ELEVATORS FOR DEGENERATIONS OF PEL STRUCTURES

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Abstract. We show that the maximal rank of mixed characteristic degenerations of abelian varieties parameterized by a PEL-type Shimura variety are the same as the maximal rank of equicharacteristic zero degenerations (of abelian varieties parameterized by the same Shimura variety). As a byproduct, we obtain a simple proof of Yasuo Morita's conjecture in 1975 that an abelian variety with additional structures parameterized by a compact PEL-type Shimura variety has potential good reductions everywhere.

1. Introduction

All known integral models of toroidal compactifications of PEL-type Shimura varieties have natural stratifications characterized by the degeneration patterns of the universal abelian schemes. It is common wishful thinking that the (better understood) characteristic zero story should give us a fairly suggestive picture about the boundary stratification in general, including cases of bad reduction.

In this article, we shall partially justify this wishful thinking (for all PEL-type Shimura varieties at once) by showing (at least) the *ranks* (of torus parts of degenerations) match in *all* characteristics. Let us be more precise about this statement.

Let $\mathcal{O}$ be an order in a semisimple algebra, finite-dimensional over $\mathbb{Q}$, together with a positive involution $\star$. By a PEL-type $\mathcal{O}$-lattice $(L, \langle \cdot , \cdot \rangle, h_0)$ (as in [10, Def. 1.2.1.3]), we mean the following data:

1. An $\mathcal{O}$-lattice, namely a $\mathbb{Z}$-lattice L with the structure of an $\mathcal{O}$-module.
2. An alternating pairing $\langle \cdot , \cdot \rangle : L \times L \to \mathbb{Z}(1)$ satisfying $\langle bx, y \rangle = \langle x, b^\star y \rangle$ for any $x, y \in L$ and $b \in \mathcal{O}$, together with an $\mathbb{R}$-algebra homomorphism $h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ satisfying $\langle h_0(z)x, y \rangle = \langle x, h_0(z^c)y \rangle$ for any $x, y \in L$ and $z \in \mathbb{C}$, and satisfying $(2\pi\sqrt{-1})^{-1}\langle x, h_0(\sqrt{-1})x \rangle > 0$ for any nonzero $x \in L$. (In [10, Def. 1.2.1.3] h_0 was denoted by h.)

The datum of $(\mathcal{O}, \star, L, \langle \cdot , \cdot \rangle, h_0)$ defines a group functor G over $\mathrm{Spec}(\mathbb{Z})$ (as in [10, Def. 1.2.1.5]), and defines the reflex field F_0 (as in [10, Def. 1.2.5.4]). Let $\mathcal{H}$ be

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a *neat* open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}})$ (as in [15, 0.6] or [10, Def. 1.4.1.8]). Then the data of $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ and $\mathcal{H}$ define a moduli problem $\mathsf{M}_{\mathcal{H}}$ over $\mathrm{Spec}(F_0)$ by [10, Def. 1.4.1.4] (with $\Box = \emptyset$ and so that $\mathcal{O}_{F_0,(\Box)} = F_0$ there), parameterizing tuples $(A, \lambda, i, \alpha_{\mathcal{H}})$ over schemes S over $\mathrm{Spec}(F_0)$ of the following form:

(1) $A \to S$ is an abelian scheme.
(2) $\lambda : A \to A^\vee$ is a polarization.
(3) $i : \mathcal{O} \hookrightarrow \mathrm{End}_S(A)$ is an $\mathcal{O}$-endomorphism structure as in [10, Def. 1.3.3.1].
(4) $\underline{\mathrm{Lie}}_{A/S}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-module structure given naturally by i satisfies the determinantal condition in [10, Def. 1.3.4.2] given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$.
(5) $\alpha_{\mathcal{H}}$ is an (integral) level-$\mathcal{H}$ structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \langle\,\cdot\,,\,\cdot\,\rangle)$ as in [10, Def. 1.3.7.8].

By [10, Thm. 1.4.1.12 and Cor. 7.2.3.10], $\mathsf{M}_{\mathcal{H}}$ is representable by a (smooth) quasi-projective scheme over $\mathrm{Spec}(F_0)$ under the assumption that $\mathcal{H}$ is neat.

Consider the set $\mathsf{X} = \mathrm{G}(\mathbb{R})h_0$ of $\mathrm{G}(\mathbb{R})$-conjugates $h : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ of the polarization $h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$. Then it is well known (see [9, §8] or [11, §2]) that there exists a quasi-projective variety $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$ over $\mathbb{C}$, together with a canonical open and closed immersion $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}} \hookrightarrow \mathsf{M}_{\mathcal{H}} \underset{F_0}{\otimes} \mathbb{C}$ when $\mathcal{H}$ is neat, such that the double-coset space $\mathsf{Sh}_{\mathcal{H}} := \mathrm{G}(\mathbb{Q}) \backslash \mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/\mathcal{H}$ can be identified with the analytification of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$. Moreover, if we denote by $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$ the schematic image of $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}} \to \mathsf{M}_{\mathcal{H}}$, the latter being defined over $\mathrm{Spec}(F_0)$, then $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$ is a scheme defined over $\mathrm{Spec}(F_0)$ such that $\mathsf{Sh}_{\mathcal{H},\mathrm{can}} \underset{F_0}{\otimes} \mathbb{C} \cong \mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$. This allows us to talk about tuples $(A, \lambda, i, \alpha_{\mathcal{H}})$ parameterized by $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$.

Consider *semi-abelian schemes* defined as in [4, Ch. I, §2]. By the *rank* of a fiber of a semi-abelian scheme, we mean the rank (as a free abelian group) of the character group of the torus part of any geometric fiber above the fiber. By a *degeneration* over a discrete valuation ring R based at $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$, we mean a semi-abelian scheme $A \to \mathrm{Spec}(R)$, together with a morphism $\mathrm{Spec}(\mathrm{Frac}(R)) \to \mathsf{Sh}_{\mathcal{H},\mathrm{can}}$, such that $A \underset{R}{\otimes} \mathrm{Frac}(R)$ is the pullback of the universal abelian variety over $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$ under $\mathrm{Spec}(\mathrm{Frac}(R)) \to \mathsf{Sh}_{\mathcal{H},\mathrm{can}}$. By the *rank* of a degeneration $A \to \mathrm{Spec}(R)$ as above, we mean the rank of the special fiber of A. Note that we allow the rank to be zero, in which case A is an abelian scheme over $\mathrm{Spec}(R)$.

Definition 1.1. *For any characteristic $p \geq 0$ (of fields), we define r_p to be the maximal rank among degenerations over discrete valuation rings R based at $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$ with* **residue characteristic** *p. (The number p is allowed to be zero.)*

The invariant r_0 can be calculated explicitly, because of the following facts:

(1) The relation between the theory of degeneration and the algebraic construction of toroidal compactifications in the PEL-type cases is well understood. (See [4] and [10].)
(2) The algebraic and analytic constructions of toroidal compactifications are known to be compatible over $\mathbb{C}$. (See [11] for an explanation using explicit identifications of theta functions. We will not need this fact in this article.)
(3) The analytic compactifications over $\mathbb{C}$ can be described group-theoretically. (See [2], [1], [7], and [15].)

Concretely, a degeneration of rank r corresponds to an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$-submodule of $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, totally isotropic under $\langle \cdot, \cdot \rangle$ and of dimension r over $\mathbb{Q}$, determining a rational parabolic subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. The relation between such parabolic subgroups of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ and the rational boundary components of X is well known.

On the other hand, there is no obvious way to calculate the invariant r_p for all characteristics $p \geq 0$. When p is *not* a so-called *bad prime* in the sense of [10, Def. 1.4.1.1], the invariant r_p agrees with r_0 because of the existence of a smooth toroidal compactification (with boundary stratified by smooth locally closed subschemes) over $\mathbb{Z}_{(p)}$ constructed using the theory of degeneration. (See [10, Thm. 6.4.1.1].) However, this does not cover the cases when p is a bad prime. Our motivation for this article is to supply an argument independent of the residue characteristics. (We emphasize that the residue characteristic is irrelevant not only in the statement of the result, but also in the proof. No p is special in this problem!)

The main result of this article is the following:

Theorem 1.2. *With the setting as above, we have $r_0 = r_p$ for every $p \geq 0$.*

Let us sketch the proof of Theorem 1.2 (and give an outline of the article) as follows. (In this sketch, for simplicity, we will abbreviate degenerations over discrete valuations based at $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$ as *degenerations*.)

Our first objective will be to prove Theorem 4.1, which asserts the inequality $r_p \leq r_0$ for every $p \geq 0$. This will follow from the existence of what we will call the *elevators*. Roughly speaking, for a given mixed characteristics degeneration of rank r, an elevator is a semi-abelian scheme over a noetherian normal *domain*, such that the given degeneration is a pullback of such a semi-abelian scheme, and such that at least one of its one characteristic zero fibers has rank equal to r. We will review the background terminologies in Sections 2.1 and 2.2, and give the precise definition of elevators in Section 2.3. We will then explain the construction of elevators in Section 3, using techniques developed in the construction of boundary charts of toroidal compactifications in [4] and [10].

Once Theorem 4.1 is established, the special case $r_p = r_0 = 0$ for every $p \geq 0$ of the above inequality implies Yasuo Morita's original conjecture in [12]. (Note that $r_p = 0$ for every $p \geq 0$ implies that *all* abelian varieties parameterized by $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$ have *potential good reductions everywhere.*) We shall explain this implication in Section 4.2, giving also a criterion of properness for integral models of PEL-type Shimura varieties in Theorem 4.9, with *no* assumption on the residue characteristics.

Then we will prove Theorem 5.1, which asserts the opposite inequality $r_0 \leq r_p$ for every characteristic $p \geq 0$. The preparatory Proposition 5.2 (for proving Theorem 5.1) is the *only* step we use results in [10] not essentially known in [4], which allows us to choose an equicharacteristic zero degeneration of rank r_0, such that the abelian part of its (characteristic zero) special fiber extends to abelian schemes over discrete valuation rings of all possible mixed characteristics. (This extension uses Theorem 4.9, which we have just proved.) Starting with such mixed characteristics abelian schemes, by a construction similar to the one (of elevators) in Section 3, we obtain degenerations of rank r_0 over discrete valuation rings of all possible mixed characteristics, as desired.

Finally, Theorem 1.2 follows as a combination of Theorems 4.1 and 5.1.

Following the referee's suggestion, we have included several commutative diagrams summarizing the constructions. Although these diagrams will not be logically needed in the proofs, we hope they will be helpful for understanding the arguments.

2. Terminologies

We shall follow [10, Notations and Conventions] unless otherwise specified. (The references to [10] uses the original numbering in the submitted thesis, but the reader is encouraged to consult the revision available on the author's website, sometimes with slightly modified numberings, for corrections and improvements.)

2.1. **Degenerating families.** Let $\mathcal{O}$ be as in Section 1. We shall denote pullbacks of objects to rings or schemes by subscripts when there is no confusion.

Definition 2.1. *Let S be any normal locally noetherian scheme over $\mathrm{Spec}(\mathbb{Z})$. A* **degenerating family of type** $(\mathrm{PE}, \mathcal{O})$ *is a tuple (A, λ, i) over S such that:*

1. *A is a semi-abelian scheme over S.*
2. *There exists an open dense subscheme S_1 of S such that A_{S_1} is an abelian scheme. In this case, there is a unique semi-abelian scheme $A^\vee$ (up to unique isomorphism), called the dual semi-abelian scheme of A, such that $A^\vee_{S_1}$ is the dual abelian scheme of A_{S_1}.*
3. *$\lambda : A \to A^\vee$ is a group homomorphism that induces by restriction a polarization λ_{S_1} of A_{S_1}.*
4. *$i : \mathcal{O} \to \mathrm{End}_S(A)$ is a map that defines by restriction an $\mathcal{O}$-structure $i_{S_1} : \mathcal{O} \to \mathrm{End}_{S_1}(A_{S_1})$ of (A_{S_1}, λ_{S_1}). (See [10, Def. 1.3.3.1].)*

2.2. **Theory of degeneration data.** Let $\mathcal{O}$ be as above.

Let R be a noetherian normal domain complete with respect to an ideal I, with $\mathrm{rad}(I) = I$ for convenience. Let $S := \mathrm{Spec}(R)$, $K := \mathrm{Frac}(R)$, $\eta := \mathrm{Spec}(K)$ the generic point of S, $R_0 := R/I$, and $S_0 := \mathrm{Spec}(R_0)$. We shall denote the pullbacks to η or S_0 by subscripts η or 0, respectively.

Definition 2.2 (cf. [10, Def. 5.1.1.4])**.** *With notations and assumptions as above, the category $\mathrm{DEG}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(R, I)$ has objects consisting of degenerating families (A, λ, i) of type $(\mathrm{PE}, \mathcal{O})$ (over $S = \mathrm{Spec}(R)$) such that A_0 is an extension of an abelian scheme B_0 by a* **split** *torus T_0 over S_0.*

By the theory of degeneration data for polarized abelian varieties in [4, Ch. II and III] (explained in [10, Ch. 4]), generalized by functoriality for polarized abelian varieties with endomorphisms (see [10, Sec. 5.1.1]), the so-called *Mumford's construction* induces an equivalence of categories

$$\begin{aligned} \mathrm{M}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(R, I) : \mathrm{DD}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(R, I) &\to \mathrm{DEG}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(R, I) \\ (B, \lambda_B, i_B, X, Y, \phi, c, c^\vee, \tau) &\mapsto (A, \lambda, i) \end{aligned}$$

realizing (A, λ, i) (up to isomorphism) as the image of an object in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(R, I)$ given by the following data:

1. An abelian scheme B over S, a polarization $\lambda_B : B \to B^\vee$ of B, and an $\mathcal{O}$-endomorphism structure $i_B : \mathcal{O} \hookrightarrow \mathrm{End}_S(B)$ of (B, λ_B).

(2) Two $\mathcal{O}$-lattices X and Y of the same rank over S, with an $\mathcal{O}$-equivariant embedding $\phi : Y \to X$. (Here X and Y correspond, respectively, to the character groups of the torus parts of A and $A^\vee$. We shall denote the actions of an element $b \in \mathcal{O}$ on X and Y by $i_X(b)$ and $i_Y(b)$, respectively. When the context is clear, we shall simply denote the actions by b.)

(3) Two $\mathcal{O}$-equivariant morphisms $c : X \to B^\vee$ and $c^\vee : Y \to B$ of group schemes over S, satisfying the compatibility $c\phi = \lambda_B c^\vee$. (By abuse of notation, we denote by X and Y the constant group schemes X_S and Y_S over S, respectively.)

(4) A trivialization of biextensions $\tau : \mathbf{1}_{Y \times X, \eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_{B,\eta}^{\otimes -1}$ with symmetric pullback under $\mathrm{Id}_Y \times \phi : Y \times Y \to Y \times X$, satisfying the following conditions:
   (a) (*Compatibility with $\mathcal{O}$-actions*:) For any $b \in \mathcal{O}$, we have a canonical identification of sections $(i_Y(b) \times \mathrm{Id}_X)^* \tau = (\mathrm{Id}_Y \times i_X(b^\star))^* \tau$ under the canonical isomorphism $(i_B(b) \times \mathrm{Id}_{B^\vee})^* \mathcal{P}_B \cong (\mathrm{Id}_B \times (i_B(b))^\vee)^* \mathcal{P}_B$.
   (b) (*Positivity*:) For any $y \in Y$ and $\chi \in X$, the trivialization $\tau(y, \chi)$ defines an isomorphism of invertible sheaves from $(c^\vee(y), c(\chi))^* \mathcal{P}_{B,\eta}$ to $\mathbf{1}_\eta$. Under this isomorphism (which we again denote by $\tau(y, \chi)$), the canonical R-integral structure $(c^\vee(y), c(\chi))^* \mathcal{P}_B$ of $(c^\vee(y), c(\chi))^* \mathcal{P}_{B,\eta}$ determines an invertible R-submodule $I_{y,\chi}$ of K. Then the positivity condition is that $I_{y,\phi(y)} \subset I$ for all nonzero y in Y. (Clearly, $I_{0,0} = R$.)

We say that $(B, \lambda_B, i_B, X, Y, \phi, c, c^\vee, \tau)$ is the *degeneration data* of (A, λ, i). The theory works even when X and Y are zero. (Then, by [4, Ch. I, 2.8], $A \cong B$ is an abelian scheme over S, and the positivity condition for τ is trivially verified.)

We shall suppress I from the notation when it is clear from the context. (This is the case, for example, when R is a discrete valuation ring.)

2.3. **Elevators for degenerations of PEL structures.**

Definition 2.3. *Let $R^\diamond$ be a complete discrete valuation ring such that the characteristic of $K^\diamond := \mathrm{Frac}(R^\diamond)$ is zero. Let $k^\diamond$ be the residue field of $R^\diamond$. Suppose $(A^\diamond, \lambda^\diamond, i^\diamond) \to \mathrm{Spec}(R^\diamond)$ is a degenerating family of type* $(\mathrm{PE}, \mathcal{O})$. *An* **elevator** *for $(A^\diamond, \lambda^\diamond, i^\diamond)$ is the following set of data:*

(1) *A complete discrete valuation ring R with residue field k, together with a finite étale morphism* $\mathrm{Spec}(R) \to \mathrm{Spec}(R^\diamond)$.

(2) *A noetherian normal domain $\widetilde{R}$, together with a degenerating family $(\widetilde{A}, \widetilde{\lambda}, \widetilde{i})$ of type* $(\mathrm{PE}, \mathcal{O})$ *over $\widetilde{R}$.*

(3) *A morphism* $\mathrm{Spec}(R) \to \mathrm{Spec}(\widetilde{R})$, *together with an isomorphism*
$$(A^\diamond, \lambda^\diamond, i^\diamond) \underset{R^\diamond}{\otimes} R \xrightarrow{\sim} (\widetilde{A}, \widetilde{\lambda}, \widetilde{i}) \underset{\widetilde{R}}{\otimes} R.$$
(*In particular, the characteristic of $\widetilde{K} := \mathrm{Frac}(\widetilde{R})$ is zero.*)

(4) *There is at least one (functorial) point* $\mathrm{Spec}(\widetilde{k}) \to \mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ *such that $\widetilde{A}_{\widetilde{k}}$ and $A^\diamond_{k^\diamond}$ have the same rank.*

Note that an elevator is not quite a deformation. The ring $\widetilde{R}$ can stay the same when we replace $R^\diamond$ or R with a much larger ring. The intuition is that an elevator should be part of something like a moduli space of degenerations of a similar pattern.

3. Construction of elevators

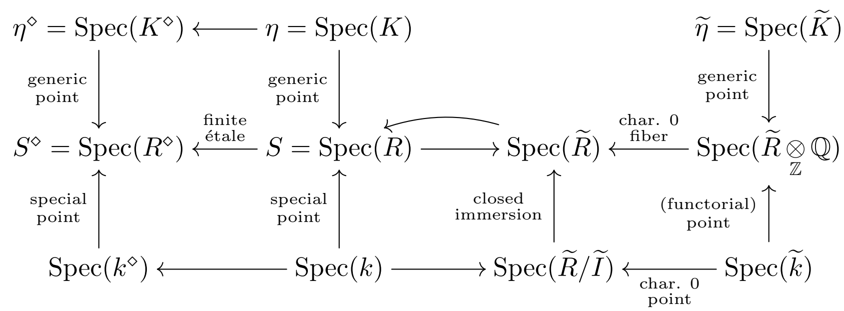
Theorem 3.1. *Let $R^\diamond$ and $(A^\diamond, \lambda^\diamond, i^\diamond)$ be as in Definition* 2.3. *Then there exists an elevator for $(A^\diamond, \lambda^\diamond, i^\diamond)$ satisfying the following additional conditions:*

(1) *The ring $\widetilde{R}$ is a noetherian normal domain over R, complete with respect to some ideal $\widetilde{I}$ such that* $\mathrm{rad}(\widetilde{I}) = \widetilde{I}$, *and $(\widetilde{A}, \widetilde{\lambda}, \widetilde{i})$ is an object of* $\mathrm{DEG}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(\widetilde{R}, \widetilde{I})$.

(2) *Suppose $(B, \lambda_B, i_B, X, Y, \phi, c, c^\vee, \tau)$ is the degeneration data associated with $(A^\diamond, \lambda^\diamond, i^\diamond) \underset{R^\diamond}{\otimes} R$, and suppose $(\widetilde{B}, \lambda_{\widetilde{B}}, i_{\widetilde{B}}, \widetilde{X}, \widetilde{Y}, \widetilde{\phi}, \widetilde{c}, \widetilde{c}^\vee, \widetilde{\tau})$ is the degeneration data associated with $(\widetilde{A}, \widetilde{\lambda}, \widetilde{i})$. Then $(\widetilde{B}, \lambda_{\widetilde{B}}, i_{\widetilde{B}}, \widetilde{X}, \widetilde{Y}, \widetilde{\phi}, \widetilde{c}, \widetilde{c}^\vee)$ (with $\widetilde{\tau}$ omitted) is isomorphic to $(B, \lambda_B, i_B, X, Y, \phi, c, c^\vee) \underset{R}{\otimes} \widetilde{R}$.*

The two additional conditions in Theorem 3.1 mean that only the *period* $\widetilde{\tau}$ is being *elevated*. All other data are essentially intact.

The relations among the rings $R^\diamond$, $K^\diamond$, $k^\diamond$, R, K, k, $\widetilde{R}$, $\widetilde{K}$, and $\widetilde{k}$ can be summarized by the following commutative diagram (of their spectra):



The curly arrow from $\mathrm{Spec}(\widetilde{R})$ to $S = \mathrm{Spec}(R)$ in this diagram is not required in the definition of an elevator (see Definition 2.3), but will be a byproduct of the construction.

The proof of Theorem 3.1 will be carried out in subsequent subsections. Our main references will be [4] and [10]. Our formulations follow mainly [10, Sec. 6.2], but our approach follows [4, Ch. IV] very closely. The main new step is the trick in the beginning of Section 3.4, which allows us to work without worrying about bad reductions.

3.1. Choice of R and degeneration data.

Let $R^\diamond \to R$ be a finite étale morphism of complete discrete valuation rings such that the pullbacks of the torus parts of $A^\diamond$ and $A^{\diamond\vee}$ are split over R. Let $S := \mathrm{Spec}(R)$, $K := \mathrm{Frac}(R)$, $\eta := \mathrm{Spec}(K)$ the generic point of S, and k the residue field of R.

The degenerating family $(A^\diamond_S, \lambda^\diamond_S, i^\diamond_S)$ of type $(\mathrm{PE}, \mathcal{O})$ over S defines an object in $\mathrm{DEG}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(R)$, with degeneration data $(B, \lambda_B, i_B, X, Y, \phi, c, c^\vee, \tau)$ in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(R)$.

3.2. Chart for the datum τ without positivity condition.

Let $\mathbf{S}$ denote the finitely generated abelian group

$$\mathbf{S} := (Y \underset{\mathbb{Z}}{\otimes} X) / \begin{pmatrix} y \otimes \phi(y') - y' \otimes \phi(y) \\ (by) \otimes \chi - y \otimes (b^\star \chi) \end{pmatrix}_{\substack{y, y' \in Y, \\ \chi \in X, b \in \mathcal{O}}} .$$

Let $\mathbf{S}_{\mathrm{free}}$ denote the free quotient of $\mathbf{S}$, namely the quotient of $\mathbf{S}$ by its torsion subgroup $\mathbf{S}_{\mathrm{tor}}$. Let $E := \underline{\mathrm{Hom}}(\mathbf{S}, \mathbf{G}_{\mathrm{m}})$, $E_{\mathrm{tor}} := \underline{\mathrm{Hom}}(\mathbf{S}_{\mathrm{tor}}, \mathbf{G}_{\mathrm{m}})$, and $E_{\mathrm{free}} := \underline{\mathrm{Hom}}(\mathbf{S}_{\mathrm{free}}, \mathbf{G}_{\mathrm{m}})$ denote the group schemes of multiplicative type of finite type over $\mathrm{Spec}(\mathbb{Z})$. Then E_{free} is a split torus, and E_{tor} is finite and of multiplicative type.

For each $y \in Y$ and each $\chi \in X$, the pullback $(c^\vee(y), c(\chi))^* \mathcal{P}_B$ is an invertible sheaf over S. The (functorial) biextension structure of $\mathcal{P}_B$, the symmetry of $(\mathrm{Id}_B \times \lambda_B)^* \mathcal{P}_B$, and the compatibility $c\phi = \lambda_B c^\vee$, allow us to associate a well defined invertible sheaf $\Psi(\ell)$ with each $\ell \in \mathbf{S}$, satisfying the following properties:

(1) If $\ell = [y \otimes \chi]$, the image of $y \otimes \chi \in Y \underset{\mathbb{Z}}{\otimes} X$ under the canonical morphism $Y \underset{\mathbb{Z}}{\otimes} X \to \mathbf{S}$, then there is a canonical isomorphism $\Psi(\ell) \cong (c^\vee(y), c(\chi))^* \mathcal{P}_B$.

(2) For $\ell, \ell' \in \mathbf{S}$, we have a canonical isomorphism $\Delta^*_{\ell,\ell'} : \Psi(\ell) \underset{\mathscr{O}_S}{\otimes} \Psi(\ell') \cong \Psi(\ell + \ell')$ of invertible sheaves over S.

(3) The collection of isomorphisms $\Delta^*_{\ell,\ell'}$ satisfy necessary conditions making the $\mathscr{O}_S$-module $\underset{\ell \in \mathbf{S}}{\oplus} \Psi(\ell)$ an $\mathscr{O}_S$-algebra, and so that

$$\Xi := \underline{\mathrm{Spec}}_{\mathscr{O}_S} \Big(\underset{\ell \in \mathbf{S}}{\oplus} \Psi(\ell) \Big)$$

has a canonical structure of an E-torsor over S.

(4) The same isomorphisms $\Delta^*_{\ell,\ell'}$ for $\ell, \ell' \in \mathbf{S}_{\mathrm{tor}}$ defines similarly an E_{tor}-torsor

$$\Xi_{\mathrm{tor}} := \underline{\mathrm{Spec}}_{\mathscr{O}_S} \Big(\underset{\ell \in \mathbf{S}_{\mathrm{tor}}}{\oplus} \Psi(\ell) \Big)$$

over S, together with a canonical (surjective) morphism $\Xi \to \Xi_{\mathrm{tor}}$ having the structure of an E_{free}-torsor. In particular, this morphism is smooth and of finite type.

By construction, the scheme $\Xi \to S$ is the universal space for trivializations $\mathbf{1}_{Y \times X} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_B^{\otimes -1}$ of biextensions with symmetric pullbacks under $\mathrm{Id}_Y \times \phi : Y \times Y \to Y \times X$, and with compatibility with $\mathcal{O}$-actions (but without the positivity condition). Let $\breve{\tau}$ be the universal object over Ξ. Then the universal property of $\Xi \to S$ determines a canonical morphism

$$\eta \to \Xi \tag{3.2}$$

lifting the canonical morphism $\eta \to S$, such that the trivialization $\tau : \mathbf{1}_{Y \times X, \eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}_{B,\eta}^{\otimes -1}$ over η is the pullback of the universal object $\breve{\tau}$ over Ξ.

3.3. **Chart for the datum τ with positivity condition.** Let $\mathbf{S}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}, \mathbb{Z})$ and $\mathbf{S}^\vee_{\mathbb{R}} := \mathbf{S}^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{R} \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbf{S}, \mathbb{R})$. By definition of $\mathbf{S}$, the $\mathbb{R}$-vector space $\mathbf{S}^\vee_{\mathbb{R}}$ is isomorphic to the space of symmetric pairings $(\,\cdot\,,\,\cdot\,) : (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}$ satisfying $(bx, y) = (x, b^\star y)$ for any $x, y \in Y \underset{\mathbb{Z}}{\otimes} \mathbb{R}$ and $b \in \mathcal{O}$. Then we define an element in $\mathbf{S}^\vee_{\mathbb{R}}$ to be *positive definite* if the associated pairing is.

A *cone* in $\mathbf{S}^\vee_{\mathbb{R}}$ is a subset stable under the natural multiplication action of the group $\mathbb{R}_{>0}$. A rational polyhedral cone in $\mathbf{S}^\vee_{\mathbb{R}}$ is a cone of the form $\sigma = \mathbb{R}_{>0} v_1 + \ldots + \mathbb{R}_{>0} v_n$ with $v_1, \ldots, v_n \in \mathbf{S}^\vee_{\mathbb{Q}} = \mathbf{S}^\vee \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. We say that σ is *nondegenerate* if its closure does not contain any nonzero $\mathbb{R}$-vector subspace. We say a nondegenerate σ is *smooth* if the $v_1, \ldots, v_n$ can be chosen to be part of a $\mathbb{Z}$-basis of $\mathbf{S}^\vee$. The canonical pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}} : \mathbf{S}\times\mathbf{S}^\vee\to\mathbb{Z}$ defines by extension of scalar a canonical pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}} : \mathbf{S}\times\mathbf{S}^\vee_{\mathbb{R}}\to\mathbb{R}$. For any rational polyhedral cone σ in $\mathbf{S}^\vee_{\mathbb{R}}$, we define its dual in $\mathbf{S}$ to be the semi-group (with unit 0)

$$\sigma^\vee := \{x\in\mathbf{S} : \langle x,y\rangle_{\mathrm{can.}}\geq 0, \forall y\in\sigma\},$$

and we define $\sigma_0^\vee$ to be the semi-subgroup (without unit 0) of $\sigma^\vee$ defined by

$$\sigma_0^\vee := \{x\in\mathbf{S} : \langle x,y\rangle_{\mathrm{can.}} > 0, \forall y\in\sigma\}.$$

Note that $\sigma^\vee$ always contains $\mathbf{S}_{\mathrm{tor}}$.

For each nondegenerate rational polyhedral cone σ in $\mathbf{S}^\vee_{\mathbb{R}}$, we define the affine toroidal embedding of Ξ along σ to be

$$\Xi(\sigma) := \underline{\mathrm{Spec}}_{\mathscr{O}_S}\Big(\underset{\ell\in\sigma^\vee}{\oplus}\Psi(\ell)\Big).$$

Here the $\mathscr{O}_S$-algebra structure of $\underset{\ell\in\sigma^\vee}{\oplus}\Psi(\ell)$ is induced by the same morphisms $\Delta^*_{\ell,\ell'}$ used in the definition Ξ. By construction, we have an open embedding $\Xi\hookrightarrow\Xi(\sigma)$ extending the action of E. Since $\sigma^\vee$ always contains $\mathbf{S}_{\mathrm{tor}}$, there is a canonical surjection $\Xi(\sigma)\to\Xi_{\mathrm{tor}}$. Then the usual theory (as in [8]) of toroidal embeddings for the torus $\mathbf{S}_{\mathrm{free}}$ implies that the morphism $\Xi(\sigma)\to\Xi_{\mathrm{tor}}$ is smooth if σ is smooth.

The scheme $\Xi(\sigma)$ has a natural closed subscheme Ξ_σ (often not having the same underlying topological space as the complement $\Xi(\sigma)-\Xi$), called the σ-stratum, defined by the $\mathscr{O}_S$-sheaf of ideals $\underset{\ell\in\sigma_0^\vee}{\oplus}\Psi(\ell)$ in $\underset{\ell\in\sigma^\vee}{\oplus}\Psi(\ell)$, and is mapped to itself under the action of E on $\Xi(\sigma)$. Let us define a subgroup $\sigma^\perp$ of $\mathbf{S}$ by

$$\sigma^\perp := \{x\in\mathbf{S} : \langle x,y\rangle_{\mathrm{can.}} = 0, \forall y\in\sigma\}.$$

Then $E_\sigma := \underline{\mathrm{Hom}}(\sigma^\perp,\mathbf{G}_{\mathrm{m}})$ is a quotient group of E. The induced action of E on Ξ_σ factors through E_σ, and makes $\Xi_\sigma\to S$ a torsor under E_σ. Since $\sigma^\perp$ contains $\mathbf{S}_{\mathrm{tor}}$, its torsion subgroup is exactly $\mathbf{S}_{\mathrm{tor}}$. Then the canonical morphism $\Xi_\sigma\to\Xi_{\mathrm{tor}}$ is surjective, and it is smooth because it is a torsor under the split torus $E_{\sigma,\mathrm{free}}$ with character group $\sigma^\perp_{\mathrm{free}} := \sigma^\perp/\mathbf{S}_{\mathrm{tor}}$.

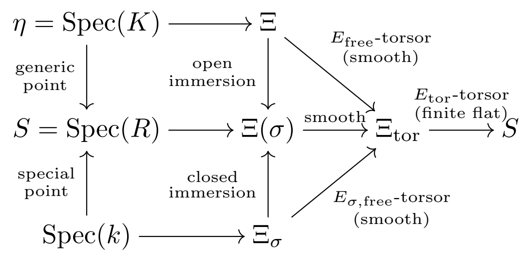
Let $\mathbf{P}^+$ be the cone in $\mathbf{S}^\vee_{\mathbb{R}}$ corresponding to positive definite pairings. Let υ denote the discrete valuation of K with valuation ring R. For each $y\in Y$ and $\chi\in X$, let $I_{y,\chi}$ be determined by the trivialization $\tau : \mathbf{1}_{Y\times X,\eta}\overset{\sim}{\to}(c^\vee\times c)^*\mathcal{P}^{\otimes -1}_{B,\eta}$ as in Section 2.2. Then the positivity of τ means $\upsilon(I_{y,\phi(y)})>0$ for any $y\neq 0$, and the association $(y,y')\mapsto\upsilon(I_{y,\phi(y')})\in\mathbb{Z}$ defines by extension of scalar a *positive definite* pairing $(Y\underset{\mathbb{Z}}{\otimes}\mathbb{R})\times(Y\underset{\mathbb{Z}}{\otimes}\mathbb{R})\to\mathbb{R}$, corresponding to an element b_τ in $\mathbf{P}^+$.

Let us take any *smooth* nondegenerate rational polyhedral cone σ in $\mathbf{P}^+$ that contains b_τ. Define the affine toroidal embedding $\Xi\hookrightarrow\Xi(\sigma)$ and the closed σ-stratum Ξ_σ of $\Xi(\sigma)$ over S as explained above. Then both $\Xi(\sigma)$ and Ξ_σ are smooth over Ξ_{tor}. Since σ contains b_τ, the morphism (3.2) extends to a morphism

$$S = \mathrm{Spec}(R)\to\Xi(\sigma) \tag{3.3}$$

mapping the special point $\mathrm{Spec}(k)$ of S to Ξ_σ.

Let us summarize the construction thus far in the following commutative diagram:



The composition of the arrows in the middle row is the identity morphism $\mathrm{Id}_S : S \to S$. (However, Ξ is not a scheme over η, and Ξ_σ is not a scheme over $\mathrm{Spec}(k)$. It is important that both Ξ and Ξ_σ are schemes over S.)

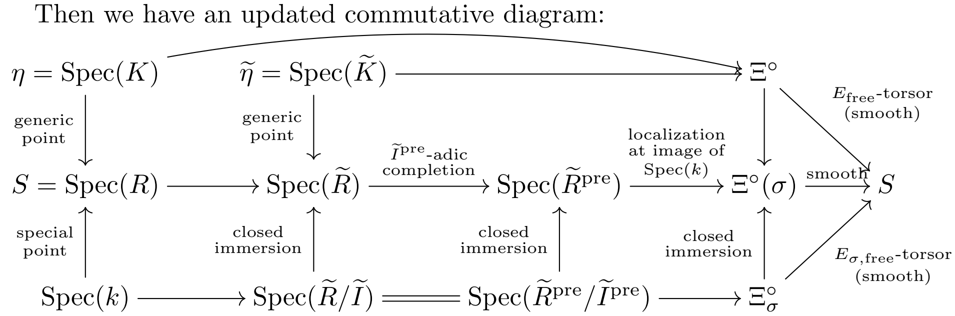
3.4. **Choice of the elevator.** The morphism $S \to \Xi(\sigma) \to \Xi_{\mathrm{tor}}$ induced by (3.3) gives in particular a section of the structural morphism $\Xi_{\mathrm{tor}} \to S$. Let us denote by Ξ°, $\Xi^\circ(\sigma)$, and Ξ°_σ the pullbacks of the smooth schemes Ξ, $\Xi(\sigma)$, and Ξ_σ over Ξ_{tor} along this section $S \to \Xi_{\mathrm{tor}}$. By abuse of notation, let us denote the pullback of the universal object $\breve{\tau}$ over Ξ to Ξ° by the same notation $\breve{\tau}$. Then the morphism (3.3) induces a morphism

$$S = \mathrm{Spec}(R) \to \Xi^\circ(\sigma) \tag{3.4}$$

mapping $\eta = \mathrm{Spec}(K)$ to Ξ° (resp. $\mathrm{Spec}(k)$ to Ξ°_σ), such that the trivialization $\tau : \mathbf{1}_{Y \times X, \eta} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}^{\otimes -1}_{B,\eta}$ over η is the pullback of $\breve{\tau}$ over Ξ°.

Let $\widetilde{R}^{\mathrm{pre}}$ be the local ring of $\Xi^\circ(\sigma)$ at the image of $\mathrm{Spec}(k) \to \Xi^\circ(\sigma)$. Let $\widetilde{I}^{\mathrm{pre}}$ be the ideal of definition of the σ-stratum of $\mathrm{Spec}(\widetilde{R}^{\mathrm{pre}})$ induced by the σ-stratum Ξ°_σ of $\Xi^\circ(\sigma)$. (The ideal $\widetilde{I}^{\mathrm{pre}}$ is not supposed to be the same as the maximal ideal defining the image of $\mathrm{Spec}(k)$.) Let $\widetilde{R}$ be the completion of $\widetilde{R}^{\mathrm{pre}}$ with respect to $\widetilde{I}^{\mathrm{pre}}$, and let $\widetilde{I}$ be the induced ideal of definition. Since $\Xi^\circ(\sigma)$ is smooth over $\mathrm{Spec}(R)$, the ring $\widetilde{R}$ is a noetherian normal domain. Then the data of $\widetilde{R}$ and $\widetilde{I}$ allow us to talk about the categories $\mathrm{DEG}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(\widetilde{R}, \widetilde{I})$ and $\mathrm{DD}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(\widetilde{R}, \widetilde{I})$. (See Section 2.2.) Let $\widetilde{K} := \mathrm{Frac}(\widetilde{R})$ and $\widetilde{\eta} := \mathrm{Spec}(\widetilde{K})$.

Then we have an updated commutative diagram:



Let $(\widetilde{B}, \lambda_{\widetilde{B}}, i_{\widetilde{B}}, X, Y, \phi, \widetilde{c}, \widetilde{c}^\vee)$ be the pullback of $(B, \lambda_B, i_B, X, Y, \phi, c, c^\vee)$ to $\widetilde{R}$, and let $\widetilde{\tau} : \mathbf{1}_{Y \times X, \widetilde{\eta}} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}^{\otimes -1}_{B,\widetilde{\eta}}$ be the pullback of $\breve{\tau}$ to $\widetilde{\eta}$. Then we obtain a tautological tuple $(\widetilde{B}, \lambda_{\widetilde{B}}, i_{\widetilde{B}}, X, Y, \phi, \widetilde{c}, \widetilde{c}^\vee, \widetilde{\tau})$ over $\widetilde{R}$, which defines an object in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(\widetilde{R}, \widetilde{I})$. Accordingly, Mumford's construction $\mathrm{M}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(\widetilde{R}, \widetilde{I})$ gives an object

$(\widetilde{A},\widetilde{\lambda},\widetilde{i})$ of $\mathrm{DEG}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(\widetilde{R},\widetilde{I})$. By functoriality of Mumford's construction, the pullback of $(\widetilde{A},\widetilde{\lambda},\widetilde{i}) \to \mathrm{Spec}(\widetilde{R})$ along the morphism $\mathrm{Spec}(R) \to \mathrm{Spec}(\widetilde{R})$ induced by (3.4) is isomorphic to $(A^\diamond,\lambda^\diamond,i^\diamond) \underset{R^\diamond}{\otimes} R \to \mathrm{Spec}(R)$.

Since Ξ°_σ is smooth over $S = \mathrm{Spec}(R)$, the σ-stratum of $\mathrm{Spec}(\widetilde{R})$ has at least one (functorial) point $\mathrm{Spec}(\widetilde{k})$ in characteristic zero. Since the character groups of the torus parts of $\widetilde{A}$ are constant and equal to X for points on the σ-stratum of $\mathrm{Spec}(\widetilde{R})$, the dimensions of the torus parts of $\widetilde{A}_{\widetilde{k}}$ and A_k are the same.

Thus, the data of $\mathrm{Spec}(R) \to \mathrm{Spec}(R^\diamond)$, $\mathrm{Spec}(R) \to \mathrm{Spec}(\widetilde{R})$, $\mathrm{Spec}(\widetilde{k}) \to \mathrm{Spec}(\widetilde{R})$, and $(\widetilde{A},\widetilde{\lambda},\widetilde{i}) \to \mathrm{Spec}(\widetilde{R})$ define an elevator for $(A^\diamond,\lambda^\diamond,i^\diamond) \to \mathrm{Spec}(R^\diamond)$ as in Definition 2.3. This completes the proof of Theorem 3.1.

4. Upper-bounds for maximal ranks of degenerations

4.1. **Application of existence of elevators.**

Theorem 4.1. *With the setting as in Theorem* 1.2*, we have the inequality* $r_p \leq r_0$ *for every* $p \geq 0$.

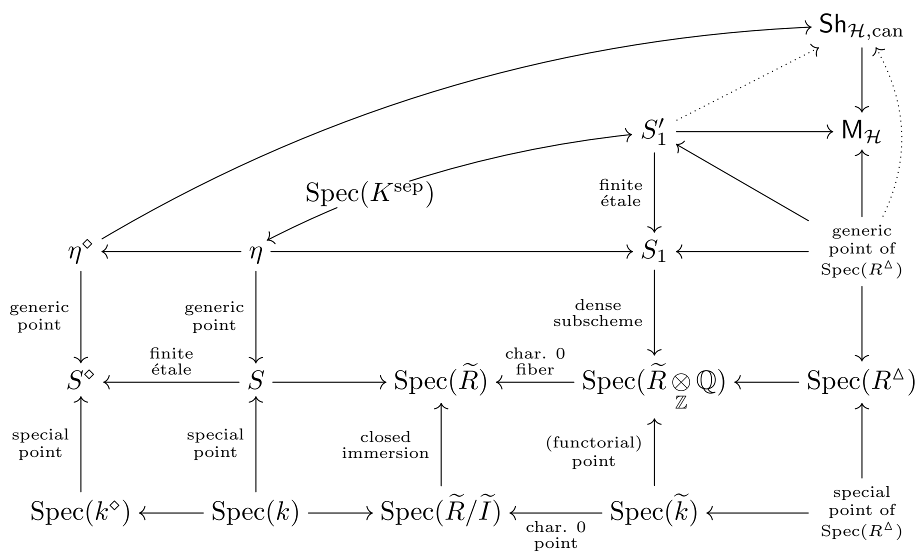
Proof. Suppose that there is a degeneration $A^\diamond$ of rank r over some discrete valuation ring $R^\diamond$ based at $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$. The induced morphism $\mathrm{Spec}(K^\diamond) \to \mathsf{Sh}_{\mathcal{H},\mathrm{can}} \hookrightarrow \mathsf{M}_{\mathcal{H}}$ defines an object $(A_1^\diamond,\lambda_1^\diamond,i_1^\diamond,\alpha_{\mathcal{H}}^\diamond)$ of $\mathsf{M}_{\mathcal{H}}(\mathrm{Spec}(K^\diamond))$, such that $A_1^\diamond$ is isomorphic to the pullback of $A^\diamond$ to $K^\diamond$. By a result of Raynaud [4, Ch. I, 2.8] (or [10, Prop. 3.3.1.7]), the additional structures $(\lambda_1^\diamond,i_1^\diamond)$ of $A_1^\diamond$ extend uniquely to structures $(\lambda^\diamond,i^\diamond)$ of $A^\diamond$ over $R^\diamond$. Then we have a degenerating family $(A^\diamond,\lambda^\diamond,i^\diamond) \to \mathrm{Spec}(R^\diamond)$ of type $(\mathrm{PE},\mathcal{O})$ extending $(A_1^\diamond,\lambda_1^\diamond,i_1^\diamond)$. Moreover, Theorem 3.1 guarantees the existence of a degenerating family $(\widetilde{A},\widetilde{\lambda},\widetilde{i}) \to \mathrm{Spec}(\widetilde{R})$ such that the pullback of $\widetilde{A}$ to $\mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ has a fiber of rank r.

Let S_1 be the maximal dense subscheme of $\mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ over which $\widetilde{A}$ is an abelian scheme. Let K^{sep} denote a separable closure of K. Then there exists an affine integral scheme S_1' finite étale over S_1, together with a morphism $\mathrm{Spec}(K^{\mathrm{sep}}) \to S_1'$ lifting $\mathrm{Spec}(K) \to S_1$, such that $(\widetilde{A}_{S_1'},\widetilde{\lambda}_{S_1'},\widetilde{i}_{S_1'})$ satisfies the Lie algebra condition given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ and is equipped with a level $\mathcal{H}$-structure $\widetilde{\alpha}_{\mathcal{H}}$ of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}, \langle\,\cdot\,,\,\cdot\,\rangle)$, and such that the pullbacks of $(\widetilde{A}_{S_1'},\widetilde{\lambda}_{S_1'},\widetilde{i}_{S_1'},\widetilde{\alpha}_{\mathcal{H}}) \to S_1'$ and $(A_1^\diamond,\lambda_1^\diamond,i_1^\diamond,\alpha_{\mathcal{H}}^\diamond) \to \mathrm{Spec}(K^\diamond)$ to $\mathrm{Spec}(K^{\mathrm{sep}})$ are isomorphic to each other. (See [10, Def. 1.3.4.2, Lem. 1.2.5.13, and Cor. 1.3.6.7].) Let $S_1' \to \mathsf{M}_{\mathcal{H}}$ be the morphism determined by $(\widetilde{A}_{S_1'},\widetilde{\lambda}_{S_1'},\widetilde{i}_{S_1'},\widetilde{\alpha}_{\mathcal{H}})$ by the universal property of $\mathsf{M}_{\mathcal{H}}$. By construction, the two compositions of morphisms $\mathrm{Spec}(K^{\mathrm{sep}}) \to S_1' \to \mathsf{M}_{\mathcal{H}}$ and $\mathrm{Spec}(K^{\mathrm{sep}}) \to \mathrm{Spec}(K) \to \mathrm{Spec}(K^\diamond) \to \mathsf{Sh}_{\mathcal{H},\mathrm{can}} \hookrightarrow \mathsf{M}_{\mathcal{H}}$ define the same point. Since S_1' is connected, the canonical morphism $S_1' \to \mathsf{M}_{\mathcal{H}}$ factors through $S_1' \to \mathsf{Sh}_{\mathcal{H},\mathrm{can}} \hookrightarrow \mathsf{M}_{\mathcal{H}}$.

Let $R^\triangle$ be a discrete valuation ring, with a morphism $\mathrm{Spec}(R^\triangle) \to \mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ whose restriction to the generic point factors through $S_1' \to \mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ and whose restriction to the special point factors through $\mathrm{Spec}(\widetilde{k}) \to \mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$. Such an $R^\triangle$ exists because $\mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ is noetherian and integral. Then the pullback of

$(\widetilde{A}, \widetilde{\lambda}, \widetilde{i}) \to \operatorname{Spec}(\widetilde{R})$ to $\operatorname{Spec}(R^{\Delta})$ is a degeneration of rank r based at $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$, showing that $r \leq r_0$ as desired. □

The relations among the base schemes involved in the proof of Theorem 4.1 can be summarized by the following commutative diagram (cf. the diagram following Theorem 3.1), in which the dotted arrows are induced by the solid arrows by the connectedness of S_1':



Corollary 4.2 (of Theorem 4.1)**.** *With the setting as in Theorem* 1.2*, if* $r_0 = 0$*, then* $r_p = 0$ *for every* $p \geq 0$*.*

4.2. **Morita's conjecture.** Yasuo Morita's original conjecture in [12], concerning potential good reduction everywhere of abelian varieties with PEL structures, can be reformulated in our language as follows:

Conjecture 4.3 (Yasuo Morita)**.** *With the setting as above, if the analytic space* $\mathsf{Sh}_{\mathcal{H}}$ *is compact, then* $r_p = 0$ *for every* $p \geq 0$*.*

Remark 4.4. By [3, 11.4 and 11.6], the assumption that the analytic space $\mathsf{Sh}_{\mathcal{H}}$ is compact is equivalent to the assumption that the group $\mathrm{G}(\mathbb{Q})$ contains no unipotent element other than the identity.

Let us begin with a simple reduction step:

Lemma 4.5. *The analytic space* $\mathsf{Sh}_{\mathcal{H}}$ *is compact if and only if the scheme* $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$ *is projective* (*and hence proper*) *over* $\operatorname{Spec}(F_0)$.

Proof. By [2, 10.11], $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}}$ is projective over $\operatorname{Spec}(\mathbb{C})$ if $\mathsf{Sh}_{\mathcal{H}}$ is compact. By [6, IV-3, 9.1.5], $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$ is projective over $\operatorname{Spec}(F_0)$ because $\mathsf{Sh}_{\mathcal{H},\mathrm{alg}} \cong \mathsf{Sh}_{\mathcal{H},\mathrm{can}} \underset{F_0}{\otimes} \mathbb{C}$. □

Theorem 4.6. *Conjecture* 4.3 *is true.*

Proof. By Lemma 4.5, we see that $r_0 = 0$ if the analytic space $\mathsf{Sh}_{\mathcal{H}}$ is compact. Then Conjecture 4.3 follows from Corollary 4.2. □

Remark 4.7. Although our proof of Conjecture 4.3 might appear to be simple, we would like to mention that our approach is (as yet) the only known one that does not involve assumptions of the following kinds:

(1) The size of L is *small* in the sense that $L \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ is (or is close to being) a simple module of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$. (See [12] or [9, end of §5] for more precise statements.)

(2) For some (archimedean or non-archimedean) place v of $\mathbb{Q}$, the group $\mathrm{G}(\mathbb{Q}_v)$ (or some analogue using some Mumford–Tate group) contains no unipotent element other than the identity. (See [12], [14], [16].)

Remark 4.8. Since the proof of [16] uses [4, Ch. IV, Thm. 6.7], which is in turn based on a delicate gluing argument beyond the application of the theory of degeneration data in [4, Ch. II and III], our approach is logically the simplest among the known ones on Morita's original conjecture in [12]. Nevertheless, we do not claim that our approach can be modified to tackle the stronger version of Morita's conjecture formulated using Mumford–Tate groups as in [14] and [16].

Let us include a criterion of properness strengthening [10, Thm. 5.3.3.1]:

Theorem 4.9. *Let $\mathcal{O}'_{F_0}$ be the localization of $\mathcal{O}_{F_0}$ at some (possibly empty) multiplicative subset of non-units. Let M' be a scheme separated of finite type over $\mathrm{Spec}(\mathcal{O}'_{F_0})$. Suppose the following conditions are satisfied:*

(1) *There is an open* **dense** *subscheme M'' of M' such that $\mathsf{M}'' \to \mathrm{Spec}(\mathcal{O}'_{F_0})$ is smooth.*

(2) *There exists a morphism $f : \mathsf{M}'_{F_0} := \mathsf{M}' \underset{\mathcal{O}'_{F_0}}{\otimes} F_0 \to \mathsf{M}_{\mathcal{H}}$ (over $\mathrm{Spec}(F_0)$) such that, for any complete discrete valuation ring $R^\diamond$ with fraction field $K^\diamond$ of characteristic zero, and with algebraically closed residue field, a morphism $\xi_1 : \mathrm{Spec}(K^\diamond) \to \mathsf{M}'_{F_0}$ defining an object $(A_1^\diamond, \lambda_1^\diamond, i_1^\diamond, \alpha_{\mathcal{H}}^\diamond)$ of $\mathsf{M}_{\mathcal{H}}(\mathrm{Spec}(K^\diamond))$ by composition with f extends to a morphism $\xi : \mathrm{Spec}(R^\diamond) \to \mathsf{M}'$ whenever the abelian scheme $A_1^\diamond$ extends to an abelian scheme over $\mathrm{Spec}(R^\diamond)$.*

(3) *The morphism f in (2) factors through some open and closed subscheme $\mathsf{M}^\circ \hookrightarrow \mathsf{M}_{\mathcal{H}}$.*

(4) *The scheme M° is proper over $\mathrm{Spec}(F_0)$.*

Then $\mathsf{M}' \to \mathrm{Spec}(\mathcal{O}'_{F_0})$ is proper.

Proof. To show that $\mathsf{M}' \to \mathrm{Spec}(\mathcal{O}'_{F_0})$ is proper, we need to verify the valuative criterion for it. By (1), it suffices to show that, for any $\mathrm{Spec}(R^\diamond) \to \mathrm{Spec}(\mathcal{O}'_{F_0})$ where $R^\diamond$ is a complete discrete valuation ring $R^\diamond$ with fraction field $K^\diamond$ (of characteristic zero) and with algebraically closed residue field $k^\diamond$, any morphism $\xi_1 : \mathrm{Spec}(K^\diamond) \to \mathsf{M}'_{F_0}$ extends to a morphism $\xi : \mathrm{Spec}(R^\diamond) \to \mathsf{M}'$.

By composition with f in (2), the morphism $\xi_1 : \mathrm{Spec}(K^\diamond) \to \mathsf{M}'_{F_0}$ induces a morphism $f \circ \xi_1 : \mathrm{Spec}(K^\diamond) \to \mathsf{M}_{\mathcal{H}}$ defining an object $(A_1^\diamond, \lambda_1^\diamond, i_1^\diamond, \alpha_{\mathcal{H}}^\diamond)$ of $\mathsf{M}_{\mathcal{H}}(\mathrm{Spec}(K^\diamond))$. Then, as in the first paragraph of the proof of Theorem 4.1, we have a degenerating family $(A^\diamond, \lambda^\diamond, i^\diamond)$ of type $(\mathrm{PE}, \mathcal{O})$ over $\mathrm{Spec}(R^\diamond)$ extending $(A_1^\diamond, \lambda_1^\diamond, i_1^\diamond)$.

By Theorem 3.1, we obtain a degenerating family $(\widetilde{A}, \widetilde{\lambda}, \widetilde{i})$ of type $(\mathrm{PE}, \mathcal{O})$ over $\widetilde{R}$, together with morphisms $\mathrm{Spec}(R) \to \mathrm{Spec}(R^\diamond)$ and $\mathrm{Spec}(R) \to \mathrm{Spec}(\widetilde{R})$ satisfying the additional properties in Definition 2.3. Let S_1 be the maximal dense subscheme of $\mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ over which $\widetilde{A}$ is an abelian scheme, and let K^{sep} denote a separable closure of K. As in the second paragraph of the proof of Theorem

4.1, there exists a finite étale morphism $S'_1 \twoheadrightarrow S_1$ from an *integral* scheme, such that $(\widetilde{A}_{S'_1}, \widetilde{\lambda}_{S'_1}, \widetilde{i}_{S'_1})$ is equipped with the additional structures defining a canonical morphism $S'_1 \to \mathsf{M}_{\mathcal{H}}$, and such that the composition of morphisms $\mathrm{Spec}(K^{\mathrm{sep}}) \to \mathrm{Spec}(K) \to \mathrm{Spec}(K^\diamond) \stackrel{f\circ\xi_1}{\to} \mathsf{M}_{\mathcal{H}}$ factors through this $S'_1 \to \mathsf{M}_{\mathcal{H}}$. Then $S'_1 \to \mathsf{M}_{\mathcal{H}}$ factors through $S'_1 \to \mathsf{M}^\circ \hookrightarrow \mathsf{M}_{\mathcal{H}}$ by (3) and the connectedness of S'_1.

Consider an arbitrary spectrum of a discrete valuation ring with generic point above S_1 and with special point above a point of $\mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$. The pullback of $(\widetilde{A}, \widetilde{\lambda}, \widetilde{i})$ to this spectrum defines a degeneration of an object of $\mathsf{M}^\circ \hookrightarrow \mathsf{M}_{\mathcal{H}}$. By the usual theory of Néron models, or by a result of Raynaud [4, Ch. I, 2.7] (or [10, Prop. 3.3.1.7]), the degeneration is uniquely determined by its generic fiber. By (4), such a degeneration has to be rank zero, which means its special fiber is above a point of S_1. Hence S_1 has to be the whole $\mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$.

Since there exists at least one point $\mathrm{Spec}(\widetilde{k}) \to \mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ such that the torus part of $\widetilde{A}_{\widetilde{k}}$ has the same dimension as the torus part of $A^\diamond_{k^\diamond}$, we see that the torus part of $A^\diamond_{k^\diamond}$ is trivial. This shows that $A^\diamond \to \mathrm{Spec}(R^\diamond)$ is an abelian scheme, and the theorem follows from (2). □

Corollary 4.10. *Let* M° *be an open and closed subscheme of* $\mathsf{M}_{\mathcal{H}}$. (*For example, we can take* M° *to be the image of the canonical embedding* $\mathsf{Sh}_{\mathcal{H},\mathrm{can}} \hookrightarrow \mathsf{M}_{\mathcal{H}}$.) *Consider the natural* (*quasi-finite*) *morphism from* M° *to a suitable coarse Siegel moduli scheme* (*with possibly non-principal polarizations*) *over* $\mathrm{Spec}(\mathbb{Z})$ (*constructed in* [13]). *Take the closure of the schematic image of this natural morphism, and take the normalization* M' *of this closure in the "function field" of* M°. (*We allow the "function field" to be a product of fields.*) *If* $\mathsf{M}^\circ \to \mathrm{Spec}(F_0)$ *is proper, then the scheme* M' *is* **projective** *over* $\mathrm{Spec}(\mathcal{O}_{F_0})$.

Proof. By construction, the conditions in Theorem 4.9 are all satisfied by M', with $\mathcal{O}'_{F_0} = \mathcal{O}_{F_0}$ (localizing at the empty set). Thus M' is proper over $\mathrm{Spec}(\mathcal{O}_{F_0})$ by Theorem 4.9, and hence projective over $\mathrm{Spec}(\mathcal{O}_{F_0})$ by quasi-projectivity of the coarse Siegel moduli scheme over $\mathrm{Spec}(\mathbb{Z})$. □

5. Lower-bounds for maximal ranks of degenerations

Since the ranks of fibers of a semi-abelian scheme are non-deceasing under specialization (by arguments using torsion points, as in [5, IX, 2.2.1 and 2.2.3]), it is natural to expect the following result:

Theorem 5.1. *With the setting as in Theorem* 1.2, *we have the equality* $r_0 \leq r_p$ *for every characteristic* $p \geq 0$.

To prove Theorem 5.1, it suffices to construct a degeneration of rank r_0 over a discrete valuation ring with residue characteristic p for each characteristic $p \geq 0$. Although alternative approaches might exist, we prefer to exploit an implication of the main results of [10], so that a construction similar to the one for elevators in Section 2.3 can be carried out. (However, we will only need the conclusion of [10] in *characteristic zero*. That is, although we will use the algebraic construction of the toroidal compactifications over $\mathrm{Spec}(F_0)$, we will never use their smooth integral models.)

5.1. **Boundary stratum of toroidal compactifications.** Let us fix a characteristic $p \geq 0$.

According to [10, Thm. 6.4.1.1 and Thm. 7.3.3.4], $\mathsf{M}_{\mathcal{H}} \to \mathrm{Spec}(F_0)$ admits a *projective smooth toroidal compactification* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} = \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}} \to \mathrm{Spec}(F_0)$, subject to some combinatorial choice of Σ, with a stratification by locally closed subschemes. The universal object over $\mathsf{M}_{\mathcal{H}}$ extends (after forgetting level structures) to a degenerating family (A, λ, i) of type $(\mathrm{PE}, \mathcal{O})$ over $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, and the restriction of A to each stratum of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ has constant (split) torus parts. Let $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}^{\mathrm{tor}}$ denote the schematic closure of $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$ in $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$. Since $\mathsf{Sh}_{\mathcal{H},\mathrm{can}} \hookrightarrow \mathsf{M}_{\mathcal{H}}$ is open and closed, and since $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is regular, $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}^{\mathrm{tor}} \hookrightarrow \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ is again open and closed. The stratification of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ thus induces a stratification of $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}^{\mathrm{tor}}$ with similar properties. By the semistable reduction theorem [4, Ch. I, 2.6] (or [10, Thm. 3.3.2.4]), the properness of $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}^{\mathrm{tor}} \to \mathrm{Spec}(F_0)$ shows that the maximal rank r_0 among degenerations over equicharacteristic zero discrete valuation rings based at $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$ is achieved over at least one (locally closed) stratum of $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}^{\mathrm{tor}}$. Moreover, since ranks of fibers of a semi-abelian scheme are non-deceasing under specialization (as mentioned above), we may require that this stratum is *closed* (and hence proper over $\mathrm{Spec}(F_0)$). Let us pick any such (closed) stratum Z° of $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}^{\mathrm{tor}}$, which is the intersection of $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}^{\mathrm{tor}}$ with a (locally closed) stratum Z of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$.

Proposition 5.2. *There exists a degeneration $(A^\circ, \lambda^\circ, i^\circ)$ over a complete discrete valuation ring R° based at $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$, which is up to isomorphism the image under $\mathrm{M}_{\mathrm{PE},\mathcal{O}}^{\mathrm{split}}(R^\circ)$ of some tuple $(B^\circ, \lambda_{B^\circ}, i_{B^\circ}, X, Y, \phi, c^\circ, c^{\circ\vee}, \tau^\circ)$ in $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}^{\mathrm{split}}(R^\circ)$, satisfying the following conditions:*

1. *There exists a morphism $S^\circ := \mathrm{Spec}(R^\circ) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, with generic point of S° mapped to $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$ and special point of S° mapped to Z°, such that $(A^\circ, \lambda^\circ, i^\circ)$ is isomorphic to the pullback of (A, λ, i) under this morphism. Then $\mathrm{rk}_{\mathbb{Z}} X = \mathrm{rk}_{\mathbb{Z}} Y = r_0$ according to the choice of Z°.*
2. *There exists a complete discrete valuation ring R of residue characteristic p, with fraction field $K := \mathrm{Frac}(R)$ the residue field of R°, together with a tuple $(B, \lambda_B, i_B, X, Y, \phi, c, c^\vee)$ forming part of an object in $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}^{\mathrm{split}}(R)$, such that*

$$(B^\circ, \lambda_{B^\circ}, i_{B^\circ}, X, Y, \phi, c^\circ, c^{\circ\vee}) \cong (B, \lambda_B, i_B, X, Y, \phi, c, c^\vee) \underset{R}{\otimes} R^\circ. \tag{5.3}$$

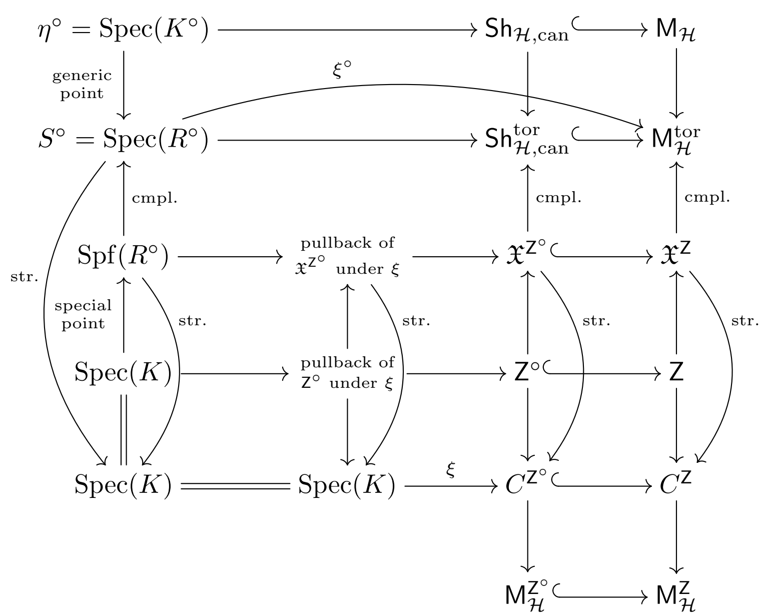
Proof. According to the construction in [10, Sec. 6.2], Z is canonically a torus-torsor over a scheme C^{Z} proper smooth (and surjective) over some moduli problem $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}}$ (whose definition is analogous to that of $\mathsf{M}_{\mathcal{H}}$). Moreover, the formal completion of $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ along Z is isomorphic to a formal scheme $\mathfrak{X}^{\mathsf{Z}}$ over C^{Z}. (To form the formal completion along a locally closed stratum, we first remove the other strata appearing in the closure of this stratum from the total space, and then form the formal completion of the remaining space along this stratum.) Then Z° is also a torus-torsor (under the same torus) over a scheme C^{Z° proper smooth (and surjective) over an open and closed subscheme $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}^\circ}$ of $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}}$, and the formal completion of $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}^{\mathrm{tor}}$ along Z° is isomorphic to a formal scheme $\mathfrak{X}^{\mathsf{Z}^\circ}$ over C^{Z°.

Let R be any complete discrete valuation ring with algebraically closed residue field of characteristic p, with (characteristic zero) fraction field $K = \mathrm{Frac}(R)$, such that there exists a morphism $\mathrm{Spec}(K) \to \mathsf{Z}^\circ$. This induces canonically a morphism $\xi : \mathrm{Spec}(K) \to C^{\mathsf{Z}^\circ}$.

Let R° be any complete discrete valuation ring with fraction field K° and residue field K, together with a morphism from $\mathrm{Spf}(R^\circ)$ to the pullback of $\mathfrak{X}^{\mathsf{Z}^\circ}$ under ξ, which defines canonically a morphism $\xi^\circ : \mathrm{Spec}(R^\circ) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ whose restriction to the generic point $\eta^\circ := \mathrm{Spec}(K^\circ)$ of $\mathrm{Spec}(R^\circ)$ factors through $\mathsf{Sh}_{\mathcal{H},\mathrm{can}} \hookrightarrow \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$, and whose restriction to the special point $\mathrm{Spec}(K)$ of $\mathrm{Spec}(R^\circ)$ factors through $\mathrm{Spec}(K) \to \mathsf{Z}^\circ \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$.

Let $(A^\circ, \lambda^\circ, i^\circ)$ be the pullback of (A, λ, i) under $\xi^\circ : \mathrm{Spec}(R^\circ) \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$. Since the torus part of the special fiber of A° is constant, $(A^\circ, \lambda^\circ, i^\circ)$ is up to isomorphism the image under $\mathrm{M}_{\mathrm{PE},\mathcal{O}}^{\mathrm{split}}(R^\circ)$ of some tuple $(B^\circ, \lambda_{B^\circ}, i_{B^\circ}, X, Y, \phi, c^\circ, c^{\circ\vee}, \tau^\circ)$ in $\mathrm{DD}_{\mathrm{PE},\mathcal{O}}^{\mathrm{split}}(R^\circ)$. Since the canonical morphism $\mathrm{Spf}(R^\circ) \to \mathfrak{X}^{\mathsf{Z}^\circ} \to C^{\mathsf{Z}^\circ}$ factors through the pullback of $\mathfrak{X}^{\mathsf{Z}^\circ}$ under ξ, by the universal property of C^{Z}, there exists a tuple $(B_K, \lambda_{B_K}, i_{B_K}, c_K, c_K^\vee)$ over K such that $(B_K, \lambda_{B_K}, i_{B_K}, c_K, c_K^\vee) \underset{K}{\otimes} R^\circ \cong (B^\circ, \lambda_{B^\circ}, i_{B^\circ}, c^\circ, c^{\circ\vee})$.

The relations among the base schemes involved in characteristic zero can be summarized in the following commutative diagram, in which "str." means structural morphisms and "cmpl." means formal completions:



Since $\mathsf{Z}^\circ \to \mathrm{Spec}(F_0)$ is proper, $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}^\circ} \to \mathrm{Spec}(F_0)$ is also proper. By Corollary 4.10, the abelian scheme B_K parameterized by $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}^\circ}$ has potential good reduction everywhere. In particular, it extends to an abelian scheme B over R. By a result of Raynaud [4, Ch. I, 2.8] (or [10, Prop. 3.3.1.7]), the additional structures (λ_{B_K}, i_{B_K}) of B_K extend uniquely to structures (λ_B, i_B) of B over R. Since X (resp. Y) is finitely generated, the morphism $c_K : X \to B_K^\vee$ (resp. $c_K^\vee : Y \to B_K$) is determined by finitely many $\mathrm{Spec}(K)$-valued points of $B_K^\vee$ (resp. B_K), and each of these points extends to a unique $\mathrm{Spec}(R)$-valued point of $B^\vee$ (resp. B) by properness of $B^\vee \to \mathrm{Spec}(R)$ (resp. $B \to \mathrm{Spec}(R)$). Thus we obtain a morphism $c : X \to B^\vee$ (resp. $c^\vee : Y \to B$) extending c_K (resp. $c_K^\vee$), and the tuple $(B, \lambda_B, i_B, X, Y, \phi, c, c^\vee)$ over R satisfies (5.3), as desired. $\square$

Remark 5.4. In the proof of Proposition 5.2, we referred to Corollary 4.10 rather than the seemingly more relevant Theorem 4.6, because $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}^\circ}$ might not be the Shimura variety (analogous to $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$) attached to the data defining the moduli problem $\mathsf{M}_{\mathcal{H}}^{\mathsf{Z}}$.

5.2. **Elevators in the opposite direction.** Let us keep the setting in Proposition 5.2. Let $S := \mathrm{Spec}(R)$ and $\eta := \mathrm{Spec}(K)$.

Using the tuple $(B, \lambda_B, i_B, X, Y, \phi, c, c^\vee) \to S$, let $\mathbf{S}$, $\mathbf{S}_{\mathrm{free}}$, $\mathbf{S}_{\mathrm{tor}}$, E, E_{free}, E_{tor}, Ξ, Ξ_{tor}, and $\breve{\tau}$ be constructed as in Section 3.2. Let $b_{\tau^\circ} \in \mathbf{P}^+$ be defined by τ° as in Section 3.3. Let us take any *top-dimensional* smooth nondegenerate rational polyhedral cone σ in $\mathbf{P}^+$ that contains b_{τ°, and construct $\Xi(\sigma)$ and Ξ_σ (as in Section 3.3) using the cone σ.

The degeneration data of $(A^\circ, \lambda^\circ, i^\circ)$ defines a canonical morphism $\mathrm{Spec}(R^\circ) \to \Xi(\sigma)$ mapping the special point $\eta = \mathrm{Spec}(K)$ of $\mathrm{Spec}(R^\circ)$ to the σ-stratum Ξ_σ. Since σ is top-dimensional, $\sigma^\perp = \mathbf{S}_{\mathrm{tor}}$ in $\mathbf{S}$. Hence the structural morphism $\Xi_\sigma \to \Xi_{\mathrm{tor}}$ is an isomorphism, and we obtain a morphism $\eta \to \Xi_{\mathrm{tor}}$ lifting the canonical morphism $\eta \to S$. Since $\Xi_{\mathrm{tor}} \to S$ is finite, the morphism $\eta \to \Xi_{\mathrm{tor}}$ extends to a section of the structural morphism $\Xi_{\mathrm{tor}} \to S$. By taking pullbacks along the section $S \to \Xi_\sigma$, we obtain schemes Ξ°, $\Xi^\circ(\sigma)$, and Ξ°_σ smooth over S as in Section 3.4, together with the universal object $\breve{\tau}$ over Ξ° by abuse of notation.

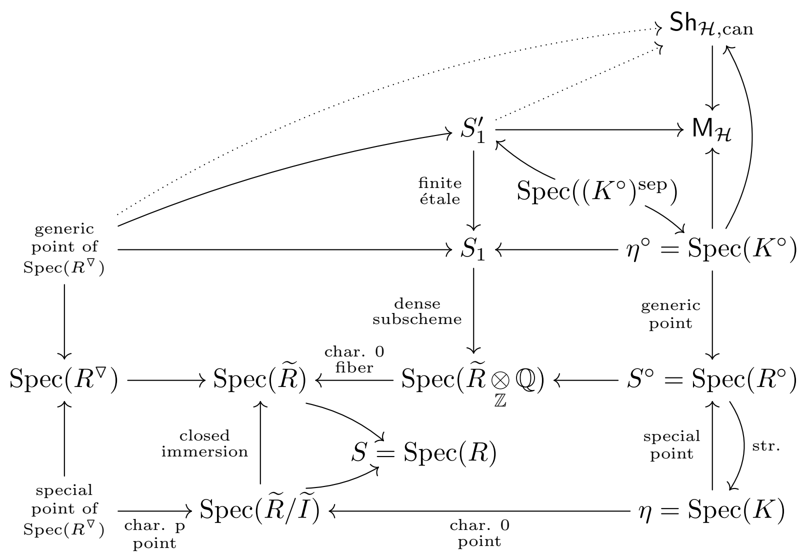
Let $\widetilde{R}^{\mathrm{pre}}$ be the noetherian normal domain underlying the affine scheme $\Xi^\circ(\sigma)$ (smooth over $S = \mathrm{Spec}(R)$), and let $\widetilde{I}^{\mathrm{pre}}$ be the ideal of definition of the σ-stratum Ξ°_σ of $\Xi^\circ(\sigma)$. Let $\widetilde{R}$ be the completion of $\widetilde{R}^{\mathrm{pre}}$ with respect to $\widetilde{I}^{\mathrm{pre}}$, and let $\widetilde{I}$ be the induced ideal of definition. Since $\Xi^\circ(\sigma)$ is smooth over $S = \mathrm{Spec}(R)$, $\widetilde{R}^{\mathrm{pre}}$ is excellent, and hence the ring $\widetilde{R}$ remains to be a noetherian normal domain. Then the data of $\widetilde{R}$ and $\widetilde{I}$ allow us to talk about the categories $\mathrm{DEG}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(\widetilde{R}, \widetilde{I})$ and $\mathrm{DD}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(\widetilde{R}, \widetilde{I})$. (See Section 2.2.) Let $\widetilde{K} := \mathrm{Frac}(\widetilde{R})$ and $\widetilde{\eta} := \mathrm{Spec}(\widetilde{K})$. Let $(\widetilde{B}, \lambda_{\widetilde{B}}, i_{\widetilde{B}}, X, Y, \phi, \widetilde{c}, \widetilde{c}^\vee)$ be the pullback of $(B, \lambda_B, i_B, X, Y, \phi, c, c^\vee)$ to $\widetilde{R}$, and let $\widetilde{\tau} : \mathbf{1}_{Y \times X, \widetilde{\eta}} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}^{\otimes -1}_{B,\widetilde{\eta}}$ be the pullback of $\breve{\tau}$ to $\widetilde{\eta}$. Then we obtain a tautological tuple $(\widetilde{B}, \lambda_{\widetilde{B}}, i_{\widetilde{B}}, X, Y, \phi, \widetilde{c}, \widetilde{c}^\vee, \widetilde{\tau})$ over $\widetilde{R}$, which defines an object in $\mathrm{DD}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(\widetilde{R}, \widetilde{I})$, and hence an object $(\widetilde{A}, \widetilde{\lambda}, \widetilde{i})$ of $\mathrm{DEG}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(\widetilde{R}, \widetilde{I})$ by Mumford's construction $\mathrm{M}^{\mathrm{split}}_{\mathrm{PE},\mathcal{O}}(\widetilde{R}, \widetilde{I})$.

Let S_1 be the maximal dense subscheme of $\mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$ over which $\widetilde{A}$ is an abelian scheme. By [4, Ch. III, Prop. 5.10 and Cor. 5.11] (or [10, Prop. 4.5.3.10 and Cor. 4.5.3.11]), $\widetilde{\tau} : \mathbf{1}_{Y \times X, \widetilde{\eta}} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}^{\otimes -1}_{B,\widetilde{\eta}}$ extends to a trivialization of biextensions $\tau_{S_1} : \mathbf{1}_{Y \times X, S_1} \xrightarrow{\sim} (c^\vee \times c)^* \mathcal{P}^{\otimes -1}_{B,S_1}$. By the above construction, there is a morphism from $S^\circ = \mathrm{Spec}(R^\circ)$ to $\mathrm{Spec}(\widetilde{R} \underset{\mathbb{Z}}{\otimes} \mathbb{Q})$, mapping the generic point $\eta^\circ = \mathrm{Spec}(K^\circ)$ to S_1, such that the pullback of the tuple $(\widetilde{B}, \lambda_{\widetilde{B}}, i_{\widetilde{B}}, X, Y, \phi, \widetilde{c}, \widetilde{c}^\vee, \tau_{S_1})$ to $\mathrm{Spec}(R^\circ)$ is isomorphic to $(B^\circ, \lambda_{B^\circ}, i_{B^\circ}, X, Y, \phi, c^\circ, c^{\circ\vee}, \tau^\circ)$. By functoriality of Mumford's construction, this shows that the pullbacks of $(\widetilde{A}, \widetilde{\lambda}, \widetilde{i}) \to \mathrm{Spec}(\widetilde{R})$ and $(A^\circ, \lambda^\circ, i^\circ) \to \mathrm{Spec}(R^\circ)$ to $\mathrm{Spec}(K^\circ)$ are isomorphic to each other. Let $(K^\circ)^{\mathrm{sep}}$ denote a separable closure of K°. Let $S'_1 \twoheadrightarrow S_1$ be a finite étale morphism from an *integral* scheme, such that the composition of canonical morphisms $\mathrm{Spec}((K^\circ)^{\mathrm{sep}}) \to \mathrm{Spec}(K^\circ) \to S_1$ lifts to a canonical morphism $\mathrm{Spec}((K^\circ)^{\mathrm{sep}}) \to S'_1$ (matching pullbacks of all tautological data). Then, as in the second paragraph of the proof of Theorem 4.1, the canonical morphism $S'_1 \to \mathsf{M}_{\mathcal{H}}$ factors through $S'_1 \to \mathsf{Sh}_{\mathcal{H},\mathrm{can}} \hookrightarrow \mathsf{M}_{\mathcal{H}}$.

5.3. **Ends of proofs.** Let $R^{\triangledown}$ be a mixed characteristic discrete valuation ring of residue characteristic p, with a morphism $\mathrm{Spec}(R^{\triangledown}) \to \mathrm{Spec}(\widetilde{R})$ whose restriction to the generic point factors through $S'_1 \to \mathrm{Spec}(\widetilde{R})$ and whose restriction to the special point factors through $\mathrm{Spec}(\widetilde{R}/\widetilde{I}) \to \mathrm{Spec}(\widetilde{R})$. Such an $R^{\triangledown}$ exists because $\mathrm{Spec}(\widetilde{R})$ is noetherian and integral. Since the pullback of $\widetilde{A}$ to $\mathrm{Spec}(\widetilde{R}/\widetilde{I})$ is the extension of the pullback of the abelian scheme B by the torus with constant character group X, the pullback of $\widetilde{A}$ to $\mathrm{Spec}(R^{\triangledown})$ is a degeneration of rank $\mathrm{rk}_{\mathbb{Z}} X = r_0$ based at $\mathsf{Sh}_{\mathcal{H},\mathrm{can}}$, showing that $r_0 \leq r_p$. This completes the proof of Theorem 5.1. □

Now Theorem 1.2 follows as a combination of Theorems 4.1 and 5.1. □

The relations among the base schemes involved in the proof of Theorem 5.1 can be summarized in the following commutative diagram (cf. the diagram following the proof of Theorem 4.1), in which the dotted arrows are induced by the solid arrows by the connectedness of S'_1:



(The arrows from $\mathrm{Spec}(\widetilde{R})$ and $\mathrm{Spec}(\widetilde{R}/\widetilde{I})$ to $S = \mathrm{Spec}(R)$ exist because of their construction; cf. the commutative diagram in Section 3.4.)

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