CLOSED IMMERSIONS OF TOROIDAL COMPACTIFICATIONS OF SHIMURA VARIETIES

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ABSTRACT. We explain that any closed immersion between Shimura varieties defined by morphisms of Shimura data extends to some closed immersion between their projective smooth toroidal compactifications, up to refining the choices of cone decompositions. We also explain that the same holds for many closed immersions between integral models of Shimura varieties and their toroidal compactifications available in the literature.

CONTENTS

1. INTRODUCTION

Given any closed immersion between Shimura varieties or their integral models defined by some morphism of Shimura data (and some additional data, in the case of integral models), it is natural to ask whether it extends to a closed immersion between their toroidal compactifications. Since the construction of toroidal compactifications depends on the choices of some compatible collections of *cone decompositions*, part of the question is whether this can be achieved by some good choices of them, which we might want to be *refinements* of some given ones.

This question is not as trivial as it seems to be. Already in characteristic zero, the analogous question for minimal compactifications is subtle. In fact, in Scholze's groundbreaking work [27], for Hodge-type Shimura varieties, his "perfectoid minimal compactifications" at infinite levels were first constructed using the closures in the minimal compactifications of Siegel modular varieties, rather than the minimal

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compactifications of the Hodge-type Shimura varieties themselves; but the morphism from the minimal compactification of the Shimura variety to the closure in the minimal compactification of the Siegel modular variety is generally not even injective on geometric points. As for toroidal compactifications, if the ambient toroidal compactification is prescribed, then the closure of the Shimura subvariety is generally not normal (and hence cannot be a toroidal compactification by itself), and it might also happen that there exists no morphism that is injective on geometric points from any toroidal compactification of the Shimura subvariety. (See Remarks 4.1 and 4.2 for a related counter-example.)

In this article, we shall show that, under reasonable assumptions, there exist compatible collections of cone decompositions, up to refinements, such that the morphisms between the associated toroidal compactifications are indeed closed immersions (see Theorem 2.2, and Propositions 4.9 and 4.10). We expect this to be useful for studying cycles of Shimura varieties defined by special subvarieties (see Section 5 for some examples). As an application, we shall generalize the construction of "perfectoid toroidal compactifications" from the Siegel case in [25, Appendix] to all Hodge-type cases, and verify [9, Hypothesis 2.18] (see Section 6).

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2. Main results

Let us assume we are in one of the following cases:

Assumption 2.1. *(1) For each $i = 0, 1$, let $(\mathrm{G}_i, \mathrm{D}_i)$ be a Shimura datum (see [8, 1.2.1]), where D_i is a $\mathrm{G}_i(\mathbb{R})$-conjugacy class of a homomorphism $h_i : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{G}_{i,\mathbb{R}}$. Let $\rho : \mathrm{G}_0 \to \mathrm{G}_1$ be an injective homomorphism of algebraic groups over $\mathbb{Q}$ such that $\big(\rho(\mathbb{R})\big)(\mathrm{D}_0) \subset \mathrm{D}_1$. Let $\mathcal{H}_i \subset \mathrm{G}_i(\mathbb{A}^\infty)$ be neat (see [26, 0.6]) open compact subgroups, for $i = 0, 1$, such that $\mathcal{H}_0 = \big(\rho(\mathbb{A}^\infty)\big)^{-1}(\mathcal{H}_1)$. Let F denote a subfield of $\mathbb{C}$ containing the reflex field of $(\mathrm{G}_0, \mathrm{D}_0)$ (which then also contains that of $(\mathrm{G}_1, \mathrm{D}_1)$ by [8, 2.2.1]), and let $\mathsf{S}_0 := \mathrm{Spec}(F)$. For each $i = 0, 1$, let X_i denote the base change to F of the canonical model of the Shimura variety associated with $(\mathrm{G}_i, \mathrm{D}_i)$ at level $\mathcal{H}_i$. Then we have a canonical morphism $f : \mathsf{X}_0 \to \mathsf{X}_1$ over S_0, which we* **assume** *to be a closed immersion. (This can be achieved up to replacing $\mathcal{H}_1$ with a finite index subgroup still containing $\big(\rho(\mathbb{A}^\infty)\big)(\mathcal{H}_0)$, by [7, 1.15].)*

(2) For each $i = 0, 1$, let $(\mathcal{O}_i, \star_i, L_i, \langle \cdot\,, \cdot \rangle_i, h_i)$ be an integral PEL datum (see [18, Def. 1.1.1.1]). Assume that $\mathcal{O}_1$ is a subring of $\mathcal{O}_0$ preserved by $\star_0$, that $\star_1 = \star_0|_{\mathcal{O}_1}$, and that $(L_0, \langle \cdot\,, \cdot \rangle_0, h_0) \cong (L_1, \langle \cdot\,, \cdot \rangle_1, h_1)$ as PEL-type $\mathcal{O}_1$-lattices (see [16, Def. 1.2.1.3]). For each $i = 0, 1$, let G_i denote the associated group functor over $\mathrm{Spec}(\mathbb{Z})$, as in [16, Def. 1.2.1.6], so that we have a canonical injective homomorphism $\rho : \mathrm{G}_0 \to \mathrm{G}_1$ by definition. Let F denote a subfield of $\mathbb{C}$ that is a finite extension of the reflex field F_0 of

$(\mathcal{O}_0, \star_0, L_0, \langle\,\cdot\,,\,\cdot\,\rangle_0, h_0)$ (*see* [16, Def. 1.2.5.4]) (*which is also the reflex field of* $(\mathrm{G}_0 \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \mathrm{G}_0(\mathbb{R}) \cdot h_0)$, *and hence also that of* F_1 *of* $(\mathcal{O}_1, \star_1, L_1, \langle\,\cdot\,,\,\cdot\,\rangle_1, h_1)$ *or* $(\mathrm{G}_1 \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \mathrm{G}_1(\mathbb{R}) \cdot h_1)$, *by* [8, 2.2.1]). *Let* $\Box$ *be a set of rational primes* (*see* [16, Notation and Convetions]) *that are good* (*see* [16, Def. 1.4.1.1]) *for both* $(\mathcal{O}_i, \star_i, L_i, \langle\,\cdot\,,\,\cdot\,\rangle_i, h_i)$, *for* $i = 0, 1$, *and let* $\mathsf{S}_0 := \mathrm{Spec}(\mathcal{O}_{F,(\Box)})$. *Let* $\mathcal{H}_i \subset \mathrm{G}_i(\mathbb{A}^{\infty,\Box})$ *be neat* (*see* [16, Def. 1.4.1.8]) *open compact subgroups, for* $i = 0, 1$, *such that* $\mathcal{H}_0 = \big(\rho(\mathbb{A}^{\infty,\Box})\big)^{-1}(\mathcal{H}_1)$. *For each* $i = 0, 1$, *let* $\mathsf{M}_{\mathcal{H}_i}$ *denote the* (*smooth*) *moduli scheme over* $\mathrm{Spec}(\mathcal{O}_{F_i,\Box})$ *associated with* $(\mathcal{O}_i, \star_i, L_i, \langle\,\cdot\,,\,\cdot\,\rangle_i, h_i)$ *at* $\mathcal{H}_i$ (*see* [16, Def. 1.4.1.4, Thm. 1.4.1.11, and Cor. 7.2.3.10]). *By restricting the* $\mathcal{O}_0$-*endomorphism structures parameterized by* $\mathsf{M}_{\mathcal{H}_0}$ *to* $\mathcal{O}_1$-*endomorphism structures, we obtain a canonical morphism* $\mathsf{M}_{\mathcal{H}_0} \underset{\mathcal{O}_{F_0,\Box}}{\otimes} \mathcal{O}_{F,(\Box)} \to \mathsf{M}_{\mathcal{H}_1} \underset{\mathcal{O}_{F_1,\Box}}{\otimes} \mathcal{O}_{F,(\Box)}$ *over* S_0. *Then we take* X_0 *and* X_1 *to be open-and-closed subschemes of* $\mathsf{M}_{\mathcal{H}_0} \underset{\mathcal{O}_{F_0,\Box}}{\otimes} \mathcal{O}_{F,(\Box)}$ *and* $\mathsf{M}_{\mathcal{H}_1} \underset{\mathcal{O}_{F_1,\Box}}{\otimes} \mathcal{O}_{F,(\Box)}$, *respectively, such that the above morphism induces a morphism* $f : \mathsf{X}_0 \to \mathsf{X}_1$ *over* S_0, *which we* **assume** *to be a closed immersion.*

(3) *For* $i = 0, 1$, *suppose that we have integral PEL data* $(\mathcal{O}_i, \star_i, L_i, \langle\,\cdot\,,\,\cdot\,\rangle_i, h_i)$ (*for which* p *might not be good*), *together with some suitable choices of* $(\mathcal{O}_i, \star_i, L_{i,\mathrm{j}}, \langle\,\cdot\,,\,\cdot\,\rangle_{i,\mathrm{j}}, h_{i,\mathrm{j}})$ *and a shared choice of a collection of auxiliary integral PEL data* $\{(\mathcal{O}_{\mathrm{aux}}, \star_{\mathrm{aux}}, L_{\mathrm{j,aux}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{j,aux}}, h_{\mathrm{j,aux}})\}_{\mathrm{j} \in \mathrm{J}}$ (*for which* p *is good*), *as in* [17, Sec. 2 and 4]; *and that* $(\mathcal{O}_1, \star_1, L_1, \langle\,\cdot\,,\,\cdot\,\rangle_1, h_1)$ *also serves as a choice of an auxiliary integral PEL datum for* $(\mathcal{O}_0, \star_0, L_0, \langle\,\cdot\,,\,\cdot\,\rangle_0, h_0)$ (*but without requiring that* p *is good for either of these two*). *Then we have homomorphisms* $\mathrm{G}_0 \xrightarrow{\rho} \mathrm{G}_1 \xrightarrow{\rho_{\mathrm{j,aux}}} \mathrm{G}_{\mathrm{j,aux}}$, *for all* $\mathrm{j} \in \mathrm{J}$. *Suppose that we have neat open compact subgroups* $\mathcal{H}_0 \subset \mathrm{G}(\hat{\mathbb{Z}})$, $\mathcal{H}_1 \subset \mathrm{G}(\hat{\mathbb{Z}})$, *and* $\mathcal{H}_{\mathrm{j,aux}} \subset \mathrm{G}_{\mathrm{j,aux}}(\hat{\mathbb{Z}}^p)$ *such that* $\mathcal{H}_0 = \big(\rho(\mathbb{A}^\infty)\big)^{-1}(\mathcal{H}_1)$ *and such that the images of* $\mathcal{H}_1$ *under* $\mathrm{G}_1(\hat{\mathbb{Z}}) \to \mathrm{G}_{\mathrm{j,aux}}(\hat{\mathbb{Z}}^p)$ *are neat and contained in* $\mathcal{H}_{\mathrm{j,aux}}$, *for all* $\mathrm{j} \in \mathrm{J}$. *Let* F *denote a subfield of* $\mathbb{C}$ *that is a finite extension of the reflex field of* $(\mathcal{O}_0, \star_0, L_0, \langle\,\cdot\,,\,\cdot\,\rangle_0, h_0)$, *and hence also those of* $(\mathcal{O}_1, \star_1, L_1, \langle\,\cdot\,,\,\cdot\,\rangle_1, h_1)$ *and* $(\mathcal{O}_{\mathrm{aux}}, \star_{\mathrm{aux}}, L_{\mathrm{j,aux}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{j,aux}}, h_{\mathrm{j,aux}})$, *for all* $\mathrm{j} \in \mathrm{J}$. *With the above data, we have associated moduli problems* $\mathsf{M}_{\mathcal{H}_0}$ *and* $\mathsf{M}_{\mathcal{H}_1}$ *over* $\mathrm{Spec}(F)$, *and associated auxiliary moduli problems* $\mathsf{M}_{\mathcal{H}_{\mathrm{j,aux}}}$ *over* $\mathsf{S}_0 := \mathrm{Spec}(\mathcal{O}_{F,(p)})$, *together with canonical finite morphisms* $\mathsf{M}_{\mathcal{H}_0} \to \mathsf{M}_{\mathcal{H}_1} \to \prod_{\mathrm{j} \in \mathrm{J}} \mathsf{M}_{\mathcal{H}_{\mathrm{j,aux}}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *over* $\mathrm{Spec}(F)$, *which extend to canonical finite morphisms* $\vec{\mathsf{M}}_{\mathcal{H}_0} \to \vec{\mathsf{M}}_{\mathcal{H}_1} \to \prod_{\mathrm{j} \in \mathrm{J}} \mathsf{M}_{\mathcal{H}_{\mathrm{j,aux}}}$ *over* S_0 *by taking normalizations as in* [17, Sec. 4]. *Then we take* X_0 *and* X_1 *to be open-and-closed subschemes of* $\vec{\mathsf{M}}_{\mathcal{H}_0}$ *and* $\vec{\mathsf{M}}_{\mathcal{H}_1}$, *respectively, such that* $\vec{\mathsf{M}}_{\mathcal{H}_0} \to \vec{\mathsf{M}}_{\mathcal{H}_1}$ *induces a morphism* $f : \mathsf{X}_0 \to \mathsf{X}_1$ *over* S_0, *which we* **assume** *to be a closed immersion.*

(4) *Suppose that we have a morphism of Shimura data* $(\mathrm{G}_0, \mathrm{D}_0) \to (\mathrm{G}_1, \mathrm{D}_1)$ *defined by some injective homomorphism* $\rho : \mathrm{G}_0 \to \mathrm{G}_1$ *as in* (1), *and suppose that we have a Siegel embedding* $(\mathrm{G}_1, \mathrm{D}_1) \hookrightarrow (\mathrm{G}_{\mathrm{aux}}, \mathrm{D}_{\mathrm{aux}})$ *defined by some injective homomorphism* $\mathrm{G}_1 \to \mathrm{G}_{\mathrm{aux}}$, *with* $\mathrm{G}_{\mathrm{aux}} \cong \mathrm{GSp}_{2g,\mathbb{Q}}$, *for some* $g \geq 0$. *Suppose that we have neat open compact subgroups* $\mathcal{H}_0 \subset \mathrm{G}_0(\mathbb{A}^\infty)$, $\mathcal{H}_1 \subset \mathrm{G}_1(\mathbb{A}^\infty)$, *and* $\mathcal{H}_{\mathrm{aux}} \subset \mathrm{G}_{\mathrm{aux}}(\mathbb{A}^{\infty,p})$ *such that* $\mathcal{H}_0 = \big(\rho(\mathbb{A}^\infty)\big)^{-1}(\mathcal{H}_1)$ *and such that the image of* $\mathcal{H}_1$ *under* $\mathrm{G}_1(\mathbb{A}^\infty) \to \mathrm{G}_{\mathrm{aux}}(\mathbb{A}^{\infty,p})$ *is neat and*

contained in $\mathcal{H}_{\mathrm{aux}}$. Let F denote a subfield of $\mathbb{C}$ that is a finite extension of the reflex field of $(\mathrm{G}_0, \mathrm{D}_0)$*, and hence also that of* $(\mathrm{G}_1, \mathrm{D}_1)$*. Let* X_0 *and* X_1 *be integral models over* $\mathsf{S}_0 := \mathrm{Spec}(\mathcal{O}_{F,(p)})$ *of the Shimura varieties associated with* $(\mathrm{G}_0, \mathrm{D}_0)$ *and* $(\mathrm{G}_1, \mathrm{D}_1)$ *at levels* $\mathcal{H}_0$ *and* $\mathcal{H}_1$*, respectively, defined by taking normalizations of the characteristic zero models over* F (*which are base changes of the corresponding canonical models to* F) *over the Siegel moduli over* $\mathrm{Spec}(\mathbb{Z}_{(p)})$ *associated with* $(\mathrm{G}_{\mathrm{aux}}, \mathrm{D}_{\mathrm{aux}})$ *and the prime-to-p level* $\mathcal{H}_{\mathrm{aux}}$*, as in* [24, Introduction]. *Then we have a canonically induced morphism* $f : \mathsf{X}_0 \to \mathsf{X}_1$ *over* S_0*, which we* **assume** *to be a closed immersion.*

We shall say that we are in Cases (1), (2), (3), or (4) depending on the case we are in Assumption 2.1. In each case, we have good toroidal compactifications $\mathsf{X}_i \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i}$ associated with some compatible collections of cone decompositions Σ_i, for $i = 0, 1$, whose properties we will review in more detail in the next section.

Our main result is the following:

Theorem 2.2. *Let* $f : \mathsf{X}_0 \to \mathsf{X}_1$ *be as in Assumption* 2.1*. Then there exist toroidal compactifications* $\mathsf{X}_i \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i}$*, for* $i = 0, 1$*, associated with some compatible collections* Σ_i *of projective smooth cone decompositions* (*see* [2, 3, 26] *in Case* (1)*; see* [16, Thm. 6.4.1.1 and 7.3.3.4] *in Case* (2)*; see* [19, Thm. 6.1] *in Case* (3)*; and see* [24, Thm. 4.1.5 and Rem. 4.1.6] *in Case* (4)) *such that* f *extends to a closed immersion* $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1} : \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \to \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$*. Moreover, if we denote by* $\mathcal{I}_{\Sigma_i}$ *the* $\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i}}$*-ideal defining the boundary* $\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i} - \mathsf{X}_i$ (*with its reduced subscheme structure*)*, for* $i = 0, 1$*, then we may require that* $f^{\mathrm{tor},*}_{\Sigma_0,\Sigma_1}(\mathcal{I}_{\Sigma_1}) \cong \mathcal{I}_{\Sigma_0}$ *as* $\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0}}$*-ideals. We may require that* Σ_0 *and* Σ_1 *refine any finite number of prescribed compatible collections of cone decompositions.*

The proof of Theorem 2.2 will be completed in Section 4.

Remark 2.3.
(1) In Cases (2) and (3), for example, we can take X_i to be the schematic closure of the base change to $\mathrm{Spec}(F)$ of the canonical model of the Shimura variety associated with the Shimura datum $(\mathrm{G}_i \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \mathrm{G}_i(\mathbb{R}) \cdot h_i)$ (see [14, Sec. 8], [15, Sec. 2], and [23, Sec. 1.2]), for $i = 0, 1$, when $\mathrm{G}_i \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is connected and $(\mathrm{G}_i \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \mathrm{G}_i(\mathbb{R}) \cdot h_i)$ qualifies as a Shimura datum.
(2) In Case (2), in order to show that $f : \mathsf{X}_0 \to \mathsf{X}_1$ is indeed a closed immersion, we often have to resort to the moduli interpretations of $\mathsf{M}_{\mathcal{H}_0}$ and $\mathsf{M}_{\mathcal{H}_1}$.
(3) In Case (3), when the levels $\mathcal{H}_0$ and $\mathcal{H}_1$ differ at p from the stabilizers of L_0 and L_1, it is generally more difficult to verify that the morphism $f : \mathsf{X}_0 \to \mathsf{X}_1$ defined abstractly by taking normalizations is a closed immersion. Practically, when the levels are parahoric at p (and satisfies some technical assumptions), we can still define X_0 and X_1 using some explicit moduli problems—see, for example, [17, Ex. 2.4 and 13.12, and Rem. 16.5]. However, we do not (yet) have a method to study higher levels in general.
(4) In Case (4), the similar verification that $f : \mathsf{X}_0 \to \mathsf{X}_1$ is a closed immersion is subtle already when the levels are hyperspecial at p as in [13].
(5) Nevertheless, Theorem 2.2 provides closed immersions $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1} : \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \to \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$ as long as the input $f : \mathsf{X}_0 \to \mathsf{X}_1$ is a closed immersion, and we included all four cases (which in theory allows arbitrarily high levels at p in Cases (3) and (4)) even when the assumption of being a closed immersion cannot be easily verified in general.

(6) Certainly, we expect Theorem 2.2 to extend to integral models of abelian-type Shimura varieties, generalizing those constructed in Cases (2), (3), and (4) in Assumption 2.1, as soon as the their toroidal compactifications are constructed and shown to have desired properties as in Propositions 3.1 and 3.4 below. However, we do not expect it to be any easier to verify that $f : \mathsf{X}_0 \to \mathsf{X}_1$ is indeed a closed immersion.

Remark 2.4. In Theorem 2.2, the main reason to consider the projectivity of the cone decompositions is that it ensures that the toroidal compactifications we obtained are schemes rather than merely algebraic spaces.

Remark 2.5. In Theorem 2.2, the assertion that $f^{\mathrm{tor},*}_{\Sigma_0,\Sigma_1}(\mathcal{I}_{\Sigma_1}) \cong \mathcal{I}_{\Sigma_0}$ does not follow from the assertion that $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1}$ is a closed immersion. (See Example 5.1 below.)

Remark 2.6. Since base changes of closed immersions are still closed immersions, by using [20, Thm. 2.3.2], Theorem 2.2 implies similar results for partial toroidal compactifications of *well-positioned subschemes* of base changes of integral models of Shimura varieties. We shall leave the precise statements to interested readers.

3. Morphisms between toroidal compactifications

In all cases in Assumption 2.1, we have good toroidal and minimal compactifications $\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i} \to \mathsf{S}_0$ and $\mathsf{X}^{\min}_i \to \mathsf{S}_0$, for $i = 0, 1$, whose qualitative properties we shall summarize as follows, based on the constructions in [4, 2, 3, 26, 16, 17, 19] (as in [21, Prop. 2.2] and [20, Prop. 2.1.2 and 2.1.3, and Cor. 2.1.7] and their proofs):

Proposition 3.1. *For each $i = 0, 1$, there is a canonical minimal compactification*

$$J^{\min}_i : \mathsf{X}_i \hookrightarrow \mathsf{X}^{\min}_i$$

over S_0, together with a canonical collection of toroidal compactifications

$$J^{\mathrm{tor}}_{i,\Sigma_i} : \mathsf{X}_i \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i}$$

over S_0, labeled by certain compatible collections Σ_i of cone decompositions, satisfying the following properties:

(1) For each Σ_i, there is a proper surjective structural morphism

$$\oint_{i,\Sigma_i} : \mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i} \to \mathsf{X}^{\min}_i,$$

compatible with $J^{\min}_i$ and $J^{\mathrm{tor}}_{i,\Sigma_i}$ in the sense that $J^{\min}_i = \oint_{i,\Sigma_i} \circ J^{\mathrm{tor}}_{i,\Sigma_i}$.

(2) The scheme $\mathsf{X}^{\min}_i$ admits a stratification by locally closed subschemes Z_i flat over S_0, each of which is isomorphic to a finite quotient of an analogue of X_i. (*Nevertheless, in Cases* (2) *and* (3), *we can still identity each Z_i with an analogue of X_i.*)

(3) Each Σ_i is a set $\{\Sigma_{\mathsf{Z}_i}\}_{\mathsf{Z}_i}$ of cone decompositions Σ_{Z_i} with the same index set as that of the strata of $\mathsf{X}^{\min}_i$. (*In* [16], *the elements of this index set was called* **cusp labels**.) *For simplicity, we shall suppress such cusp labels and denote the associated objects with subscripts given by the strata Z_i.*

(4) For each stratum Z_i, the cone decomposition Σ_{Z_i} is a cone decomposition of some $\mathbf{P}_{\mathsf{Z}_i}$, where $\mathbf{P}_{\mathsf{Z}_i}$ is the union of the interior $\mathbf{P}^+_{\mathsf{Z}_i}$ of a homogenous self-adjoint cone (*see* [3, Ch. 2]) *and its rational boundary components, which is admissible with respect to some arithmetic group Γ_{Z_i} acting on $\mathbf{P}_{\mathsf{Z}_i}$* (*and hence also on Σ_{Z_i}*). *Then Σ_{Z_i} has a subset $\Sigma^+_{\mathsf{Z}_i}$ forming a cone*

decomposition of $\mathbf{P}^+_{\mathsf{Z}_i}$*. If* τ *is a cone in* Σ_{Z_i} *that is not in* $\Sigma^+_{\mathsf{Z}_i}$*, then there exist a stratum* Z'_i *of* $\mathsf{X}^{\min}_i$*, whose closure in* $\mathsf{X}^{\min}_i$ *contains* Z_i*, and a cone* τ' *in* $\Sigma^+_{\mathsf{Z}'_i}$*, whose* $\Gamma_{\mathsf{Z}'_i}$*-orbit is uniquely determined by the* Γ_{Z_i}*-orbit of* τ*.*

We may and we shall assume that Σ_i *is smooth, and that, for each* Z_i *and each* $\sigma \in \Sigma^+_{\mathsf{Z}_i}$*, the stabilizer* $\Gamma_{\mathsf{Z}_i,\sigma}$ *of* σ *in* Γ_{Z_i} *is trivial.*

(5) *For each* Σ_i*, the associated* $\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i}$ *admits a stratification by locally closed subschemes* $\mathsf{Z}_{i,[\sigma]}$ *flat over* S_0*, labeled by the strata* Z_i *of* $\mathsf{X}^{\min}_i$ *and the orbits* $[\sigma] \in \Sigma^+_{\mathsf{Z}_i}/\Gamma_{\mathsf{Z}_i}$*. The stratifications of* $\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i}$ *and* $\mathsf{X}^{\min}_i$ *are compatible with each other in a precise sense, which we summarize as follows: The preimage of a stratum* Z_i *of* $\mathsf{X}^{\min}_{\mathcal{H}_i}$ *is the (set-theoretic) disjoint union of the strata* $\mathsf{Z}_{i,[\sigma]}$ *of* $\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i}$ *with* $[\sigma] \in \Sigma^+_{\mathsf{Z}_i}/\Gamma_{\mathsf{Z}_i}$*. If* τ *is a face of a representative* σ *of* $[\sigma]$*, which is identified (as in the property (4) above) with the* $\Gamma_{\mathsf{Z}'_i}$*-orbit* $[\tau']$ *of some cone* τ' *in* $\Sigma^+_{\mathsf{Z}'_i}$*, where* Z'_i *is a stratum whose closure in* $\mathsf{X}^{\min}_i$ *contains* Z_i*, then* $\mathsf{Z}_{i,[\sigma]}$ *is contained in the closure of* $\mathsf{Z}'_{i,[\tau']}$*.*

(6) *For each stratum* Z_i *of* $\mathsf{X}^{\min}_i$*, there is a proper surjective morphism*

$$C_{\mathsf{Z}_i} \to \mathsf{Z}_i$$

(whose precise description is not important for our purpose), together with a morphism

$$\Xi_{\mathsf{Z}_i} \to C_{\mathsf{Z}_i}$$

of schemes which is a torsor under the pullback of a split torus E_{Z_i} *with some character group* $\mathbf{S}_{\mathsf{Z}_i}$ *over* $\mathrm{Spec}(\mathbb{Z})$*, so that we have*

$$\Xi_{\mathsf{Z}_i} \cong \underline{\mathrm{Spec}}_{\mathscr{O}_{C_{\mathsf{Z}_i}}}\Big(\mathop{\oplus}_{\ell\in\mathbf{S}_{\mathsf{Z}_i}} \Psi_{\mathsf{Z}_i}(\ell)\Big),$$

for some invertible sheaves $\Psi_{\mathsf{Z}_i}(\ell)$*. (Each* $\Psi_{\mathsf{Z}_i}(\ell)$ *can be viewed as the subsheaf of* $(\Xi_{\mathsf{Z}_i} \to C_{\mathsf{Z}_i})_*\mathscr{O}_{\Xi_{\mathsf{Z}_i}}$ *on which* E_{Z_i} *acts via the character* $\ell \in \mathbf{S}_{\mathsf{Z}_i}$*.) This character group* $\mathbf{S}_{\mathsf{Z}_i}$ *admits a canonical action of* Γ_{Z_i}*, and its* $\mathbb{R}$*-dual* $\mathbf{S}^\vee_{\mathsf{Z}_i,\mathbb{R}} := \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}_{\mathsf{Z}_i}, \mathbb{R})$ *canonically contains the above sets* $\mathbf{P}_{\mathsf{Z}_i}$ *and* $\mathbf{P}^+_{\mathsf{Z}_i}$ *as subsets with compatible* Γ_{Z_i}*-actions.*

(7) *For each* $\sigma \in \Sigma_{\mathsf{Z}_i}$*, consider the canonical pairing* $\langle\,\cdot\,,\,\cdot\,\rangle : \mathbf{S}_{\mathsf{Z}_i} \times \mathbf{S}^\vee_{\mathsf{Z}_i,\mathbb{R}} \to \mathbb{R}$ *and* $\sigma^\vee := \{\ell \in \mathbf{S}_{\mathsf{Z}_i} : \langle \ell, y\rangle \geq 0,\ \forall y \in \sigma\}$*,* $\sigma^\vee_0 := \{\ell \in \mathbf{S}_{\mathsf{Z}_i} : \langle \ell, y\rangle > 0,\ \forall y \in \sigma\}$*, and* $\sigma^\perp := \{\ell \in \mathbf{S}_{\mathsf{Z}_i} : \langle \ell, y\rangle = 0,\ \forall y \in \sigma\} \cong \sigma^\vee/\sigma^\vee_0$*. Then we have the affine toroidal embedding*

$$\Xi_{\mathsf{Z}_i} \hookrightarrow \Xi_{\mathsf{Z}_i}(\sigma) := \underline{\mathrm{Spec}}_{\mathscr{O}_{C_{\mathsf{Z}_i}}}\Big(\mathop{\oplus}_{\ell\in\sigma^\vee} \Psi_{\mathsf{Z}_i}(\ell)\Big).$$

The scheme $\Xi_{\mathsf{Z}_i}(\sigma)$ *has a closed subscheme* $\Xi_{\mathsf{Z}_i,\sigma}$ *defined by the ideal sheaf corresponding to* $\mathop{\oplus}\limits_{\ell\in\sigma^\vee_0} \Psi_{\mathsf{Z}_i}(\ell)$*, so that*

$$\Xi_{\mathsf{Z}_i,\sigma} \cong \underline{\mathrm{Spec}}_{\mathscr{O}_{C_{\mathsf{Z}_i}}}\Big(\mathop{\oplus}_{\ell\in\sigma^\perp} \Psi_{\mathsf{Z}_i}(\ell)\Big).$$

Then $\Xi_{\mathsf{Z}_i}(\sigma)$ *admits a natural stratification by locally closed subschemes* $\Xi_{\mathsf{Z}_i,\tau}$ *(i.e., the closed subscheme as above of the open subscheme* $\Xi_{\mathsf{Z}_i}(\tau)$ *of* $\Xi_{\mathsf{Z}_i}(\sigma)$*), where* τ *runs over all the faces of* σ *in* Σ_{Z_i}*.*

(8) *For each given* Σ_i*, and for each* Z_i*, consider the full toroidal embedding*

$$\overline{\Xi}_{\mathsf{Z}_i,\Sigma_{\mathsf{Z}_i}} = \mathop{\cup}_{\sigma\in\Sigma_{\mathsf{Z}_i}} \Xi_{\mathsf{Z}_i}(\sigma)$$

defined by the cone decomposition Σ_{Z_i} (*cf.* [16, Thm. 6.1.2.8 and Sec. 6.2.5]), *and consider the formal completion*

$$\mathfrak{X}_{\mathsf{Z}_i,\Sigma_{\mathsf{Z}_i}} := (\overline{\Xi}_{\mathsf{Z}_i,\Sigma_{\mathsf{Z}_i}})^\wedge_{\underset{\tau\in\Sigma^+_{\mathsf{Z}_i}}{\cup}\Xi_{\mathsf{Z}_i,\tau}}$$

of $\overline{\Xi}_{\mathsf{Z}_i,\Sigma_{\mathsf{Z}_i}}$ *along its closed subscheme* $\underset{\tau\in\Sigma^+_{\mathsf{Z}_i}}{\cup}\Xi_{\mathsf{Z}_i,\tau}$. *Consider, for each* $\sigma\in\Sigma^+_{\mathsf{Z}_i}$, *the formal completion*

$$\mathfrak{X}^\circ_{\mathsf{Z}_i,\sigma} := \big(\Xi_{\mathsf{Z}_i}(\sigma)\big)^\wedge_{\Xi_{\mathsf{Z}_i}(\sigma)^+}$$

of $\Xi_{\mathsf{Z}_i}(\sigma)$ *along its closed subscheme* $\Xi_{\mathsf{Z}_i}(\sigma)^+ := \underset{\tau\in\Sigma^+_{\mathsf{Z}_i},\,\overline{\tau}\subset\overline{\sigma}}{\cup}\Xi_{\mathsf{Z}_i,\tau}$. *Then* $\mathfrak{X}_{\mathsf{Z}_i,\Sigma_{\mathsf{Z}}}$ *admits an open covering by* $\mathfrak{X}^\circ_{\mathsf{Z}_i,\sigma}$ *for* σ *running through elements of* $\Sigma^+_{\mathsf{Z}_i}$, *and we have canonical flat morphisms* $\mathfrak{X}^\circ_{\mathsf{Z}_i,\sigma}\hookrightarrow\mathfrak{X}_{\mathsf{Z}_i,\Sigma_{\mathsf{Z}_i}}\to\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i}$ (*of locally ringed spaces*) *inducing isomorphisms*

$$\mathfrak{X}^\circ_{\mathsf{Z}_i,\sigma}\xrightarrow{\sim}(\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i})^\wedge_{\underset{\tau\in\Sigma^+_{\mathsf{Z}_i},\,\overline{\tau}\subset\overline{\sigma}}{\cup}\mathsf{Z}_{i,[\tau]}} \tag{3.2}$$

and

$$\mathfrak{X}_{\mathsf{Z}_i,\Sigma_{\mathsf{Z}_i}}/\Gamma_{\mathsf{Z}_i}\xrightarrow{\sim}(\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i})^\wedge_{\underset{[\tau]\in\Sigma^+_{\mathsf{Z}_i}/\Gamma_{\mathsf{Z}_i}}{\cup}\mathsf{Z}_{i,[\tau]}}. \tag{3.3}$$

More precisely, for each $\sigma\in\Sigma^+_{\mathsf{Z}_i}$, *and for each affine open formal subscheme* $\mathfrak{W}=\mathrm{Spf}(R)$ *of* $\mathfrak{X}^\circ_{\mathsf{Z}_i,\sigma}$, *under the canonically induced (flat) morphisms* $W:=\mathrm{Spec}(R)\to\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i}$ *and* $\mathrm{Spec}(R)\to\Xi_{\mathsf{Z}_i}(\sigma)$ *induced by* (3.2), *the stratification of* W *induced by that of* $\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i}$ *coincides with the stratification of* W *induced by that of* $\Xi_{\mathsf{Z}_i}(\sigma)$. *In particular, the preimages of* X_i *and* Ξ_{Z_i} *coincide as an open subscheme* W^0 *of* W.

As for the morphism $f:\mathsf{X}_0\to\mathsf{X}_1$, we have the following:

Proposition 3.4. *Assume slightly more generally (than in Assumption 2.1) that* $\big(\rho(\mathbb{A}^\infty)\big)(\mathcal{H}_0)\subset\mathcal{H}_1$ *and hence that the morphism* $f:\mathsf{X}_0\to\mathsf{X}_1$ *is finite. Then there exists a canonical finite morphism*

$$f^{\min}:\mathsf{X}_0^{\min}\to\mathsf{X}_1^{\min}$$

such that $f^{\min}\circ J_0^{\min}=J_1^{\min}\circ f$ *over* S_0, *together with a canonical collection of proper morphisms*

$$f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1}:\mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0}\to\mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$$

such that $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1}\circ J^{\mathrm{tor}}_{0,\Sigma_0}=J^{\mathrm{tor}}_{1,\Sigma_1}\circ f$ *and* $f^{\min}\circ\oint_{0,\Sigma_0}=\oint_{1,\Sigma_1}\circ f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1}$ *over* S_0, *labeled by certain pairs* (Σ_0,Σ_i) *of compatible collections of cone decompositions that are compatible with each other in a sense that we shall explain below, satisfying the following properties:*

(1) *For each stratum* Z_0 *of* $\mathsf{X}_0^{\min}$, *there exists a (unique) stratum* Z_1 *of* $\mathsf{X}_1^{\min}$ *such that* $f^{\min}(\mathsf{Z}_0)\subset\mathsf{Z}_1$ (*as subsets of* $\mathsf{X}_1^{\min}$). *Moreover,* Z_0 *is both open and closed in* $(f^{\min})^{-1}(\mathsf{Z}_1)$, *and* $f^{\min}$ *induces a finite morphism* $\mathsf{Z}_0\to\mathsf{Z}_1$.

(2) *Over any* $\mathsf{Z}_0\to\mathsf{Z}_1$ *as above, we have a finite morphism*

$$C_{\mathsf{Z}_0}\to C_{\mathsf{Z}_1},$$

over which we have a finite morphism

$$\Xi_{\mathsf{Z}_0} \to \Xi_{\mathsf{Z}_1},$$

which induces a finite morphism $\Xi_{\mathsf{Z}_0} \to \Xi_{\mathsf{Z}_1} \underset{C_{\mathsf{Z}_1}}{\times} C_{\mathsf{Z}_0}$ *which is equivariant with the pullback of a group homomorphism of tori*

$$E_{\mathsf{Z}_0} \to E_{\mathsf{Z}_1}$$

with finite kernel over $\mathrm{Spec}(\mathbb{Z})$ *that is dual to a homomorphism*

$$\mathbf{S}_{\mathsf{Z}_1} \to \mathbf{S}_{\mathsf{Z}_0}$$

of character groups with finite cokernel. The $\mathbb{R}$*-dual of this last homomorphism is an injective homomorphism* $\mathbf{S}^\vee_{\mathsf{Z}_0,\mathbb{R}} \hookrightarrow \mathbf{S}^\vee_{\mathsf{Z}_1,\mathbb{R}}$ *of* $\mathbb{R}$*-vector spaces, inducing a Cartesian diagram of injective maps*

$$\begin{tikzcd} \mathbf{P}^+_{\mathsf{Z}_0} \arrow[r, hook] \arrow[d, hook] & \mathbf{P}^+_{\mathsf{Z}_1} \arrow[d, hook] \\ \mathbf{P}_{\mathsf{Z}_0} \arrow[r, hook] & \mathbf{P}_{\mathsf{Z}_1}. \end{tikzcd}$$

All the above maps from objects associated with Z_0 *to the corresponding ones associated with* Z_1 *are equivariant with a canonical homomorphism* $\Gamma_{\mathsf{Z}_0} \to \Gamma_{\mathsf{Z}_1}$*. If* $\ell_1 \in \mathbf{S}_{\mathsf{Z}_1}$ *is mapped to* $\ell_0 \in \mathbf{S}_{\mathsf{Z}_0}$ *under* $\mathbf{S}_{\mathsf{Z}_1} \to \mathbf{S}_{\mathsf{Z}_0}$*, then the invertible sheaf* $\Psi_{\mathsf{Z}_0}(\ell_0)$ *over* C_{Z_0} *is canonically isomorphic to the pullback of the invertible sheaf* $\Psi_{\mathsf{Z}_1}(\ell)$ *over* C_{Z_1} *under the above morphism* $C_{\mathsf{Z}_0} \to C_{\mathsf{Z}_1}$*.*

When $\mathcal{H}_0 = \big(\rho(\mathbb{A}^\infty)\big)^{-1}(\mathcal{H}_1)$*, the homomorphism* $\mathbf{S}_{\mathsf{Z}_1} \to \mathbf{S}_{\mathsf{Z}_0}$ *is surjective, and hence the dual homomorphism* $E_{\mathsf{Z}_0} \to E_{\mathsf{Z}_1}$ *is a closed immersion.*

(3) If the image of $\sigma \in \Sigma_{\mathsf{Z}_0}$ *under* $\mathbf{P}_{\mathsf{Z}_0} \hookrightarrow \mathbf{P}_{\mathsf{Z}_1}$ *is contained in some* $\tau \in \Sigma_{\mathsf{Z}_1}$*, then we have a canonical morphism*

$$\Xi_{\mathsf{Z}_0}(\sigma) = \underline{\mathrm{Spec}}_{\mathscr{O}_{C_{\mathsf{Z}_0}}}\Big(\underset{\ell\in\sigma^\vee}{\oplus} \Psi_{\mathsf{Z}_0}(\ell)\Big) \to \Xi_{\mathsf{Z}_1}(\tau) = \underline{\mathrm{Spec}}_{\mathscr{O}_{C_{\mathsf{Z}_1}}}\Big(\underset{\ell\in\tau^\vee}{\oplus} \Psi_{\mathsf{Z}_1}(\ell)\Big)$$

extending $\Xi_{\mathsf{Z}_0} \to \Xi_{\mathsf{Z}_1}$*, and inducing a canonical morphism* $\Xi_{\mathsf{Z}_0}(\sigma) \to \Xi_{\mathsf{Z}_1}(\tau) \underset{C_{\mathsf{Z}_1}}{\times} C_{\mathsf{Z}_0}$ *which is equivariant with the pullback of* $E_{\mathsf{Z}_0} \to E_{\mathsf{Z}_1}$*. Moreover, there is an induced morphism*

$$\Xi_{\mathsf{Z}_0,\sigma} = \underline{\mathrm{Spec}}_{\mathscr{O}_{C_{\mathsf{Z}_0}}}\Big(\underset{\ell_0\in\sigma^\perp}{\oplus} \Psi_{\mathsf{Z}_0}(\ell_0)\Big) \to \Xi_{\mathsf{Z}_1,\tau} = \underline{\mathrm{Spec}}_{\mathscr{O}_{C_{\mathsf{Z}_1}}}\Big(\underset{\ell_1\in\tau^\perp}{\oplus} \Psi_{\mathsf{Z}_1}(\ell_1)\Big).$$

(4) We say that the collections $\Sigma_0 = \{\Sigma_{\mathsf{Z}_0}\}_{\mathsf{Z}_0}$ *and* $\Sigma_1 = \{\Sigma_{\mathsf{Z}_1}\}_{\mathsf{Z}_i}$ *are* **compatible with each other** *or simply* **compatible** *if, when* Z_0 *is mapped to* Z_1 *as above, the image of each* $\sigma \in \Sigma^+_{\mathsf{Z}_0}$ *under the map* $\mathbf{P}^+_{\mathsf{Z}_0} \hookrightarrow \mathbf{P}^+_{\mathsf{Z}_1}$ *is contained in some* $\tau \in \Sigma^+_{\mathsf{Z}_1}$*. We say that* Σ_0 **is induced by** Σ_1 *if each* $\sigma \in \Sigma^+_{\mathsf{Z}_0}$ *is exactly the preimage of some* $\tau \in \Sigma^+_{\mathsf{Z}_1}$*. (If* Σ_0 *is induced by* Σ_1*, then they are necessarily compatible.)*

(5) The morphism $f : \mathsf{X}_0 \to \mathsf{X}_1$ *extends to a proper (resp. finite) morphism* $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1} : \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \to \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$ *as above if and only if* Σ_0 *and* Σ_1 *are compatible (resp.* Σ_0 *is induced by* Σ_1*). When* Σ_0 *and* Σ_1 *are compatible, if the image of* $\sigma \in \Sigma^+_{\mathsf{Z}_0}$ *under* $\mathbf{P}^+_{\mathsf{Z}_0} \hookrightarrow \mathbf{P}^+_{\mathsf{Z}_1}$ *is contained in* $\tau \in \Sigma^+_{\mathsf{Z}_1}$*, then the morphism* $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_i}$ *induces a morphism* $\mathsf{Z}_{0,[\sigma]} \to \mathsf{Z}_{1,[\tau]}$ *(which is not necessarily proper), which can be canonically identified with the morphism* $\Xi_{\mathsf{Z}_0,\sigma} \to \Xi_{\mathsf{Z}_1,\tau}$ *above.*

For each $\tau \in \Sigma^+_{\mathsf{Z}_1}$, the preimage of $\mathsf{Z}_{1,[\tau]}$ is the (set-theoretic) disjoint union of the strata $\mathsf{Z}_{0,[\sigma]}$ labeled by $\sigma \in \Sigma^+_{\mathsf{Z}_0}$ that are mapped into τ under $\mathbf{P}^+_{\mathsf{Z}_0} \hookrightarrow \mathbf{P}^+_{\mathsf{Z}_1}$. If there is a unique such σ, which is the case exactly when σ is the preimage of τ, then the induced morphism $\mathsf{Z}_{0,[\sigma]} \to \mathsf{Z}_{1,[\tau]}$ is finite.

(6) *Suppose that Σ_0 and Σ_1 are compatible. Then there is a proper morphism*

$$\overline{\Xi}_{\mathsf{Z}_0,\Sigma_{\mathsf{Z}_0}} \to \overline{\Xi}_{\mathsf{Z}_1,\Sigma_{\mathsf{Z}_1}},$$

whose formal completion gives a proper morphism

$$\mathfrak{X}_{\mathsf{Z}_0,\Sigma_{\mathsf{Z}_0}} \to \mathfrak{X}_{\mathsf{Z}_1,\Sigma_{\mathsf{Z}_1}}. \tag{3.5}$$

These two morphisms are equivariant with the homomorphism $\Gamma_{\mathsf{Z}_0} \to \Gamma_{\mathsf{Z}_1}$ and induces a proper morphism $\mathfrak{X}_{\mathsf{Z}_0,\Sigma_{\mathsf{Z}_0}}/\Gamma_{\mathsf{Z}_0} \to \mathfrak{X}_{\mathsf{Z}_1,\Sigma_{\mathsf{Z}_1}}/\Gamma_{\mathsf{Z}_i}$, which can be identified (via isomorphisms as in (3.3)) with

$$(\mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0})^\wedge_{\underset{[\sigma]\in\Sigma^+_{\mathsf{Z}_0}/\Gamma_{\mathsf{Z}_0}}{\cup} \mathsf{Z}_{0,[\sigma]}} \to (\mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1})^\wedge_{\underset{[\tau]\in\Sigma^+_{\mathsf{Z}_1}/\Gamma_{\mathsf{Z}_1}}{\cup} \mathsf{Z}_{1,[\tau]}}.$$

If the image of $\sigma \in \Sigma^+_{\mathsf{Z}_0}$ under $\mathbf{P}^+_{\mathsf{Z}_0} \to \mathbf{P}^+_{\mathsf{Z}_1}$ is contained in some $\tau \in \Sigma^+_{\mathsf{Z}_1}$, we have an induced morphism $\mathfrak{X}^\circ_{\mathsf{Z}_0,\sigma} \to \mathfrak{X}^\circ_{\mathsf{Z}_1,\tau}$, which can be identified (via isomorphisms as in (3.2)) with

$$(\mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0})^\wedge_{\underset{\sigma'\in\Sigma^+_{\mathsf{Z}_0},\,\overline{\sigma}'\subset\overline{\sigma}}{\cup} \mathsf{Z}_{1,[\sigma']}} \to (\mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1})^\wedge_{\underset{\tau'\in\Sigma^+_{\mathsf{Z}_1},\,\overline{\tau}'\subset\overline{\tau}}{\cup} \mathsf{Z}_{1,[\tau']}}.$$

For a fixed $\tau \in \Sigma^+_{\mathsf{Z}_i}$, the pullback of (3.5) to the open formal subscheme $\mathfrak{X}^\circ_{\mathsf{Z}_1,\tau}$ on the target gives a proper morphism

$$\underset{\sigma\in\Sigma^+_{\mathsf{Z}_0},\,(\mathbf{P}_{\mathsf{Z}_0}\to\mathbf{P}_{\mathsf{Z}_1})(\sigma)\subset\tau}{\cup} \mathfrak{X}^\circ_{\mathsf{Z}_0,\sigma} \to \mathfrak{X}^\circ_{\mathsf{Z}_1,\tau}. \tag{3.6}$$

Suppose moreover that Σ_0 is induced by Σ_1. Then both morphisms (3.5) and (3.6) are **finite**. *For each $\tau \in \Sigma^+_{\mathsf{Z}_1}$ as above, with $\sigma \in \Sigma^+_{\mathsf{Z}_0}$ the preimage of τ, which is the unique element in $\Sigma^+_{\mathsf{Z}_0}$ such that $(\mathbf{P}^+_{\mathsf{Z}_0} \hookrightarrow \mathbf{P}^+_{\mathsf{Z}_1})(\sigma) \subset \tau$; and for each affine open formal subscheme $\mathfrak{W}_1 = \mathrm{Spf}(R_1)$ of $\mathfrak{X}^\circ_{\mathsf{Z}_1,\sigma}$, let $\mathfrak{W}_0 = \mathrm{Spf}(R_0)$ denote its pullback to $\mathfrak{X}^\circ_{\mathsf{Z}_0,\sigma}$. Under the morphisms $W_1 := \mathrm{Spec}(R_1) \to \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$, $W_1 \to \Xi_{\mathsf{Z}_1}(\tau)$, $W_0 := \mathrm{Spec}(R_0) \to \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0}$, and $W_0 \to \Xi_{\mathsf{Z}_0}(\sigma)$ induced by morphisms as in (3.2), the preimages of X_1 and Ξ_{Z_1} coincide as an open subscheme W^0_1 of W_1, and their further preimages in W_0 coincide with the preimages of X_0 and Ξ_{Z_0} as an open subscheme W^0_0.*

Proof. Except for the first assertion in (5), these follow from the same arguments as in [24, Sec. 2.1.28 and 4.1.12] (which are based on [26, Sec. 4.16, 6.25, and 12.4] and [11, Sec. 3.3]) in Cases (1) and (4), and as in [17, Sec. 8–11] and [20, the proof of Prop. 2.1.3] in Cases (2) and (3). As for the first assertion in (5), it follows from the universal or functorial properties of toroidal compactifications in terms of the associated cone decompositions, as in [2, 3, Ch. II, Sec. 7], [26, Prop. 6.25], [16, Thm. 6.4.1.1(6)], [19, Thm. 6.1(6)], and [24, Prop. 4.1.13]. □

Corollary 3.7. *In Proposition 3.4, suppose that Σ_0 is induced by Σ_1. Let Z_1 be a stratum of $\mathsf{X}^{\min}_1$, and let $\{\mathsf{Z}_{0,j}\}_j$ be all the strata of $\mathsf{X}^{\min}_0$ such that $f^{\min}(\mathsf{Z}_{0,j}) \subset \mathsf{Z}_1$ (as subsets of $\mathsf{X}^{\min}_1$). Consider any $\tau \in \Sigma^+_{\mathsf{Z}_1}$. For each j, let*

$$\sigma_j := (\mathbf{P}^+_{\mathsf{Z}_{0,j}} \hookrightarrow \mathbf{P}^+_{\mathsf{Z}_1})^{-1}(\tau) \in \Sigma^+_{\mathsf{Z}_{0,j}}.$$

Then the pullback of the finite morphism

$$f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1} : \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \to \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$$

under the composition of the canonical morphisms $\mathfrak{X}^{\circ}_{\mathsf{Z}_1,\tau} \xrightarrow{\sim} (\mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1})^{\wedge}_{\bigcup\limits_{\tau' \in \Sigma^+_{\mathsf{Z}_1},\, \overline{\tau}' \subset \overline{\tau}} \mathsf{Z}_{1,[\tau']}}$ *(as in* (3.2)*) and* $(\mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1})^{\wedge}_{\bigcup\limits_{\tau' \in \Sigma^+_{\mathsf{Z}_1},\, \overline{\tau}' \subset \overline{\tau}} \mathsf{Z}_{1,[\tau']}} \to \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$ *can be identified with the finite morphism*

$$\coprod_j \mathfrak{X}^{\circ}_{\mathsf{Z}_{0,j},\sigma_j} \to \mathfrak{X}^{\circ}_{\mathsf{Z}_1,\tau}$$

(defined by combining morphisms as in (3.6)*).*

Proof. This follows from (1) and (6) of Proposition 3.4. $\square$

Corollary 3.8. *In Corollary* 3.7*, with any* $\tau \in \Sigma^+_{\mathsf{Z}_1}$ *there inducing* $\sigma_j \in \Sigma^+_{\mathsf{Z}_{0,j}}$*, for each* j*, we have a commutative diagram of canonical morphisms*

$$\begin{tikzcd} E_{\mathsf{Z}_{0,j}} \arrow[r, hook] \arrow[d] & E_{\mathsf{Z}_{0,j}}(\sigma_j) \arrow[d] \\ E_{\mathsf{Z}_1} \arrow[r, hook] & E_{\mathsf{Z}_1}(\tau) \end{tikzcd} \tag{3.9}$$

over $\mathrm{Spec}(\mathbb{Z})$*, in which the horizontal morphisms are affine toroidal embeddings, which are open immersions, and where the vertical morphisms are finite. Let* x_1 *be any point of* $\mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$ *that lies on the stratum* $\mathsf{Z}_{1,[\tau]}$*. Then, étale locally at* x_1*, the commutative diagram*

$$\begin{tikzcd} \mathsf{X}_0 \arrow[r, "J^{\mathrm{tor}}_{0,\Sigma_0}"] \arrow[d, "f"'] & \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \arrow[d, "f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1}"] \\ \mathsf{X}_1 \arrow[r, "J^{\mathrm{tor}}_{1,\Sigma_1}"'] & \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1} \end{tikzcd}$$

can be identified with a commutative diagram

$$\begin{tikzcd} \coprod\limits_j \big(E_{\mathsf{Z}_{0,j}} \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_{0,j}}\big) \arrow[r, hook] \arrow[d] & \coprod\limits_j \big(E_{\mathsf{Z}_{0,j}}(\sigma_j) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_{0,j}}\big) \arrow[d] \\ E_{\mathsf{Z}_1} \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_1} \arrow[r, hook] & E_{\mathsf{Z}_1}(\tau) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_1} \end{tikzcd} \tag{3.10}$$

induced by taking fiber products of some translations of the vertical morphisms in the diagram (3.9) *by sections of* E_{Z_1} *and of the canonical morphisms* $C_{\mathsf{Z}_{0,j}} \to C_{\mathsf{Z}_1}$*. More precisely, there exists an étale neighborhood*

$$\overline{U}_1 \to \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$$

of x_1 *and an étale morphism*

$$\overline{U}_1 \to E_{\mathsf{Z}_1}(\tau) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_1}, \tag{3.11}$$

which induce by pullback under the finite morphisms $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1} : \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \to \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$ *and* $\coprod_j \big(E_{\mathsf{Z}_{0,j}}(\sigma_j) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_{0,j}}\big) \to E_{\mathsf{Z}_1}(\tau) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_1}$ *(as in* (3.10)*) some étale morphisms* $\overline{U}_0 \to \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0}$ *and* $\overline{U}_0 \to \coprod_j \big(E_{\mathsf{Z}_{0,j}}(\sigma_j) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_{0,j}}\big)$*, respectively, such that the preimage* U_1 *of* X_1 *in* $\overline{U}_1$ *coincides with the preimage of* E_{Z_1}*, and such that the preimage* U_0 *of* U_1 *in* $\overline{U}_0$ *coincides with the preimages of* X_0 *and of* $\coprod_j \big(E_{\mathsf{Z}_{0,j}} \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_{0,j}}\big)$*. Therefore, the pullback of* $\mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} - \mathsf{X}_0$ *(with its reduced subscheme structure) to* $\overline{U}_0$ *coincides (as a subscheme) with the pullback of* $\coprod_j \big(\partial E_{\mathsf{Z}_{0,j}}(\sigma_j) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_{0,j}}\big)$*, where* $\partial E_{\mathsf{Z}_{0,j}}(\sigma_j) := E_{\mathsf{Z}_{0,j}}(\sigma_j) - E_{\mathsf{Z}_{0,j}}$ *(with its reduced subscheme structure), for each* j*; and the pullback of* $\mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1} - \mathsf{X}_1$ *(with its reduced subscheme structure) to* $\overline{U}_1$ *coincides (as a subscheme) with the pullback of* $\partial E_{\mathsf{Z}_1}(\tau) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_1}$*, where* $\partial E_{\mathsf{Z}_1}(\tau) := E_{\mathsf{Z}_1}(\tau) - E_{\mathsf{Z}_1}$ *(with its reduced subscheme structure).*

Proof. These follow from Corollary 3.7 and Artin's approximation (see [1, Thm. 1.12, and the proof of the corollaries in Sec. 2]) as in the proofs of [21, Prop. 2.2(9) and Cor. 2.4], [20, Cor. 2.1.7], and [22, Prop. 5.1], which are applicable because we only need to approximate finitely many formal schemes finite over $\mathfrak{X}^{\circ}_{\mathsf{Z}_1,\tau}$, and because the formation of Henselizations of semi-local rings is compatible with base change under finite morphisms by [10, IV-4, 18.6.8]; and from the fact that all the torus torsors are already Zariski locally trivial, as in the proof of [21, Lem. 2.3]. (Note that the torus torsors might be trivialized by incompatible sections. Hence, we need to allow the canonical morphisms $E_{\mathsf{Z}_{0,j}} \to E_{\mathsf{Z}_1}$ to be translated by some possibly different sections of E_{Z_1}, when there are more than one j.) □

Remark 3.12. In Proposition 3.4, and in Corollaries 3.7 and 3.8, we only need the weaker assumption that $\big(\rho(\mathbb{A}^\infty)\big)(\mathcal{H}_0) \subset \mathcal{H}_1$. When $\mathcal{H}_0 = \big(\rho(\mathbb{A}^\infty)\big)^{-1}(\mathcal{H}_1)$, we already know in Proposition 3.4(2) that the morphism $E_{\mathsf{Z}_{0,j}} \to E_{\mathsf{Z}_1}$ in (3.9) is a closed immersion, without assuming that f is a closed immersion; but it is generally not true that the morphism $E_{\mathsf{Z}_{0,j}}(\sigma_j) \to E_{\mathsf{Z}_1}(\tau)$ is a closed immersion when $E_{\mathsf{Z}_{0,j}} \to E_{\mathsf{Z}_1}$ is (cf. Remark 4.1 below), regardless of whether f is.

We shall reinstate the full Assumption 2.1 from now on.

4. Conditions on cone decompositions

Motivated by Corollary 3.8, with the goal of proving Theorem 2.2 in mind, we would like to show the existence of collections Σ_0 and Σ_1 such that Σ_0 is induced by Σ_1 as in Proposition 3.4(4) and such that, for each $\sigma \in \Sigma^+_{\mathsf{Z}_0}$ that is the preimage under $\mathbf{P}^+_{\mathsf{Z}_0} \to \mathbf{P}^+_{\mathsf{Z}_1}$ of some $\tau \in \Sigma^+_{\mathsf{Z}_1}$, the canonical morphism $E_{\mathsf{Z}_0}(\sigma) \to E_{\mathsf{Z}_1}(\tau)$ (cf. (3.9)) is a closed immersion.

Remark 4.1. This condition of being a closed immersion is not satisfied in general. For example, it is possible to choose the linear algebraic data such that $\mathbf{S}_{\mathsf{Z}_1} \cong \mathbb{Z}^{\oplus 3} \twoheadrightarrow \mathbf{S}_{\mathsf{Z}_0} \cong \mathbb{Z}^{\oplus 2}$ corresponds to the projection to the first two factors, in which case $\mathbf{S}^\vee_{\mathsf{Z}_0,\mathbb{R}} \cong \mathbb{R}^{\oplus 2} \hookrightarrow \mathbf{S}^\vee_{\mathsf{Z}_1,\mathbb{R}} \cong \mathbb{R}^{\oplus 3}$ is the inclusion of the first two coordinates, and such that we have the following:

- $\tau \subset \mathbf{S}^\vee_{\mathsf{Z}_1,\mathbb{R}}$ is $\mathbb{R}_{>0}$-spanned by $\{(0,0,1),(-1,0,2),(1,1,-2)\}$, in which case $\tau^\vee$ is $\mathbb{Z}_{\geq 0}$-spanned by the $\mathbb{Z}$-basis $\{(-1,1,0),(0,1,0),(2,0,1)\}$ of $\mathbb{Z}^{\oplus 3}$.

- $\sigma \subset \mathbf{S}^\vee_{\mathsf{Z}_0,\mathbb{R}}$ is $\mathbb{R}_{>0}$-spanned by $\{(1,1),(0,1)\}$, in which case $\sigma^\vee$ is $\mathbb{Z}_{\geq 0}$-spanned by the $\mathbb{Z}$-basis $\{(-1,1),(1,0)\}$ of $\mathbb{Z}^{\oplus 2}$.
- $\sigma = (\mathbf{S}^\vee_{\mathsf{Z}_0,\mathbb{R}} \hookrightarrow \mathbf{S}^\vee_{\mathsf{Z}_1,\mathbb{R}})^{-1}(\tau)$. However, $\tau^\vee \to \sigma^\vee$ is not surjective, because the $\mathbb{Z}_{\geq 0}$-span of $\{(-1,1),(0,1),(2,0)\}$ cannot contain $(1,0)$.
- The morphism $E_{\mathsf{Z}_0}(\sigma) \to E_{\mathsf{Z}_1}(\tau)$ is given by the morphism $\mathrm{Spec}(\mathbb{Z}[\sigma^\vee]) \to \mathrm{Spec}(\mathbb{Z}[\tau^\vee])$ induced by $\tau^\vee \to \sigma^\vee$, and hence is not a closed immersion.

Remark 4.2. In fact, in Remark 4.1, even the induced map $E_{\mathsf{Z}_0}(\sigma)(\mathbb{C}) \to E_{\mathsf{Z}_1}(\tau)(\mathbb{C})$ on $\mathbb{C}$-points is not injective: For $? = \pm 1$, if $x_? : \mathbb{Z}[\sigma^\vee] \to \mathbb{C}$ is the ring homomorphism sending $(-1,1)$ and $(1,0)$ in $\sigma^\vee$ to 0 and ?, respectively, then the induced homomorphism $y : \mathbb{Z}[\tau^\vee] \to \mathbb{C}$ sends $(-1,1,0)$, $(0,1,0)$, and $(2,0,1)$ to 0, 0, and 1, respectively. That is, both the $\mathbb{C}$-points defined by x_1 and x_{-1} are sent to the same $\mathbb{C}$-point defined by y. This shows that, already in characteristic zero, the induced morphism $E_{\mathsf{Z}_0}(\sigma) \to E_{\mathsf{Z}_1}(\tau)$ is not universally injective, and hence cannot induce a universal homeomorphism between the source and its image in the target. Moreover, for any rational polyhedral cone $\sigma' \subset \sigma$, the induced morphism $E_{\mathsf{Z}_0}(\sigma') \to E_{\mathsf{Z}_1}(\tau)$ is not universally injective either.

Nevertheless, we have the following:

Lemma 4.3. *Let $\sigma \subset \mathbf{S}^\vee_{\mathsf{Z}_0,\mathbb{R}}$ and $\tau \subset \mathbf{S}^\vee_{\mathsf{Z}_1,\mathbb{R}}$ be any rational polyhedral cones such that $\tau = (\mathbf{S}^\vee_{\mathsf{Z}_0,\mathbb{R}} \hookrightarrow \mathbf{S}^\vee_{\mathsf{Z}_1,\mathbb{R}})(\sigma)$. Then the canonical morphism $E_{\mathsf{Z}_0}(\sigma) \cong \mathrm{Spec}(\mathbb{Z}[\sigma^\vee]) \to E_{\mathsf{Z}_1}(\tau) \cong \mathrm{Spec}(\mathbb{Z}[\tau^\vee])$ is a closed immersion.*

Proof. Given an arbitrary $\ell_0 \in \sigma^\vee$, take any lift ℓ_1 of it in $\mathbf{S}_{\mathsf{Z}_1}$, which exists because $\mathbf{S}_{\mathsf{Z}_1} \to \mathbf{S}_{\mathsf{Z}_0}$ is surjective. Given an arbitrary $y_1 \in \tau$, by assumption, there exists some $y_0 \in \sigma$ such that $y_1 = (\mathbf{S}^\vee_{\mathsf{Z}_0,\mathbb{R}} \hookrightarrow \mathbf{S}^\vee_{\mathsf{Z}_1,\mathbb{R}})(y_0)$, and so that $\langle \ell_1, y_1 \rangle = \langle \ell_0, y_0 \rangle \geq 0$. Consequently, $\ell_1 \in \tau^\vee$, and $\tau^\vee \to \sigma^\vee$ is surjective, as desired. □

Lemma 4.4. *In Lemma 4.3, let us identify $\mathbf{S}^\vee_{\mathsf{Z}_0,\mathbb{R}}$ with a subspace of $\mathbf{S}^\vee_{\mathsf{Z}_1,\mathbb{R}}$ for simplicity, so that $\tau = \sigma$ under this identification; and let $\mathbf{S}^\vee := \mathbf{S}^\vee_{\mathsf{Z}_1} \cap (\mathbb{R} \cdot \sigma)$ and $\mathbf{S} := \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}^\vee, \mathbb{Z})$, so that we have surjective homomorphisms $\mathbf{S}_{\mathsf{Z}_1} \twoheadrightarrow \mathbf{S}_{\mathsf{Z}_0} \twoheadrightarrow \mathbf{S}$ corresponding to injective homomorphisms of tori $E \hookrightarrow E_{\mathsf{Z}_0} \hookrightarrow E_{\mathsf{Z}_1}$. For the sake of clarity, let us denote by ς the same cone σ in $\mathbf{S}^\vee_{\mathbb{R}} = \mathbb{R} \cdot \sigma$. Let E, $E^\perp_{\mathsf{Z}_0}$, and $E^\perp_{\mathsf{Z}_1}$ be the split tori over $\mathrm{Spec}(\mathbb{Z})$ with character groups $\mathbf{S}$, $\mathbf{S}^\perp_{\mathsf{Z}_0} := \ker(\mathbf{S}_{\mathsf{Z}_0} \twoheadrightarrow \mathbf{S})$, and $\mathbf{S}^\perp_{\mathsf{Z}_1} := \ker(\mathbf{S}_{\mathsf{Z}_1} \twoheadrightarrow \mathbf{S})$, respectively. Let us pick any splitting $\mathbf{S}_{\mathsf{Z}_1} \cong \mathbf{S} \oplus \mathbf{S}^\perp_{\mathsf{Z}_1}$ (as $\mathbb{Z}$-modules) which induces a splitting $\mathbf{S}_{\mathsf{Z}_0} \cong \mathbf{S} \oplus \mathbf{S}^\perp_{\mathsf{Z}_0}$. Then these splittings are dual to compatible fiber products $E_{\mathsf{Z}_1} \cong E \underset{\mathrm{Spec}(\mathbb{Z})}{\times} E^\perp_{\mathsf{Z}_1}$ and $E_{\mathsf{Z}_0} \cong E \underset{\mathrm{Spec}(\mathbb{Z})}{\times} E^\perp_{\mathsf{Z}_0}$, respectively; and the canonical injective homomorphism $E_{\mathsf{Z}_0} \hookrightarrow E_{\mathsf{Z}_1}$ factors as a fiber product of the identity homomorphism of E with the canonical injective homomorphism $E^\perp_{\mathsf{Z}_0} \hookrightarrow E^\perp_{\mathsf{Z}_1}$ dual to $\mathbf{S}^\perp_{\mathsf{Z}_1} \twoheadrightarrow \mathbf{S}^\perp_{\mathsf{Z}_0}$. Moreover, these splittings extend to compatible fiber products $E_{\mathsf{Z}_1}(\tau) \cong E(\varsigma) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} E^\perp_{\mathsf{Z}_1}$ and $E_{\mathsf{Z}_0}(\sigma) \cong E(\varsigma) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} E^\perp_{\mathsf{Z}_0}$, respectively; and the canonical closed immersion $E_{\mathsf{Z}_0}(\sigma) \hookrightarrow E_{\mathsf{Z}_1}(\tau)$ factors as the fiber product of the identity morphism of $E(\varsigma)$ with the same injective group homomorphism $E^\perp_{\mathsf{Z}_0} \hookrightarrow E^\perp_{\mathsf{Z}_1}$ as above. Furthermore, any closed immersion $E_{\mathsf{Z}_0}(\sigma) \hookrightarrow E_{\mathsf{Z}_1}(\tau)$ that is a translation of the canonical one by some section of E_{Z_1} can be identified with the product of an isomorphism $E(\varsigma) \xrightarrow{\sim} E(\varsigma)$ that is the translation of the identity morphism on $E(\varsigma)$ by some section of E with a closed immersion $E^\perp_{\mathsf{Z}_0} \hookrightarrow E^\perp_{\mathsf{Z}_1}$ that is the translation of the canonical one by some section of $E^\perp_{\mathsf{Z}_1}$.*

Proof. These follow from the identification $\tau^\vee = (\mathbf{S}_{\mathsf{Z}_1} \twoheadrightarrow \mathbf{S}_{\mathsf{Z}_0})^{-1}(\sigma^\vee)$ in the proof of Lemma 4.3, and from the various definitions introduced in this lemma. $\square$

Lemma 4.5. *In Lemma* 4.3*, let* $\partial E_{\mathsf{Z}_0}(\sigma) := E_{\mathsf{Z}_0}(\sigma) - E_{\mathsf{Z}_0}$ *and* $\partial E_{\mathsf{Z}_1}(\tau) := E_{\mathsf{Z}_1}(\tau) - E_{\mathsf{Z}_1}$*, as reduced closed subschemes of* $E_{\mathsf{Z}_0}(\sigma)$ *and* $E_{\mathsf{Z}_1}(\tau)$*, respectively. Then the canonical morphism* $E_{\mathsf{Z}_0}(\sigma) \to E_{\mathsf{Z}_1}(\tau)$ *induces a canonical morphism* $\partial E_{\mathsf{Z}_0}(\sigma) \to \partial E_{\mathsf{Z}_1}(\tau)$ *and a canonical isomorphism* $\partial E_{\mathsf{Z}_0}(\sigma) \xrightarrow{\sim} \partial E_{\mathsf{Z}_1}(\tau) \underset{E_{\mathsf{Z}_1}(\tau)}{\times} E_{\mathsf{Z}_0}(\sigma)$*. If we denote by* $\mathcal{I}_\sigma$ *(resp.* $\mathcal{I}_\tau$*) the* $\mathscr{O}_{E_{\mathsf{Z}_0}(\sigma)}$*-ideal (resp.* $\mathscr{O}_{E_{\mathsf{Z}_1}(\tau)}$*-ideal) defining* $\partial E_{\mathsf{Z}_0}(\sigma)$ *(resp.* $\partial E_{\mathsf{Z}_1}(\tau)$*), then* $\mathcal{I}_\sigma \cong \big(E_{\mathsf{Z}_0}(\sigma) \to E_{\mathsf{Z}_1}(\tau)\big)^*(\mathcal{I}_\tau)$ *as* $\mathscr{O}_{E_{\mathsf{Z}_0}(\sigma)}$*-ideals.*

Proof. In the setting of Lemma 4.4, consider the reduced closed subscheme $\partial E(\varsigma) := E(\varsigma) - E$ of $E(\varsigma)$. Since $E^{\perp}_{\mathsf{Z}_0}$ is smooth as a torus, $\partial E_{\mathsf{Z}_0}(\sigma)$ coincides with the pullback of $\partial E(\varsigma)$ under the first projection in the fiber product $E_{\mathsf{Z}_0}(\sigma) \cong E(\varsigma) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} E^{\perp}_{\mathsf{Z}_0}$ as reduced subschemes of $E_{\mathsf{Z}_0}(\sigma)$, because they coincide as subsets. Similarly, $\partial E_{\mathsf{Z}_1}(\tau)$ coincides with the pullback of $\partial E(\varsigma)$ under the first projection in the fiber product $E_{\mathsf{Z}_1}(\tau) \cong E(\varsigma) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} E^{\perp}_{\mathsf{Z}_1}$ as reduced subschemes of $E_{\mathsf{Z}_1}(\tau)$. Since these two fiber products are compatible with each other, $\partial E_{\mathsf{Z}_0}(\sigma)$ coincides with the pullback of $\partial E_{\mathsf{Z}_1}(\tau)$ as subschemes, and the lemma follows. $\square$

These justify the following:

Definition 4.6. *We say that two compatible collections* Σ_0 *and* Σ_1 *of cone decompositions as in Proposition* 3.4(4) *are* **strictly compatible with each other** *or simply* **strictly compatible** *if, for each* $\mathsf{Z}_0 \to \mathsf{Z}_1$ *as in Proposition* 3.4(1)*, the image of each* $\sigma \in \Sigma^+_{\mathsf{Z}_0}$ *under* $\mathbf{P}^+_{\mathsf{Z}_0} \hookrightarrow \mathbf{P}^+_{\mathsf{Z}_1}$ *is exactly some* $\tau \in \Sigma^+_{\mathsf{Z}_1}$.

Remark 4.7. Certainly, if Σ_0 and Σ_1 are strictly compatible as in Definition 4.6, then Σ_0 is induced by Σ_1, and they are compatible, as in Proposition 3.4(4).

Lemma 4.8. *Under the assumption that* $f : \mathsf{X}_0 \to \mathsf{X}_1$ *is a closed immersion, the morphism* $\coprod_j \big(E_{\mathsf{Z}_{0,j}} \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_{0,j}}\big) \to E_{\mathsf{Z}_1} \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_1}$ *in Corollary* 3.8 *is a closed immersion over the open image of* U_1 *under* (3.11)*. Since* $E_{\mathsf{Z}_{0,j}}$ *and* E_{Z_1} *are separated group schemes with sections which are closed immersions,* $C_{\mathsf{Z}_{0,j}} \to C_{\mathsf{Z}_1}$ *(and hence* $\Xi_{\mathsf{Z}_{0,j}} \to \Xi_{\mathsf{Z}_1}$*) are also closed immersions over the further image of* U_1 *in* C_{Z_1}*, for all* j*. Moreover, if* Σ_0 *and* Σ_1 *are strictly compatible as in Definition* 4.6*, then the morphism* $\coprod_j \big(E_{\mathsf{Z}_{0,j}}(\sigma_j) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_{0,j}}\big) \to E_{\mathsf{Z}_1}(\tau) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_1}$ *in Corollary* 3.8 *is also a closed immersion over the open image of* $\overline{U}_1$ *under* (3.11)*.*

Proof. The first two assertions follow immediately from Corollary 3.8. By Lemma 4.3, the morphism $E_{\mathsf{Z}_{0,j}}(\sigma_j) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_{0,j}} \to E_{\mathsf{Z}_1}(\tau) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_1}$ is a closed immersion over the open image of $\overline{U}_1$, for each j. It remains to show that any point x in the image of $\overline{U}_1$ and in the image of $\coprod_j \big(E_{\mathsf{Z}_{0,j}}(\sigma_j) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_{0,j}}\big) \to E_{\mathsf{Z}_1}(\tau) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_1}$ lies on at most one of the images of the above closed immersions. Suppose to the contrary that there are two distinct indices j and j', together with points y and y' of $E_{\mathsf{Z}_{0,j}}(\sigma_j) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_{0,j}}$ and $E_{\mathsf{Z}_{0,j'}}(\sigma_{j'}) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_{0,j'}}$, respectively, which are mapped to the point x of $E_{\mathsf{Z}_1}(\tau) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_1}$. Then x, y, and y' have the same image z in

C_{Z_1}, which is also in the images of the closed immersions from $C_{\mathsf{Z}_{0,j}}$ and $C_{\mathsf{Z}_{0,j'}}$, and we obtain (by pullback to z) closed immersions $\phi_j : E_{\mathsf{Z}_{0,j}}(\sigma_j)_z \to E_{\mathsf{Z}_1}(\tau)_z$ and $\phi_{j'} : E_{\mathsf{Z}_{0,j'}}(\sigma_{j'})_z \to E_{\mathsf{Z}_1}(\tau)_z$ over z, which are translations of the canonical ones by some sections of $(E_{\mathsf{Z}_1})_z$, whose images overlap at x (also viewed as a point of $E_{\mathsf{Z}_1}(\tau)_z$). By Lemma 4.4, in the notation there, ϕ_j and $\phi_{j'}$ are, respectively, fiber products over z of some isomorphisms $E(\varsigma)_z \xrightarrow{\sim} E(\varsigma)_z$ that are translations of the identity morphism of $E(\varsigma)_z$ by some sections of E_z with closed immersions $\psi_j : (E^{\perp}_{\mathsf{Z}_{0,j}})_z \to (E^{\perp}_{\mathsf{Z}_1})_z$ and $\psi_{j'} : (E^{\perp}_{\mathsf{Z}_{0,j'}})_z \to (E^{\perp}_{\mathsf{Z}_1})_z$ that are translations of the canonical ones by some sections of $(E^{\perp}_{\mathsf{Z}_1})_z$. The images of ψ_j and $\psi_{j'}$ overlap at the image w of x in $(E^{\perp}_{\mathsf{Z}_1})_z$, exactly because the images of ϕ_j and $\phi_{j'}$ do at x, regardless of the above translations of the identity morphism of $E(\varsigma)_z$ by sections of E_z. Hence, the images of the restrictions $(E_{\mathsf{Z}_{0,j}})_z \to (E_{\mathsf{Z}_1})_z$ and $(E_{\mathsf{Z}_{0,j'}})_z \to (E_{\mathsf{Z}_1})_z$ of ϕ_j and $\phi_{j'}$, respectively, overlap at all points of the preimage W of $\overline{x}$ in $(E_{\mathsf{Z}_1})_z$. When canonically viewed as a subset of $E_{\mathsf{Z}_1}(\tau) \underset{\mathrm{Spec}(\mathbb{Z})}{\times} C_{\mathsf{Z}_1}$, this W contains x in its closure. Since x is a point of the open image of $\overline{U}_1$ by assumption, W must overlap with the open image of $\overline{U}_1$ at some point in the open image of U_1. Thus, we obtain a contradiction with the first assertion of this lemma, as desired. □

By Corollary 3.8 and Lemmas 4.5 and 4.8, we obtain the following:

Proposition 4.9. *If there exist compatible collections Σ_0 and Σ_1 that are strictly compatible as in Definition* 4.6*, then the induced morphism $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1} : \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \to \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$ as in Proposition* 3.4 *is a closed immersion extending $f : \mathsf{X}_0 \to \mathsf{X}_1$. Moreover, if we denote by $\mathcal{I}_{\Sigma_i}$ the $\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i}}$-ideal defining the boundary $\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_i} - \mathsf{X}_i$ (with its reduced subscheme structure), for $i = 0, 1$, then we have $f^{\mathrm{tor},*}_{\Sigma_0,\Sigma_1}(\mathcal{I}_{\Sigma_1}) \cong \mathcal{I}_{\Sigma_0}$ as $\mathscr{O}_{\mathsf{X}^{\mathrm{tor}}_{i,\Sigma_0}}$-ideals.*

In order to prove Theorem 2.2, it remains to establish the following:

Proposition 4.10. *There exist compatible collections Σ_0 and Σ_1 that are strictly compatible as in Definition* 4.6*, which we may assume to be projective and smooth and satisfy the condition that, for $i = 0, 1$, and for each Z_i and each $\sigma \in \Sigma^+_{\mathsf{Z}_i}$, the stabilizer $\Gamma_{\mathsf{Z}_i,\sigma}$ of σ in Γ_{Z_i} is trivial. Moreover, we may assume that Σ_0 and Σ_1 refine any finite number of prescribed compatible collections of cone decompositions.*

Proof. Let us temporarily ignore the assumption on projectivity and smoothness, and take Σ_0 to be induced by Σ_1 as in Proposition 3.4(4) (cf. [11, Sec. 3.3]). Note that, given any Z_1 and any $[\tau] \in \Sigma^+_{\mathsf{Z}_1}/\Gamma_{\mathsf{Z}_1}$, there exist only finitely many Z_0 mapped to Z_1; and for each such Z_0, there exist only finitely many $[\sigma] \in \Sigma^+_{\mathsf{Z}_0}/\Gamma_{\mathsf{Z}_0}$ mapped to $[\tau]$ under the map $\Sigma^+_{\mathsf{Z}_0}/\Gamma_{\mathsf{Z}_0} \to \Sigma^+_{\mathsf{Z}_1}/\Gamma_{\mathsf{Z}_1}$ (simply because there are only finitely many possible Z_0 and $[\sigma]$). Since Σ_0 is induced by Σ_1, for any $\tau \in \Sigma^+_{\mathsf{Z}_1}$ representing some $[\tau]$ as above, each $[\sigma]$ that is mapped to $[\tau]$ as above is represented by some $\sigma \in \Sigma^+_{\mathsf{Z}_0}$ that is the preimage of τ under the injection $\mathbf{S}^{\vee}_{\mathsf{Z}_0,\mathbb{R}} \hookrightarrow \mathbf{S}^{\vee}_{\mathsf{Z}_1,\mathbb{R}}$ as in Proposition 3.4(2). In this case, the image of σ is the intersection of τ with the image of $\mathbf{S}^{\vee}_{\mathsf{Z}_0,\mathbb{R}} \hookrightarrow \mathbf{S}^{\vee}_{\mathsf{Z}_1,\mathbb{R}}$. As a result, up to refining each such τ by intersections with finitely many hyperplanes, and up to refining all the finitely many σ involved accordingly, we may assume that Σ_0 and Σ_1 are strictly compatible (but still not necessarily projective and smooth). We may also refine both of them, and assume that they refine any finite number of prescribed compatible collections and satisfy the condition in the end of the first sentence of the proposition. Finally, up to further refinements, we may assume that Σ_0 and Σ_1 are both projective and smooth, because as soon as Σ_0 and Σ_1 are strictly compatible and satisfy the last condition of the proposition, any further refinements will remain so; and because, when Σ_0 and Σ_1 are strictly compatible, both the projectivity and smoothness of Σ_1 are automatically inherited by Σ_0, and hence it suffices to refine Σ_1. (However, note that such an inheritance is not necessarily true in general, when Σ_0 is merely induced by Σ_1.) $\square$

The proof of Theorem 2.2 is now complete.

5. Some examples

Example 5.1. In Case (1), suppose that $\mathrm{G}_0 = \mathrm{GL}_{2,\mathbb{Q}}$ and $\mathrm{G}_1 := \mathrm{GL}_{2,\mathbb{Q}} \underset{\mathbf{G}_{\mathrm{m},\mathbb{Q}}}{\times} \mathrm{GL}_{2,\mathbb{Q}}$, where the two structure morphisms in the fiber product are both the determinant homomorphism. Then G_1 is naturally a subgroup scheme of $\mathrm{G}_0 \times \mathrm{G}_0$, and the diagonal morphism of G_0 factors through a homomorphism $\rho : \mathrm{G}_0 \to \mathrm{G}_1$. Let $\mathcal{H}_+$ and $\mathcal{H}_-$ denote the Poincaré upper and lower half-planes, respectively, and let i denote the $\sqrt{-1}$ in $\mathcal{H}_+$. Let $h_0 : \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{G}_{0,\mathbb{R}} = \mathrm{GL}_{2,\mathbb{R}}$ be defined by $a + bi \mapsto \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$, and let h_1 the composition of h_0 with $\rho_{\mathbb{R}} : \mathrm{G}_{0,\mathbb{R}} \to \mathrm{G}_{1,\mathbb{R}}$. Then $\mathrm{G}_0(\mathbb{R}) \cdot h_0 = \mathcal{H}_\pm = \mathcal{H}_+ \coprod \mathcal{H}_-$, and $\mathrm{G}_1(\mathbb{R}) \cdot h_1 = (\mathcal{H}_+ \times \mathcal{H}_+) \coprod (\mathcal{H}_- \times \mathcal{H}_-)$. Let $\mathcal{H}_0 \subset \mathrm{G}_0(\mathbb{A}^\infty) = \mathrm{GL}_2(\mathbb{A}^\infty)$ be a principal congruence subgroup of some level $n \geq 3$, and let $\mathcal{H}_1 := (\mathcal{H}_0 \times \mathcal{H}_0) \cap \mathrm{G}_1(\mathbb{A}^\infty)$. Then X_0 is the modular curve of principal level n over $\mathsf{S}_0 = \mathrm{Spec}(\mathbb{Q})$, and X_1 is an open-and-closed subscheme of $\mathsf{X}_0 \underset{\mathsf{S}_0}{\times} \mathsf{X}_0$. In this case, the morphism $f : \mathsf{X}_0 \to \mathsf{X}_1$ is the closed immersion induced by the diagonal morphism of X_0, and all possible maps $\mathbf{P}_{\mathsf{Z}_0} \to \mathbf{P}_{\mathsf{Z}_1}$ can be identified with either $\{0\} \to \{0\}$ or the diagonal map $\mathbb{R}_{\geq 0} \to \mathbb{R}^2_{\geq 0}$. There is a unique choice of Σ_0, and $\mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0}$ is the usual compactified modular curve. Let Σ'_1 denote the compatible collection of cone decompositions for X_1 induced by $\Sigma_0 \times \Sigma_0$, which is given by either $\{0\}$ or the faces of the whole cone $\mathbb{R}^2_{>0}$. Then $\mathsf{X}^{\mathrm{tor}}_{1,\Sigma'_1}$ is an open-and-closed subscheme of $\mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \underset{\mathsf{S}_0}{\times} \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0}$, and the morphism $f^{\mathrm{tor}}_{\Sigma_0,\Sigma'_1} : \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \to \mathsf{X}^{\mathrm{tor}}_{1,\Sigma'_1}$ is the closed immersion induced by the diagonal morphism of $\mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0}$. However, Σ_0 and Σ'_1 are *not* strictly compatible, and the pullback of $\mathcal{I}_{\Sigma'_1}$ is $\mathcal{I}^{\otimes 2}_{\Sigma_0}$ rather than $\mathcal{I}_{\Sigma_0}$ (which means the image of $f^{\mathrm{tor}}_{\Sigma_0,\Sigma'_1}$ does not meet the boundary of $\mathsf{X}^{\mathrm{tor}}_{1,\Sigma'_1}$ *transversally*). (See Remark 2.5.) Nevertheless, by Theorem 2.2, there exists a refinement Σ_1 of Σ'_1 such that $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1} : \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \to \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$ is a closed immersion and such that the pullback of $\mathcal{I}_{\Sigma_1}$ is $\mathcal{I}_{\Sigma_0}$. In practice, the difference between Σ'_1 and its refinement Σ_1 is given by some subdivisions of cones of the form $\mathbb{R}^2_{>0}$, which correspond to (possibly repeated) blowups at some (possibly nonreduced closed subschemes over) products of cusps, after which the image of $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1}$ meets the boundary of $\mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$ transversally.

Example 5.2. In Case (2), suppose that we have the following:

(1) $\mathcal{O}_0 = \mathbb{Z} \times \mathbb{Z}$ and $\mathcal{O}_1 = \mathbb{Z}$ is diagonally embedded in $\mathcal{O}_0$, and $\star_0$ and $\star_1$ are trivial.
(2) $L_1 = \mathbb{Z}^{\oplus 4}$, with the first (resp. second) factor of $\mathcal{O}_0 = \mathbb{Z} \times \mathbb{Z}$ acting naturally on the first and third (resp. second and fourth) factors of $L_1 = \mathbb{Z}^{\oplus 4}$ and trivially on the remaining factors.
(3) Let $\langle \cdot , \cdot \rangle_1 : L_1 \times L_1 \to \mathbb{Z}(1)$ be the self-dual pairing defined by composing the standard symplectic pairing $\big((x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4)\big) \mapsto x_1 y_3 + x_2 y_4 - x_3 y_1 - x_4 y_2$ with a fixed choice of isomorphism $2\pi i : \mathbb{Z} \to \mathbb{Z}(1)$,

where i is the $\sqrt{-1}$ in $\mathcal{H}_+$ as in Example 5.1, and let $h_1(a+bi)$ act on $L_{1,\mathbb{R}} \cong \mathbb{R}^{\oplus 4}$ via the left multiplication by $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ on the first and third factors, and similarly on the second and fourth factors.

Then $(\mathrm{G}_0 \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \mathrm{G}_0(\mathbb{R}) \cdot h_0)$ is the same as the $(\mathrm{G}_1, \mathrm{G}_1(\mathbb{R}) \cdot h_1)$ in Example 5.1, and $(\mathrm{G}_1 \underset{\mathbb{Z}}{\otimes} \mathbb{Q}, \mathrm{G}_1(\mathbb{R}) \cdot h_1) = (\mathrm{GSp}_{4,\mathbb{Q}}, \mathcal{H}_{2,\pm})$, where $\mathcal{H}_{2,\pm}$ is the union of the Siegel upper and lower half-spaces of genus two. In both cases, the reflex field is $\mathbb{Q}$, so that we can take $F = \mathbb{Q}$, and there are no bad primes for the integral PEL data.

Let $\mathcal{H}_1 \subset \mathrm{G}_1(\hat{\mathbb{Z}}^\Box)$ be a principal congruence subgroup of some level $n \geq 3$ that is prime-to-$\Box$, and let $\mathcal{H}_0 := \mathcal{H}_1 \cap \mathrm{G}_0(\hat{\mathbb{Z}}^\Box)$. Then the moduli problem defined by $(\mathcal{O}_1, \star_1, L_1, \langle\,\cdot\,,\,\cdot\,\rangle_1, h_1)$ and $\mathcal{H}_1$ is a smooth integral model X_1 of the Siegel threefold over $\mathsf{S}_0 = \mathrm{Spec}(\mathbb{Z}_{(\Box)})$ parameterizing principally polarized abelian surfaces with symplectic principal level-n structures; and the moduli problem defined by $(\mathcal{O}_0, \star_0, L_0, \langle\,\cdot\,,\,\cdot\,\rangle_0, h_0)$ and $\mathcal{H}_0$ is the closed moduli subscheme X_0 of X_1 parameterizing principally polarized abelian surfaces of the form $(E_1 \underset{S}{\times} E_2, \lambda_1 \underset{S}{\times} \lambda_2)$, where (E_1, λ_1) and (E_2, λ_2) are canonically principally polarized elliptic curves, with principal level-n structures satisfying some conditions. At the level of connected components, X_0 can be viewed as the product of two smooth integral models of modular curves of principal level n. In this case, we have a closed immersion $f : \mathsf{X}_0 \hookrightarrow \mathsf{X}_1$, and Theorem 2.2 guarantees the existence of some closed immersion of toroidal compactifications $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1} : \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$ extending f, defined by some collections Σ_0 and Σ_1 of cone decompositions that are strictly compatible.

The map $\overline{\mathbf{P}}^+_{\mathsf{Z}_0} := (\mathbf{P}^+_{\mathsf{Z}_0} - \{0\})/\mathbb{R}^\times_{>0} \to \overline{\mathbf{P}}^+_{\mathsf{Z}_1} := (\mathbf{P}^+_{\mathsf{Z}_1} - \{0\})/\mathbb{R}^\times_{>0}$ can be from the empty set to the empty set; from a single point to a single point; or from the vertical half-line $i\mathbb{R}_{>0}$ to $\mathcal{H}_+$ (up to some identifications). In the last case, Γ_{Z_0} acts trivially on $i\mathbb{R}_{>0}$ because of neatness, while Γ_{Z_1} acts via a neat congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}_+$ (with trivial stabilizers). Then $\Sigma^+_{\mathsf{Z}_0}$ gives a subdivision of $i\mathbb{R}_{>0}$, while $\Sigma^+_{\mathsf{Z}_1}$ gives a triangularization of $\mathcal{H}_+$ that is compatible with Γ_{Z_1} and descends to a triangularization of $\mathcal{H}_+/\Gamma_{\mathsf{Z}_1}$. Note that any nontrivial subdivision of $i\mathbb{R}_{>0}$ means, when we view the connected components of X_0 as products of those of two smooth integral models of modular curves, we have (possibly repeated) blowups at some subschemes over products of cusps. (This is the end of Example 5.2.)

Example 5.3. In Case (2), suppose that we have the following:

(1) $n \geq 2$ is any integer.
(2) K is an imaginary quadratic extension of $\mathbb{Q}$, with maximal order $\mathcal{O}_K$.
(3) $\mathcal{O}_0 = \mathcal{O}_K \times \mathcal{O}_K$ and $\mathcal{O}_1 = \mathcal{O}_K$ is diagonally embedded in $\mathcal{O}_0$, and $\star_0$ and $\star_1$ are the complex conjugations (simultaneously on both factors of $\mathcal{O}_0$).
(4) $L_1 = \mathcal{O}_K^{\oplus n+1}$, with the first (resp. second) factor of $\mathcal{O}_0 = \mathcal{O}_K \times \mathcal{O}_K$ acting naturally on the first n factors (resp. last factor) of $L_1 = \mathcal{O}_K^{\oplus n+1}$ and trivially on the remaining factors.
(5) Let $\varepsilon \in \mathrm{Diff}^{-1}_{\mathcal{O}_K/\mathbb{Z}}$ be any element in the inverse different that is invariant under the complex conjugation, and let $\langle\,\cdot\,,\,\cdot\,\rangle_1 : L_1 \times L_1 \to \mathbb{Z}(1)$ be the pairing defined by composing the pairing $\big((x_1, x_2, \ldots, x_{n+1}), (y_1, y_2, \ldots, y_{n+1})\big) \mapsto \mathrm{Tr}_{\mathcal{O}_K/\mathbb{Z}}\big(\varepsilon \cdot (-x_1 y_1 + x_2 y_2 + \cdots + x_{n+1} y_{n+1})\big)$ with a fixed choice of isomorphism $2\pi\sqrt{-1} : \mathbb{Z} \to \mathbb{Z}(1)$, and let $h_1(z)$ act on $L_{1,\mathbb{R}} \cong \mathbb{C}^{\oplus n+1}$ via

the left multiplication by the complex conjugate $\overline{z}$ on the first factor, and by z itself on the remaining factors.

Then $\mathrm{G}_0 \underset{\mathbb{Z}}{\otimes} \mathbb{R} \cong \mathrm{G}(\mathrm{U}_{n-1,1} \times \mathrm{U}_1) \cong \mathrm{GU}_{n-1,1} \underset{\mathbf{G}_{\mathrm{m},\mathbb{R}}}{\times} \mathrm{GU}_1$, where the two structure morphisms in the fiber product are similitude homomorphisms; and $\mathrm{G}_1 \underset{\mathbb{Z}}{\otimes} \mathbb{R} \cong \mathrm{GU}_{n,1}$. In both cases, the reflex field is K because $n \geq 2$, so that we can take $F = K$; and the bad primes are those ramified in K and divides $\mathrm{Tr}_{\mathcal{O}_K/\mathbb{Z}}(\varepsilon)$, and we can take $\Box$ to be any set of rational primes that are not bad. Let us choose $\mathcal{H}_0$ and $\mathcal{H}_1$ suitably, so that we have smooth integral models X_0 and X_1 over S_0, with a closed immersion $f : \mathsf{X}_0 \to \mathsf{X}_1$, which can be interpreted as mapping a smooth integral model of a $\mathrm{GU}_{n-1,1}$ Shimura variety to a smooth integral model of a $\mathrm{GU}_{n,1}$ Shimura variety defined by taking fiber products of the universal abelian scheme with some CM elliptic curves (which explains the U_1 part). (It is perhaps better to work with abelian-type Shimura varieties and arrange $\mathrm{G}_0 \underset{\mathbb{Z}}{\otimes} \mathbb{R} \to \mathrm{G}_1 \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ to be $\mathrm{U}_{n-1,1} \to \mathrm{U}_{n,1}$, but the difference is on the centers and hence unimportant for our purpose.)

By Theorem 2.2, there exists some closed immersion of toroidal compactifications $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1} : \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$ extending f, defined by some strictly compatible collections Σ_0 and Σ_1 of cone decompositions. But note that we have no choice to make for Σ_0 and Σ_1. All possible maps $\mathbf{P}_{\mathsf{Z}_0} \to \mathbf{P}_{\mathsf{Z}_1}$ can be identified with either $\{0\} \to \{0\}$ or $\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and in all cases the cone decompositions are uniquely determined and trivial (and satisfy all the usual conditions we impose). Hence, Theorem 2.2 just says that the canonical morphism $f^{\mathrm{tor}} : \mathsf{X}^{\mathrm{tor}}_0 \to \mathsf{X}^{\mathrm{tor}}_1$ between smooth integral models of toroidal compactifications over S_0, where all the collections of cone decompositions are now justifiably omitted from the notation, is a closed immersion.

Nevertheless, such a discussion is not completely meaningless. The fact that smooth toroidal compactifications of X_0 and X_1 uniquely exist is well known, but the fact that closed immersions $f : \mathsf{X}_0 \to \mathsf{X}_1$ extend as above to closed immersions $f^{\mathrm{tor}} : \mathsf{X}^{\mathrm{tor}}_0 \to \mathsf{X}^{\mathrm{tor}}_1$ is probably less so. Also, as soon as we have such a f^{tor}, we can consider the closed immersion $(\mathrm{Id}_{\mathsf{X}_0}, f) : \mathsf{X}_0 \to \mathsf{X}_0 \underset{\mathsf{S}_0}{\times} \mathsf{X}_1$, which then extends to the closed immersion $(\mathrm{Id}_{\mathsf{X}^{\mathrm{tor}}_0}, f^{\mathrm{tor}}) : \mathsf{X}^{\mathrm{tor}}_0 \to \mathsf{X}^{\mathrm{tor}}_0 \underset{\mathsf{S}_0}{\times} \mathsf{X}^{\mathrm{tor}}_1$, which provides the justification for some usual geometric considerations related to the Gan–Gross–Prasad conjecture.

We have similar assertions in Case (3). (This is the end of Example 5.3.)

Example 5.4. In Case (1), suppose that G_0 is the special orthogonal group over $\mathbb{Q}$ defined by a quadratic space V_0 of signature $(n-1, 2)$ at ∞, for some $n \geq 2$, and let G_1 be the special orthogonal group over $\mathbb{Q}$ defined by $V_1 := (\mathbb{Q} \cdot e) \overset{\perp}{\oplus} V_0$, where the quadratic form is defined to have value $+1$ on the additional basis vector e. Then $\mathrm{G}_0 \underset{\mathbb{Q}}{\otimes} \mathbb{R} \cong \mathrm{SO}_{n-1,2}$ and $\mathrm{G}_1 \underset{\mathbb{Q}}{\otimes} \mathbb{R} \cong \mathrm{SO}_{n,2}$. Let i be the same $\sqrt{-1}$ as in Example 5.1. Up to suitable choices of the above isomorphisms, we can arrange that h_0 and h_1 are defined by mapping $\mathbf{G}_{\mathrm{m},\mathbb{C}} \to \mathrm{SO}_{2,\mathbb{R}} : r(\cos\theta + i \sin\theta) \mapsto \left(\begin{smallmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{smallmatrix}\right)$ into the second factors of the diagonally embedded compact subgroups $\mathrm{SO}_{n-1,\mathbb{R}} \times \mathrm{SO}_{2,\mathbb{R}}$ and $\mathrm{SO}_{n,\mathbb{R}} \times \mathrm{SO}_{2,\mathbb{R}}$ of $\mathrm{SO}_{n-1,2}$ and $\mathrm{SO}_{n,2}$, respectively. Then the reflex fields of both Shimura data $(\mathrm{G}_0, \mathrm{G}_0(\mathbb{R}) \cdot h_0)$ and $(\mathrm{G}_1, \mathrm{G}_1(\mathbb{R}) \cdot h_1)$ are $\mathbb{Q}$, and we can take F to be $\mathbb{Q}$ (or any field extension in $\mathbb{C}$). Let $\mathcal{H}_0$ and $\mathcal{H}_1$ be chosen such that $f : \mathsf{X}_0 \to \mathsf{X}_1$ is a closed immersion over $\mathsf{S}_0 = \mathrm{Spec}(F)$. Then $\overline{\mathbf{P}}^+_{\mathsf{Z}_0} \to \overline{\mathbf{P}}^+_{\mathsf{Z}_1}$ can be either from the empty set to the empty set; from a single point to a single point; or from the hyperbolic $(n-1)$-space to the hyperbolic n-space (equivariant with $\mathrm{SO}_{n-1,1} \to \mathrm{SO}_{n,1}$, up to some identifications). (The map $i\mathbb{R}_{>0} \to \mathcal{H}_+$ in Example 5.2 can be viewed as a special case of the last possibility, with $n = 2$.) By Theorem 2.2, there exists some closed immersion of toroidal compactifications $f^{\mathrm{tor}}_{\Sigma_0,\Sigma_1} : \mathsf{X}^{\mathrm{tor}}_{0,\Sigma_0} \hookrightarrow \mathsf{X}^{\mathrm{tor}}_{1,\Sigma_1}$ extending f, for some strictly compatible Σ_0 and Σ_1.

We have similar assertions in Cases (1) and (4) if we replace special orthogonal groups above with the corresponding general spin groups, with suitable associated Shimura data and Siegel embeddings. (This is the end of Example 5.4.)

6. Perfectoid toroidal compactifications

Finally, as an application, let us verify [9, Hypothesis 2.18]. As explained in [9], this allows for a substantial simplification of the proof of the main theorems in [9].

Let us explain how [9, Hypothesis 2.18] fits into our setting. In Case (1), suppose that $\mathrm{G}_1 = \mathrm{GSp}_{2g,\mathbb{Q}}$, for some $g \geq 0$, so that $\rho : \mathrm{G}_0 \to \mathrm{G}_1$ induces a Siegel embedding $(\mathrm{G}_0, \mathrm{D}_0) \hookrightarrow (\mathrm{G}_1, \mathrm{D}_1)$, making $(\mathrm{G}_0, \mathrm{D}_0)$ a Hodge-type Shimura datum. We shall fix the choice of a rational prime $p > 0$, and assume that the base field $F = C$ is the completion of an algebraic closure of $\mathbb{Q}_p$. Let $\mathcal{H}^p_1 \subset \mathrm{G}_1(\mathbb{A}^{\infty,p})$ be a neat open compact subgroup, and let $\mathcal{H}^p_0 := \big(\rho(\mathbb{A}^{\infty,p})\big)^{-1}(\mathcal{H}^p_1)$. For each $r \geq 0$, consider the principal congruence subgroup $\mathcal{H}^{(r)}_{1,p} := \ker\big(\mathrm{GSp}_{2g}(\mathbb{Z}_p) \to \mathrm{GSp}_{2g}(\mathbb{Z}/p^r)\big)$ at p, and let $\mathcal{H}^{(r)}_1 := \mathcal{H}^p_1\mathcal{H}^{(r)}_{1,p}$. Let $\mathcal{H}^{(r)}_{0,p} := \big(\rho(\mathbb{Q}_p)\big)^{-1}(\mathcal{H}^{(r)}_{1,p})$ and $\mathcal{H}^{(r)}_0 := \mathcal{H}^p_0\mathcal{H}^{(r)}_{0,p} = \big(\rho(\mathbb{A}^\infty)\big)^{-1}(\mathcal{H}^{(r)}_1)$. Then we have morphisms between the associated Shimura varieties $f^{(r)} : \mathsf{X}^{(r)}_0 \to \mathsf{X}^{(r)}_1$ at levels $\mathcal{H}^{(r)}_0$ and $\mathcal{H}^{(r)}_1$, respectively, which are compatible with each other when we vary r. We shall similarly denote other objects at $\mathcal{H}^{(r)}_0$ and $\mathcal{H}^{(r)}_1$ with superscripts "(r)". By [13, Lem. 2.1.2], up to replacing $\mathcal{H}^p_1$ with a finite index subgroup, we may assume that $f^{(r)}$ is a closed immersion, for all $r \geq 0$.

By Proposition 4.10, there exist collections $\Sigma^{(0)}_0$ and $\Sigma^{(0)}_1$ for $\mathsf{X}^{(0)}_0$ and $\mathsf{X}^{(0)}_1$, respectively, that are strictly compatible with each other as in Definition 4.6, which we assume to be projective and smooth and satisfy the condition that, for $i = 0, 1$, and for each $\mathsf{Z}^{(0)}_i$ and each $\sigma \in \Sigma^+_{\mathsf{Z}^{(0)}_i}$, the stabilizer $\Gamma_{\mathsf{Z}^{(0)}_i,\sigma}$ of σ in $\Gamma_{\mathsf{Z}^{(0)}_i}$ is trivial. Note that Proposition 3.4 can also be applied to morphisms between Shimura varieties associated with the same Shimura datum, but with possibly different levels. For each $r \geq 0$, let $\Sigma^{(r)}_0$ and $\Sigma^{(r)}_1$ denote the induced collections at levels $\mathcal{H}^{(r)}_0$ and $\mathcal{H}^{(r)}_1$, respectively. Then they are projective and satisfy the analogue of the above condition on stabilizers, and are strictly compatible with each other. Since the levels $\mathcal{H}^{(r)}_{1,p}$ at p are principal, for all $r \geq 0$, and since $\mathcal{H}^{(r)}_{0,p} = \big(\rho(\mathbb{Q}_p)\big)^{-1}(\mathcal{H}^{(r)}_{1,p})$, the canonical homomorphisms $\mathbf{S}_{\mathsf{Z}^{(0)}_i} \to \mathbf{S}_{\mathsf{Z}^{(r)}_i}$ can be identified with $\mathbf{S}_{\mathsf{Z}^{(0)}_i} \hookrightarrow \frac{1}{p^r}\mathbf{S}_{\mathsf{Z}^{(0)}_i}$, for $i = 1, 2$. In particular, the smoothness condition on cone decompositions remains the same when we vary $r \geq 0$. Thus, $\Sigma^{(r)}_0$ and $\Sigma^{(r)}_1$ are also smooth, and we have verified all the conditions we would like to impose on these collections. By Proposition 3.4(5), the canonical morphisms $\mathsf{X}^{(r),\mathrm{tor}}_{i,\Sigma^{(r)}_i} \to \mathsf{X}^{(r'),\mathrm{tor}}_{i,\Sigma^{(r')}_i}$, for $i = 1, 2$ and $r \geq r' \geq 0$, are all *finite*. Note that each such finite morphism is automatically flat (by [10, IV-3, 15.4.2 e′)⇒b)]) and therefore universally open (by [10, IV-2, 2.4.6]), because both its source and target are smooth and of the same equi-dimension.

For simplicity, we shall omit the subscripts "$\Sigma_i^{(r)}$" in the following. We shall change the font from X to $\mathcal{X}$ when we denote the associated adic spaces.

The case $i = 0$ of the following proposition verifies [9, Hypothesis 2.18]:

Proposition 6.1. *For $i = 0, 1$, there is a perfectoid space $\mathcal{X}_i^{(\infty),\mathrm{tor}}$ over C such that $\mathcal{X}_i^{(\infty),\mathrm{tor}} \sim \varprojlim_r \mathcal{X}_i^{(r),\mathrm{tor}}$, where "$\sim$" has the same meaning as in* [28, Def. 2.4.1].

Proof. We shall imitate the proof of [27, Thm. 4.1.1(i)]. Recall that the assertion $\mathcal{X}_i^{(\infty),\mathrm{tor}} \sim \varprojlim_r \mathcal{X}_i^{(r),\mathrm{tor}}$ means there are compatible morphisms $\mathcal{X}_i^{(\infty),\mathrm{tor}} \to \mathcal{X}_i^{(r),\mathrm{tor}}$ inducing a homeomorphism of topological spaces $|\mathcal{X}_i^{(\infty),\mathrm{tor}}| \xrightarrow{\sim} \varprojlim_r |\mathcal{X}_i^{(r),\mathrm{tor}}|$, as well as an open covering of $\mathcal{X}_i^{(\infty),\mathrm{tor}}$ by affinoid adic spaces $\mathrm{Spa}(R_i^{(\infty)}, R_i^{(\infty),+})$ inducing a homomorphism $\varinjlim R_i^{(r)} \to R_i^{(\infty)}$ with dense image, where the direct limit runs over all $r \geq 0$ and all affinoid open subspaces $\mathrm{Spa}(R_i^{(r)}, R_i^{(r),+}) \subset \mathcal{X}_i^{(r),\mathrm{tor}}$ through which the compositions of $\mathrm{Spa}(R_i^{(\infty)}, R_i^{(\infty),+}) \hookrightarrow \mathcal{X}_i^{(\infty),\mathrm{tor}} \to \mathcal{X}_i^{(r),\mathrm{tor}}$ factor.

By [25, Cor. A.19 and its proof], the above holds when $i = 1$, and we may assume that each member $\mathrm{Spa}(R_1^{(\infty)}, R_1^{(\infty),+})$ in the open covering of $\mathcal{X}_1^{(\infty),\mathrm{tor}}$ is affinoid perfectoid and is the preimage of some $\mathrm{Spa}(R_1^{(r)}, R_1^{(r),+})$, for all sufficiently large r. Since $f^{(r),\mathrm{tor}} : \mathsf{X}_0^{(r),\mathrm{tor}} \to \mathsf{X}_1^{(r),\mathrm{tor}}$ is a closed immersion by Proposition 4.9 (and the constructions of $\Sigma_0^{(r)}$ and $\Sigma_1^{(r)}$), the associated morphism $\mathcal{X}_0^{(r),\mathrm{tor}} \to \mathcal{X}_1^{(r),\mathrm{tor}}$ is a closed immersion of adic spaces. Hence, we have $\mathcal{X}_0^{(r),\mathrm{tor}} \underset{\mathcal{X}_1^{(r),\mathrm{tor}}}{\times} \mathrm{Spa}(R_1^{(r)}, R_1^{(r),+}) \cong \mathrm{Spa}(R_0^{(r)}, R_0^{(r),+})$ for some Huber pair $(R_0^{(r)}, R_0^{(r),+})$ such that $R_1^{(r)} \to R_0^{(r)}$ is surjective. Let $I^{(r)}$ denote the kernel of this homomorphism. Let $Z^{(r)}$ denote the Zariski closed subset of $\mathrm{Spa}(R_1^{(\infty)}, R_1^{(\infty),+})$, as in [27, Def. 2.2.1], defined by the image of $I^{(r)}$ in $R_1^{(\infty)}$. By comparing definitions, we can identify $Z^{(r)}$ with $|\mathrm{Spa}(R_0^{(r)}, R_0^{(r),+})| \underset{|\mathrm{Spa}(R_1^{(r)}, R_1^{(r),+})|}{\times} |\mathrm{Spa}(R_1^{(\infty)}, R_1^{(\infty),+})|$ as closed subsets of $|\mathrm{Spa}(R_1^{(\infty)}, R_1^{(\infty),+})|$. By [27, Lem. 2.2.2], there is a canonical affinoid perfectoid space $\mathrm{Spa}(R_0^{(\infty),(r)}, R_0^{(\infty),(r),+})$, with a morphism $\mathrm{Spa}(R_0^{(\infty),(r)}, R_0^{(\infty),(r),+}) \to \mathrm{Spa}(R_1^{(\infty)}, R_1^{(\infty),+})$, induced by a canonical homomorphism $R_1^{(\infty)} \to R_0^{(\infty),(r)}$ with dense image, inducing a homeomorphism $|\mathrm{Spa}(R_0^{(\infty),(r)}, R_0^{(\infty),(r),+})| \xrightarrow{\sim} Z^{(r)}$. Moreover, by the construction in the proof of [27, Lem. 2.2.2], the composition of $R_1^{(r)} \to R_1^{(\infty)} \to R_0^{(\infty),(r)}$ factors through $R_1^{(r)} \to R_0^{(r)}$. By the universal property explained in [27, Rem. 2.2.3], for all $r' \geq r$, we have compatible canonical homomorphisms $(R_0^{(\infty),(r)}, R_0^{(\infty),(r),+}) \to (R_0^{(\infty),(r')}, R_0^{(\infty),(r'),+})$ over $(R_1^{(\infty)}, R_1^{(\infty),+})$.

Let $(R_0^{(\infty)}, R_0^{(\infty),+})$ denote the p-adic completion of $\varinjlim_r (R_0^{(\infty),(r)}, R_0^{(\infty),(r),+})$, where the direct limit runs over all sufficiently large r such that $(R_0^{(\infty),(r)}, R_0^{(\infty),(r),+})$ are defined as above, which is canonically a Huber pair over $(R_1^{(\infty)}, R_1^{(\infty),+})$. Since the homomorphisms $R_1^{(\infty)} \to R_0^{(\infty),(r)}$ have dense images, so does the composition of $R_1^{(\infty)} \to \varinjlim_r R_0^{(\infty),(r)} \to R_0^{(\infty)}$. Since the p-th power homomorphism $R_0^{(\infty),+}/p \to R_0^{(\infty),+}/p$ is surjective because the p-th power homomorphisms $R_0^{(\infty),(r),+}/p \to R_0^{(\infty),(r),+}/p$ are,

$R_0^{(\infty)}$ is a perfectoid C-algebra, by [12, Prop. 3.6.2]. Thus, we have obtained an affinoid perfectoid space $\mathrm{Spa}(R_0^{(\infty)}, R_0^{(\infty),+})$ over $\mathrm{Spa}(R_1^{(\infty)}, R_1^{(\infty),+})$. Moreover, for all sufficiently large r, we have compatible homomorphisms $R_0^{(r)} \to R_0^{(\infty)}$, and the composition of $\varinjlim_r R_1^{(r)} \to R_1^{(\infty)} \to R_0^{(\infty)}$ factors through the induced homomorphism $\varinjlim_r R_0^{(r)} \to R_0^{(\infty)}$. Since the homomorphisms $\varinjlim_r R_1^{(r)} \to R_1^{(\infty)} \to R_0^{(\infty)}$ have dense images, so do their composition and the induced homomorphism $\varinjlim_r R_0^{(r)} \to R_0^{(\infty)}$. The corresponding morphisms of adic spaces induce homeomorphisms of topological spaces $|\mathrm{Spa}(R_0^{(\infty),(r)}, R_0^{(\infty),(r),+})| \xrightarrow{\sim} \varprojlim_r |\mathrm{Spa}(R_0^{(\infty),(r)}, R_0^{(\infty),(r),+})| \xrightarrow{\sim} \varprojlim_r \Big(|\mathrm{Spa}(R_0^{(r)}, R_0^{(r),+})| \underset{|\mathrm{Spa}(R_1^{(r)}, R_1^{(r),+})|}{\times} |\mathrm{Spa}(R_1^{(\infty)}, R_1^{(\infty),+})| \Big)$. Since the induced map $|\mathrm{Spa}(R_i^{(\infty)}, R_i^{(\infty),+})| \to \varprojlim_r |\mathrm{Spa}(R_i^{(r)}, R_i^{(r),+})|$ is a homeomorphism when $i = 1$, the same is true when $i = 0$, by canonically identifying these topological spaces as subspaces of $\prod_r |\mathrm{Spa}(R_1^{(r)}, R_1^{(r),+})|$. Thus, the affinoid perfectoid space $\mathrm{Spa}(R_0^{(\infty)}, R_0^{(\infty),+})$ satisfies $\mathrm{Spa}(R_0^{(\infty)}, R_0^{(\infty),+}) \sim \varprojlim_r \mathrm{Spa}(R_0^{(r)}, R_0^{(r),+})$. By gluing such $\mathrm{Spa}(R_0^{(\infty)}, R_0^{(\infty),+})$ using [28, Prop. 2.4.3 and 2.4.5] over an open covering of $\mathcal{X}_1^{(\infty),\mathrm{tor}}$ by affinoid perfectoid spaces $\mathrm{Spa}(R_1^{(\infty)}, R_1^{(\infty),+})$ as above, we obtain a perfectoid space $\mathcal{X}_0^{(\infty),\mathrm{tor}}$ over C such that $\mathcal{X}_0^{(\infty),\mathrm{tor}} \sim \varprojlim_r \mathcal{X}_0^{(r),\mathrm{tor}}$, as desired. $\square$

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