpha$ along the pullback of $\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}$ is étale (in the same sense as above) over $\mathfrak{U}_{\breve{\tau}}$.

Lemma 5.16. *The pullback of $\mathcal{L}_\alpha$ to $(\overline{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ is isomorphic to the pullback of $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q,\breve{\tau}})$ from $\mathfrak{U}_{\breve{\tau}}$.*

Proof. The canonical morphisms

$$(\overline{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \to \overline{U}_\alpha \to \mathsf{N}^{\mathrm{tor}} \quad \text{and} \quad (\overline{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \to \mathfrak{U}_{\breve{\tau}} \to \mathsf{N}^{\mathrm{tor}}$$

are induced respectively by morphisms

$$(\widetilde{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \to \widetilde{U}_\alpha \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}} \quad \text{and} \quad (\widetilde{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \to \widetilde{\mathfrak{U}}_{\breve{\tau}} \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$$

over $\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$. Under both these morphisms, the pullback of $(\widetilde{G},\widetilde{\lambda},\widetilde{i},\widetilde{\alpha}_{\widetilde{\mathcal{H}}}) \to \widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}}$ is canonically isomorphic to the pullback of the Mumford family (as in the proof of Proposition 4.3). Since the isomorphism class of the pullback of $\mathcal{L}_\alpha$ to $(\overline{U}_\alpha)^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}}$ is determined by the pullback of $\underline{B}(\widetilde{G}) : \underline{\mathbf{S}}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}(\widetilde{G})} \to \underline{\mathrm{Inv}}(\widetilde{\mathsf{M}}^{\mathrm{tor}}_{\widetilde{\mathcal{H}}})$ (as in the proof of Lemma 5.10), we can pullback along $(\overline{U}_\alpha)^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma)]}} \to \mathfrak{U}_{\check{\tau}} \to \mathsf{N}^{\mathrm{tor}}$ and conclude that $\mathcal{L}_\alpha$ is isomorphic to the pullback of $\widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{j_Q,\check{\tau}})$ from $\mathfrak{U}_{\check{\tau}}$. $\square$

By Lemma 4.29, we have

$$\imath^* f_*^{\mathrm{tor}}(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}) \cong f_*^{\mathrm{tor}}(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}) \cong H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})),$$

and $\imath^* R^1 f_*^{\mathrm{tor}}(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}) \cong R^1 f_*^{\mathrm{tor}}(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})$ is equipped with a decreasing filtration with (locally free) graded pieces

$$\mathrm{Gr}^0(\imath^* R^1 f_*^{\mathrm{tor}}(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})) \cong H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^1(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}))$$

and

$$\mathrm{Gr}^1(\imath^* R^1 f_*^{\mathrm{tor}}(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}})) \cong H^1(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})).$$

Thus, to show that (5.15) is an isomorphism over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, it suffices (by comparison of ranks of locally free sheaves) to show that it induces surjections from subquotients of $\imath^* f_*^{\mathrm{tor}}(\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ to these graded pieces over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

By tensoring the above filtration with $\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ (and by (3.15)), we obtain a decreasing filtration on $\imath^* R^1 f_*^{\mathrm{tor}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$ with

$$\mathrm{Gr}^0(\imath^* R^1 f_*^{\mathrm{tor}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})) \cong H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^1(\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$$

and

$$\mathrm{Gr}^1(\imath^* R^1 f_*^{\mathrm{tor}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})) \cong H^1(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})).$$

Since $\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \cong (f^{\mathrm{tor}})^*(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, \underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}))$, we have

$$\imath^* f_*^{\mathrm{tor}}(\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \cong H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\imath^*\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})),$$

and the morphism

$$\imath^* f_*^{\mathrm{tor}}(\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \to H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^1(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}))$$

induced by (5.15) can be identified with the morphism

$$H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^0(\imath^*\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})) \to H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^1(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^\wedge_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})) \tag{5.17}$$

given by cup product with the image of $\imath^*(D^{\mathrm{tor}}_{\widetilde{\ell}_{j_Q}})$ in $\mathrm{Gr}^0(\imath^* R^1 f_*^{\mathrm{tor}}(\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})) \cong H^0(\mathfrak{N}_{\check{\sigma},\tau}, \underline{\mathscr{H}}^1(\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$.

For simplicity, let us define $\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} := \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \underset{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}{\times} \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}$. Then the structural morphism $\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ factors as $\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \to \widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \to \mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$. Over $\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$, there is an exact sequence

$$\begin{aligned} 0 \to (\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \to \widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}})^*(\Omega^1_{\widetilde{C}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}) & \\ \to \imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to \overline{\Omega}^1_{\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \to 0 & \end{aligned}$$

of locally free sheaves, where $\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \cong \overline{\Omega}^1_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$. By taking duals, we obtain an exact sequence

$$0 \to \overline{\underline{\mathrm{Der}}}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \to \imath^*\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$$
$$\to (\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau} \to \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})^*(\underline{\mathrm{Der}}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}) \to 0.$$

We have similar sequences with $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$ replaced with the locally isomorphic quotient $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. (For simplicity, in the notation of such differentials, we shall suppress the locally isomorphic quotients by $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$.)

Since $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{jQ,\breve{\tau}})$ is the pullback of an invertible sheaf on $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$, the image of $\imath^*(D^{\mathrm{tor}}_{\widetilde{\ell}_{jQ}})$ in $H^0(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathscr{H}}^1(\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$ lies locally over each $\mathfrak{U}_{\breve{\tau}}$ in the image of

$$(\mathfrak{U}_{\breve{\tau}} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^*R^1h_*(\Omega^1_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}})$$
$$\xrightarrow{\sim} \underline{\mathscr{H}}^1((\mathfrak{U}_{\breve{\tau}} \to \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})^*(\Omega^1_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}))$$
$$\to \underline{\mathscr{H}}^1(\imath^*\overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}).$$

Hence (5.17) factors as

$$\begin{aligned}
&H^0(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathscr{H}}^0(\imath^*\overline{\underline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))\\
&\twoheadrightarrow H^0(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathscr{H}}^0((\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau} \to \widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}})^*(\underline{\mathrm{Der}}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}})))\\
&\xrightarrow{\sim} (\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^*R^0h_*(\underline{\mathrm{Der}}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}})\\
&\to (\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^*R^1h_*(\mathscr{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}})\\
&\xrightarrow{\sim} H^0(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathscr{H}}^1(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})).
\end{aligned}$$

Lemma 5.18. *The morphism*

$$R^0h_*(\underline{\mathrm{Der}}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}) \to R^1h_*(\mathscr{O}_{\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}})$$

defined by cup product with $d\log(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{jQ,\breve{\tau}}))$ *depends only on the image* $\widetilde{\ell}_{jQ}$ *of* $\breve{\ell}_{jQ,\breve{\tau}}$ *in* $\mathbf{S}_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}}$ *under the second morphism in* (3.6) (*and hence is independent of the choice of* $\breve{\ell}_{jQ,\breve{\tau}}$)*. Moreover, this morphism is surjective over* $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$*.*

Proof. By forming equivariant quotients and invariants, we may (and we shall) replace $\widetilde{\mathcal{H}}$ and $\mathcal{H}$ with principal level subgroups of some level n, as in the proof of Lemma 4.9. Then the morphism $h : \widetilde{C}_{\breve{\Phi}_n,\breve{\delta}_n} \to C_{\Phi_n,\delta_n}$ is a torsor under its kernel C, which is an abelian scheme $\mathbb{Z}^{\times}_{(\Box)}$-isogenous to $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q,A)^\circ \to \mathsf{M}^{\mathsf{Z}_n}_n$. The restriction of $\widetilde{\Psi}_{\breve{\Phi}_n,\breve{\delta}_n}(\breve{\ell}_{jQ,\breve{\tau}})$ to C depends only on the image $\widetilde{\ell}_{jQ}$ of $\breve{\ell}_{jQ,\breve{\tau}}$ in $\widetilde{\sigma}_0^\vee$, and is *relatively ample* by the same proofs of Corollary 2.12 and Lemma 5.5 (with $G_{\mathsf{M}_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}$ replaced with $A \to \mathsf{M}^{\Phi_n}_n$). Hence the lemma follows. $\Box$

Corollary 5.19. *The morphism* (5.17) *is surjective over* $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. *Its kernel is the subsheaf* $H^0(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^0(\underline{\overline{\mathrm{Der}}}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}))$ *of* $H^0(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^0(\imath^*\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$.

Now consider the induced morphism

$$\begin{aligned} H^0(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^0(\underline{\overline{\mathrm{Der}}}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}})) &\hookrightarrow H^0(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^0(\imath^*\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})) \\ &\xrightarrow{\sim} R^0 f^{\mathrm{tor}}_*(\imath^*\underline{\overline{\mathrm{Der}}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}) \to R^1 f^{\mathrm{tor}}_*(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}) \end{aligned}$$

defined by cup product with $\imath^*(D^{\mathrm{tor}}_{\widetilde{\ell}_{j_Q}})$. This composition has image in

$$H^1(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^0(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})),$$

because its further composition with

$$R^1 f^{\mathrm{tor}}_*(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}) \twoheadrightarrow H^0(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^1(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}))$$

is zero (by Corollary 5.19). Thus the question is whether cup product with $\imath^*(D^{\mathrm{tor}}_{\widetilde{\ell}_{j_Q}})$ induces a morphism

(5.20)
$$H^0(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^0(\underline{\overline{\mathrm{Der}}}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}})) \to H^1(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^0(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}))$$

surjective over $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

Lemma 5.21. *Suppose* $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, *and* $\breve{\ell} \in \breve{\sigma}^{\perp}$. *Suppose* $\mathfrak{V}$ *is an affine open formal subscheme of* $\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ *over which the pullback of* $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$ *is a principal ideal of* $\mathscr{O}_{\mathfrak{V}}$ *generated by some section* x. *Let* $\mathfrak{U} := \mathfrak{U}_{\breve{\tau}} \underset{\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}{\times} \mathfrak{V}$ *and let* $\overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathfrak{U}}$ *be the pullback of* $\overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathsf{N}^{\mathrm{tor}}}$ *to* $\mathfrak{U}$. *Let*

$$\overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathfrak{V}} := (\mathfrak{U} \to \mathfrak{V})_*(\overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathfrak{U}}).$$

Then there exists a canonical injection $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}) \hookrightarrow \overline{\mathscr{O}}^{\times,\mathrm{gp}}_{\mathfrak{V}}$ *over* $\mathfrak{V}$, *and the value of the section* $d\log(x)$ *of* $(\mathfrak{U} \to \mathfrak{V})_*\overline{\Omega}^1_{\mathfrak{U}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$ *determines a canonical section of* $\overline{\Omega}^1_{\mathfrak{U}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$ *(which is independent of the choice of the generator* x*).*

Proof. If we replace x with ax, for some $a \in \mathscr{O}^{\times}_{\mathfrak{V}}$, then $d\log(ax) = d\log(a) + d\log(x) = d\log(x)$ because $d\log(a) = 0$ in $(\mathfrak{U} \to \mathfrak{V})_*\overline{\Omega}^1_{\mathfrak{U}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$. □

Corollary 5.22. *Suppose* $\breve{\tau} \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, *and* $\breve{\ell} \in \breve{\sigma}^{\perp}$. *Then the local generators of* $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$ *in Lemma* 5.21 *determine a well-defined section of* $\overline{\Omega}^1_{\mathfrak{U}_{\breve{\tau}}/\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$, *which we denote by* $d\log(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}))$.

Proof. Since $\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell})$ is defined over $\widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}$ (or rather $\widetilde{C}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}$), we can always cover $\mathfrak{U}_{\breve{\tau}}$ by open formal subschemes $\mathfrak{U}$ as in Lemma 5.21. □

Lemma 5.23. *For any $\breve{\tau}, \breve{\tau}' \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \tau}$ such that $\breve{\tau}$ and $\breve{\tau}'$ are adjacent to each other, let us define the section $u_{[\breve{\tau}],[\breve{\tau}']}$ of $\underline{\mathscr{H}}^0(\overline{\Omega}^1_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \tau} / \widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau}})([\breve{\tau}]^{\mathrm{cl}} \cap [\breve{\tau}']^{\mathrm{cl}})$ to be*

$$d\log(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q, \breve{\tau}} - \breve{\ell}_{j_Q, \breve{\tau}'}))$$

(*as in Corollary* 5.22). *Then this is well defined and determines a section u of $H^1(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^0(\imath^* \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$ that induces by cup product the same morphism as* (5.20).

Proof. If $\breve{\tau}$ and $\breve{\tau}'$ are adjacent, then $\gamma\breve{\tau}$ and $\gamma'\breve{\tau}'$ are adjacent for $\gamma, \gamma' \in \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \Phi_{\mathcal{H}}}$ only when $\gamma = \gamma'$ (by Condition 1.29; cf. Lemma 3.1), in which case

$$\breve{\ell}_{j_Q, \gamma\breve{\tau}} - \breve{\ell}_{j_Q, \breve{\tau}} = \gamma\breve{\ell}_{j_Q, \breve{\tau}} - \breve{\ell}_{j_Q, \breve{\tau}} = \gamma\breve{\ell}_{j_Q, \breve{\tau}'} - \breve{\ell}_{j_Q, \breve{\tau}'} = \breve{\ell}_{j_Q, \gamma\breve{\tau}'} - \breve{\ell}_{j_Q, \breve{\tau}'}$$

(because $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \Phi_{\mathcal{H}}}$ acts by the same translation on $\breve{\ell}_{j_Q, \breve{\tau}}$ and $\breve{\ell}_{j_Q, \breve{\tau}'}$). This shows that the assignment of $u_{[\breve{\tau}],[\breve{\tau}']}$ is independent of the choices of the respective representatives $\breve{\tau}$ and $\breve{\tau}'$ of $[\breve{\tau}]$ and $[\breve{\tau}']$, and that u is well defined.

Cup product with u induces the same morphism as (5.20) because the canonical morphism

$$\underline{\overline{\mathrm{Der}}}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \tau} / \widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau}} \otimes \imath^* \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to \mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau)]}}}$$

factors through

$$\underline{\overline{\mathrm{Der}}}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \tau} / \widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau}} \otimes \overline{\Omega}^1_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\delta}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \tau} / \widetilde{\mathfrak{X}}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau}} \to \mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau)]}}},$$

and because cup product with the image of $\imath^*(D^{\mathrm{tor}}_{\widetilde{\ell}_{j_Q}})$ in $H^0(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^1(\imath^* \overline{\Omega}^1_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$ induces the zero morphism (cf. the paragraph preceding Lemma 5.18). $\square$

Consider any sequence $\breve{\tau}_1, \breve{\tau}_2, \ldots, \breve{\tau}_k$ of adjacent cones in $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \tau}$, such that $\breve{\tau}_k = \gamma\breve{\tau}_1$ for some $\gamma \in \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \Phi_{\mathcal{H}}}$. The union of the cones in any such sequence form a subset of $\widetilde{\mathfrak{N}}_{\breve{\sigma},\tau}$ contractible to a path joining a point in $\breve{\tau}$ with its translation by γ in $\gamma\breve{\tau}$, whose image in $\mathfrak{N}_{\breve{\sigma},\tau}$ defines a loop. Suppose we have a class s in $H^1(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^0(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau)]}}}))$ represented by a collection of sections

$$s_{[\breve{\tau}],[\breve{\tau}']} \in \underline{\mathscr{H}}^0(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau)]}}})([\breve{\tau}]^{\mathrm{cl}} \cap [\breve{\tau}']^{\mathrm{cl}})$$

for $[\breve{\tau}], [\breve{\tau}'] \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \tau} / \Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}}, \Phi_{\mathcal{H}}}$, and suppose we define formally $s_{\breve{\tau}, \breve{\tau}'} = s_{[\breve{\tau}],[\breve{\tau}']}$ for any $\breve{\tau}, \breve{\tau}' \in \Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}}, \breve{\sigma}, \tau}$. Then we can define the *path integral* of s along the sequence $\breve{\tau}_1, \breve{\tau}_2, \ldots, \breve{\tau}_k$ to be the sum

$$\sum_{i=1}^{k-1} s_{\breve{\tau}_i, \breve{\tau}_{i+1}}.$$

This defines a morphism

$$H^1(\mathfrak{N}_{\breve{\sigma},\tau}, \underline{\mathscr{H}}^0(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau)]}}})) \to \mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}}, \delta_{\mathcal{H}}, \tau}}. \tag{5.24}$$

Note that this is a realization of the cap product

$$H_1(\mathfrak{N}_{\breve{\sigma},\tau},\mathbb{Z})\times H^1(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathscr{H}}^0(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}))\\ \to H_0(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathscr{H}}^0(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}))\cong\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}.$$

Lemma 5.25. *For any $\breve{\ell}\in\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ that is mapped to $\widetilde{\ell}_{j_Q}$ in $\widetilde{\sigma}_0^\vee$ under the second morphism in* (3.6)*, the assignment $\gamma\mapsto d\log(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\gamma\breve{\ell}-\breve{\ell}))$ for $\gamma\in\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$ induces a morphism*

$$\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}\underset{\mathbb{Z}}{\otimes}\mathscr{O}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}\to\overline{\Omega}^1_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}},$$

which is an isomorphism over $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$.

Proof. Since $\gamma\breve{\ell}$ and $\breve{\ell}$ have the same image $\widetilde{\ell}_{j_Q}$ in $\widetilde{\sigma}_0^\vee$ under the second morphism in (3.6), the difference $\gamma\breve{\ell}-\breve{\ell}$ lands in $\breve{\sigma}^\perp$. For any $\breve{\ell}'\in\breve{\sigma}^\perp$, an elementary matrix calculation (using any splitting of $s_{\breve{X}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}:\breve{X}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}\twoheadrightarrow\widetilde{X}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$) shows that $\gamma\breve{\ell}'-\breve{\ell}'$ lies in $\mathbf{S}_{\Phi_{\mathcal{H}}}=(\mathbf{S}_{\Phi_{\mathcal{H}}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q})\cap\mathbf{S}_{\breve{\Phi}_{\widetilde{\mathcal{H}}}}$ (identified as the image of the first morphism in (3.6)). Therefore, we have $(\gamma_1\gamma_2\breve{\ell}-\breve{\ell})-(\gamma_1\breve{\ell}-\breve{\ell})-(\gamma_2\breve{\ell}-\breve{\ell})=\gamma_1(\gamma_2\breve{\ell}-\breve{\ell})-(\gamma_2\breve{\ell}-\breve{\ell})\in\mathbf{S}_{\Phi_{\mathcal{H}}}$, which shows that the assignment $\gamma\mapsto\gamma\breve{\ell}-\breve{\ell}$ defines a group homomorphism $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}\to(\breve{\sigma}^\perp/\mathbf{S}_{\Phi_{\mathcal{H}}})$. By the choice of j_Q, the element $\widetilde{\ell}_{j_Q}$ is represented by a positive definite matrix with respect to any choice of basis, and hence the homomorphism $\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}\to(\breve{\sigma}^\perp/\mathbf{S}_{\Phi_{\mathcal{H}}})$ induced by $\gamma\mapsto\gamma\breve{\ell}-\breve{\ell}$ is injective (by another elementary matrix calculation over $\mathbb{Q}$). By comparison of dimensions, this shows that the induced injective homomorphism

$$\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}\to(\breve{\sigma}^\perp/\mathbf{S}_{\Phi_{\mathcal{H}}})\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$$

is bijective. Since $\overline{\Omega}^1_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$ is generated over $\mathscr{O}_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}}$ by

$$\{d\log(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}')):\breve{\ell}'\text{ representatives of }\breve{\sigma}^\perp/\mathbf{S}_{\Phi_{\mathcal{H}}}\},$$

the lemma follows. □

Lemma 5.26. *Let $\breve{\tau}_1$, $\breve{\tau}_2$, …, $\breve{\tau}_k$ be a sequence of adjacent cones in $\Sigma_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}$, such that $\breve{\tau}_k=\gamma\breve{\tau}_1\neq\breve{\tau}_1$ for some $\gamma\in\Gamma_{\widetilde{\Phi}_{\widetilde{\mathcal{H}}},\Phi_{\mathcal{H}}}$. Then the composition of* (5.20) *and* (5.24) *is surjective over $\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$.*

Proof. If $\gamma\breve{\tau}_1\neq\breve{\tau}_1$, then $\breve{\ell}_{j_Q,\gamma\breve{\tau}_1}=\gamma\breve{\ell}_{j_Q,\breve{\tau}_1}\neq\breve{\ell}_{j_Q,\breve{\tau}_1}$ by the proof of Lemma 5.25. By Lemma 5.25, this implies that $d\log(\widetilde{\Psi}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}}}(\breve{\ell}_{j_Q,\breve{\tau}_1}-\breve{\ell}_{j_Q,\breve{\tau}_k}))$ defines a nonzero section of $\overline{\Omega}^1_{\widetilde{\mathfrak{X}}_{\breve{\Phi}_{\widetilde{\mathcal{H}}},\breve{\delta}_{\widetilde{\mathcal{H}}},\breve{\sigma},\tau}/\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}}$ over every $\mathfrak{U}_{[\breve{\tau}]}\underset{\mathbb{Z}}{\otimes}\mathbb{Q}$. Let t be any section of $H^0(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathscr{H}}^0(\imath^*\underline{\mathrm{Der}}_{\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}))$. Cup product with u (see Lemma 5.23) sends t to the class s in $H^1(\mathfrak{N}_{\breve{\sigma},\tau},\underline{\mathscr{H}}^0(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}}))$ represented (up to a sign convention) by the collection of sections

$$s_{[\breve{\tau}],[\breve{\tau}']}\in\underline{\mathscr{H}}^0(\mathscr{O}_{(\mathsf{N}^{\mathrm{tor}})^{\wedge}_{\mathsf{Z}_{[(\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau)]}}})([\breve{\tau}]^{\mathrm{cl}}\cap[\breve{\tau}']^{\mathrm{cl}})$$

determined by $s_{\check{\tau},\check{\tau}'} = t \cup (d\log(\widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{j_Q,\check{\tau}} - \check{\ell}_{j_Q,\check{\tau}'})))$ for any $\check{\tau},\check{\tau}' \in \Sigma_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau}$. Therefore, if locally there exists t such that $t \cup (d\log(\widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{j_Q,\check{\tau}_1} - \check{\ell}_{j_Q,\check{\tau}_k})))$ is the pullback of (local) generators of $\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, which is possible by Lemma 5.25, then the path integral

$$\begin{aligned}\sum_{i=1}^{k-1} s_{\check{\tau}_i,\check{\tau}_{i+1}} &= \sum_{i=1}^{k-1} t \cup (d\log(\widetilde{\Psi}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{j_Q,\check{\tau}_i} - \check{\ell}_{j_Q,\check{\tau}_{i+1}}))) \\ &= t \cup (d\log(\Psi_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}}}(\check{\ell}_{j_Q,\check{\tau}_1} - \check{\ell}_{j_Q,\check{\tau}_k})))\end{aligned}$$

is defined locally by generators of $\mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\tau}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. This shows that the composition of (5.20) with (5.24) is surjective over $\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, as desired. □

Corollary 5.27. *The morphism* (5.20) *is surjective over* $\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$.

Proof. By Lemma 4.21, (4.25), and Lemma 5.25, the morphism (5.20) is surjective over $\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ if its composition with (5.24) is surjective over $\widetilde{\mathfrak{X}}_{\check{\Phi}_{\widetilde{\mathcal{H}}},\check{\delta}_{\widetilde{\mathcal{H}}},\check{\sigma},\tau} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ for some collection of sequences $\check{\tau}_1, \check{\tau}_2, \ldots, \check{\tau}_k$ defining loops in $\mathfrak{N}_{\check{\sigma},\tau}$ generating $H_1(\mathfrak{N}_{\check{\sigma},\tau}, \mathbb{Z})$. Hence the corollary follows from Lemma 5.26. □

Now Proposition 5.14 follows from the combination of Corollaries 5.19 and 5.27. By Lemma 5.2 and Remark 5.4, Proposition 5.14 implies the existence of the canonical isomorphism (5.3). Thus Corollary 4.36 implies:

Corollary 5.28. *For any integer* $b \geq 0$*, we have a canonical isomorphism*

$$R^b f_*^{\mathrm{tor}}(\mathscr{O}_{\mathsf{N}^{\mathrm{tor}}}) \cong \wedge^b(\underline{\mathrm{Hom}}_{\mathcal{O}}(Q^\vee, \underline{\mathrm{Lie}}_{G^\vee/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}))$$

of locally free sheaves over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$*, compatible with cup products and exterior products, extending the composition of canonical isomorphisms* (2.19) *over* $\mathsf{M}_{\mathcal{H}}$*.*

This completes the proof of (3b) and (3d) of Theorem 2.15, using respectively (3a) and (3c) of Theorem 2.15. As explained in Section 3E, this also makes (4c) and (5c) of Theorem 2.15 unconditional. The proof of Theorem 2.15 is now complete.

6. Canonical extensions of principal bundles

6A. **Principal bundles.** Consider $(G_{\mathsf{M}_{\mathcal{H}}}, \lambda_{\mathsf{M}_{\mathcal{H}}}, i_{\mathsf{M}_{\mathcal{H}}}, \alpha_{\mathcal{H}}) \to \mathsf{M}_{\mathcal{H}}$, the restriction of the degenerating family $(G, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, which is isomorphic to the tautological tuple over $\mathsf{M}_{\mathcal{H}}$; and consider the relative de Rham cohomology $\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})$ and the relative de Rham homology $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}(\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}), \mathscr{O}_{\mathsf{M}_{\mathcal{H}}})$. We have the canonical pairing $\langle \cdot, \cdot \rangle_\lambda : \underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1)$ defined as the composition of $(\mathrm{Id} \times \lambda_{\mathsf{M}_{\mathcal{H}}})_*$ followed by the perfect pairing $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}_1^{\mathrm{dR}}(G^\vee_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1)$ defined by the first Chern class of the Poincaré invertible sheaf over $G_{\mathsf{M}_{\mathcal{H}}} \underset{\mathsf{M}_{\mathcal{H}}}{\times} G^\vee_{\mathsf{M}_{\mathcal{H}}}$. (See for example [14, 1.5].) Under the assumption that $\lambda_{\mathsf{M}_{\mathcal{H}}}$ has degree prime to $\Box$, we know that $\lambda_{\mathsf{M}_{\mathcal{H}}}$ is separable, that $(\lambda_{\mathsf{M}_{\mathcal{H}}})_*$ is an isomorphism, and hence that the pairing $\langle \cdot, \cdot \rangle_\lambda$ above is *perfect*.

Let $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ also denote the induced pairing on $\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H})\times\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H})$ by duality. By [5, Lem. 2.5.3], we have canonical short exact sequences

$$0\to\underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}\to\underline{H}^{\mathrm{dR}}_1(G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H})\to\underline{\mathrm{Lie}}_{G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}\to 0$$

and

$$0\to\underline{\mathrm{Lie}}^\vee_{G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}\to\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H})\to\underline{\mathrm{Lie}}_{G^\vee_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}\to 0.$$

The submodules $\underline{\mathrm{Lie}}^\vee_{G^\vee_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}$ and $\underline{\mathrm{Lie}}^\vee_{G_{\mathsf{M}_\mathcal{H}}/\mathsf{M}_\mathcal{H}}$ are maximal totally isotropic with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$.

Consider the $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-module

$$L\underset{\mathbb{Z}}{\otimes}\mathbb{C}\to(L\underset{\mathbb{Z}}{\otimes}\mathbb{C})/\mathsf{P}_h,\tag{6.1}$$

where $\mathsf{P}_h:=\{\sqrt{-1}x-h(\sqrt{-1})x : x\in L\underset{\mathbb{Z}}{\otimes}\mathbb{R}\}\subset L\underset{\mathbb{Z}}{\otimes}\mathbb{C}$.

Now suppose there exists a finite extension F_0' of F_0 in $\mathbb{C}$, and a subset $\Box'$ of $\Box$, such that F_0' is unramified at all primes in $\Box'$, and such that, by setting $R_0:=\mathcal{O}_{F_0',(\Box')}$, there exists an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}R_0$-module L_0 such that $L_0\underset{R_0}{\otimes}\mathbb{C}\cong(L\underset{\mathbb{Z}}{\otimes}\mathbb{C})/\mathsf{P}_h$. Once the choice of F_0' is fixed, the choice of L_0 is unique up to isomorphism because $\mathcal{O}\underset{\mathbb{Z}}{\otimes}R_0$-modules are uniquely determined by their multiranks. (See [38, Lem. 1.1.3.4 and Def. 1.1.3.5] for the notion of multiranks.) Let

$$\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}:(L_0\oplus L_0^\vee(1))\times(L_0\oplus L_0^\vee(1))\to R_0(1)$$

be the alternating pairing defined by $\langle(x_1,f_1),(x_2,f_2)\rangle_{\mathrm{can.}}:=f_2(x_1)-f_1(x_2)$ (cf. [38, Lem. 1.1.4.16]).

Definition 6.2. *For any R_0-algebra R, set*

$$\mathrm{G}_0(R):=\left\{\begin{array}{c}(g,r)\in\mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}R}((L_0\oplus L_0^\vee(1))\underset{R_0}{\otimes}R)\times\mathbf{G}_{\mathrm{m}}(R):\\ \langle gx,gy\rangle_{\mathrm{can.}}=r\langle x,y\rangle_{\mathrm{can.}},\forall x,y\in(L_0\oplus L_0^\vee(1))\underset{R_0}{\otimes}R\end{array}\right\},$$
$$\mathrm{P}_0(R):=\{(g,r)\in\mathrm{G}_0(R):g(L_0^\vee(1)\underset{R_0}{\otimes}R)=L_0^\vee(1)\underset{R_0}{\otimes}R\},$$
$$\mathrm{M}_0(R):=\mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}R}(L_0^\vee(1)\underset{R_0}{\otimes}R)\times\mathbf{G}_{\mathrm{m}}(R),$$

where we view $\mathrm{M}_0(R)$ canonically as a quotient of $\mathrm{P}_0(R)$ by

$$\mathrm{P}_0(R)\to\mathrm{M}_0(R):(g,r)\mapsto(g|_{L_0^\vee(1)\underset{R_0}{\otimes}R},r).$$

The assignments are functorial in R and define group functors G_0, P_0, and M_0 over R_0.

Lemma 6.3. *For any complete local ring R over R_0 with separably closed residue field, there is an isomorphism*

$$(L\underset{\mathbb{Z}}{\otimes}R,\langle\,\cdot\,,\,\cdot\,\rangle)\cong(L_0\oplus L_0^\vee(1),\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}})\underset{R_0}{\otimes}R,$$

and hence an isomorphism $\mathrm{G}(R)\cong\mathrm{G}_0(R)$. (Consequently, $\mathrm{P}_0(R)$ can be identified with a "parabolic" subgroup of $\mathrm{G}(R)$.)

(In practice, it is not necessary to take R to be complete local. Much smaller rings would suffice for the existence of isomorphisms as in Lemma 6.3.)

In what follows, by abuse of notation, we shall replace $\mathsf{M}_{\mathcal{H}}$ etc with their base extensions from $\mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ to $\mathrm{Spec}(R_0)$, and replace $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(\Box)})$ with $\mathrm{Spec}(R_0)$.

Definition 6.4. *The* **principal** P_0**-bundle** *over* $\mathsf{M}_{\mathcal{H}}$ *is the* P_0*-torsor*

$$\mathcal{E}_{\mathrm{P}_0} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}((\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}), \langle\,\cdot\,,\,\cdot\,\rangle_{\lambda}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1), \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}), ((L_0 \oplus L_0^{\vee}(1)) \underset{R_0}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1), L_0^{\vee}(1) \underset{R_0}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}})),$$

the sheaf of isomorphisms of $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}$*-sheaves of symplectic* $\mathcal{O}$*-modules with maximal totally isotropic* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R_0$*-submodules.* (*The group* P_0 *acts as automorphisms on* $(L \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\lambda}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1), L_0^{\vee}(1) \underset{R_0}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}})$ *by definition. The third entries in the tuples represent the values of the pairings.*)

Definition 6.5. *The* **principal** M_0**-bundle** *over* $\mathsf{M}_{\mathcal{H}}$ *is the* M_0*-torsor*

$$\mathcal{E}_{\mathrm{M}_0} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}((\underline{\mathrm{Lie}}^{\vee}_{G^{\vee}_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1)), (L_0^{\vee}(1) \underset{R_0}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1))),$$

the sheaf of isomorphisms of $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}$*-sheaves of* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R_0$*-modules.* (*We view the second entries in the pairs as an additional structure, inherited from the corresponding objects for* P_0*. The group* M_0 *acts obviously as automorphisms on* $(L_0^{\vee}(1) \underset{R_0}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1))$ *by definition.*)

These define étale torsors because, by the theory of infinitesimal deformations (cf. for example [38, Ch. 2]) and the theory of Artin's approximations (cf. [3, Thm. 1.10 and Cor. 2.5]),

$$(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}), \langle\,\cdot\,,\,\cdot\,\rangle_{\lambda}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1), \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}})$$

and

$$((L_0 \oplus L_0^{\vee}(1)) \underset{R_0}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1), L_0^{\vee}(1) \underset{R_0}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H}}})$$

are étale locally isomorphic.

Definition 6.6. *For any* R_0*-algebra* E*, we denote by* $\mathrm{Rep}_E(\mathrm{P}_0)$ (*resp.* $\mathrm{Rep}_E(\mathrm{M}_0)$) *the category of* E*-modules with algebraic actions of* $\mathrm{P}_0 \underset{R_0}{\otimes} E$ (*resp.* $\mathrm{M}_0 \underset{R_0}{\otimes} E$).

Definition 6.7. *Let* E *be any* R_0*-algebra. For any* $W \in \mathrm{Rep}_E(\mathrm{P}_0)$*, we define*

$$\mathcal{E}_{\mathrm{P}_0,E}(W) := (\mathcal{E}_{\mathrm{P}_0} \underset{R_0}{\otimes} E) \overset{\mathrm{P}_0 \underset{R_0}{\otimes} E}{\times} W,$$

called the **automorphic sheaf** *over* $\mathsf{M}_{\mathcal{H}} \underset{R_0}{\otimes} E$ *associated with* W*. It is called an* **automorphic bundle** *if* W *is locally free of finite rank over* E*. We define similarly* $\mathcal{E}_{\mathrm{M}_0,E}(W)$ *for* $W \in \mathrm{Rep}_E(\mathrm{M}_0)$ *by replacing* P_0 *with* M_0 *in the above expression.*

Lemma 6.8. *Let* E *be any* R_0*-algebra. If we view an element* $W \in \mathrm{Rep}_E(\mathrm{M}_0)$ *as an element in* $\mathrm{Rep}_E(\mathrm{P}_0)$ *via the canonical surjection* $\mathrm{P}_0 \twoheadrightarrow \mathrm{M}_0$*, then we have a canonical isomorphism* $\mathcal{E}_{\mathrm{P}_0,E}(W) \cong \mathcal{E}_{\mathrm{M}_0,E}(W)$.

6B. **Canonical extensions.** By taking $Q = \mathcal{O}$, so that $\underline{\mathrm{Hom}}_{\mathcal{O}}(Q, G_{\mathsf{M}_{\mathcal{H}}})^{\circ} \cong G_{\mathsf{M}_{\mathcal{H}}}$ and so that there exists some $\mathbb{Z}^{\times}_{(\Box)}$-isogeny $\kappa^{\mathrm{isog}} : G_{\mathsf{M}_{\mathcal{H}}} \to \mathsf{N}$ as in Theorem 2.15, the locally free sheaf $\underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \cong \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})$ extends to the locally free sheaf $\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$ over $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$. Let

$$\underline{H}_1^{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}) := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}(\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}), \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}).$$

Proposition 6.9. *There exists a* **unique** *locally free sheaf $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$ over $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ satisfying the following properties:*

(1) *The sheaf $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$, canonically identified as a subsheaf of the quasicoherent sheaf $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}))$, is self-dual under the pairing $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*\langle \cdot , \cdot \rangle_{\lambda}$. We shall denote the induced pairing by $\langle \cdot , \cdot \rangle^{\mathrm{can}}_{\lambda}$.*

(2) *$\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$ contains $\underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ as a subsheaf totally isotropic under $\langle \cdot , \cdot \rangle^{\mathrm{can}}_{\lambda}$.*

(3) *The quotient sheaf $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}/\underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ can be canonically identified with the subsheaf $\underline{\mathrm{Lie}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ of $(\mathsf{M}_{\mathcal{H}} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})_*\underline{\mathrm{Lie}}_{G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}}$.*

(4) *The pairing $\langle \cdot , \cdot \rangle^{\mathrm{can}}_{\lambda}$ induces an isomorphism $\underline{\mathrm{Lie}}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \xrightarrow{\sim} \underline{\mathrm{Lie}}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$ which coincides with $d\lambda$.*

(5) *Let $\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})$. The Gauss–Manin connection*

$$\nabla : \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H}}/\mathsf{S}_0}$$

extends to an integrable connection

$$\nabla : \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} \to \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} \tag{6.10}$$

with log poles along $\mathsf{D}_{\infty,\mathcal{H}}$, called the extended Gauss–Manin connection, such that the composition

$$\begin{aligned} \underline{\mathrm{Lie}}^{\vee}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} &\hookrightarrow \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} \\ &\xrightarrow{\nabla} \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} \twoheadrightarrow \underline{\mathrm{Lie}}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0} \end{aligned} \tag{6.11}$$

induces by duality the **extended Kodaira–Spencer morphism**

$$\underline{\mathrm{Lie}}^{\vee}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}} \to \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$$

in [38, Thm. 4.6.3.32], *which factors through* $\underline{\mathrm{KS}}$ (*in Definition* 1.40) *and induces the extended Kodaira–Spencer isomorphism* $\mathrm{KS}_{G/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}/\mathsf{S}_0}$ *in* (4) *of Theorem* 1.41.

With these characterizing properties, we say that $(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}, \nabla)$ is the **canonical extension** *of $(\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}), \nabla)$.*

Proof. The uniqueness of $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$ is clear by the first four properties. To show the existence, let us take $\underline{H}_1^{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}$ to be the sheaf $\underline{H}_1^{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$ (for $Q = \mathcal{O}$, as mentioned before this proposition). It is locally free with a Hodge filtration by (3c) of Theorem 2.15. Moreover, by taking some integer $N > 0$ prime-to-$\Box$ such that $N\,\mathrm{Diff}^{-1} \subset \mathcal{O}$, we obtain by multiplication by N a morphism $j_Q : Q^\vee \cong \mathrm{Diff}^{-1} \hookrightarrow Q = \mathcal{O}$ as in Lemma 2.5 such that pullback by κ^{isog} identifies $\langle\,\cdot\,,\,\cdot\,\rangle_{\lambda_{\mathsf{M}_{\mathcal{H}}},j_Q} : \underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}^1_{\mathrm{dR}}(\mathsf{N}/\mathsf{M}_{\mathcal{H}}) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1)$ canonically with $\langle\,\cdot\,,\,\cdot\,\rangle_{\lambda_{\mathsf{M}_{\mathcal{H}}}} : \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \times \underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}}) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H}}}(1)$. Then (1), (2), and (3) follow from (3d) of Theorem 2.15, and (4) follows from Proposition 5.14 (which is used to prove (3b) of Theorem 2.15). It remains to verify (5). By definition, $\underline{H}^1_{\mathrm{dR}}(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}} \cong \underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}})$. The existence of ∇ in (6.10) follows from (3e) of Theorem 2.15. By Remark 4.42, the pullback of (6.11) to $\mathsf{M}_{\mathcal{H}}$ is induced by the usual Kodaira–Spencer class. Since the extended Kodaira–Spencer morphism in [38, Thm. 4.6.3.32] is defined exactly as a morphism induced by the usual Kodaira–Spencer morphism (by normality of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ and local freeness of the sheaves involved), it is induced by duality by (6.11), as desired. $\Box$

Remark 6.12. The notion of *canonical extensions* is closely related to the notion of *regular singularities* of algebraic differential equations. (See [13] and [34] for the notion of regular singularities. See [45], [16, Ch. VI], [24], [25], and [40] for the notion of canonical extensions over $\mathbb{C}$, and see [42] for an earlier treatment of canonical extensions in mixed characteristics. See in particular [24, Thm. 4.2] for the explanation of why and how the two notions are related.)

Then the principal bundle $\mathcal{E}_{\mathrm{P}_0}$ extends canonically to a principal bundle $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ by setting

$$\begin{aligned}\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0} := \underline{\mathrm{Isom}}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}((&\underline{H}^{\mathrm{dR}}_1(G_{\mathsf{M}_{\mathcal{H}}}/\mathsf{M}_{\mathcal{H}})^{\mathrm{can}}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathrm{can}}_{\lambda}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}(1), \underline{\mathrm{Lie}}^\vee_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}),\\ &((L_0 \oplus L_0^\vee(1)) \underset{R_0}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}(1), L_0^\vee(1) \underset{R_0}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}})),\end{aligned}$$

and the principal bundle $\mathcal{E}_{\mathrm{M}_0}$ extends canonically to a principal bundle $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}$ by setting

$$\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0} := \underline{\mathrm{Isom}}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}((\underline{\mathrm{Lie}}^\vee_{G^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}(1)), (L_0^\vee(1) \underset{R_0}{\otimes} \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}(1))).$$

Definition 6.13. *Let E be any R_0-algebra. For any $W \in \mathrm{Rep}_E(\mathrm{P}_0)$, we define*

$$\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,E}(W) := (\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0} \underset{R_0}{\otimes} E) \overset{\mathrm{P}_0 \underset{R_0}{\otimes} E}{\times} W,$$

called the **canonical extension** *of $\mathcal{E}_{\mathrm{P}_0,E}(W)$, and define*

$$\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_0,E}(W) := \mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,E}(W) \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}}{\otimes} \mathscr{I}_{\mathsf{D}_{\infty,\mathcal{H}}},$$

called the **subcanonical extension** *of $\mathcal{E}_{\mathrm{P}_0,E}(W)$, where $\mathscr{I}_{\mathsf{D}_{\infty,\mathcal{H}}}$ is the $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H}}}$-ideal defining the relative Cartier divisor $\mathsf{D}_{\infty,\mathcal{H}}$ (with its reduced structure) in* (3) *of Theorem* 1.41*. We define similarly $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,E}(W)$ and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_0,E}(W)$ with P_0* (*and its principal bundle*) *replaced accordingly with M_0* (*and its principal bundle*)*.*

Lemma 6.14. *Let E be any R_0-algebra. If we view an element in $W \in \mathrm{Rep}_E(\mathrm{M}_0)$ as an element in $\mathrm{Rep}_E(\mathrm{P}_0)$ in the canonical way, then we have canonical isomorphisms $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_0,E}(W) \cong \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_0,E}(W)$ and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_0,E}(W) \cong \mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_0,E}(W)$.*

6C. **Fourier–Jacobi expansions.** Let us fix a representative $(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})$ of a cusp label $[(\mathsf{Z}_{\mathcal{H}}, \Phi_{\mathcal{H}}, \delta_{\mathcal{H}})]$ for $\mathsf{M}_{\mathcal{H}}$ (as in Section 1C). As usual, we shall omit $\mathsf{Z}_{\mathcal{H}}$ from the notation.

Definition 6.15. *The* **principal** M_0**-bundle** *over* $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ *is the* M_0*-torsor*

$$\begin{aligned}\mathcal{E}_{\mathrm{M}_0}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}((\underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}, \mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}(1)),&\\ (L_0^{\vee}(1) \underset{R_0}{\otimes} \mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}, \mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}(1))),&\end{aligned}$$

with conventions as in Definition 6.5.

Then we define $\mathcal{E}_{\mathrm{M}_0,E}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(W)$ for any R_0-algebra E and any $W \in \mathrm{Rep}_E(\mathrm{M}_0)$ as in Definition 6.7.

Lemma 6.16. *Let E be any R_0-algebra. For any $W \in \mathrm{Rep}_E(\mathrm{M}_0)$, there is a canonical isomorphism*

$$(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^* \mathcal{E}_{\mathrm{M}_0}^{\mathrm{can}}(W) \cong (\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^* \mathcal{E}_{\mathrm{M}_0}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(W).$$

Proof. This is because of the canonical isomorphism

$$(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^* \underline{\mathrm{Lie}}^{\vee}_{G^{\vee}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \cong (\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^* \underline{\mathrm{Lie}}^{\vee}_{G^{\vee,\natural}/C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}. \qquad \square$$

By the construction of $\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}$ as a formal completion, we have a natural morphism

$$(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})_* \mathscr{O}_{\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}} \to \prod_{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}} \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell)$$

of $\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}$-modules. By Lemma 6.16, we have the composition of canonical morphisms

$$\begin{aligned}\Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, \mathcal{E}_{\mathrm{M}_0}^{\mathrm{can}}(W)) &\to \Gamma(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}, (\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}})^* \mathcal{E}_{\mathrm{M}_0}^{\mathrm{can}}(W))\\ &\to \Gamma(\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma}, (\mathfrak{X}_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}},\sigma} \to C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}})^* \mathcal{E}_{\mathrm{M}_0}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(W))\\ &\to \prod_{\ell \in \mathbf{S}_{\Phi_{\mathcal{H}}}} \Gamma(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}, \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \mathcal{E}_{\mathrm{M}_0}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(W)),\end{aligned}$$

which we call the morphism of *algebraic Fourier–Jacobi expansions.*

Definition 6.17. *The* **ℓ-th algebraic Fourier–Jacobi morphism**

$$\Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, \mathcal{E}_{\mathrm{M}_0}^{\mathrm{can}}(W)) \to \Gamma(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}, \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \mathcal{E}_{\mathrm{M}_0}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(W))$$

is the ℓ-th factor of the morphism of algebraic Fourier–Jacobi expansions.

Remark 6.18. If $\mathrm{Gr}_{-1}^{\mathsf{Z}} = \{0\}$, then the abelian scheme $C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}} \to \mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ is trivial (i.e., the structural morphism is an isomorphism), and $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}$ is finite over $\mathsf{S}_0 = \mathrm{Spec}(R_0)$. Hence $\Gamma(C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}, \Psi_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(\ell) \underset{\mathscr{O}_{C_{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}}}{\otimes} \mathcal{E}_{\mathrm{M}_0}^{\Phi_{\mathcal{H}},\delta_{\mathcal{H}}}(W)) \cong \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}_{\mathcal{H}}}} \underset{R_0}{\otimes} W)$. In this case, the Fourier–Jacobi expansions are often called q-expansions (because no genuine “Jacobi theta functions” are involved).

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