Geometric modular forms and the cohomology of torsion automorphic sheaves

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Abstract. In this survey article, we explain how modular forms can be defined geometrically in higher dimensions and in mixed characteristics using smooth toroidal compactifications, and how this can be useful for studying the cohomology groups of PEL-type Shimura varieties valued in torsion automorphic coefficients.

Contents

1. Background and introduction

Let us begin with the *proto-example*, namely the classical theory of modular curves and modular forms. (References for facts summarized here can be found in, for example, the survey articles in [**CY97**] and [**CSS97**], where one can find perhaps the most famous application of the theory of modular forms. I personally find the experience of reading [**Shi71**], [**Del71a**], [**DR73**], and [**KM85**] most helpful.)

Consider an integer $k \geq 1$ defining the "weight", and a congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ defining the "level". Let $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) be the space of holomorphic modular (resp. cusp) forms of weight k. Let $Y_{\Gamma,\mathbb{C}} = \Gamma\backslash\mathsf{H}$ ("open modular curve") be the quotient of the Poincaré upper half plane H, and let $X_{\Gamma,\mathbb{C}}$ ("compactified modular curve") be the compact Riemann surface containing $Y_{\Gamma,\mathbb{C}}$ (by adding the "cusps"). (More details will be given in §2.1.)

When $k \geq 2$, there is the so-called *Eichler-Shimura isomorphism*

$$M_k(\Gamma) \oplus S_k(\Gamma)^c \cong H^1(Y_{\Gamma,\mathbb{C}}, \mathrm{Sym}^{k-2}(\underline{\mathbb{C}}^{\oplus 2})), \tag{1.1}$$

where the superscript c means the complex conjugation, and where $\mathrm{Sym}^{k-2}(\underline{\mathbb{C}}^{\oplus 2})$ is the local system attached to the representation $\mathrm{Sym}^{k-2}(\mathbb{C}^{\oplus 2})$ of GL_2.

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By Serre duality (realized by integration as a $\mathbb{C}$-anti-linear isomorphism), we may rewrite (1.1) as

$$H^0(X_{\Gamma,\mathbb{C}}, \overline{\omega}_{\mathbb{C}}^{\otimes k}) \oplus H^1(X_{\Gamma,\mathbb{C}}, \overline{\omega}_{\mathbb{C}}^{\otimes(-k+2)}) \cong H^1(Y_{\Gamma,\mathbb{C}}, \mathrm{Sym}^{k-2}(\underline{\mathbb{C}}^{\oplus 2})), \tag{1.2}$$

where $\overline{\omega}_{\mathbb{C}}$ is a (holomorphic) line bundle over $X_{\Gamma,\mathbb{C}}$ canonically extending the familiar Hodge line bundle $\omega_{\mathbb{C}}$ over $Y_{\Gamma,\mathbb{C}}$, such that we can define geometrically $M_k(\Gamma) := H^0(X_{\Gamma,\mathbb{C}}, \overline{\omega}_{\mathbb{C}}^{\otimes k})$. Then the Eichler-Shimura isomorphism can be reinterpreted as a *degeneracy statement* in (mixed) Hodge theory. (Note that here $X_{\Gamma,\mathbb{C}}$ is compact and $Y_{\Gamma,\mathbb{C}}$ is non-compact. The use of nice compactifications is standard in mixed Hodge theory. See [**Del71b**] and [**PS08**].)

By the Betti-étale comparison, the right-hand side of (1.2) can be replaced with the (ℓ-adic) étale cohomology (after making the coefficient ℓ-adic using some fixed choice of an isomorphism $\mathbb{C} \cong \mathbb{Q}_\ell^{\mathrm{ac}}$, where $\mathbb{Q}_\ell^{\mathrm{ac}}$ is a fixed algebraic closure of the ℓ-adic numbers $\mathbb{Q}_\ell$), carrying interesting actions of both the Hecke algebra and the absolute Galois group (of the field of definition of $X_{\Gamma,\mathbb{C}}$). This leads to, among other things, the important association of Galois representations with (Hecke) eigenforms.

On the other hand, while one can ask many interesting arithmetic questions about $M_k(\Gamma)$ and $S_k(\Gamma)$, they do not carry interesting Galois actions. In fact, the geometric objects $Y_{\Gamma,\mathbb{C}}$, $X_{\Gamma,\mathbb{C}}$, $\omega_{\mathbb{C}}$, and $\overline{\omega}_{\mathbb{C}}$ all have very natural models over the integers, defined by moduli problems of elliptic curves with level structures, and by their degenerations. Using these *integral models*, people have been able to define and study the modular forms algebro-geometrically in *mixed characteristics*, and study interesting phenomena such as *congruences* among modular forms of very different natures (e.g., among cusp forms and Eisenstein series).

Now the question is what we can do in higher dimensions.

First of all, over $\mathbb{C}$, the modular curves are generalized by the so-called *Shimura varieties*, together with a theory of compactifications and automorphic bundles. (The group GL_2 is replaced with larger reductive algebraic groups, whose irreducible representations are used in the definition of such automorphic bundles.) The Eichler-Shimura isomorphism (1.2) has precise analogues given by the *dual BGG spectral sequences* of Faltings, with *degeneracy* due to mixed Hodge theory, relating *modular forms* on the left-hand sides to the *group cohomology* on the right-hand sides. Thanks to the recent advances in trace formula and related techniques, it is probably fair to say that people have made more progresses on the side of group cohomology (with coefficients over $\mathbb{C}$) than on the side of modular forms.

On the other hand, if we consider group cohomology with *torsion coefficients*, then known transcendental methods no longer apply, except that the (torsion version of) p-adic Hodge theory allow us to compare such group cohomology with log crystalline and log de Rham cohomology in mixed characteristics $(0, p)$, and that we still have some analogues of the dual BGG spectral sequences, from *modular forms* in mixed characteristics on the left-hand side to the *log de Rham cohomology* on the right-hand side. Now we might hope to answer questions about group cohomology with torsion coefficients by studying modular forms or related geometric objects.

In what follows, we will explain the following two topics in more detail:

(1) Firstly, we will explain how modular forms can be defined and studied in higher dimensions and in *good* mixed characteristics using *good integral models* of compactifications and automorphic bundles.

(2) Secondly, we will explain some effective conditions for the spaces of such modular forms to enjoy nice properties (such as vanishing or freeness over coefficients rings at particular cohomology degrees), and deduce corresponding results for group cohomology with torsion coefficients. (This is a joint work with Junecue Suh.)

We will provide references of both original and introductory natures. (For references to [**Lan08**], although we will use the numbering in the original version, the reader is advised to consult the errata and revision for corrections of typos and minor mistakes, and for improved exposition.)

2. Geometric modular forms in higher dimensions

2.1. Review of the theory in dimension one. Let us begin with the analytic definition of modular forms of one variable. (For more information, see [**Shi71**, Ch. 2].) The group $\mathrm{GL}_2(\mathbb{R})_+ := \{\gamma \in \mathrm{GL}_2(\mathbb{R}) : \det(\gamma) > 0\}$ acts on the Poincaré upper-half plane $\mathsf{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ by the familiar Möbius transformation $z \mapsto \gamma z := \frac{az+b}{cz+d}$, defined for any $z \in \mathsf{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R})_+$. Let $k \geq 1$ be an integer, and let Γ be a congruence subgroup (namely, defined by congruence conditions) of $\mathrm{SL}_2(\mathbb{Z})$.

Definition 2.1. *A (holomorphic)* **modular form** *of* **weight** $k \geq 1$ *and* **level** Γ *is a holomorphic function $f : \mathsf{H} \to \mathbb{C}$ satisfying the following two conditions:*

(1) (**automorphy condition**) *For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, we have the functional equation $f(z) = (cz+d)^{-k} f(\gamma z)$.*
(2) (**growth condition**) *For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the function $(cz+d)^{-k} f(\gamma z)$ stays bounded as $\mathrm{Im}(z) \to \infty$. (If $(cz+d)^{-k} f(\gamma z) \to 0$ as $\mathrm{Im}(z) \to \infty$, we say that f is a* **cusp form**.*)*

We shall denote by $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) the $\mathbb{C}$-vector space of modular forms (resp. cusp forms) of weight k and level Γ.

For many applications of modular forms to number theory, it is desirable to answer the following fundamental questions:

Question 2.2. *Can we define $M_k(\Gamma; R)$ and $S_k(\Gamma; R)$ (i.e., modular forms and cusp forms "over R") for rings R other than $\mathbb{C}$? Even better, can we make the definition functorial and compatible with arbitrary flat base changes in R?*

For example, we would like to define $M_k(\Gamma; \text{“}\mathbb{Z}\text{”})$ and $M_k(\Gamma; \text{“}\mathbb{F}_p\text{”})$, where "$\mathbb{Z}$" stands for some localization of the ring of integers in some number field at a prescribed set of primes, and where "$\mathbb{F}_p$" is a residue field of "$\mathbb{Z}$" of characteristic $p > 0$. Then we want $M_k(\Gamma) \cong M_k(\Gamma; \mathbb{C}) \cong M_k(\Gamma; \text{“}\mathbb{Z}\text{”}) \underset{\text{“}\mathbb{Z}\text{”}}{\otimes} \mathbb{C}$ and $S_k(\Gamma) \cong S_k(\Gamma; \mathbb{C}) \cong S_k(\Gamma; \text{“}\mathbb{Z}\text{”}) \underset{\text{“}\mathbb{Z}\text{”}}{\otimes} \mathbb{C}$, with the reduction maps $M_k(\Gamma; \text{“}\mathbb{Z}\text{”}) \to M_k(\Gamma; \text{“}\mathbb{F}_p\text{”})$ for each p.

Since modular forms are a priori defined by transcendental conditions, it is not clear how these questions should be answered. Thus the first step would be to revise the definition of modular forms, by introducing the so-called *modular curves*.

For each point $z \in \mathsf{H}$, we can define a lattice $L_z := \mathbb{Z}z + \mathbb{Z}$ in $\mathbb{C}$, and an elliptic curve $E_z := \mathbb{C}/L_z$. By varying z, we obtain a holomorphic family $E \to \mathsf{H}$ of elliptic curves. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, then $L_z = \mathbb{Z}z + \mathbb{Z} = \mathbb{Z}(az+b) + \mathbb{Z}(cz+d) = (cz+d)L_{\gamma z}$, and hence the multiplication $(cz+d)^{-1} : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$ defines an isomorphism $E_z = \mathbb{C}/L_z \xrightarrow{\sim} E_{\gamma z} = \mathbb{C}/L_{\gamma z}$. This shows that:

(1) If Γ is a *torsion-free* congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then $E \to \mathsf{H}$ descends to a holomorphic family $E_{\Gamma,\mathbb{C}} \to Y_{\Gamma,\mathbb{C}} := \Gamma\backslash\mathsf{H}$ of elliptic curves.

(2) If $\omega_{\Gamma,\mathbb{C}} := \underline{\mathrm{Lie}}^\vee_{E_{\Gamma,\mathbb{C}}/Y_{\Gamma,\mathbb{C}}}$ is the relative cotangent bundle along the identity section, then sections of $\omega_{\Gamma,\mathbb{C}}^{\otimes k}$ can be represented by holomorphic functions $f : \mathsf{H} \to \mathbb{C}$ satisfying $f(\gamma z) = (cz+d)^k f(z)$ for any $z \in \mathsf{H}$ and $\gamma \in \Gamma$.

Therefore, we see that the automorphy condition in Definition 2.1 can be redefined geometrically. To take care of the growth condition, let $X_{\Gamma,\mathbb{C}} := \Gamma\backslash\mathsf{H}^*$, where $\mathsf{H}^* = \mathsf{H} \cup \mathbb{P}^1(\mathbb{Q})$ (as subsets of $\mathbb{P}^1(\mathbb{C})$, with natural actions of $\mathrm{GL}_2(\mathbb{R})_+$ induced by the canonical action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{P}^1(\mathbb{C})$) is given a topology such that the quotient $X_{\Gamma,\mathbb{C}}$ has the structure of a compact Riemann surface, and such that $\omega_{\Gamma,\mathbb{C}}$ extends naturally to a line bundle $\overline{\omega}_{\Gamma,\mathbb{C}}$ over $X_{\Gamma,\mathbb{C}}$. We shall call $Y_{\Gamma,\mathbb{C}}$ (resp. $X_{\Gamma,\mathbb{C}}$) the *modular curve* (resp. *compactified modular curve*) of level Γ. By its very construction, $Y_{\Gamma,\mathbb{C}}$ parameterizes isomorphism classes of pairs (E, α_Γ) over $\mathbb{C}$, where E is an elliptic curve over $\mathbb{C}$, and α_Γ is a level Γ structure, namely a Γ-orbit of isomorphisms $\alpha : \mathbb{Z}^{\oplus 2} \xrightarrow{\sim} H_1(E,\mathbb{Z})$. Then we have canonical isomorphisms $M_k(\Gamma;\mathbb{C}) \cong H^0(X_{\Gamma,\mathbb{C}}, \overline{\omega}_{\Gamma,\mathbb{C}}^{\otimes k})$ and $S_k(\Gamma;\mathbb{C}) \cong H^0(X_{\Gamma,\mathbb{C}}, \overline{\omega}_{\Gamma,\mathbb{C}}^{\otimes k}(-\infty))$, where $(-\infty)$ means vanishing at the cusps $X_{\Gamma,\mathbb{C}} - Y_{\Gamma,\mathbb{C}}$.

In [**DR73**], Deligne and Rapoport defined the moduli problem of elliptic curves with level Γ structures over "$\mathbb{Z}$", which in this case can be $\mathbb{Z}[\frac{1}{N}]$ for any integer N such that Γ contains $\Gamma(N) := \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$, and showed that the moduli problem is representable by a scheme Y_Γ (over "$\mathbb{Z}$"). (Later in [**KM85**], by introducing the so-called *Drinfeld level structures*, Katz and Mazur could ensure that "$\mathbb{Z}$" is indeed $\mathbb{Z}$.) Moreover, Y_Γ has a natural compactification X_Γ (over "$\mathbb{Z}$") given by the moduli of generalized elliptic curves (which are certain degenerations of elliptic curves with ordinary double points as singularities) with additional structures. The universal elliptic curve $E_\Gamma \to Y_\Gamma$ extends to a smooth group scheme $E_\Gamma^{\mathrm{ext}} \to X_\Gamma$ given by the smooth part of the universal generalized elliptic curve, and hence the line bundle $\omega_\Gamma := \underline{\mathrm{Lie}}^\vee_{E_\Gamma/Y_\Gamma}$ over Y_Γ extends to the line bundle $\overline{\omega}_\Gamma := \underline{\mathrm{Lie}}^\vee_{E_\Gamma^{\mathrm{ext}}/X_\Gamma}$.

According to [**DR73**, Ch. VII, §4], the base extensions of the geometric objects Y_Γ, X_Γ, ω_Γ, and $\overline{\omega}_\Gamma$ from "$\mathbb{Z}$" to $\mathbb{C}$ are canonically isomorphic to $Y_{\Gamma,\mathbb{C}}$, $X_{\Gamma,\mathbb{C}}$, $\omega_{\Gamma,\mathbb{C}}$, and $\overline{\omega}_{\Gamma,\mathbb{C}}$, respectively, which induce canonical isomorphisms $M_k(\Gamma;\mathbb{C}) \cong H^0(X_{\Gamma,\mathbb{C}}, \overline{\omega}_{\Gamma,\mathbb{C}}^{\otimes k})$ and $S_k(\Gamma;\mathbb{C}) \cong H^0(X_{\Gamma,\mathbb{C}}, \overline{\omega}_{\Gamma,\mathbb{C}}^{\otimes k}(-\infty))$.

For any "$\mathbb{Z}$"-algebra R, let us denote the base extensions to R by subscripts (which is justified when $R = \mathbb{C}$ thanks to the previous paragraph). Then we can define modular forms and cusp forms of weight k and level Γ over R by

$$M_k(\Gamma, R) := H^0(X_{\Gamma,R}, \overline{\omega}_{\Gamma,R}^{\otimes k}) \tag{2.3}$$

and

$$S_k(\Gamma, R) := H^0(X_{\Gamma,R}, \overline{\omega}_{\Gamma,R}^{\otimes k}(-\infty)). \tag{2.4}$$

Moreover, for any ring homomorphism $R \to R'$, we have the desired functoriality $M_k(\Gamma, R) \to M_k(\Gamma, R')$ and $S_k(\Gamma, R) \to S_k(\Gamma, R')$, and for flat ring extensions $R \to R'$, we have $M_k(\Gamma, R) \underset{R}{\otimes} R' \xrightarrow{\sim} M_k(\Gamma, R')$ and $S_k(\Gamma, R) \underset{R}{\otimes} R' \xrightarrow{\sim} S_k(\Gamma, R')$.

Hence Question 2.2 has been answered.

2.2. Summary of key ingredients. Before moving on, let us summarize the key ingredients in the geometric definition of modular forms:

(1) The analytic definition of $Y_{\Gamma,\mathbb{C}}$ and $X_{\Gamma,\mathbb{C}}$ over $\mathbb{C}$.
(2) The *integral model* Y_Γ of $Y_{\Gamma,\mathbb{C}}$ defined using the moduli of elliptic curves with level Γ structures. (The precise ring of definition "$\mathbb{Z}$" depends on the definition of level Γ structures.)
(3) The integral model X_Γ of $X_{\Gamma,\mathbb{C}}$ defined using degenerations of elliptic curves into curves with ordinary double points as singularities.
(4) The definition of $\omega_{\Gamma,\mathbb{C}}$ using the holomorphic family $E_{\Gamma,\mathbb{C}} \to Y_{\Gamma,\mathbb{C}}$.
(5) The definition of ω_Γ using the universal elliptic curve $E_\Gamma \to Y_\Gamma$.
(6) The definition of $\overline{\omega}_\Gamma$ using the extended object $E_\Gamma^{\mathrm{ext}} \to X_\Gamma$. (This uses implicitly some compactified object $\overline{E}_\Gamma \to X_\Gamma$.)
(7) The definitions of $M_k(\Gamma; R)$ and $S_k(\Gamma; R)$ (in (2.3) and (2.4)) using H^0 of tensor powers of $\overline{\omega}_\Gamma$ (with vanishing along the cusps in the case of S_k).

In the following subsections, we will discuss generalizations of these one by one.

2.3. Shimura varieties and their compactifications over the complex numbers. The modular curve $Y_{\Gamma,\mathbb{C}}$ over $\mathbb{C}$ is generalized by the *Shimura varieties*.

One starts with a *Shimura datum* (G, X), where G is a reductive algebraic group over $\mathbb{Q}$ and X is a finite union of *Hermitian symmetric domains* carrying an action of $\mathrm{G}(\mathbb{R})$ and satisfying some conditions. (We will not make these conditions precise because we will not need them.) Let H be an open compact subgroup of $\mathrm{G}(\mathbb{A}^\infty)$. (Here $\mathbb{A}^\infty = \hat{\mathbb{Z}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$, where $\hat{\mathbb{Z}} = \varprojlim_N (\mathbb{Z}/N\mathbb{Z})$.) Then the Shimura variety at level H attached to this datum is the double quotient

$$\mathsf{Sh}_{H,\mathbb{C}} := \mathrm{G}(\mathbb{Q})\backslash(\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/H). \tag{2.5}$$

A priori, this is only a complex analytic space, which is a finite union of quotients of X by arithmetic subgroups of $\mathrm{G}(\mathbb{Q})$. (The finiteness of the union is due to the finiteness of $\mathrm{G}(\mathbb{Q})\backslash\mathrm{G}(\mathbb{A}^\infty)/H$, by [**Bor63**, Thm. 5.1].)

According to Satake and Baily-Borel [**BB66**, 10.11], there is a *minimal compactification* $\mathsf{Sh}_{H,\mathbb{C}}^{\min}$ of $\mathsf{Sh}_{H,\mathbb{C}}$, which is canonical and given by a finite union of complex normal projective varieties. According to Mumford and his coworkers [**AMRT75**], there is a collection of (non-canonical) nonsingular and/or projective compactifications $\mathsf{Sh}_{H,\mathbb{C}}^{\mathrm{tor}}$ in the category of complex algebraic spaces, called *toroidal compactifications*, parameterized by certain combinatorial data of compatible choices of cone decompositions, which resolve the (generally very complicated) singularities of the minimal compactification $\mathsf{Sh}_{H,\mathbb{C}}^{\min}$. Thus, $\mathsf{Sh}_{H,\mathbb{C}}$ is a nice geometric object with nice compactifications (in the category of algebraic varieties, if we only consider projective toroidal compactifications). This is important for defining and studying, for example, the Hodge structure on the de Rham cohomology. (Later in mixed characteristics even the very existence of nice compactifications is unclear, and we can say very little about relations among different compactifications, because we do not have analogues of Hironaka's embedded resolution of singularities [**Hir64a, Hir64b**].)

Moreover, by the theory of canonical models, $\mathsf{Sh}_{H,\mathbb{C}}$ has a canonical model Sh_H over a number field F_0 called the *reflex field*. (It is then customary to also call Sh_H the *Shimura variety* over F_0.) According to Harris [**Har89**] and Pink [**Pin89**], $\mathsf{Sh}_{H,\mathbb{C}}^{\min}$ and $\mathsf{Sh}_{H,\mathbb{C}}^{\mathrm{tor}}$ also have canonical models $\mathsf{Sh}_H^{\min}$ and $\mathsf{Sh}_H^{\mathrm{tor}}$, respectively, over (the same) F_0.

The upshot is that the reflex field depends on the Shimura datum (G, X) but *not* on the level H. As a result, the inductive limit of (ℓ-adic) étale cohomology groups of Sh_H (as the level H varies) carries both the Hecke action (defined by an action of $\mathrm{G}(\mathbb{A}^\infty)$ on the limit) and the Galois action (of $\mathrm{Gal}(\mathbb{Q}^{\mathrm{ac}}/F_0)$, where $\mathbb{Q}^{\mathrm{ac}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$), and these commute with each other. (Similar statements can be made about the intersection cohomology defined using the minimal compactifications.) This is why people look for relations among automorphic representations and Galois representations in the cohomology of Shimura varieties.

The theory of Shimura varieties has its origin in, as suggested by its name, the works of Shimura [**Shi02**]. (See also [**Shi71**] and [**Shi98**].) The prevailing formulation is due to Deligne, whose papers [**Del71a**] and [**Del79**] are the canonical references in this subject. A helpful introduction is [**Mil05**]. (For the purpose of this article, we will *not* need the theory of canonical models.)

The theory of compactifications of arithmetic quotients of Riemannian or Hermitian symmetric spaces is an important subject by itself. Apart from the minimal and toroidal compactifications, there are many other compactifications as topological spaces, often far beyond the category of algebraic varieties. (See Borel-Ji [**BJ06**] for a nice overall treatment on the subject.) We emphasize the minimal and toroidal compactifications because they are the only ones that turned out to have nice models over the reflex fields, and even over integers. (See §2.5 below.)

2.4. Integral models of Shimura varieties. Not all Shimura varieties are known to have nice integral models. (Here integral means defined over $\mathrm{Spec}(\mathcal{O}_{F_0}[\frac{1}{N}])$ for some explicit integer N.) Although by abstract nonsense algebraic varieties defined over number fields have *some* integral model over the ring of integers, such abstract models are not useful when we need to have precise control on the local structures. To the best of our knowledge, all useful integral models of Shimura varieties involve (either directly or indirectly) moduli spaces of abelian varieties with certain additional structures.

Among Shimura varieties that do possess useful integral models, the *PEL-type Shimura varieties* are those that admit interpretations as moduli spaces of abelian varieties with the *PEL structures*, namely the *polarizations* ("P"), the *endomorphism structures* ("E"), and the *level structures* ("L"). (This certainly has its origin in works of Shimura.) One reason for these structures to be useful is because it is easy to define moduli problems with these structures over very general rings. Definitions of moduli problems over "$\mathbb{Z}$" in the *good reduction case* (namely cases where we know the moduli problem is smooth over a base scheme with prescribed residue characteristics) can be found in Zink [**Zin82**, §1], Langlands-Rapoport [**LR87**, §6], and in utmost generality in Kottwitz [**Kot92**, §5], following earlier ideas of Grothendieck and Deligne (using objects *up to isogeny*). (I recommend reading the definition given by Kottwitz.)

The *bad reduction case* (namely cases where the model can have singularities in fibers of positive characteristics) can be very difficult in general, and the justification for the theory (i.e., "How bad can we allow the models to be?") may depend heavily on the applications. People might have the impression from low dimensions and a few very successful examples in arbitrary dimensions (such as [**HT01**]) that there should be some optimal theory, but we cannot rule out the possibility that different applications might deserve very different treatments. In what follows, we will leave the bad reduction case to specialists and focus mainly on the good reduction case.

For the purpose of compactifications later, we will prefer to define the moduli problems by parameterizing objects *up to isomorphism*. Since many readers today might be more familiar with the definition by isogeny classes, and since the definition by isomorphism classes is less canonical, we shall supply more detail. (The definition by isomorphism classes is certainly not new. This is the approach taken in many famous special cases, such as [**DR73, KM85, Rap78, DP94, MFK94, FC90, Lar88, Lar92**] and many later works.)

Let $(\mathcal{O}, \star, L, \langle\cdot\,,\cdot\rangle, h_0)$ be an *integral PEL datum* in the following sense:

(1) $\mathcal{O}$ is an order in a (nonzero) semisimple algebra, finite-dimensional over $\mathbb{Q}$, together with a positive involution *.
(2) L is an $\mathcal{O}$-lattice, namely a $\mathbb{Z}$-lattice with the structure of an $\mathcal{O}$-module.
(3) $\langle\cdot\,,\cdot\rangle : L \times L \to \mathbb{Z}(1)$ is an alternating pairing satisfying $\langle bx, y\rangle = \langle x, b^\star y\rangle$ for any $x, y \in L$ and $b \in \mathcal{O}$.
(4) $h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ is an $\mathbb{R}$-algebra homomorphism satisfying:
   (a) For any $z \in \mathbb{C}$ and $x, y \in L \underset{\mathbb{Z}}{\otimes} \mathbb{R}$, we have $\langle h_0(z)x, y\rangle = \langle x, h_0(z^c)y\rangle$, where $\mathbb{C} \to \mathbb{C} : z \mapsto z^c$ is the complex conjugation.
   (b) For any choice of $\sqrt{-1}$ in $\mathbb{C}$, the pairing $-\sqrt{-1}\,\langle\cdot\,, h_0(\sqrt{-1})\,\cdot\rangle$: $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \times (L \underset{\mathbb{Z}}{\otimes} \mathbb{R}) \to \mathbb{R}$ is symmetric and positive definite. (This last condition forces $\langle\cdot\,,\cdot\rangle$ to be nondegenerate.)

(In [**Lan08**, Def. 1.2.1.3], h_0 was denoted by h.) The tuple $(\mathcal{O},^\star, L, \langle\cdot\,,\cdot\rangle, h_0)$ then gives us an integral version of the $(B,^\star, V, \langle\cdot\,,\cdot\rangle, h_0)$ in [**Kot92**] and related works. We shall denote the center of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ by F. (Then F is a product of number fields.)

Definition 2.6 (cf. [**Lan08**, Def. 1.2.1.5]). *Let $\mathcal{O}$ and $(L, \langle\cdot\,,\cdot\rangle)$ be given as above. Then we define for any $\mathbb{Z}$-algebra R*

$$\mathrm{G}(R) := \left\{ \begin{array}{c} (g, r) \in \mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}R}(L \underset{\mathbb{Z}}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R) : \\ \langle gx, gy\rangle = r\langle x, y\rangle,\ \forall x, y \in L \underset{\mathbb{Z}}{\otimes} R \end{array} \right\}.$$

The assignment is functorial in R and defines a group functor G *over* $\mathrm{Spec}(\mathbb{Z})$*.*

The homomorphism $h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ defines a Hodge structure of weight -1 on L, with Hodge decomposition

$$L \underset{\mathbb{Z}}{\otimes} \mathbb{C} = V_0 \oplus V_0^c, \tag{2.7}$$

such that $h_0(z)$ acts as $1 \otimes z$ on V_0, and as $1 \otimes z^c$ on V_0^c. Let F_0 be the reflex field, namely the field of definition of the *isomorphism class* of the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-module V_0. (See [**Kot92**, §5] and [**Lan08**, Def. 1.2.5.4]. Note that this does not mean there exists an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} F_0$-module whose base extension from F_0 to $\mathbb{C}$ is isomorphic to V_0.) We shall denote the ring of integers in F (resp. F_0) by $\mathcal{O}_F$ (resp. $\mathcal{O}_{F_0}$).

We say that a rational prime number $p > 0$ is *good* if it satisfies the following conditions (cf. [**Kot92**, §5] and [**Lan08**, Def. 1.4.1.1]):

(1) p is unramified in $\mathcal{O}$ (as in [**Lan08**, Def. 1.1.1.14]).
(2) $p \neq 2$ if $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ involves simple factors of type D (as in [**Lan08**, Def. 1.2.1.15]).
(3) The pairing $\langle\cdot\,,\cdot\rangle$ is perfect after base change to $\mathbb{Z}_p$.

Let us fix a choice of a good prime $p > 0$.

Let $\mathcal{H}$ be a *neat* open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^p)$. (See [**Pin89**, 0.6] or [**Lan08**, Def. 1.4.1.8] for the definition of neatness. The assumption of neatness corresponds to the assumption that the congruence subgroup Γ is torsion-free when defining modular curves of level Γ.)

By [**Lan08**, Def. 1.4.1.4] (with $\Box = \{p\}$ there), the data of $(L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ and $\mathcal{H}$ define a moduli problem $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, parameterizing tuples $(A, \lambda, i, \alpha_{\mathcal{H}})$ over schemes S over S_0 of the following form:

(1) $A \to S$ is an abelian scheme.
(2) $\lambda : A \to A^\vee$ is a polarization of degree prime to p.
(3) $i : \mathcal{O} \hookrightarrow \mathrm{End}_S(A)$ is an $\mathcal{O}$-endomorphism structure as in [**Lan08**, Def. 1.3.3.1].
(4) $\underline{\mathrm{Lie}}_{A/S}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}$-module structure given naturally by i satisfies the determinantal condition in [**Lan08**, Def. 1.3.4.1] given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$. (The idea and the formulation are due to [**Kot92**, §5] and [**RZ96**, 3.23(a)], respectively.)
(5) $\alpha_{\mathcal{H}}$ is an (integral) level-$\mathcal{H}$ structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p, \langle\,\cdot\,,\,\cdot\,\rangle)$ as in [**Lan08**, Def. 1.3.7.8]. (In general this condition is nontrivial even if $\mathcal{H} = \mathrm{G}(\hat{\mathbb{Z}}^p)$ is maximal, unlike in certain famous special cases.)

(The definition can be identified with the one in [**Kot92**, §5] by [**Lan08**, Prop. 1.4.3.3].) By [**Lan08**, Thm. 1.4.1.12 and Cor. 7.2.3.10], $\mathsf{M}_{\mathcal{H}}$ is representable by a (smooth) quasi-projective scheme over S_0 (under the assumption that $\mathcal{H}$ is neat). (The proof of quasi-projectivity uses results of Moret-Bailly [**MB85**], but does not require Mumford's geometric invariant theory [**MFK94**]. It is intriguing and somewhat mysterious, but certainly no surprise, that both involve algebraic theta functions.)

Consider the (real analytic) set $\mathsf{X} = \mathrm{G}(\mathbb{R})h_0$ of $\mathrm{G}(\mathbb{R})$-conjugates $h : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ of $h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$. Let $H^p := \mathcal{H}$ and $H_p := \mathrm{G}(\mathbb{Z}_p)$ be open compact subgroups of $\mathrm{G}(\hat{\mathbb{Z}}^p)$ and $\mathrm{G}(\mathbb{Q}_p)$, respectively, and let H be the open compact subgroup $H^p H_p$ of $\mathrm{G}(\hat{\mathbb{Z}})$. Define $\mathsf{Sh}_{H,\mathbb{C}}$ by forming the double quotient as in (2.5), which we also view (by abuse of language) as a finite union of quasi-projective varieties (using Baily and Borel's theorem [**BB66**, 10.11]).

For any $h \in \mathsf{X}$, the real analytic torus $A_h := (L \underset{\mathbb{Z}}{\otimes} \mathbb{R})/L$ with complex structure given by h has the structure of a polarized abelian variety with endomorphism and level structures defining a $\mathbb{C}$-valued point of $\mathsf{M}_{\mathcal{H}}$. More generally, any element of $\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)$ defines some $\mathbb{C}$-valued point of $\mathsf{M}_{\mathcal{H}}$, and two elements define the same point if they lie in the same double orbit (for $\mathrm{G}(\mathbb{Q})$ and H). In other words, $\mathsf{Sh}_{H,\mathbb{C}}$ can be viewed as an analytic moduli space of abelian varieties with PEL structures defined by $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$. By arguing carefully as in [**Kot92**, §8] and [**Lana**, §2], one can show that there is a canonical isomorphism between $\mathsf{Sh}_{H,\mathbb{C}}$ and an *open and closed* subscheme of $\mathsf{M}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathbb{C}$. (In general, $\mathsf{Sh}_{H,\mathbb{C}}$ might not be isomorphic to the whole of $\mathsf{M}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathbb{C}$, due to the so-called *failure of Hasse's principle*. See for example [**Kot92**, §8] and [**Lan08**, Rem. 1.4.3.11].)

2.5. Integral models of compactifications. Just as the compactification X_Γ of Y_Γ (see §2.1) is constructed using moduli of generalized elliptic curves (which are certain degenerations of elliptic curves), to compactify integral models of PEL-type Shimura varieties, we would like to study the degeneration of the objects parameterized by the corresponding moduli problems.

Based on the work of Mumford [**Mum72**], Faltings and Chai (see [**Fal85**], [**Cha85**], and especially [**FC90**]) studied the theory of *semi-abelian* degenerations for polarized abelian varieties over noetherian normal complete adic rings satisfying certain reasonable conditions, and constructed smooth integral models of (smooth) toroidal compactifications of the Siegel modular varieties (parameterizing principally polarized abelian schemes with principal level structures, over base schemes over which the primes dividing the level are invertible).

In their construction, they did not use any moduli problem of degenerating objects. Instead, they glue boundary charts (whose construction depends on choices of cone decompositions) to the moduli problem in the étale topology. Such a process is feasible because the sheaves of relative log differentials can be explicitly calculated and compared over the charts. As a byproduct, they obtained integral models of the minimal compactifications of Siegel modular varieties by taking the projective spectra of certain graded algebras of (scalar-valued) algebraic automorphic forms. (Unlike in the theory over $\mathbb{C}$, we do not know any direct construction of the minimal compactifications.) The theory is generalized in Larsen's thesis [**Lar88**] (see also [**Lar92**]) for certain Picard modular varieties.

In my thesis [**Lan08**], I constructed smooth integral models of (smooth) compactifications of all types of PEL-type Shimura varieties (as defined in §2.4 above), based on a generalization of the method of Faltings and Chai's in [**FC90**] with a new emphasis on the degeneration of *level structures*. (As remarked above, the definition of level structures in general is nontrivial even if $\mathcal{H} = \mathrm{G}(\hat{\mathbb{Z}}^p)$ is maximal.) This involves some calculation of Weil pairings that has been unnecessary in earlier works.

The case of Siegel modular varieties with bad reductions of parahoric levels at p is treated in the thesis of Stroh [**Str08**] (see also [**Str10a, Str10b**]). It should be possible to treat similar parahoric levels for most symplectic and unitary cases by combining Stroh's ideas with ours. (Maybe Stroh will carry this out.) It is less clear what one should expect in cases of deeper levels, because (without applications in mind) it is already unclear how the integral models should be defined.

There are also the important *canonical compactifications* constructed by Alexeev and Nakamura [**AN99, Ale02**] and by Olsson [**Ols08**], which indeed use some *moduli problem* of degenerating objects. However, while such compactifications are very interesting for algebraic geometers, they have no known applications to the study of automorphic representations and related topics in number theory. (Although it might seem a disadvantage that the old-fashioned toroidal compactifications are noncanonical and have to be constructed by gluing boundary charts, it is also an advantage that we have a very precise description of the boundary!)

In what follows we will focus on good reduction models of (smooth) toroidal compactifications (which will eventually be the only ones useful for the strategy in §3). Let us summarize some results in [**Lan08**, Ch. 6–7].

Theorem 2.8 (see [**Lan08**, Thm. 6.4.1.1, 7.2.4.1, and 7.3.3.4] for more details). *When $\mathcal{H}$ is neat, $\mathsf{M}_{\mathcal{H}}$ admits a* **toroidal compactification** $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}} = \mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$*, a scheme* *projective and smooth over* S_0*, depending on a compatible collection* Σ (*of the so-called* **cone decompositions**) *that is* **projective** *and* **smooth** *in the sense of* [**Lan08**, Def. 6.3.3.2 and 7.3.1.3]. (*By abuse of language we sometimes simply call* Σ *a cone decomposition, even though it is technically a compatible collection of cone decompositions along various boundary components.*) *It satisfies the following properties:*

(1) *The universal abelian scheme* $A \to \mathsf{M}_{\mathcal{H}}$ *extends to a semi-abelian scheme* $A^{\mathrm{ext}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$*, the polarization* $\lambda : A \to A^\vee$ *extends to a prime-to-p isogeny* $\lambda^{\mathrm{ext}} : A^{\mathrm{ext}} \to (A^{\mathrm{ext}})^\vee$ *between semi-abelian schemes, and the endomorphism structure* $i : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathsf{M}_{\mathcal{H}}}(A)$ *extends to an endomorphism structure* $i^{\mathrm{ext}} : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{ext}}}(A^{\mathrm{ext}})$. (*Because the base is normal, once the semi-abelian extension* $A^{\mathrm{ext}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *exists, it is unique, and the remaining structures extend uniquely. See* [**FC90**, Ch. I, Prop. 2.7].)

(2) *The complement of* $\mathsf{M}_{\mathcal{H}}$ *in* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ (*with its reduced structure*) *is a relative Cartier divisor* $\mathsf{D} = \mathsf{D}_{\infty,\mathcal{H}}$ *with* **simple** *normal crossings, and the intersections of components of* D *define a stratification of* $\mathsf{M}_{\mathcal{H}}$ *with locally closed strata given by schemes smooth over* S_0.

(3) *There is an* **extended Kodaira-Spencer isomorphism** *from a quotient of* $\underline{\mathrm{Lie}}^\vee_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}}{\otimes} \underline{\mathrm{Lie}}^\vee_{(A^{\mathrm{ext}})^\vee/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ (*by precise relations defined by* λ^{ext} *and* i^{ext}) *to*

$$\overline{\Omega}^1_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}/\mathsf{S}_0} := \Omega^1_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}/\mathsf{S}_0}(\log \mathsf{D}) = \Omega^1_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}/\mathsf{S}_0}[\mathrm{d}\log \mathsf{D}],$$

the sheaf of modules of log 1-differentials on $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *over* S_0*, with respect to the relative Cartier divisor* D.

(4) *Let* $\omega := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^\vee_{A^{\mathrm{ext}}/\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$. *Then the scheme* $\mathrm{Proj}(\underset{r\geq 0}{\oplus} \Gamma(\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}, \omega^{\otimes r}))$ *is normal and projective over* S_0*, contains* $\mathsf{M}_{\mathcal{H}}$ *as an open dense subscheme, and defines the* **minimal compactification** $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ *of* $\mathsf{M}_{\mathcal{H}}$ (*independent of the choice of* Σ). *Moreover, the line bundle* ω *descends to an* **ample line bundle** *over* $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$.

(5) $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *is the normalization of the blow-up of* $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ *along a coherent sheaf of ideals* $\mathcal{J}$ *of* $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}}$ *whose pullback* $\jmath$ *to* $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}$ *is of the form* $\mathscr{O}_{\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}}(-\mathsf{D}')$*, for some relative Cartier divisor* D' *with normal crossings on* $\mathsf{M}_{\mathcal{H}}^{\mathrm{tor}}$ *such that* $\mathsf{D}'_{\mathrm{red}} = \mathsf{D}$. *In particular:*

$$\exists\, r_0 > 0 \text{ such that } \omega^{\otimes r}(-\mathsf{D}') \text{ is ample for every } r \geq r_0. \tag{2.9}$$

(*Here we use* [**Lan08**, Thm. 7.3.3.4] *and the assumption that* Σ *is projective.*)

In what follows, we shall sometimes omit Σ when the choice is clear.

Let Sh_H denote the schematic image of the canonical morphism $\mathsf{Sh}_{H,\mathbb{C}} \to \mathsf{M}_{\mathcal{H}}$. Let $\mathsf{M}_{\mathcal{H},0}$ (resp. $\mathsf{M}_{\mathcal{H},\Sigma,0}^{\mathrm{tor}}$) denote the schematic closure of Sh_H in $\mathsf{M}_{\mathcal{H}}$ (resp. $\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}}$). Then $\mathsf{M}_{\mathcal{H},0}$ is smooth over S_0, and $\mathsf{M}_{\mathcal{H},\Sigma,0}^{\mathrm{tor}} \to \mathsf{S}_0$ is proper and smooth and shares the properties of $\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}} \to \mathsf{S}_0$ listed above. By abuse of notation, we denote the pullback of D to $\mathsf{M}_{\mathcal{H},\Sigma,0}^{\mathrm{tor}}$ still by D. Similarly, let $\mathsf{M}_{\mathcal{H},0}^{\mathrm{min}}$ denote the schematic closure of Sh_H in $\mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$. Then $\mathsf{M}_{\mathcal{H},\Sigma,0}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H},0}^{\mathrm{min}}$ enjoys the same properties of $\mathsf{M}_{\mathcal{H},\Sigma}^{\mathrm{tor}} \to \mathsf{M}_{\mathcal{H}}^{\mathrm{min}}$ described in Theorem 2.8.

REMARK 2.10. Although not logically related, (2) of Theorem 2.8 asserts that our choices of cone decompositions are consistent with those called *SNC* in [**Har89**] and subsequent works such as [**HZ94a**], [**HZ94b**], and [**HZ01**].

In [**Lana**], by using comparison between spaces of analytic and algebraic theta functions, it is shown that $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,0} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathbb{C}$ is *indeed* a toroidal compactification $\mathsf{Sh}^{\mathrm{tor}}_{H,\mathbb{C}}$ of $\mathsf{Sh}_{H,\mathbb{C}} \cong \mathsf{M}_{\mathcal{H},0} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathbb{C}$ (constructed by the method of [**AMRT75**], [**Har89**], and [**Pin89**]), and as a consequence that $\mathsf{M}^{\mathrm{min}}_{\mathcal{H},0} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} \mathbb{C}$ is *the* minimal compactification $\mathsf{Sh}^{\mathrm{min}}_{H}$ (constructed by the method of [**BB66**] and [**Pin89**]). (We have to point out that the claims in certain works that this is true because the local charts of the analytic and algebraic constructions look similar, is not justified, because the construction methods are not logically related to each other.)

Thanks to its stratification by *smooth* locally closed subschemes, the (smooth) toroidal compactification $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,0}$ (with its long list of nice properties inherited from $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ in Theorem 2.8) is a nice geometric object suitable for the study of Hodge, de Rham, and crystalline cohomology. (We will see some applications in §3.)

An intriguing (yet less noticed) fact is that the construction of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,0}$ is also useful for *compact Shimura varieties*. (If a boundary stratum is empty in characteristic zero, it has to be empty in positive characteristic too.) This leads to:

COROLLARY 2.11. *If* $\mathsf{Sh}_{H,\mathbb{C}}$ (*with its real analytic structure inherited from* X) *is* **compact**, *then* $\mathsf{M}_{\mathcal{H},0}$ *is proper* (*and hence projective*) *over* S_0.

(It is possible to prove this using *only* part of Faltings and Chai [**FC90**], without the gluing construction of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$. See [**Lanb**, §4].)

2.6. Automorphic bundles and canonical extensions over the complex numbers. For simplicity, let us maintain the PEL-type setup in this subsection.

Consider the subgroup $\mathrm{P}_{0,\mathbb{C}}$ of $\mathrm{G}_{\mathbb{C}}$ stabilizing the Hodge filtration given by the Hodge decomposition (2.7). (This $\mathrm{P}_{0,\mathbb{C}}$ will be compatible with the P_0 to be defined in §2.8 below.) The Hodge decomposition (2.7) itself then induces a splitting of the Levi quotient $\mathrm{M}_{0,\mathbb{C}}$ as a subgroup of $\mathrm{P}_{0,\mathbb{C}}$. By varying h in X, the varying Hodge filtration then defines an embedding of X in the flag variety $\mathrm{G}_{\mathbb{C}}/\mathrm{P}_{0,\mathbb{C}}$. By abuse of language, we shall also identify this flag variety with its complex points $\mathrm{G}(\mathbb{C})/\mathrm{P}_0(\mathbb{C})$. Given any algebraic representation $W_{\mathbb{C}}$ of $\mathrm{P}_{0,\mathbb{C}}$, the sheaf of holomorphic sections of

$$(\mathrm{G}(\mathbb{C}) \times W_{\mathbb{C}})/\mathrm{P}_0(\mathbb{C}) \to \mathrm{G}(\mathbb{C})/\mathrm{P}_0(\mathbb{C}) \tag{2.12}$$

defines a $\mathrm{G}(\mathbb{C})$-equivariant holomorphic vector bundle on $\mathrm{G}(\mathbb{C})/\mathrm{P}_0(\mathbb{C})$, whose restriction to X descends to its (smooth) arithmetic quotients and defines a holomorphic vector bundle $\underline{W}_{\mathbb{C}}$ on $\mathsf{Sh}_{H,\mathbb{C}}$. The (holomorphic) vector bundles like $\underline{W}_{\mathbb{C}}$ are generalizations of the line bundles $\omega_{\mathbb{C}}^{\otimes k}$ in §1.

On the other hand, given any algebraic representation $V_{\mathbb{C}}$ of $\mathrm{G}_{\mathbb{C}} := \mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$, one can consider the sheaf of locally constant (resp. holomorphic) sections of

$$\mathrm{G}(\mathbb{Q})\backslash((\mathsf{X} \times V_{\mathbb{C}}) \times \mathrm{G}(\mathbb{A}^{\infty})/H) \to \mathsf{Sh}_{H,\mathbb{C}} = \mathrm{G}(\mathbb{Q})\backslash(\mathsf{X} \times \mathrm{G}(\mathbb{A}^{\infty})/H), \tag{2.13}$$

which we denote by ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ (resp. $\underline{V}_{\mathbb{C}}$). (Here "B" means the Betti version of the automorphic sheaves we consider.) The cohomology of ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ can be computed by the de Rham cohomology (and hence can be computed using the Lie algebra cohomology, see [**BW00**, Ch. VII]). Moreover, $\underline{V}_{\mathbb{C}}$ is naturally equipped with an integrable connection $\nabla : \underline{V}_{\mathbb{C}} \to \underline{V}_{\mathbb{C}} \underset{\mathscr{O}_{\mathsf{Sh}_{H,\mathbb{C}}}}{\otimes} \Omega^1_{\mathsf{Sh}_{H,\mathbb{C}}}$ (inducing the algebraic de Rham complex $(\underline{V}_{\mathbb{C}} \underset{\mathscr{O}_{\mathsf{Sh}_{H,\mathbb{C}}}}{\otimes} \Omega^\bullet_{\mathsf{Sh}_{H,\mathbb{C}}}, \nabla))$, such that ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ is canonically isomorphic to the sheaf of horizontal sections of $(\underline{V}_{\mathbb{C}}, \nabla)$, and so that we have

$$H^i(\mathsf{Sh}_{H,\mathbb{C}}, {}_{\mathrm{B}}\underline{V}_{\mathbb{C}}) \cong H^i_{\mathrm{dR}}(\mathsf{Sh}_{H,\mathbb{C}}, \underline{V}_{\mathbb{C}}) := H^i(\mathsf{Sh}_{H,\mathbb{C}}, (\underline{V}_{\mathbb{C}} \underset{\mathscr{O}_{\mathsf{Sh}_{H,\mathbb{C}}}}{\otimes} \Omega^\bullet_{\mathsf{Sh}_{H,\mathbb{C}}}, \nabla)).$$

(The last term is a hypercohomology of complexes.) The sheaves ${}_{\mathrm{B}}\underline{V}_{\mathbb{C}}$ are generalizations of the sheaves $\mathrm{Sym}^{k-2}(\underline{\mathbb{C}}^{\oplus 2})$ in §1.

However, for obtaining mixed Hodge structures on $H^i_{\mathrm{dR}}(\mathsf{Sh}_{H,\mathbb{C}}, \underline{V}_{\mathbb{C}})$, so that an analogue of (1.2) is possible, it is desirable to introduce the so-called *canonical extensions* of both bundles like $\underline{V}_{\mathbb{C}}$ and $\underline{W}_{\mathbb{C}}$ over toroidal compactifications of $\mathsf{Sh}_{H,\mathbb{C}}$. (See [**Mum77**], [**FC90**, Ch. VI], [**Har89**], [**Har90**], and [**Mil90**]. See in particular [**Har89**, Thm. 4.2] for the relation between canonical extensions and the notion of *regular singularities* of algebraic differential equations in [**Del70**] and [**Kat71**].) We shall denote canonical extensions by the superscript "can".

For simplicity, assume that $\mathrm{G}_{\mathbb{C}}$ has no type D factors (so that it is connected, and so that its irreducible representations are uniquely determined by their highest weights; otherwise, we need to group together several highest weights sharing the same irreducible representation, as in [**LSb, LSc**]). By choosing a suitable common maximal torus of $\mathrm{G}_{\mathbb{C}}$ and $\mathrm{M}_{0,\mathbb{C}}$, and by choosing suitable Borel subgroups of $\mathrm{G}_{\mathbb{C}}$ and $\mathrm{M}_{0,\mathbb{C}}$, we can compare the weights of $\mathrm{G}_{\mathbb{C}}$ and $\mathrm{M}_{0,\mathbb{C}}$ and assume that the dominant weights $\mathrm{X}^+_{\mathrm{G}_{\mathbb{C}}}$ for $\mathrm{G}_{\mathbb{C}}$ form a subset of the dominant weights $\mathrm{X}^+_{\mathrm{M}_{0,\mathbb{C}}}$ for $\mathrm{M}_{0,\mathbb{C}}$. Let $\mathrm{W}_{\mathrm{G}_{\mathbb{C}}}$ (resp. $\mathrm{W}_{\mathrm{M}_{0,\mathbb{C}}}$) denote the Weyl group of $\mathrm{G}_{\mathbb{C}}$ (resp. $\mathrm{M}_{0,\mathbb{C}}$), and let

$$\mathrm{W}^{\mathrm{M}_{0,\mathbb{C}}} := \{w \in \mathrm{W}_{\mathrm{G}_{\mathbb{C}}} : w(\mathrm{X}^+_{\mathrm{G}_{\mathbb{C}}}) \subset \mathrm{X}^+_{\mathrm{M}_{0,\mathbb{C}}}\}.$$

Let ρ be the half sum of positive roots of $\mathrm{G}_{\mathbb{C}}$, and let $w \cdot \mu = w(\mu + \rho) - \rho$ be the familiar dot action (respecting the infinitesimal weight of Harish-Chandra). Let $V_{\mu,\mathbb{C}}$ (resp. $W_{\nu,\mathbb{C}}$) denote the irreducible representation of $\mathrm{G}_{\mathbb{C}}$ (resp. $\mathrm{M}_{0,\mathbb{C}}$) of highest weight μ (resp. ν).

In [**Fal83**], using older ideas of Bernstein-Gelfand-Gelfand [**BGG75**], Faltings showed that there is the *dual BGG spectral sequence*, which is of the form

$$E_1^{a,i-a} := \bigoplus_{w \in \mathrm{W}^{\mathrm{M}_{0,\mathbb{C}}}} H^{i-l(w)}(\mathsf{Sh}^{\mathrm{tor}}_{H,\mathbb{C}}, \mathrm{Gr}^a_{\mathtt{F}}((\underline{W}^\vee_{w\cdot\mu,\mathbb{C}})^{\mathrm{can}})) \Rightarrow H^i_{\mathrm{dR}}(\mathsf{Sh}_{H,\mathbb{C}}, \underline{V}^\vee_{\mu,\mathbb{C}}). \tag{2.14}$$

The degeneracy of (2.14) is shown in certain special cases in [**Fal83**] (the anisotropic case) and [**FC90**] (the Siegel case), and in general in [**HZ01**, Cor. 4.2.3], all using some kinds of mixed Hodge theory.

We shall consider cohomology classes in $H^i(\mathsf{Sh}^{\mathrm{tor}}_{H,\mathbb{C}}, (\underline{W}_{\nu,\mathbb{C}})^{\mathrm{can}})$, where $i \in \mathbb{Z}$ and ν is any dominant weight of $\mathrm{M}_0(\mathbb{C})$, *modular forms* or rather *automorphic forms* of weight ν and level H (for our PEL datum). (If need be, we can also specify the cohomology degree i we are using.) Then (2.14) (with its degeneracy) asserts that there is a filtration on $H^i_{\mathrm{dR}}(\mathsf{Sh}_{H,\mathbb{C}}, \underline{V}^\vee_{\mu,\mathbb{C}})$ with graded pieces given by automorphic forms of some specific weights (and level H). This is the desired generalization of (1.1), or rather (1.2). (We encourage the reader to work out the details and understand why this generalizes (1.2), and what this does generalize.)

2.7. Automorphic bundles in mixed characteristics. In mixed characteristics, the integral models of Shimura varieties are *not* defined by quotients of symmetric spaces, and hence the definition of automorphic bundles will require more algebraic considerations. The idea we learned from Milne [**Mil90**] is that it is better to consider the so-called *principal bundles*, which are geometric objects which take care of the needed twisting without having to go through the quotients as in (2.13) and (2.12) for each individual representations. (The idea is then successfully applied to the mixed characteristics setup in, for example, [**MT02**].)

Let p be the good prime chosen in §2.4.

LEMMA 2.15. *There exists a finite extension F_0' of F_0 in $\mathbb{C}$, unramified at p, together with an $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathcal{O}_{F_0',(p)}$-module L_0 such that $L_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} \mathbb{C} \cong V_0$ as $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-modules.*

See [**Lan08**, Lem. 1.2.5.9 in the revision] for a proof. For each fixed F_0', the choice of L_0 is unique up to isomorphism.

Let us denote by $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}} : (L_0 \oplus L_0^\vee(1)) \times (L_0 \oplus L_0^\vee(1)) \to \mathcal{O}_{F_0',(p)}(1)$ (cf. [**Lan08**, Lem. 1.1.4.16]) the alternating pairing $\langle (x_1, f_1), (x_2, f_2)\rangle_{\mathrm{can.}} := f_2(x_1) - f_1(x_2)$. The natural right action of $\mathcal{O}$ on $L_0^\vee(1)$ defines a natural left action of $\mathcal{O}$ by composition with the involution $\star : \mathcal{O} \overset{\sim}{\to} \mathcal{O}$. Then (2.7) canonically induces an isomorphism $L_0^\vee(1) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \cong V_0^c$ of $\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathbb{C}$-modules.

DEFINITION 2.16. *For any $\mathcal{O}_{F_0',(p)}$-algebra R, set*

$$\mathrm{G}_0(R) := \left\{ \begin{array}{c} (g, r) \in \mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}R}((L_0 \oplus L_0^\vee(1)) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R) : \\ \langle gx, gy\rangle_{\mathrm{can.}} = r\langle x, y\rangle_{\mathrm{can.}},\ \forall x, y \in (L_0 \oplus L_0^\vee(1)) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R \end{array} \right\},$$

$$\mathrm{P}_0(R) := \left\{ (g, r) \in \mathrm{G}_0(R) : g(L_0^\vee(1) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R) = L_0^\vee(1) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R \right\},$$

$$\mathrm{M}_0(R) := \mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}R}(L_0^\vee(1) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R),$$

where we view $\mathrm{M}_0(R)$ canonically as a quotient of $\mathrm{P}_0(R)$ by

$$\mathrm{P}_0(R) \to \mathrm{M}_0(R) : (g, r) \mapsto (g|_{L_0^\vee(1) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R}, r).$$

The assignments are functorial in R, and define group functors G_0, P_0, and M_0 over $\mathrm{Spec}(\mathcal{O}_{F_0',(p)})$.

By [**Lan08**, Prop. 1.1.1.17, Cor. 1.2.5.7, and Cor. 1.2.3.10], there exists a discrete valuation ring R_1 over $\mathcal{O}_{F_0',(p)}$ satisfying the following conditions:

(1) The maximal ideal of R_1 is generated by p, and the residue field k_1 of R_1 is a *finite field* of characteristic p. In this case, the p-adic completion of R_1 is isomorphic to the Witt vectors $W(k_1)$ over k_1.
(2) The ring $\mathcal{O}_F$ is split over R_1, in the sense that $\mathrm{Hom}_{\mathbb{Z}\text{-alg.}}(\mathcal{O}_F, R_1)$ has cardinality $[F : \mathbb{Q}]$.
(3) There exists an isomorphism

$$(L \underset{\mathbb{Z}}{\otimes} R_1, \langle\,\cdot\,,\,\cdot\,\rangle) \cong (L_0 \oplus L_0^\vee(1), \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}) \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1 \tag{2.17}$$

inducing an isomorphism $\mathrm{G} \underset{\mathbb{Z}}{\otimes} R_1 \cong \mathrm{G}_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1$ realizing $\mathrm{P}_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1$ as a subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} R_1$. (The existence of the isomorphism (2.17) follows from [**Lan08**, Cor. 1.2.3.10] by comparing multi-ranks.)

Remark 2.18. For the purpose of studying arithmetic questions, it is often harmless (and helpful) to enlarge the coefficient ring.

From now on, let us fix the choices of R_1 and the isomorphism (2.17), and set $\mathcal{O}_{F,1} := \mathcal{O}_F \underset{\mathbb{Z}}{\otimes} R_1$, $\mathcal{O}_1 := \mathcal{O} \underset{\mathbb{Z}}{\otimes} R_1$, $L_1 := L \underset{\mathbb{Z}}{\otimes} R_1$, $L_{0,1} := L_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1$, $\mathrm{G}_1 := \mathrm{G}_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1 \cong \mathrm{G} \underset{\mathbb{Z}}{\otimes} R_1$, $\mathrm{P}_1 := \mathrm{P}_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1$, and $\mathrm{M}_1 := \mathrm{M}_0 \underset{\mathcal{O}_{F_0',(p)}}{\otimes} R_1$. We shall also denote base changes of geometric objects such as $\mathsf{M}_{\mathcal{H},0}$ (from $\mathcal{O}_{F_0,(p)}$) to R_1 by replacing 0 with 1.

Definition 2.19. *The* **principal** G_1**-bundle** *over* $\mathsf{M}_{\mathcal{H},1}$ *is the* G_1*-torsor*

$$\mathcal{E}_{\mathrm{G}_1} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}((\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}), \langle \cdot , \cdot \rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)), \\ ((L_{0,1} \oplus L_{0,1}^\vee(1)) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \langle \cdot , \cdot \rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1))),$$

the sheaf of isomorphisms of $\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}$*-sheaves of symplectic* $\mathcal{O}$*-modules. (The pairing* $\langle \cdot , \cdot \rangle_\lambda$ *on* $\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$ *is defined by the polarization* λ *as in* [**DP94**, 1.5]. *The group* G_1 *acts as automorphisms on* $(L \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \langle \cdot , \cdot \rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1))$ *by definition. The third entries in the tuples represent the values of the pairings. We allow isomorphisms of symplectic modules to modify the pairings up to units.)*

Definition 2.20. *The* **principal** P_1**-bundle** *over* $\mathsf{M}_{\mathcal{H},1}$ *is the* P_1*-torsor*

$$\mathcal{E}_{\mathrm{P}_1} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}((\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}), \langle \cdot , \cdot \rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}}), \\ ((L_{0,1} \oplus L_{0,1}^\vee(1)) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \langle \cdot , \cdot \rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), L_{0,1}^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}})),$$

the sheaf of isomorphisms of $\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}$*-sheaves of symplectic* $\mathcal{O}$*-modules with maximal totally isotropic* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R_1$*-submodules. (The sheaf* $\underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}}$ *is a subsheaf of* $\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$ *totally isotropic under the pairing* $\langle \cdot , \cdot \rangle_\lambda$. *The group* P_1 *acts as automorphisms on* $(L \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \langle \cdot , \cdot \rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), L_{0,1}^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}})$ *by definition.)*

These are torsors because $(\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}), \langle \cdot , \cdot \rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}})$ and $((L_{0,1} \oplus L_{0,1}^\vee(1)) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \langle \cdot , \cdot \rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), L_{0,1}^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}})$ are étale locally isomorphic by the theory of infinitesimal deformations (cf. for example [**Lan08**, Ch. 2]) and the theory of Artin's approximations (cf. [**Art69**, Thm. 1.10 and Cor. 2.5]).

Definition 2.21. *The* **principal** M_1**-bundle** *over* $\mathsf{M}_{\mathcal{H},1}$ *is the* M_1*-torsor*

$$\mathcal{E}_{\mathrm{M}_1} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}((\underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)), (L_{0,1}^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1))),$$

the sheaf of isomorphisms of $\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}$*-sheaves of* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R_1$*-modules. (We view the second entries in the pairs as an additional structure, inherited from* *the corresponding objects for* P_1*. The group* M_1 *acts as automorphisms on* $(L_{0,1}^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1))$ *by definition.)*

DEFINITION 2.22. *For any* R_1*-algebra* R*, we denote by* $\mathrm{Rep}_R(\mathrm{G}_1)$ *(resp.* $\mathrm{Rep}_R(\mathrm{P}_1)$*, resp.* $\mathrm{Rep}_R(\mathrm{M}_1)$*) the category of* **finite** R*-modules with algebraic actions of* $\mathrm{G}_1 \underset{R_1}{\otimes} R$ *(resp.* $\mathrm{P}_1 \underset{R_1}{\otimes} R$*, resp.* $\mathrm{M}_1 \underset{R_1}{\otimes} R$*).*

DEFINITION 2.23. *Let* R *be any* R_1*-algebra. For any* $W \in \mathrm{Rep}_R(\mathrm{G}_1)$*, we define*

$$\mathcal{E}_{\mathrm{G}_1,R}(W) := (\mathcal{E}_{\mathrm{G}_1} \underset{R_1}{\otimes} R) \overset{\mathrm{G}_1 \underset{R_1}{\otimes} R}{\times} W, \tag{2.24}$$

and call it the **automorphic sheaf** *over* $\mathsf{M}_{\mathcal{H},1} \underset{R_1}{\otimes} R$ *associated with* W*. It is called an* **automorphic bundle** *if* W *is locally free as an* R*-module. We define similarly* $\mathcal{E}_{\mathrm{P}_1,R}(W)$ *(resp.* $\mathcal{E}_{\mathrm{M}_1,R}(W)$*) for* $W \in \mathrm{Rep}_R(\mathrm{P}_1)$ *(resp.* $W \in \mathrm{Rep}_R(\mathrm{M}_1)$*) by replacing* G_1 *with* P_1 *(resp. with* M_1*) in the above expression* (2.24)*.*

EXAMPLE 2.25. We have $\mathcal{E}_{\mathrm{G}_1,R_1}(L_1) \cong \mathcal{E}_{\mathrm{P}_1,R_1}(L_1) \cong \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$, with Hodge filtration defined by the submodule $\mathcal{E}_{\mathrm{P}_1,R_1}(L_{0,1}^\vee(1)) \cong \mathcal{E}_{\mathrm{M}_1,R_1}(L_{0,1}^\vee(1)) \cong \underline{\mathrm{Lie}}_{A^\vee/\mathsf{M}_{\mathcal{H},1}}^\vee$, and with top graded piece $\mathcal{E}_{\mathrm{P}_1,R_1}(L_{0,1}) \cong \mathcal{E}_{\mathrm{M}_1,R_1}(L_{0,1}) \cong \underline{\mathrm{Lie}}_{A/\mathsf{M}_{\mathcal{H},1}}$.

The Hodge filtration on $\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$ can be (compatibly) generalized by defining a *Hodge filtration* $\mathtt{F}$ on any object $W \in \mathrm{Rep}_R(\mathrm{P}_1)$, which induces the Hodge filtration on $\mathcal{E}_{\mathrm{P}_1,R}(W)$, still denoted by $\mathtt{F}$. (For $W \in \mathrm{Rep}_R(\mathrm{G}_1)$ one considers $\mathcal{E}_{\mathrm{G}_1,R}(W) \cong \mathcal{E}_{\mathrm{P}_1,R}(W|_{\mathrm{P}_1})$; for $W \in \mathrm{Rep}_R(\mathrm{M}_1)$, the Hodge filtration on $\mathcal{E}_{\mathrm{M}_1,R}(W)$ is always canonically split.)

There is a canonical way to define the *Gauss-Manin connections*

$$\nabla : \mathcal{E}_{\mathrm{G}_1,R}(W) \to \mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$$

using the Gauss-Manin connection of $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$. The complex

$$(\mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$$

it induces is called the *de Rham complex* attached to $\mathcal{E}_{\mathrm{G}_1,R}(W)$. (Good places to learn about connections and de Rham complexes in the algebraic setup are [**KO68, Kat71, Kat70, Kat72**].)

2.8. Canonical extensions in mixed characteristics; compactifications of Kuga families. Our treatment of canonical extensions in mixed characteristics follow mainly [**FC90**, Ch. VI] and [**MT02**] (although we do have a different construction of (good toroidal) compactifications of Kuga families in [**Lanc**], using toroidal boundary strata of *larger* Shimura varieties).

Let $m \geq 0$ be any integer, and let $\mathsf{N}_m := A^m$ be the m-fold fiber product of $A \to \mathsf{M}_{\mathcal{H}}$. By [**Lanc**, Thm. 2.15], by taking $Q := \mathcal{O}^{\oplus m}$ there (cf. [**Lanc**, Ex. 2.2]), the abelian scheme $\mathsf{N}_m \to \mathsf{M}_{\mathcal{H}}$ (which we call a *Kuga family*) admits a collection of (non-canonical) toroidal compactifications $\mathsf{N}^{\mathrm{tor}}_{m,\kappa}$, indexed by a directed partially ordered set $\mathbf{K}_{m,\mathcal{H},\Sigma}$ of κ's, such that the (smooth) structural morphism $f_m : \mathsf{N}_m \to \mathsf{M}_{\mathcal{H}}$ extends to a proper *log smooth* morphism $f^{\mathrm{tor}}_{m,\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ for each $\kappa \in \mathbf{K}_{m,\mathcal{H},\Sigma}$. This collection $\{\mathsf{N}^{\mathrm{tor}}_{m,\kappa}\}_{\kappa \in \mathbf{K}_{m,\mathcal{H},\Sigma}}$ enjoys a long list of nice properties (see the statements of [**Lanc**, Thm. 2.15]); we will give precise references to them when needed.

By abuse of notation, we shall denote the pullbacks of $f^{\mathrm{tor}}_{m,\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ to $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,0}$ and $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ by $f^{\mathrm{tor}}_{m,\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ and $f^{\mathrm{tor}}_{m,\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$, respectively (and similarly over other base schemes).

Proposition 2.26. *The locally free sheaf $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})$ extends to a* **unique** *locally free sheaf $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$, satisfying the following properties:*

(1) *The sheaf $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}$, canonically identified as a subsheaf of the quasi-coherent sheaf $(\mathsf{M}_{\mathcal{H},1} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1})_*(\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1}))$, is self-dual under the pairing $(\mathsf{M}_{\mathcal{H},1} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1})_*(\langle\cdot,\cdot\rangle_\lambda)$. We shall denote the induced pairing by $\langle\cdot,\cdot\rangle^{\mathrm{can}}_\lambda$.*
(2) *$\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}$ contains $\underline{\mathrm{Lie}}^\vee_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}$ as a subsheaf that is totally isotropic under $\langle\cdot,\cdot\rangle^{\mathrm{can}}_\lambda$.*
(3) *The quotient sheaf $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}/\underline{\mathrm{Lie}}^\vee_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}$ can be canonically identified with the subsheaf $\underline{\mathrm{Lie}}_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}$ of $(\mathsf{M}_{\mathcal{H},1} \hookrightarrow \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1})_*\underline{\mathrm{Lie}}_{A/\mathsf{M}_{\mathcal{H},1}}$.*
(4) *The pairing $\langle\cdot,\cdot\rangle^{\mathrm{can}}_\lambda$ induces canonical an isomorphism $\underline{\mathrm{Lie}}_{A^{\mathrm{ext}}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}} \xrightarrow{\sim} \underline{\mathrm{Lie}}_{(A^{\mathrm{ext}})^\vee/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}$ which coincides with $\mathrm{d}\,\lambda^{\mathrm{ext}}$.*
(5) *Let*

$$\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}} := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}(\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}).$$

The Gauss-Manin connection of $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$ extends to an integrable connection

$$\nabla : \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}} \to \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}{\otimes} \overline{\Omega}^1_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}/\mathsf{S}_1}$$

with log poles along D, called the extended Gauss-Manin connection, compatible with the extended Kodaira-Spencer morphism in Theorem 2.8.

With these properties, we say that $(\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}, \nabla)$ is the **canonical extension** *of $(\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1}), \nabla)$.*

The locally free sheaf $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}$ (with all these stated properties) is unique once it exists. To show the existence, we use the morphism $f^{\mathrm{tor}}_{1,\kappa} : \mathsf{N}^{\mathrm{tor}}_{1,\kappa} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}$ for some $\kappa \in \mathbf{K}_{1,\mathcal{H},\Sigma}$ (i.e., $m = 1$ in above), and show that the $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}$-dual of the locally free sheaf $\underline{H}^1_{\text{log-dR}}(\mathsf{N}^{\mathrm{tor}}_{1,\kappa}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}) := R^1(f^{\mathrm{tor}}_{1,\kappa})_*(\overline{\Omega}^\bullet_{\mathsf{N}^{\mathrm{tor}}_{1,\kappa}/\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma}})$ satisfies all the properties of $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}$ stated in Proposition 2.26. (See [**Lanc**, proof of Prop. 6.9, based on Thm. 2.15] for details. The compactifications of Kuga families with $m \geq 1$ will be useful for other purposes as well. See §3.4 below.)

Remark 2.27. We formulated Proposition 2.26 in this somewhat axiomatic way to emphasize that any construction achieving these properties would serve the same purpose for the construction of canonical extensions (of automorphic bundles). Therefore, one can refer to [**FC90**, Ch. VI] and related works in special cases, without having to explain the consistency with [**Lanc**]. (This is desirable because the methods in [**FC90**, Ch. VI] and [**Lanc**] are different.)

Then the principal bundle $\mathcal{E}_{\mathrm{G}_1}$ (defined above in §2.7) extends canonically to a principal bundle $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1}$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$ by setting

$$\begin{aligned}(2.28)\quad \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1} := \underline{\mathrm{Isom}}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes}\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}((\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\mathrm{can}}, \langle\,\cdot\,,\,\cdot\,\rangle^{\mathrm{can}}_{\lambda}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}(1)),\\ ((L_{0,1}\oplus L^{\vee}_{0,1}(1))\underset{\mathcal{O}_{F_0,(p)}}{\otimes}\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}(1))).\end{aligned}$$

Similarly, the principal bundle $\mathcal{E}_{\mathrm{P}_1}$ (resp. $\mathcal{E}_{\mathrm{M}_1}$) extends canonically to a principal bundle $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1}$ (resp. $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_1}$) over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}$. As before, these are torsors by Artin's theory of approximations (cf. [**Art69**, Thm. 1.10 and Cor. 2.5]), because they have sections over completions of strict local rings.

Definition 2.29. *Let R be any R_1-algebra. For any $W \in \mathrm{Rep}_R(\mathrm{G}_1)$, we define*

$$(2.30)\qquad \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W) := (\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1}\underset{R_1}{\otimes} R) \overset{\mathrm{G}_1\underset{R_1}{\otimes}R}{\times} W,$$

called the **canonical extension** *of $\mathcal{E}_{\mathrm{G}_1,R}(W)$, and define accordingly $\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W) := \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W)\underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}}{\otimes}\mathscr{I}_{\mathsf{D}}$, called the* **subcanonical extension** *of $\mathcal{E}_{\mathrm{G}_1,R}(W)$, where $\mathscr{I}_{\mathsf{D}}$ is the $\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,1}}$-ideal defining the relative Cartier divisor D (in (2) of Theorem 2.8). Using the extended Gauss-Manin connection in Proposition 2.26, the Gauss-Manin connection of $\mathcal{E}_{\mathrm{G}_1,R}(W)$ extends (uniquely) to integral connections of $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W)$ and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(W)$ (with log poles along the boundary divisor D). We define similarly $\mathcal{E}^{\mathrm{can}}_{\mathrm{P}_1,R}(W)$, $\mathcal{E}^{\mathrm{sub}}_{\mathrm{P}_1,R}(W)$, $\mathcal{E}^{\mathrm{can}}_{\mathrm{M}_1,R}(W)$, and $\mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_1,R}(W)$ with G_1 (and its principal bundle) replaced accordingly with P_1 and M_1 (and their respective principal bundles).*

2.9. Geometric modular forms in higher dimensions. Now we are ready to give definitions of modular forms in higher dimensions and in mixed characteristics. We will call them *algebraic automorphic forms.*

Let R be any R_1-algebra. For any $W \in \mathrm{Rep}_R(\mathrm{G}_1)$, we can then define the graded modules of R-valued *algebraic automorphic forms of weight W* using cohomology of the coherent sheaf $\mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(W)$ over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R} := \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,0}\underset{\mathcal{O}_{F_0,(p)}}{\otimes} R$.

The theory is more useful if we have parameters for the weight modules W to be used. In mixed characteristics $(0,p)$, it is helpful to at least introduce the notion of *p-small weights*: (In what follows, we will use the obvious notations such as the weights $\mathrm{X}_{\mathrm{G}_1}$ and roots Φ_{G_1} for G_1, although we have not defined them formally. The dominant weights $\mathrm{X}^+_{\mathrm{G}_1}$ and $\mathrm{X}^+_{\mathrm{M}_1}$ will be chosen compatibly with each other, as in the case over $\mathbb{C}$.)

Definition 2.31. *We say $\mu \in \mathrm{X}^+_{\mathrm{G}_1}$ is p-***small** *if $(\mu+\rho,\alpha^{\vee}) \leq p$ for every $\alpha \in \Phi^+_{\mathrm{G}_1}$. We say $\mu \in \mathrm{X}^+_{\mathrm{M}_1}$ is p-***small** *if $(\mu+\rho,\alpha^{\vee}) \leq p$ for every $\alpha \in \Phi^+_{\mathrm{M}_1}$. We denote the subset of $\mathrm{X}^+_{\mathrm{G}_1}$ (resp. $\mathrm{X}^+_{\mathrm{M}_1}$) that are p-small by $\mathrm{X}^{<p}_{\mathrm{G}_1}$ (resp. $\mathrm{X}^{<p}_{\mathrm{M}_1}$).*

Since G_1 (resp. M_1) is split over R_1, there exists a split reductive algebraic group $\mathrm{G}_{\mathrm{split}}$ (resp. $\mathrm{M}_{\mathrm{split}}$) over $\mathbb{Z}$ such that $\mathrm{G}_1 \cong \mathrm{G}_{\mathrm{split}}\underset{\mathbb{Z}}{\otimes} R_1$ (resp. $\mathrm{M}_1 \cong \mathrm{M}_{\mathrm{split}}\underset{\mathbb{Z}}{\otimes} R_1$). Using the *Weyl modules* (over $\mathbb{Z}$), namely the span of a highest weight vector under the action of the group scheme and the distribution algebra over $\mathbb{Z}$, we can define canonically (by base extension from $\mathbb{Z}$ to R_1) the R_1-module V_μ (resp. W_ν) of highest weight $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$ (resp. $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$). We set $V_{\mu,R} := V_\mu \underset{R_1}{\otimes} R$ (resp.

$W_{\nu,R} := W_\nu \underset{R_1}{\otimes} R$) for any R_1-algebra R, and we set $\underline{V}_{\mu,R} := \mathcal{E}_{\mathrm{G}_1,R}(V_{\mu,R})$, $\underline{V}^{\mathrm{can}}_{\mu,R} := \mathcal{E}^{\mathrm{can}}_{\mathrm{G}_1,R}(V_{\mu,R})$, $\underline{V}^{\mathrm{sub}}_{\mu,R} := \mathcal{E}^{\mathrm{sub}}_{\mathrm{G}_1,R}(V_{\mu,R})$, $\underline{W}_{\nu,R} := \mathcal{E}_{\mathrm{M}_1,R}(W_{\nu,R})$, $\underline{W}^{\mathrm{can}}_{\nu,R} := \mathcal{E}^{\mathrm{can}}_{\mathrm{M}_1,R}(W_{\nu,R})$, and $\underline{W}^{\mathrm{sub}}_{\nu,R} := \mathcal{E}^{\mathrm{sub}}_{\mathrm{M}_1,R}(W_{\nu,R})$. (For more details concerning the Weyl modules, see [**LSb**, §2.6], its references to [**PT02**], and the references of [**PT02**] to other works. Here for ease of exposition we assumed that G_1 has no type D factors, but this assumption is not necessary.)

Definition 2.32. *Let $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$. Let R be any R_1-algebra. Consider the following graded modules of R-valued* **algebraic automorphic forms of weight** *ν:*

1. *$A^\bullet_{\nu,\mathrm{can}}(\mathcal{H};R) := H^\bullet(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R},\underline{W}^{\mathrm{can}}_{\nu,R})$. We call these forms* **canonical***.*
2. *$A^\bullet_{\nu,\mathrm{sub}}(\mathcal{H};R) := H^\bullet(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R},\underline{W}^{\mathrm{sub}}_{\nu,R})$. We call these forms* **subcanonical***.*
3. *$A^\bullet_{\nu,\mathrm{int}}(\mathcal{H};R) := \mathrm{image}(H^\bullet(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R},\underline{W}^{\mathrm{sub}}_{\nu,R}) \to H^\bullet(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},\Sigma,R},\underline{W}^{\mathrm{can}}_{\nu,R}))$. We call these forms* **interior***.*

(*The modifier "canonical" will often be suppressed.*) *In all three cases, the choice of Σ is immaterial* (*cf.* [**Lanc**, (4) of Thm. 2.15] *or rather* [**Lan08**, proof of Lem. 7.1.1.3])*.*

However, our terminology (canonical, subcanonical, and interior) are *not* standard; there do not seem to be standard names for these spaces, except when the cohomology degree is 0 or d (thanks to the prototypical case $R = \mathbb{C}$). In degree 0, forms in $A^0_\nu(\mathcal{H};R)_{\mathrm{can}}$ can be called *holomorphic*, while forms in $A^0_\nu(\mathcal{H};R)_{\mathrm{sub}}$ can be called *cuspidal holomorphic*. In degree d, forms in $A^d_\nu(\mathcal{H};R)_{\mathrm{sub}}$ can be called *anti-holomorphic* (thanks to Hodge theory over $\mathbb{C}$), while forms in $A^d_\nu(\mathcal{H};R)_{\mathrm{can}}$ can be called *cuspidal anti-holomorphic* (thanks to Serre duality, following the case of degree 0). We refrain from calling $A^\bullet_\nu(\mathcal{H};R)_{\mathrm{sub}}$ cuspidal because this is not justified in degrees higher than 0. In general, $A^\bullet_\nu(\mathcal{H};R)_{\mathrm{sub}}$ is not a submodule of $A^\bullet_\nu(\mathcal{H};R)_{\mathrm{can}}$.

The main justification we have for these terminologies is that these algebraic automorphic forms can be used to define filtrations on the algebraic log de Rham cohomology in mixed characteristics. More precisely, the usual *Hodge spectral sequence* for the log de Rham cohomology groups

$$H^{a+b}_{\text{log-dR}}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R},(\underline{V}^\vee_{\mu,R})^{\mathrm{can}}) := H^{a+b}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R},(\underline{V}^\vee_{\mu,R})^{\mathrm{can}} \underset{\mathscr{O}_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}}}{\otimes} \overline{\Omega}^\bullet_{\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}/\mathsf{S}_R}) \tag{2.33}$$

can be replaced with the *dual BGG spectral sequence*

(2.34)
$$E^{a,b}_1 := \underset{w\in\mathrm{W}^{\mathrm{M}_1}}{\oplus} H^{a+b-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R},\mathrm{Gr}^a_{\mathtt{F}}((\underline{W}^\vee_{w\cdot\mu,R})^{\mathrm{can}})) \Rightarrow H^{a+b}_{\text{log-dR}}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R},(\underline{V}^\vee_{\mu,R})^{\mathrm{can}}).$$

There is also a "compactly supported" analogue with $(\underline{W}^\vee_{w\cdot\mu,R})^{\mathrm{can}}$ replaced with $(\underline{W}^\vee_{w\cdot\mu,R})^{\mathrm{sub}}$. (See Faltings [**Fal83**], Faltings-Chai [**FC90**], Mokrane-Tilouine-Polo [**MPT02**], and my article with Polo [**LP**] for details in the dual BGG construction.)

Remark 2.35. If R is a field of characteristic zero, then the log de Rham cohomology $H^\bullet_{\text{log-dR}}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R},(\underline{V}^\vee_{\mu,R})^{\mathrm{can}})$ calculates the usual de Rham cohomology $H^\bullet_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},R},\underline{V}^\vee_{\mu,R})$. However, this is in general not true when R has a residue field of characteristic $p > 0$. Moreover, because we do not have Hironaka's embedded resolution of singularities [**Hir64a, Hir64b**] in mixed characteristics, we do not know if any non-toroidal smooth compactifications of $\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R$ with a simple normal crossings boundary divisor would yield the same cohomology groups.

The obvious advantage of (2.34) is that its left-hand side is given by a direct sum of spaces of algebraic automorphic forms, for which we might have better methods, rather than by some abstract hypercohomology.

3. Cohomology of torsion automorphic sheaves and vanishing theorems

Since Taylor's thesis (see [**Tay88**, Thm. 4.2]), people know that the torsion in the cohomology of (unions of) locally symmetric spaces (including Shimura varieties) can be abundant and can have number-theoretic significance. Recent works such as Bergeron-Venkatesh [**BV**] have even shown examples where the torsion grows exponentially with levels. On the other hand, although it is speculated that there is *little or no torsion* in the case of Shimura varieties, not much seemed to be known.

In Mokrane-Tilouine [**MT02**] and Dimitrov [**Dim05**], they explored the idea in Faltings-Chai [**FC90**] of studying the $\mathbb{Z}_p$-valued Betti or étale cohomology using log crystalline and log de Rham cohomology in characteristic p, and proved some vanishing and freeness results for the special cases of Siegel modular threefolds and Hilbert modular varieties, *after* localization at a prime of the Hecke algebra with *non-effective conditions* on the image of the associated mod p Galois representation. Their work is based on combinatorial comparisons of patterns of Hodge-Tate weights, and the non-effective conditions they used are to guarantee that the desired patterns can only appear in certain preferred cohomology degrees. A fundamental question is whether there is a method which does not rely on such non-effective conditions.

In my joint work with Junecue Suh [**LSa, LSb, LSc**], we discovered a way to translate the (geometric) "Kodaira type conditions" in vanishing theorems of Deligne-Illusie [**DI87**], Illusie [**Ill90**], Esnault-Viehweg [**EV92**], and Ogus [**Ogu94**] to the (representation-theoretic) "sufficient regularity conditions" in vanishing theorems of Faltings [**Fal83**], Vogan-Zuckerman [**VZ84**], Li-Schwermer [**LS04**], Saper [**Sap05**], and others, and proved new vanishing theorems with *torsion coefficients.* As a byproduct, we obtained freeness for the *interior cohomology*, namely the canonical image of H_c in H, with coefficients in $\mathbb{Z}_p$-valued automorphic sheaves, under a list of mild and *effective* conditions, for *all* PEL-type Shimura varieties.

Our approach does not require any non-effective assumptions (such as those in Mokrane-Tilouine [**MT02**]), and gives after base change to $\mathbb{C}$ a *purely algebraic proof* of several vanishing results so far only proved by transcendental methods.

3.1. Setup. Let us maintain the setup in §2 (and in particular the choices of p and R_1). An important running assumption is that there is no level at p.

The structural homomorphism $\mathcal{O}_{F_0} \to R_1$ determines a p-adic place of F_0, and we will denote the completion of $\mathcal{O}_{F_0}$ at this place by W. Since p is unramified in $\mathcal{O}$ and hence in $\mathcal{O}_{F_0}$, we can identify W with the ring of Witt vectors of its residue field. By passing to the completions, W embeds canonically into the p-adic completion of R_1.

3.2. Vanishing theorem for torsion Betti cohomology.

DEFINITION 3.1. *We say that* $\mu \in \mathrm{X}^+_{\mathrm{G}_\mathbb{C}}$ *is* **sufficiently regular** *if, for any positive root* $\alpha \in \Phi^+_{\mathrm{G}_\mathbb{C}}$*, we have* $(\mu, \alpha^\vee) \geq 1$ *for any* α *coming from a type* A *factor of* $\mathrm{G}_\mathbb{C}$*, and* $(\mu, \alpha^\vee) \geq 2$ *otherwise.* (*We define similarly for* $\mu \in \mathrm{X}^+_{\mathrm{G}_1}$ *when we work in mixed characteristics later.*)

For any cohomology theory with the notion of compactly supported cohomology, we define the *interior cohomology* by setting $H_{\mathrm{int}} := \mathrm{image}(H_{\mathrm{c}} \to H)$. Note that H_{int} is *not* defined as a derived functor. Therefore, for example, there are no long exact sequences attached to short exact sequences.

THEOREM 3.2. *There exists an explicit function $C(\mu)$ (depending on μ and also on the PEL datum, but not on the level) such that, for any* **sufficiently regular** *weight $\mu \in \mathrm{X}^+_{\mathrm{G}_{\mathbb{C}}}$ as above, for any prime p good for the PEL datum (defining $\mathsf{Sh}_{H,\mathbb{C}}$) such that $p > C(\mu)$, and for any W-algebra R, we have:*

(1) $H^i(\mathsf{Sh}_{H,\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{\mu,R}) = 0$ *for every $i < d := \dim_{\mathbb{C}}(\mathsf{Sh}_{H,\mathbb{C}})$.*
(2) $H^i_{\mathrm{c}}(\mathsf{Sh}_{H,\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{\mu,R}) = 0$ *for every $i > d$.*
(3) $H^i_{\mathrm{int}}(\mathsf{Sh}_{H,\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{\mu,R}) = 0$ *for every $i \neq d$.*
(4) *(“***liftability***”) For ? = c or int, the canonical “reduction mod p” morphism $H^d_?(\mathsf{Sh}_{H,\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{\mu,R}) \to H^d_?(\mathsf{Sh}_{H,\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{\mu,(R/pR)})$ is* **surjective***.*
(5) *(“***freeness***”) For ? = $\varnothing$ or int, if $\mathsf{Sh}_{H,\mathbb{C}}$ is compact, or if R is flat over W, then $H^d_?(\mathsf{Sh}_{H,\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{\mu,R})$ is a* **free** *R-module of finite rank.*

If $\mathsf{Sh}_{H,\mathbb{C}}$ is compact, or if one only cares about (3), then the case where $R = \mathbb{C}$ can be proved by transcendental methods (harmonic forms, L_2 methods, etc) as in, for example, Faltings [**Fal83**]. In the non-compact case, to the best of our knowledge, the first analytic proofs of (1) and (2) were given by Li and Schwermer's work on the Eisenstein cohomology of arithmetic groups (see [**LS04**, Cor. 5.6]), and roughly at the same time by Saper's work on $\mathcal{L}$-modules (see [**Sap05**, §11, Thm. 5]). Before their works, the important special case of symplectic groups with factors of rank two was treated using Franke's method in [**TU99**, Appendix A]. However, there is no known transcendental proof for the torsion case $R = k_1$ (or rather $R = \mathbb{F}_p^{\mathrm{ac}}$).

In fact, it is an elementary exercise in homological algebra that the special case $R = k_1$ implies all other cases. Therefore it suffices to focus on this special case.

3.3. Vanishing theorem for torsion log de Rham and log Hodge cohomology. Based on a series of reduction steps using (torsion) comparison theorems among Betti, étale, log crystalline, and log de Rham cohomology (see [**Del77**, Arcata, V, Cor. 3.3] and [**AGV73**, XI, Thm. 4.4] for the Betti–étale comparison; see [**LSb**, §5] and [**LSc**, §9] for the explanation of the étale–log crystalline comparison based on [**BM02**], [**FM87**], [**Fal89**], and [**Fal02**]; see [**Ber74**], [**BO78**], and [**Kat89**, Thm. 6.2] for the log crystalline–log de Rham comparison; for all of these we use the Kuga families and their good toroidal compactifications), the proof of Theorem 3.2 can be reduced to the following:

THEOREM 3.3. *For any R_1-algebra R, the analogues of Theorem 3.2 for the log Hodge and log de Rham cohomology over $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}$ are true.*

3.4. Idea of the proof. It suffices to show that $H^i_{\text{log-dR}}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},R}, (\underline{V}^\vee_{\mu,R})^{\mathrm{can}}) = 0$ for $i < d$. By (2.34) (with $R = k_1$), it suffices to show that, for any $w \in \mathrm{W}^{\mathrm{M}_1}$ and any $i < d$, we have

$$H^{i-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},k_1}, (\underline{W}^\vee_{w\cdot\mu,k_1})^{\mathrm{can}}) = 0. \tag{3.4}$$

Without some condition on μ, this cannot be true. (See [**Suh08**] for counterexamples in the context of compact Picard modular surfaces. There μ is trivial and hence violates the sufficient regularity condition in Definition 3.1.)

Now that the main point is to show that the cohomology groups of certain coherent sheaves vanish, it is natural for us to resort to generalizations of Kodaira vanishing in characteristic p, which is essentially the only vanishing we know other than the cruder Serre's vanishing. (I would recommend starting with Deligne-Illusie [**DI87**] and Esnault-Viehweg [**EV92**].)

For any such vanishing theorem, the coherent sheaves are tensored with line bundles with certain positivity condition, and we prefer such line bundles to be *canonical extensions* of *automorphic line bundles*, i.e., of the form $\underline{W}^{\mathrm{can}}_{\nu_0,k_1}$ for some $\nu_0 \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$. On integral models of PEL-type Shimura varieties, there is essentially only one source of such line bundles, namely those constructed using variants of the ω in Theorem 2.8, and we can find choices of ν_0 such that, for any $w \in \mathrm{W}_{\mathrm{G}_1}$, we have $(w(\nu_0), \alpha^\vee) \leq 1$ for any root α coming from a type A factor of G_1, and $(w(\nu_0), \alpha^\vee) \leq 2$ otherwise (cf. Definition 3.1).

In the compact case, we can use Deligne-Illusie [**DI87**] and Illusie [**Ill90**], together with (2.34), and show that

$$H^{i-l(w)}(\mathsf{M}_{\mathcal{H},k_1}, \underline{W}^\vee_{w\cdot\mu+\nu_0,k_1}) = 0$$

for $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$ satisfying some condition $p > C_0(\mu)$ and for any $w \in \mathrm{W}^{\mathrm{M}_1}$. Here $\underline{W}^\vee_{w\cdot\mu+\nu_0,k_1} \cong \underline{W}^\vee_{\nu_0,k_1} \otimes \underline{W}^\vee_{w\cdot\mu,k_1}$, where $\underline{W}_{\nu_0,k_1}$ is an ample line bundle, and where $\underline{W}^\vee_{w\cdot\mu,k_1}$ is a vector bundle with an integrable connection of a *geometric origin*, because up to Tate twist it can be constructed (via *geometric plethysm*) as a summand of the relative de Rham cohomology of some $\mathsf{N}_{m,k_1} = A^m_{k_1} \to \mathsf{M}_{\mathcal{H},k_1}$ for some $m \geq 0$. The results of Deligne-Illusie and Illusie apply because the whole setup lifts to $W_2(k_1)$ (and in fact even $W(k_1)$).

By changing our perspective a little bit, we can fix $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$ and ask for each $w \in \mathrm{W}^{\mathrm{M}_1}$ whether there exists some $\mu' \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$ such that $p > C_0(\mu')$ and $w \cdot \mu = w \cdot \mu' + \nu_0$. Equivalently, this is asking whether $\mu - w(\nu_0)$ is an element of $\mathrm{X}^{+,<p}_{\mathrm{G}_1}$ satisfying $p > C_0(\mu')$. If μ is sufficiently regular as in Definition 3.1, then $(\mu - w(\nu_0), \alpha^\vee) \geq 0$, and the condition is verified if we define $C(\mu)$ to be slightly larger than $C_0(\mu)$. This proves the key vanishing (3.4), as desired. (One can picture this as having enough room in the Weyl chamber for shifting μ towards the wall.)

To generalize the argument to the non-compact case, we need the following:

(1) We need to use the fact that the relative log de Rham cohomology $\mathscr{H}$ of $f^{\mathrm{tor}}_{m,\kappa} : \mathsf{N}^{\mathrm{tor}}_{m,\kappa} \to \mathsf{M}^{\mathrm{tor}}_{\mathcal{H},1}$ enjoys a long list of (unusually) nice properties. Note that since the morphism $f^{\mathrm{tor}}_{m,\kappa}$ is not semistable in general, most of the properties we need (in mixed characteristic) are impossible to prove by abstract methods. Fortunately, in [**Lanc**], we can prove everything we need by using explicit boundary charts of toroidal compactifications. (For experts working on varieties in general, it is perhaps surprising that such good compactifications exists at all in mixed characteristics.)

(2) By applying Ogus's result [**Ogu94**] to the log crystal attached to $\mathscr{H}$ (over $W(k_1)$), we obtain the so-called *decomposition theorem* (generalizing those of Deligne-Illusie's and Illusie's) for the push-forward of the de Rham complex (of the reduction over k_1) under the relative Frobenius morphism. (Here the main properties we need about $\mathscr{H}$ are that it is locally free, self-dual, with degenerate (relative) Hodge spectral sequence, and that the crystalline Frobenius of the attached log crystal is an isogeny.)

(3) In Esnault-Viehweg [**EV92**], they proved a Kodaira type vanishing theorem for line bundles that satisfy some positivity condition *weaker* than ampleness along the boundary, which quite luckily is satisfied in our context thanks to (2.9) in Theorem 2.8. (This allowed us to give a simple proof of a liftability theorem in [**LSa**], which nevertheless is a special case of what we are proving now.) Combining their techniques with the decomposition theorem provided by Ogus, we can show that

$$H^{i-l(w)}(\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},k_1}, (\underline{W}^{\vee}_{w\cdot\mu+\nu_0,k_1})^{\mathrm{can}}) = 0.$$

(Here the main property we need about $\mathscr{H}$ is that the Gauss-Manin connection has *nilpotent residues* along the irreducible components of the boundary divisor D of $\mathsf{M}^{\mathrm{tor}}_{\mathcal{H},k_1}$.)

(4) Then we can conclude the proof of (3.4) as in the compact case.

This gives a sketch of the proof of Theorem 3.3. As a byproduct, we also obtain a suitable analogue of Theorem 3.3 for *cohomological* algebraic automorphic forms, namely those that appear in the left-hand side of (2.34) for some μ.

3.5. Concluding remark.

REMARK 3.5 (sources of torsion). The results we have obtained suggest that the only possible sources of p-torsion in the cohomology of Shimura varieties, or more ambitiously cohomology of arithmetic groups (i.e. of arithmetic quotients of Riemannian symmetric spaces, often without Hermitian structures), are as follows:

(1) Lack of Hermitian structure (not even Shimura varieties).
(2) Boundary cohomology (the cone of the canonical morphism from H_{c} to H, complementing the interior cohomology).
(3) Ramification at p (including levels); i.e., bad reductions.
(4) Large weights (compared with p).
(5) Irregular weights.

We believe (1) and (2) are related. We are inclined to believe that (3), (4), (5) are difficult for intrinsic reasons.

There are many concurrent works on torsion in the cohomology of arithmetic groups, and we have to stop our survey here due to the limitation of our knowledge. We hope that this article is at least useful for understanding one aspect of the theory.

References

[AEK05] J. Arthur, D. Ellwood, and R. Kottwitz (eds.), *Harmonic analysis, the trace formula, and Shimura varieties*, Clay Mathematics Proceedings, vol. 4, Proceedings of the Clay Mathematics Institute 2003 Summer School, The Fields Institute, Toronto, Canada, June 2–27, 2003, American Mathematical Society, Providence, Rhode Island, Clay Mathematics Institute, Cambridge, Massachusetts, 2005.

[AGV73] M. Artin, A. Grothendieck, and J.-L. Verdier (eds.), *Théorie des topos et cohomologie étale des schémas* (*SGA 4*)*, Tome 3*, Lecture Notes in Mathematics, vol. 305, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[Ale02] V. Alexeev, *Complete moduli in the presence of semiabelian group action*, Ann. Math. (2) **155** (2002), 611–708.

[AMRT75] A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactification of locally symmetric varieties*, Lie Groups: History Frontiers and Applications, vol. 4, Math Sci Press, Brookline, Massachusetts, 1975, TEXified and republished as [**AMRT10**].

[AMRT10] ———, *Smooth compactification of locally symmetric varieties*, 2nd ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, New York, 2010.

[AN99] V. Alexeev and I. Nakamura, *On Mumford's construction of degenerating abelian varieties*, Tôhoku Math. J. (2) **51** (1999), 399–420.

[Art69] M. Artin, *Algebraization of formal moduli: I*, in Spencer and Iyanaga [**SI69**], pp. 21–71.

[BB66] W. L. Baily, Jr. and A. Borel, *Compactification of arithmetic quotients of bounded symmetric domains*, Ann. Math. (2) **84** (1966), no. 3, 442–528.

[BC79] A. Borel and W. Casselman (eds.), *Automorphic forms, representations and L-functions*, Proceedings of Symposia in Pure Mathematics, vol. 33, Part 2, held at Oregon State University, Corvallis, Oregon, July 11 – August 5, 1977, American Mathematical Society, Providence, Rhode Island, 1979.

[Ber74] P. Berthelot, *Cohomologie crystalline des schémas de caractéristique* $p > 0$, Lecture Notes in Mathematics, vol. 407, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[BFI$^+$02] P. Berthelot, J.-M. Fontaine, L. Illusie, K. Kato, and M. Rapoport (eds.), *Cohomologies p-adiques et applications arithmétiques (II)*, Astérisque, no. 279, Société Mathématique de France, Paris, 2002.

[BGG75] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, *Differential operators on the base affine space and a study of* $\mathfrak{g}$*-modules*, in Gelfand [**Gel75**], pp. 21–64.

[BJ06] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Mathematics: Theory & Applications, Birkhäuser, Boston, 2006.

[BM02] C. Breuil and W. Messing, *Torsion étale and crystalline cohomologies*, in Berthelot et al. [**BFI$^+$02**], pp. 81–124.

[BO78] P. Berthelot and A. Ogus, *Notes on crystalline cohomology*, Mathematical Notes, vol. 21, Princeton University Press, Princeton, 1978.

[Bor63] A. Borel, *Some finiteness properties of adele groups over number fields*, Publ. Math. Inst. Hautes. Étud. Sci. **16** (1963), 5–30.

[BV] N. Bergeron and A. Venkatesh, *The asymptotic growth of torsion homology for arithmetic groups*, preprint.

[BW00] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, 2nd ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, Rhode Island, 2000.

[Cha85] C.-L. Chai, *Compactification of Siegel moduli schemes*, London Mathematical Society Lecture Note Series, vol. 107, Cambridge University Press, Cambridge, New York, 1985.

[CM90a] L. Clozel and J. S. Milne (eds.), *Automorphic forms, Shimura varieties, and L-functions. Volume II*, Perspectives in Mathematics, vol. 11, Proceedings of a Conference held at the University of Michigan, Ann Arbor, July 6–16, 1988, Academic Press Inc., Boston, 1990.

[CM90b] L. Clozel and J. S. Milne (eds.), *Automorphic forms, Shimura varieties, and L-functions. Volume I*, Perspectives in Mathematics, vol. 10, Proceedings of a Conference held at the University of Michigan, Ann Arbor, July 6–16, 1988, Academic Press Inc., Boston, 1990.

[Con71] Congrès International des Mathématiciens, 1/10 Septembre 1970, Nice, France, *Actes du Congrès International des Mathématiciens, 1970, publiés sous la direction du Comité d'Organisation du Congrès*, vol. 1, Gauthier-Villars, Paris, 1971.

[CSS97] G. Cornell, J. H. Silverman, and G. Stevens (eds.), *Modular forms and Fermat's last theorem*, Springer-Verlag, Berlin, Heidelberg, New York, 1997.

[CY97] J. Coates and S.T. Yau (eds.), *Elliptic curves, modular forms & Fermat's last theorem*, Proceedings of a conference held in the Institute of Mathematics of the Chinese Univerisity of Hong Kong, International Press, Cambridge, Massachusetts, 1997.

[Del70] P. Deligne, *Equations différentielles à points singuliers réguliers*, Lecture Notes in Mathematics, vol. 163, Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[Del71a] ———, *Formes modulaires et représentations ℓ-adiques*, Séminaire Bourbaki, exposé n° 355 (février 1969), Lecture Notes in Mathematics, vol. 179, Springer-Verlag, Berlin, Heidelberg, New York, 1971, pp. 139–172.

[Del71b] ———, *Théorie de Hodge: II*, Publ. Math. Inst. Hautes. Étud. Sci. **40** (1971), 5–57.

[Del77] P. Deligne (ed.), *Cohomologie étale* (*SGA $4\frac{1}{2}$*), Lecture Notes in Mathematics, vol. 569, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[Del79] ———, *Variétés de Shimura: Interprétation modulaire, et techniques de construction de modèles canoniques*, in Borel and Casselman [**BC79**], pp. 247–290.

[DI87] P. Deligne and L. Illusie, *Relèvements modulo p^2 et décompositions du complex de de Rham*, Invent. Math. **89** (1987), 247–270.

[Dim05] M. Dimitrov, *Galois representations modulo p and cohomology of Hilbert modular varieties*, Ann. Sci. Ecole Norm. Sup. (4) **38** (2005), 505–551.

[DK73] P. Deligne and W. Kuyk (eds.), *Modular functions of one variable II*, Lecture Notes in Mathematics, vol. 349, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[DP94] P. Deligne and G. Pappas, *Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant*, Compositio Math. **90** (1994), 59–79.

[DR73] P. Deligne and M. Rapoport, *Les schémas de modules de courbes elliptiques*, in Deligne and Kuyk [**DK73**], pp. 143–316.

[EV92] H. Esnault and E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, vol. 20, Birkhäuser, Boston, 1992.

[Fal83] G. Faltings, *On the cohomology of locally symmetric hermitian spaces*, in Malliavin [**Mal83**], pp. 55–98.

[Fal85] ———, *Arithmetic Kompaktifizierung der Modulraums der abelschen Varietäten*, in Hirzebruch et al. [**HSS85**], pp. 321–383.

[Fal89] ———, *Crystalline cohomoloy and p-adic Galois-representations*, in Igusa [**Igu89**], pp. 25–80.

[Fal02] ———, *Almost étale extensions*, in Berthelot et al. [**BFI$^+$02**], pp. 185–270.

[FC90] G. Faltings and C.-L. Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 22, Springer-Verlag, Berlin, Heidelberg, New York, 1990.

[FM87] J.-M. Fontaine and W. Messing, *p-adic periods and p-adic étale cohomology*, in Ribet [**Rib87**], pp. 179–207.

[Gel75] I. M. Gelfand (ed.), *Lie groups and their representations*, Summer School of the Bolyai János Mathematical Society, (Budapest, 1971), Adam Hilger Ltd., London, 1975.

[Har89] M. Harris, *Functorial properties of toroidal compactifications of locally symmetric varieties*, Proc. London Math. Soc. (3) **59** (1989), 1–22.

[Har90] ———, *Automorphic forms and the cohomology of vector bundles on Shimura varieties*, in Clozel and Milne [**CM90a**], pp. 41–91.

[Hir64a] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I*, Ann. Math. (2) **79** (1964), no. 1, 109–203.

[Hir64b] ———, *Resolution of singularities of an algebraic variety over a field of characteristic zero: II*, Ann. Math. (2) **79** (1964), no. 2, 205–326.

[HSS85] F. Hirzebruch, J. Schwermer, and S. Suter (eds.), *Arbeitstagung Bonn 1984*, Lecture Notes in Mathematics, vol. 1111, Proceedings of the meeting held by the Max-Planck Instutut für Mathematik, Bonn, June 15–22, 1984, Springer-Verlag, Berlin, Heidelberg, New York, 1985.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, no. 151, Princeton University Press, Princeton, 2001.

[HZ94a] M. Harris and S. Zucker, *Boundary cohomology of Shimura varieties I. — Coherent cohomology on toroidal compactifications*, Ann. Sci. Ecole Norm. Sup. (4) **27** (1994), 249–344.

[HZ94b] ———, *Boundary cohomology of Shimura varieties II. Hodge theory at the boundary*, Invent. Math. **116** (1994), 243–307.

[HZ01] ———, *Boundary cohomology of Shimura varieties III. Coherent cohomology on higher-rank boundary strata and applications to Hodge theory*, Mémoires de la Société Mathématique de France. Nouvelle Série, vol. 85, Société Mathématique de France, Paris, 2001.

[Igu89] J.-I. Igusa (ed.), *Algebraic analysis, geometry, and number theory*, Proceedings of the JAMI Inaugural Conference, The Johns Hopkins University Press, Baltimore, 1989.

[Ill90] Luc Illusie, *Réduction semi-stable et décomposition de complexes de de Rham*, Duke Math. J. **60** (1990), no. 1, 139–185.

[Kat70] N. M. Katz, *Nilpotent connections and the monodromy theorem: applications of a result of Turrittin*, Publ. Math. Inst. Hautes. Étud. Sci. **39** (1970), 175–232.

[Kat71] ———, *The regularity theorem in algebraic geometry*, in *Actes du Congrès International des Mathématiciens, 1970, publiés sous la direction du Comité d'Organisation du Congrès* [**Con71**], pp. 437–443.

[Kat72] ———, *Algebraic solutions of differential equations (p-curvature and the Hodge filtration)*, Invent. Math. **18** (1972), 1–118.

[Kat89] K. Kato, *Logarithmic structures of Fontaine-Illusie*, in Igusa [**Igu89**], pp. 191–224.

[KM85] N. M. Katz and B. Mazur, *Arithmetic moduli of elliptic curves*, Annals of Mathematics Studies, no. 108, Princeton University Press, Princeton, 1985.

[KO68] N. M. Katz and T. Oda, *On the differentiation of De Rham cohomology classes with respect to parameters*, J. Math. Kyoto Univ. **8** (1968), 199–213.

[Kot92] R. E. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. **5** (1992), no. 2, 373–444.

[Lana] K.-W. Lan, *Comparison between analytic and algebraic constructions of toroidal compactifications of PEL-type Shimura varieties*, J. Reine Angew. Math., to appear, doi:10.1515/CRELLE.2011.099.

[Lanb] ———, *Elevators for degenerations of PEL structures*, Math. Res. Lett., to appear.

[Lanc] ———, *Toroidal compactifications of PEL-type Kuga families*, Algebra Number Theory, to appear.

[Lan08] ———, *Arithmetic compactification of PEL-type Shimura varieties*, Ph. D. Thesis, Harvard University, Cambridge, Massachusetts, 2008, errata and revision available online at the author's website.

[Lar88] M. J. Larsen, *Unitary groups and L-adic representations*, Ph.D. thesis, Princeton University, Princeton, 1988.

[Lar92] ———, *Arithmetic compactification of some Shimura surfaces*, in Langlands and Ramakrishnan [**LR92**], pp. 31–45.

[LP] K.-W. Lan and P. Polo, *Dual BGG complexes for automorphic bundles*, preprint.

[LR87] R. P. Langlands and M. Rapoport, *Shimuravarietäten und Gerben*, J. Reine Angew. Math. **378** (1987), 113–220.

[LR92] R. P. Langlands and D. Ramakrishnan (eds.), *The zeta functions of Picard modular surfaces*, based on lectures delivered at a CRM Workshop in the spring of 1988, Les Publications CRM, Montréal, 1992.

[LSa] K.-W. Lan and J. Suh, *Liftability of mod p cusp forms of parallel weights*, Int. Math. Res. Not. IMRN **2011** (2011), 1870–1879, doi:10.1093/imrn/rnq145.

[LSb] K.-W. Lan and J. Suh, *Vanishing theorems for torsion automorphic sheaves on compact PEL-type Shimura varieties*, Duke Math. J., to appear.

[LSc] ———, *Vanishing theorems for torsion automorphic sheaves on general PEL-type Shimura varieties*, preprint.

[LS04] J.-S. Li and J. Schwermer, *On the Eisenstein cohomology of arithmetic groups*, Duke Math. J. **123** (2004), no. 1, 141–169.

[Mal83] M.-P. Malliavin (ed.), *Séminaire d'algèbre Paul Dubreil et Marie-Paule Malliavin*, Lecture Notes in Mathematics, vol. 1029, Proceedings, Paris 1982 (35ème Année), Springer-Verlag, Berlin, Heidelberg, New York, 1983.

[MB85] L. Moret-Bailly, *Pinceaux de variétés abéliennes*, Astérisque, vol. 129, Société Mathématique de France, Paris, 1985.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, 3rd ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 34, Springer-Verlag, Berlin, Heidelberg, New York, 1994.

[Mil90] J. S. Milne, *Canonical models of (mixed) Shimura varieties and automorphic vector bundles*, in Clozel and Milne [**CM90b**], pp. 283–414.

[Mil05] ———, *Introduction to Shimura varieties*, in Arthur et al. [**AEK05**], pp. 265–378.

[MPT02] A. Mokrane, P. Polo, and J. Tilouine, *Cohomology of Siegel varieties*, Astérisque, no. 280, Société Mathématique de France, Paris, 2002.

[MT02] A. Mokrane and J. Tilouine, *Cohomology of Siegel varieties with p-adic integral coefficients and applications*, in *Cohomology of Siegel varieties* [**MPT02**], pp. 1–95.

[Mum72] D. Mumford, *An analytic construction of degenerate abelian varieties over complete rings*, Compositio Math. **24** (1972), no. 3, 239–272.

[Mum77] ———, *Hirzebruch's proportionality theorem in the non-compact case*, Invent. Math. **42** (1977), 239–272.

[Ogu94] A. Ogus, *F-crystals, Griffiths transversality, and the Hodge decomposition*, Astérisque, no. 221, Société Mathématique de France, Paris, 1994.

[Ols08] M. C. Olsson, *Compactifying moduli spaces for abelian varieties*, Lecture Notes in Mathematics, vol. 1958, Springer-Verlag, Berlin, Heidelberg, New York, 2008.

[Pin89] R. Pink, *Arithmetic compactification of mixed Shimura varieties*, Ph.D. thesis, Rheinischen Friedrich-Wilhelms-Universität, Bonn, 1989.

[PS08] C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge structures*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, vol. 52, Springer-Verlag, Berlin, Heidelberg, New York, 2008.

[PT02] P. Polo and J. Tilouine, *Bernstein-Gelfand-Gelfand complex and cohomology of nilpotent groups over $\mathbb{Z}_{(p)}$ for representations with p-small weights*, in *Cohomology of Siegel varieties* [**MPT02**], pp. 97–135.

[Rap78] M. Rapoport, *Compactifications de l'espace de modules de Hilbert-Blumenthal*, Compositio Math. **36** (1978), no. 3, 255–335.

[Rib87] K. A. Ribet (ed.), *Current trends in arithmetic algebraic geometry*, Contemporary Mathematics, vol. 67, Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference on Algebraic Geometry, August 18–24, 1985, Arcata, California, American Mathematical Society, Providence, Rhode Island, 1987.

[RZ96] M. Rapoport and Th. Zink, *Period spaces for p-divisible groups*, Annals of Mathematics Studies, no. 141, Princeton University Press, Princeton, 1996.

[Sap05] L. Saper, *$\mathcal{L}$-modules and the conjecture of Rapoport and Goresky-MacPherson*, in Tilouine et al. [**TCHV05**], pp. 319–334.

[Shi71] G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, vol. 11, Princeton University Press, Princeton, 1971.

[Shi98] ———, *Abelian varieties with complex multiplication and modular functions*, Princeton University Press, Princeton, 1998.

[Shi02] ———, *Collected papers: 1954–1966, 1967–1977, 1978–1988, 1989–2001 Volumes I–IV*, Springer-Verlag, Berlin, Heidelberg, New York, 2002.

[SI69] D. C. Spencer and S. Iyanaga (eds.), *Global analysis. Papers in honor of K. Kodaira*, Princeton University Press, Princeton, 1969.

[Str08] B. Stroh, *Compactification de variétés de Siegel aux places de mauvaise réduction*, Ph.D. thesis, Université Henri Poincaré, Nancy, France, 2008.

[Str10a] ———, *Compactification de variétés de Siegel aux places de mauvaise réduction*, Bull. Soc. Math. France **138** (2010), no. 2, 259–315.

[Str10b] ———, *Compactification minimale et mauvaise réduction*, Ann. Inst. Fourier. Gronoble **60** (2010), no. 3, 1035–1055.

[Suh08] J. Suh, *Plurigenera of general type surfaces in mixed characteristic*, Compositio Math. **144** (2008), 1214–1226.

[Tay88] R. L. Taylor, *On congruences between modular forms*, Ph.D. thesis, Princeton University, Princeton, 1988.

[TCHV05] J. Tilouine, H. Carayol, M. Harris, and M.-F. Vignéras (eds.), *Formes automorphes (I): Actes du semestre du Centre émile Borel, printemps 2000*, Astérisque, no. 298, Société Mathématique de France, Paris, 2005.

[TU99] J. Tilouine and E. Urban, *Several-variable p-adic families of Siegel-Hilbert cusp eigensystems and their Galois representations*, Ann. Sci. Ecole Norm. Sup. (4) **32** (1999), 499–574.

[VZ84] D. A. Vogan, Jr. and G. J. Zuckerman, *Unitary representations with non-zero cohomology*, Compositio Math. **53** (1984), no. 1, 51–90.

[Zin82] T. Zink, *Über die schlechte Reduktion einiger Shimuramannigfaltigkeiten*, Compositio Math. **45** (1982), no. 1, 15–107.

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