VANISHING THEOREMS FOR TORSION AUTOMORPHIC SHEAVES ON COMPACT PEL-TYPE SHIMURA VARIETIES

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Abstract. Given a compact PEL-type Shimura variety, a sufficiently regular weight (defined by mild and effective conditions), and a prime number p unramified in the linear data and larger than an effective bound given by the weight, we show that the (Betti) cohomology with $\mathbb{Z}_p$-coefficients of the given weight vanishes away from the middle degree, and hence has no p-torsion. We do not need any other assumption (such as ones on the images of the associated Galois representations).

Introduction

The cohomology of Shimura varieties (with coefficients in algebraic representations of the associated reductive groups) has been an important tool for studying the relation between the theory of automorphic forms and arithmetic. In this article, we try to answer a basic question:

Question. *Let p be a prime number. When is the* (*Betti*) *cohomology of the Shimura variety with* (*possibly non-trivial*) *integral coefficients p-torsion free?*

Certainly, when we fix both the level and the coefficient system, the answer is in the affirmative for all sufficiently large p. But to the best of our knowledge, there has been no known, effective bound that applies to general Shimura varieties. Moreover, it is a priori unclear whether such a bound can be found that is insensitive to raising the level, even if we focus only on neat and prime-to-p levels.

The main results of this article provide the following (partial) answer: Consider a compact PEL-type Shimura variety at a neat level, a weight μ that is "sufficiently regular" (a mild and effective condition which, in the unitary case, coincides with the usual notion of regularity), and a prime number p that is unramified in the linear data defining the Shimura variety. If the level is *maximal hyperspecial* at p and if p is larger than an *effective bound* that is a function of μ (but is independent of the prime-to-p level), then the Betti cohomology of the variety with coefficients

2010 *Mathematics Subject Classification.* Primary 11G18; Secondary 14F17, 14F30, 11F75.

Key words and phrases. Shimura varieties, vanishing theorems, p-adic cohomology, torsion-freeness, liftability.

The research of the first author is supported by the Qiu Shi Science and Technology Foundation, and by the National Science Foundation under agreement No. DMS-0635607. The research of the second author was supported by the National Science Foundation under agreement No. DMS-0635607. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of these organizations.

Please refer to Duke Math. J. 161 (2012), no. 6, pp. 1113–1170, doi:10.1215/00127094-1548452, for the official version. Please refer to the errata (available on at least one of the authors' websites) for a list of known errors (most of which have been corrected in this compilation, for the convenience of the reader).

in the $\mathbb{Z}_p$-module corresponding to μ is concentrated in the middle degree, and has no p-torsion. (See Theorem 8.12 and Corollary 8.13 for more precise statements. Variants in other cohomology theories are also given in Section 8.)

We stress that all the conditions we need are explicit and can be verified easily in practice. We do not make any assumptions such as the ones on the images of the associated Galois representations (which are often far from effective). (See for example the remark in [37, §9, line 6] on their "residually large image" assumption (RLI).)

Our approach to this problem is to use the de Rham cohomology of the good reduction modulo p of the Shimura variety in question. The main technical inputs are Illusie's vanishing theorem, Faltings's dual BGG construction, and a new observation relating the (geometric) "Kodaira type" conditions on the coefficient systems to the (representation-theoretic) "sufficient regularity" conditions.

Although all the techniques we use have been known for many years, their simple combination (when the level is neat and prime-to-p) has not been implemented in any special cases. By base extension to $\mathbb{C}$, we also obtain the first purely algebraic proof of certain vanishing results that had only been proved by transcendental methods.

We remark that closely related questions on (the absence of) p-torsion in the cohomology of Lubin–Tate towers have been considered in the work of Boyer. Our approach differs fundamentally from his, and does not subsume the results there.

Here is an outline of the article. In Sections 1 and 2, we review the basic setups in geometry and representation theory, which are standard but necessary. In Sections 3–4, we explain the realization of automorphic bundles and their cohomology using fiber products of the universal abelian scheme over our Shimura variety, following [13, pp. 234–235], [19, III.2], and [37, II.2]. In Section 5, we explain how the comparison among different cohomology theories with automorphic coefficients can be reduced to the standard results with constant coefficients. (We work out these sections in detail, sometimes with steps not readily available in the literature, because we want to pin down optimal bounds on the sizes of p.) In Section 6, we introduce Illusie's vanishing theorem [22] and its reformulations using Faltings's dual BGG construction. Then we explain our key observation (mentioned above) in Section 7, with an analysis on ample automorphic line bundles with weights of "minimal size". This is the most crucial part of this article. The main results will be presented in Section 8, including our vanishing theorems for cohomology with automorphic coefficients, and their obvious implications to questions of torsion-freeness and liftability.

The ideas in this article can be generalized to all PEL-type cases (including noncompact ones), which we have carried out in the article [31]. See the introduction there for more details.

The results in this article on torsion-freeness and liftability have potential applications to the study of p-adic modular forms and Taylor–Wiles systems. (For example, Michael Harris has applied our results to the study of Taylor–Wiles systems. See [18].) After all, very little has been known (or even conjectured) about the torsion in the cohomology of Shimura varieties. We naturally expect more of such interesting results and applications to appear in the future.

We shall follow [29, Notations and Conventions] unless otherwise specified.

Contents

1. Geometric setup

1.1. **Linear algebraic data.** Let $(\mathcal{O}, \star, L, \langle\,\cdot\,,\,\cdot\,\rangle, h_0)$ be an integral PEL datum in the following sense:

(1) $\mathcal{O}$ is an order in a (nonzero) semisimple algebra, finite-dimensional over $\mathbb{Q}$, together with a positive involution $\star$.
(2) $(L, \langle \cdot, \cdot \rangle, h_0)$ is a PEL-type $\mathcal{O}$-lattice as in [29, Def. 1.2.1.3]. (In [29, Def. 1.2.1.3] h_0 was denoted by h.)

We shall denote the center of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ by F. (Then F is a product of number fields.)

Definition 1.1 (cf. [29, Def. 1.2.1.5]). *Let $\mathcal{O}$ and $(L, \langle \cdot, \cdot \rangle)$ be given as above. Then we define for any $\mathbb{Z}$-algebra R*

$$\mathrm{G}(R) := \left\{ (g, r) \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(L \underset{\mathbb{Z}}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R) : \langle gx, gy \rangle = r \langle x, y \rangle, \ \forall x, y \in L \right\}.$$

The assignment is functorial in R and defines a group functor G over $\mathrm{Spec}(\mathbb{Z})$. The projection to the second factor $(g, r) \mapsto r$ defines a homomorphism $\nu : \mathrm{G} \to \mathbf{G}_{\mathrm{m}}$, which we call the **similitude character**. *For simplicity, we shall often denote elements (g, r) in G only by g, and denote by $\nu(g)$ the value of r when we need it. (This is an abuse of notation, because r is not always determined by g.)*

The homomorphism $h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ defines a Hodge structure of weight -1 on L, with Hodge decomposition

$$L \underset{\mathbb{Z}}{\otimes} \mathbb{C} = V_0 \oplus V_0^c, \tag{1.2}$$

such that $h_0(z)$ acts as $1 \otimes z$ on V_0, and as $1 \otimes z^c$ on V_0^c. Let F_0 be the reflex field (see [29, Def. 1.2.5.4]) defined by the $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-module V_0.

By abuse of notation, we shall denote the ring of integers in F (resp. F_0) by $\mathcal{O}_F$ (resp. $\mathcal{O}_{F_0}$). This is in conflict with the notation of the order $\mathcal{O}$ in the integral PEL datum, but the precise interpretation will be clear from the context.

Let Diff^{-1} be the inverse different of $\mathcal{O}$ over $\mathbb{Z}$ (see [29, Def. 1.1.1.11]), and let $\mathrm{Disc} = [\mathrm{Diff}^{-1} : \mathcal{O}]_{\mathbb{Z}}$ be the discriminant of $\mathcal{O}$ over $\mathbb{Z}$ (see [29, Def. 1.1.1.6 and Prop. 1.1.1.12]). We say that a rational prime number $p > 0$ is *good* if it satisfies the following conditions (cf. [26, §5] and [29, Def. 1.4.1.1]):

(1) p is unramified in $\mathcal{O}$, in the sense that $p \nmid \mathrm{Disc}$ (as in [29, Def. 1.1.1.14]).
(2) $p \neq 2$ if $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ involves simple factors of type D (as in [29, Def. 1.2.1.15]).
(3) The pairing $\langle \cdot, \cdot \rangle$ is perfect after base change to $\mathbb{Z}_p$.

Let us fix any choice of a good prime $p > 0$.

Lemma 1.3. *There exists a finite extension F_0' of F_0 in $\mathbb{C}$, unramified at p, together with an $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0', (p)}$-module L_0 such that $L_0 \underset{\mathcal{O}_{F_0', (p)}}{\otimes} \mathbb{C} \cong V_0$ as $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules.*

See [29, Lem. 1.2.5.9 in the revision] for a proof. For each fixed F_0', the choice of L_0 is unique up to isomorphism because $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathcal{O}_{F_0', (p)}$-modules are uniquely determined by their multi-ranks. (See [29, Lem. 1.1.3.4]. We will review the notion of multi-ranks in Section 2.1.)

Let us denote by $\langle \cdot, \cdot \rangle_{\mathrm{can.}} : (L_0 \oplus L_0^\vee(1)) \times (L_0 \oplus L_0^\vee(1)) \to \mathcal{O}_{F_0', (p)}(1)$ (cf. [29, Lem. 1.1.4.16]) the alternating pairing $\langle (x_1, f_1), (x_2, f_2) \rangle_{\mathrm{can.}} := f_2(x_1) - f_1(x_2)$. The natural right action of $\mathcal{O}$ on $L_0^\vee(1)$ defines a natural left action of $\mathcal{O}$ by composition with the involution $\star : \mathcal{O} \xrightarrow{\sim} \mathcal{O}$. Then (1.2) canonically induces an isomorphism $L_0^\vee(1) \underset{\mathbb{Z}}{\otimes} \mathbb{C} \cong V_0^c$ of $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$-modules.

Definition 1.4. *For any $\mathcal{O}_{F'_0,(p)}$-algebra R, set*

$$\mathrm{G}_0(R) := \left\{ \begin{array}{c} (g,r) \in \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}((L_0 \oplus L_0^\vee(1)) \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R) : \\ \langle gx, gy \rangle_{\mathrm{can.}} = r \langle x, y \rangle_{\mathrm{can.}},\ \forall x, y \in (L_0 \oplus L_0^\vee(1)) \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R \end{array} \right\},$$

$$\mathrm{P}_0(R) := \left\{ (g,r) \in \mathrm{G}_0(R) : g(L_0^\vee(1) \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R) = L_0^\vee(1) \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R \right\},$$

$$\mathrm{M}_0(R) := \mathrm{GL}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} R}(L_0^\vee(1) \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R) \times \mathbf{G}_{\mathrm{m}}(R),$$

where we view $\mathrm{M}_0(R)$ canonically as a quotient of $\mathrm{P}_0(R)$ by

$$\mathrm{P}_0(R) \to \mathrm{M}_0(R) : (g,r) \mapsto (g|_{L_0^\vee(1) \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R}, r).$$

The assignments are functorial in R, and define group functors G_0, P_0, and M_0 over $\mathrm{Spec}(\mathcal{O}_{F'_0,(p)})$.

By [29, Prop. 1.1.1.17, Cor. 1.2.5.7, and Cor. 1.2.3.10], there exists a discrete valuation ring R_1 over $\mathcal{O}_{F'_0,(p)}$ satisfying the following conditions:

(1) The maximal ideal of R_1 is generated by p, and the residue field κ_1 of R_1 is a *finite field* of characteristic p. In this case, the p-adic completion of R_1 is isomorphic to the Witt vectors $W(\kappa_1)$ over κ_1.

(2) The ring $\mathcal{O}_F$ is split over R_1, in the sense that $\Upsilon := \mathrm{Hom}_{\mathbb{Z}\text{-alg.}}(\mathcal{O}_F, R_1)$ has cardinality $[F : \mathbb{Q}]$. Then there is a canonical isomorphism

$$\mathcal{O}_F \underset{\mathbb{Z}}{\otimes} R_1 \cong \prod_{\tau \in \Upsilon} \mathcal{O}_{F,\tau} \tag{1.5}$$

where each $\mathcal{O}_{F,\tau}$ can be identified as the $\mathcal{O}_F$-algebra R_1 via τ.

(3) There exists an isomorphism

$$(L \underset{\mathbb{Z}}{\otimes} R_1, \langle \cdot, \cdot \rangle) \cong (L_0 \oplus L_0^\vee(1), \langle \cdot, \cdot \rangle_{\mathrm{can.}}) \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R_1 \tag{1.6}$$

inducing an isomorphism $\mathrm{G} \underset{\mathbb{Z}}{\otimes} R_1 \cong \mathrm{G}_0 \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R_1$ realizing $\mathrm{P}_0 \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R_1$ as a subgroup of $\mathrm{G} \underset{\mathbb{Z}}{\otimes} R_1$. (The existence of the isomorphism (1.6) follows from [29, Cor. 1.2.3.10] by comparing multi-ranks.)

Remark 1.7. For the purpose of studying questions such as the vanishing or freeness of cohomology with torsion coefficients, it is harmless (and helpful) to enlarge the coefficient rings.

From now on, let us fix the choices of R_1 and the isomorphism (1.6), and set $\mathcal{O}_{F,1} := \mathcal{O}_F \underset{\mathbb{Z}}{\otimes} R_1$, $\mathcal{O}_1 := \mathcal{O} \underset{\mathbb{Z}}{\otimes} R_1$, $L_1 := L \underset{\mathbb{Z}}{\otimes} R_1$, $L_{0,1} := L_0 \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R_1$, $\mathrm{G}_1 := \mathrm{G}_0 \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R_1 \cong \mathrm{G} \underset{\mathbb{Z}}{\otimes} R_1$, $\mathrm{P}_1 := \mathrm{P}_0 \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R_1$, and $\mathrm{M}_1 := \mathrm{M}_0 \underset{\mathcal{O}_{F'_0,(p)}}{\otimes} R_1$.

1.2. **PEL-type Shimura varieties.** Let $\mathcal{H}$ be a *neat* open compact subgroup of $\mathrm{G}(\hat{\mathbb{Z}}^p)$. (See [41, 0.6] or [29, Def. 1.4.1.8] for the definition of neatness.)

By [29, Def. 1.4.1.4] (with $\Box = \{p\}$ there), the data of $(L, \langle \cdot, \cdot \rangle, h_0)$ and $\mathcal{H}$ define a moduli problem $\mathsf{M}_{\mathcal{H}}$ over $\mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$, parameterizing tuples $(A, \lambda, i, \alpha_{\mathcal{H}})$ over schemes S over S_0 of the following form:

(1) $A \to S$ is an abelian scheme.
(2) $\lambda : A \to A^\vee$ is a polarization of degree prime to p.
(3) $i : \mathcal{O} \hookrightarrow \mathrm{End}_S(A)$ is an $\mathcal{O}$-endomorphism structure as in [29, Def. 1.3.3.1].
(4) $\underline{\mathrm{Lie}}_{A/S}$ with its $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Z}_{(p)}$-module structure given naturally by i satisfies the determinantal condition in [29, Def. 1.3.4.2] given by $(L \underset{\mathbb{Z}}{\otimes} \mathbb{R}, \langle \cdot\, , \cdot \rangle, h_0)$.
(5) $\alpha_{\mathcal{H}}$ is an (integral) level-$\mathcal{H}$ structure of (A, λ, i) of type $(L \underset{\mathbb{Z}}{\otimes} \hat{\mathbb{Z}}^p, \langle \cdot\, , \cdot \rangle)$ as in [29, Def. 1.3.7.8].

Remark. The definition (by isomorphism classes) can be identified with the one in [26, §5] (by prime-to-p quasi-isogeny classes) by [29, Prop. 1.4.3.3].

By [29, Thm. 1.4.1.12 and Cor. 7.2.3.10], $\mathsf{M}_{\mathcal{H}}$ is representable by a (smooth) quasi-projective scheme over S_0 (under the assumption that $\mathcal{H}$ is neat).

Consider the (real analytic) set $\mathsf{X} = \mathrm{G}(\mathbb{R})h_0$ of $\mathrm{G}(\mathbb{R})$-conjugates $h : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$ of $h_0 : \mathbb{C} \to \mathrm{End}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{R}}(L \underset{\mathbb{Z}}{\otimes} \mathbb{R})$. Let $H^p := \mathcal{H}$ and $H_p := \mathrm{G}(\mathbb{Z}_p)$ be open compact subgroups of $\mathrm{G}(\hat{\mathbb{Z}}^p)$ and $\mathrm{G}(\mathbb{Q}_p)$, respectively, and let H be the open compact subgroup $H^p H_p$ of $\mathrm{G}(\hat{\mathbb{Z}})$. It is well known (see [26, §8] or [27, §2]) that there exists a quasi-projective variety Sh_H over F_0, together with a canonical open and closed immersion $\mathsf{Sh}_H \hookrightarrow \mathsf{M}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} F_0$ (because $\mathcal{H}$ is neat), such that the analytification of $\mathsf{Sh}_H \underset{F_0}{\otimes} \mathbb{C}$ can be canonically identified with the double coset space $\mathrm{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/H$. (Note that $\mathsf{Sh}_H \hookrightarrow \mathsf{M}_{\mathcal{H}} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} F_0$ is not an isomorphism in general, due to the so-called "failure of Hasse's principle". See for example [26, §8] and [29, Rem. 1.4.3.11].)

Let $\mathsf{M}_{\mathcal{H},0}$ denote the schematic closure of Sh_H in $\mathsf{M}_{\mathcal{H}}$. Then $\mathsf{M}_{\mathcal{H},0}$ is smooth over S_0. In this article, we shall maintain from now on the following:

Assumption 1.8. *The double coset space* $\mathrm{G}(\mathbb{Q})\backslash \mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/H$*, with its real analytic structure inherited from* X*, is* **compact***.*

Theorem 1.9 (see [28, §4])**.** *Under Assumption* 1.8*,* $\mathsf{M}_{\mathcal{H},0}$ *is proper* (*and hence projective*) *over* S_0*.*

Remark 1.10. The dimension of X as a complex manifold, and hence the relative dimension of any component of the smooth scheme $\mathsf{M}_{\mathcal{H},0}$ over S_0, can be calculated easily because X is embedded as an open subset of $\mathrm{G}_0(\mathbb{C})/\mathrm{P}_0(\mathbb{C})$ (by sending any $h \in \mathsf{X}$ to the Hodge filtration it defines).

Let $\mathsf{S}_1 := \mathrm{Spec}(R_1)$, and let $\mathsf{M}_{\mathcal{H},1} := \mathsf{M}_{\mathcal{H},0} \underset{\mathsf{S}_0}{\times} \mathsf{S}_1$. By abuse of notation, we denote the pullback of the universal object over $\mathsf{M}_{\mathcal{H}}$ to $\mathsf{M}_{\mathcal{H},1}$ by $(A, \lambda, i, \alpha_{\mathcal{H}}) \to \mathsf{M}_{\mathcal{H},1}$.

Consider the relative de Rham cohomology $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$ and the relative de Rham homology $\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) := \underline{\mathrm{Hom}}_{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}(\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}), \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}})$. We have the canonical pairing $\langle \cdot\, , \cdot \rangle_\lambda : \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \times \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)$ defined as the composite of $(\mathrm{Id} \times \lambda)_*$ followed by the perfect pairing $\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \times \underline{H}_1^{\mathrm{dR}}(A^\vee/\mathsf{M}_{\mathcal{H},1}) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)$ defined by the first Chern class of the Poincaré line bundle over $A \underset{\mathsf{M}_{\mathcal{H},1}}{\times} A^\vee$. (See for example [10, 1.5].) Under the assumption that λ has degree prime-to-p, we know that

λ is separable, that λ_* is an isomorphism, and hence that the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ above is *perfect.* Let $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ also denote the induced pairing on $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \times \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$ by duality. By [4, Lem. 2.5.3], we have canonical short exact sequences $0 \to \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}} \to \underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1}) \to \underline{\mathrm{Lie}}_{A/\mathsf{M}_{\mathcal{H},1}} \to 0$ and $0 \to \underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_{\mathcal{H},1}} \to \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \to \underline{\mathrm{Lie}}_{A^\vee/\mathsf{M}_{\mathcal{H},1}} \to 0$. The submodules $\underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}}$ and $\underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_{\mathcal{H},1}}$ are maximal totally isotropic with respect to $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$.

Let $\tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1}$ be the first infinitesimal neighborhood of the diagonal image of $\mathsf{M}_{\mathcal{H},1}$ in $\mathsf{M}_{\mathcal{H},1} \underset{\mathsf{S}_1}{\times} \mathsf{M}_{\mathcal{H},1}$, and let $\mathrm{pr}_1, \mathrm{pr}_2 : \tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1} \to \mathsf{M}_{\mathcal{H},1}$ be the two projections. Then we have by definition the canonical morphism $\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}} \to \mathscr{P}^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1} := \mathrm{pr}_{1,*}\,\mathrm{pr}_2^*(\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}})$, where $\mathscr{P}^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}$ is the sheaf of principal parts of order one. The isomorphism $s : \tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1} \to \tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1}$ over $\mathsf{M}_{\mathcal{H},1}$ swapping the two components of the fiber product then defines an automorphism s^* of $\mathscr{P}^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}$. The kernel of the structural morphism $\mathrm{str}^* : \mathscr{P}^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1} \to \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}$, canonically isomorphic to $\Omega^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}$ by definition, is spanned by the image of $s^* - \mathrm{Id}^*$ (induced by $\mathrm{pr}_1^* - \mathrm{pr}_2^*$).

An important property of the relative de Rham cohomology of any smooth morphism like $A \to \mathsf{M}_{\mathcal{H},1}$ is that, for any two smooth lifts $\tilde{A}_1 \to \tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1}$ and $\tilde{A}_2 \to \tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1}$ of $A \to \mathsf{M}_{\mathcal{H},1}$, there is a canonical isomorphism $\underline{H}^1_{\mathrm{dR}}(\tilde{A}_2/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1}) \overset{\sim}{\to} \underline{H}^1_{\mathrm{dR}}(\tilde{A}_1/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1})$ lifting the identity morphism on $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$. (See for example [29, Prop. 2.1.6.4].) If we consider $\tilde{A}_1 := \mathrm{pr}_1^* A$ and $\tilde{A}_2 := \mathrm{pr}_2^* A$, then we obtain a canonical isomorphism $\mathrm{pr}_2^* \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \cong \underline{H}^1_{\mathrm{dR}}(\mathrm{pr}_2^* A/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1}) \overset{\sim}{\to} \underline{H}^1_{\mathrm{dR}}(\mathrm{pr}_1^* A/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1}) \cong \mathrm{pr}_1^* \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$, which we denote by Id^* by abuse of notation. On the other hand, the pullback by the swapping automorphism $s : \tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1} \overset{\sim}{\to} \tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1}$ defines another canonical isomorphism $s^* : \mathrm{pr}_2^* \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \cong \underline{H}^1_{\mathrm{dR}}(\mathrm{pr}_2^* A/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1}) \overset{\sim}{\to} \underline{H}^1_{\mathrm{dR}}(\mathrm{pr}_1^* A/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1}) \cong \mathrm{pr}_1^* \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$.

Definition 1.11. *The* **Gauss–Manin connection** $\nabla : \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \to \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}$ *on* $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$ *is the composition*

$$\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \overset{\mathrm{pr}_2^*}{\to} \underline{H}^1_{\mathrm{dR}}(\mathrm{pr}_2^* A/\tilde{\mathsf{M}}^{(1)}_{\mathcal{H},1}) \overset{s^*-\mathrm{Id}^*}{\to} \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}.$$

This connection coincides with the usual Gauss–Manin connection on the relative de Rham cohomology (cf. [25]).

1.3. **Automorphic bundles and de Rham complexes.**

Definition 1.12. *The* **principal** G_1**-bundle** *over* $\mathsf{M}_{\mathcal{H},1}$ *is the* G_1*-torsor*

$$\mathcal{E}_{\mathrm{G}_1} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}\big((\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1}), \langle\,\cdot\,,\,\cdot\,\rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)),$$
$$((L_{0,1} \oplus L^\vee_{0,1}(1)) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1))\big),$$

the sheaf of isomorphisms of $\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}$*-sheaves of symplectic* $\mathcal{O}$*-modules.*

The group G_1 acts as automorphisms on $(L \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1))$ by definition. The third entries in the tuples represent the values of the pairings. We allow isomorphisms of symplectic modules to modify the pairings up to units.

Definition 1.13. *The* **principal** P_1**-bundle** *over* $\mathsf{M}_{\mathcal{H},1}$ *is the* P_1*-torsor*

$$\begin{aligned}\mathcal{E}_{\mathrm{P}_1} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}((\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}), \langle\,\cdot\,,\,\cdot\,\rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}}),\\ ((L_{0,1} \oplus L_{0,1}^\vee(1)) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), L_{0,1}^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}})),\end{aligned}$$

the sheaf of isomorphisms of $\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}$*-sheaves of symplectic* $\mathcal{O}$*-modules with maximal totally isotropic* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R_1$*-submodules.*

Similarly to the previous definition, the group P_1 acts as automorphisms on $(L \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), L_{0,1}^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}})$ by definition.

The principal bundles $\mathcal{E}_{\mathrm{G}_1}$ and $\mathcal{E}_{\mathrm{P}_1}$ are (étale) torsors (of the respective group schemes G_1 and P_1) because $(\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}), \langle\,\cdot\,,\,\cdot\,\rangle_\lambda, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}})$ and $((L_{0,1} \oplus L_{0,1}^\vee(1)) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{can.}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1), L_{0,1}^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}})$ are étale locally isomorphic by the theory of infinitesimal deformations (cf. for example [29, Ch. 2]) and the theory of Artin's approximations (cf. [1, Thm. 1.10 and Cor. 2.5]).

Definition 1.14. *The* **principal** M_1**-bundle** *over* $\mathsf{M}_{\mathcal{H},1}$ *is the* M_1*-torsor*

$$\mathcal{E}_{\mathrm{M}_1} := \underline{\mathrm{Isom}}_{\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}((\underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)), (L_{0,1}^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1))),$$

the sheaf of isomorphisms of $\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}$*-sheaves of* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} R_1$*-modules.*

We view the second entries in the pairs as an additional structure, inherited from the corresponding objects for P_1. The group M_1 acts as automorphisms on $(L_{0,1}^\vee(1) \underset{R_1}{\otimes} \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}, \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1))$ by definition.

Definition 1.15. *For any* R_1*-algebra* R*, we denote by* $\mathrm{Rep}_R(\mathrm{G}_1)$ *(resp.* $\mathrm{Rep}_R(\mathrm{P}_1)$*, resp.* $\mathrm{Rep}_R(\mathrm{M}_1)$*) the category of* R*-modules of finite presentation with algebraic actions of* $\mathrm{G}_1 \underset{R_1}{\otimes} R$ *(resp.* $\mathrm{P}_1 \underset{R_1}{\otimes} R$*, resp.* $\mathrm{M}_1 \underset{R_1}{\otimes} R$*).*

Definition 1.16. *Let* R *be any* R_1*-algebra. For any* $W \in \mathrm{Rep}_R(\mathrm{G}_1)$*, we define*

$$\mathcal{E}_{\mathrm{G}_1,R}(W) := (\mathcal{E}_{\mathrm{G}_1} \underset{R_1}{\otimes} R) \overset{\mathrm{G}_1 \underset{R_1}{\otimes} R}{\times} W, \tag{1.17}$$

and call it the **automorphic sheaf** *over* $\mathsf{M}_{\mathcal{H},1} \underset{R_1}{\otimes} R$ *associated with* W*. It is called an* **automorphic bundle** *if* W *is locally free as an* R*-module. We define similarly* $\mathcal{E}_{\mathrm{P}_1,R}(W)$ *(resp.* $\mathcal{E}_{\mathrm{M}_1,R}(W)$*) for* $W \in \mathrm{Rep}_R(\mathrm{P}_1)$ *(resp.* $W \in \mathrm{Rep}_R(\mathrm{M}_1)$*) by replacing* G_1 *with* P_1 *(resp. with* M_1*) in the above expression* (1.17)*.*

Lemma 1.18. *Let* R *be any* R_1*-algebra. The assignment* $\mathcal{E}_{\mathrm{G}_1,R}(\,\cdot\,)$ *(resp.* $\mathcal{E}_{\mathrm{P}_1,R}(\,\cdot\,)$*, resp.* $\mathcal{E}_{\mathrm{M}_1,R}(\,\cdot\,)$*) defines an* **exact functor** *from* $\mathrm{Rep}_R(\mathrm{G}_1)$ *(resp.* $\mathrm{Rep}_R(\mathrm{G}_1)$*, resp.* $\mathrm{Rep}_R(\mathrm{G}_1)$*) to the category of coherent sheaves on* $\mathsf{M}_{\mathcal{H},1}$*.*

Proof. Étale locally over $\mathsf{M}_{\mathcal{H},1}$, the principal bundle $\mathcal{E}_{\mathrm{G}_1,R}$ (resp. $\mathcal{E}_{\mathrm{P}_1,R}$, resp. $\mathcal{E}_{\mathrm{M}_1,R}$) is isomorphic to the pullback of G_1 (resp. P_1, resp. M_1) from $\mathsf{S}_1 = \mathrm{Spec}(R_1)$ to $\mathsf{M}_{\mathcal{H},1}$. Therefore, $\mathcal{E}_{\mathrm{G}_1,R}(W)$ (resp. $\mathcal{E}_{\mathrm{P}_1,R}(W)$, resp. $\mathcal{E}_{\mathrm{M}_1,R}(W)$) is locally isomorphic to the pullback of W from S_1 to $\mathsf{M}_{\mathcal{H},1}$, and the assignment is functorial and exact because $\mathsf{M}_{\mathcal{H},1} \to \mathsf{S}_1$ is flat. $\square$

Lemma 1.19. *Let R be any R_1-algebra. If we consider an object $W \in \mathrm{Rep}_R(\mathrm{G}_1)$ as an object in $\mathrm{Rep}_R(\mathrm{P}_1)$ by restriction to P_1, then we have a canonical isomorphism $\mathcal{E}_{\mathrm{G}_1,R}(W) \cong \mathcal{E}_{\mathrm{P}_1,R}(W)$.*

Proof. By definition, we have a natural morphism $\mathcal{E}_{\mathrm{P}_1,R} \times W \to \mathcal{E}_{\mathrm{G}_1,R} \times W$ inducing a natural morphism $\mathcal{E}_{\mathrm{P}_1,R}(W) \to \mathcal{E}_{\mathrm{G}_1,R}(W)$. Reasoning as in the proof of Lemma 1.18, we see that this morphism is an isomorphism, because it is étale locally identified with the identity morphism $W \to W$. □

Lemma 1.20. *Let R be any R_1-algebra. If we view an object $W \in \mathrm{Rep}_R(\mathrm{M}_1)$ as an object in $\mathrm{Rep}_R(\mathrm{P}_1)$ in the canonical way (under the canonical surjection $\mathrm{P}_1 \twoheadrightarrow \mathrm{M}_1$), then we have a canonical isomorphism $\mathcal{E}_{\mathrm{P}_1,R}(W) \cong \mathcal{E}_{\mathrm{M}_1,R}(W)$.*

Proof. This follows from the very definitions of $\mathcal{E}_{\mathrm{P}_1}$ and $\mathcal{E}_{\mathrm{M}_1}$. □

Corollary 1.21. *Let R be any R_1-algebra. Suppose $W \in \mathrm{Rep}_R(\mathrm{P}_1)$ has a decreasing filtration by subobjects $\mathtt{F}^a(W) \subset W$ in $\mathrm{Rep}_R(\mathrm{P}_1)$ such that each graded piece $\mathrm{Gr}^a_{\mathtt{F}}(W) := \mathtt{F}^a(W)/\mathtt{F}^{a+1}(W)$ can be identified with an object of $\mathrm{Rep}_R(\mathrm{M}_1)$. Then $\mathcal{E}_{\mathrm{P}_1,R}(W)$ has a filtration $\mathcal{E}_{\mathrm{P}_1,R}(\mathtt{F}^a(W))$ with graded pieces $\mathcal{E}_{\mathrm{M}_1,R}(\mathrm{Gr}^a_{\mathtt{F}}(W))$.*

Proof. This follows from the exactness of the functor $\mathcal{E}_{\mathrm{P}_1,R}$ in Lemma 1.18. □

Example 1.22. We have $\mathcal{E}_{\mathrm{G}_1,R_1}(L_1) \cong \mathcal{E}_{\mathrm{P}_1,R_1}(L_1) \cong \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$, with Hodge filtration defined by the submodule $\mathcal{E}_{\mathrm{P}_1,R_1}(L_{0,1}^\vee(1)) \cong \mathcal{E}_{\mathrm{M}_1,R_1}(L_{0,1}^\vee(1)) \cong \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}}$, and with top graded piece $\mathcal{E}_{\mathrm{P}_1,R_1}(L_{0,1}) \cong \mathcal{E}_{\mathrm{M}_1,R_1}(L_{0,1}) \cong \underline{\mathrm{Lie}}_{A/\mathsf{M}_{\mathcal{H},1}}$.

In Definition 1.11, the Gauss–Manin connection is defined by the difference between the two isomorphisms $\mathrm{Id}^*, s^* : \mathrm{pr}_2^* \, \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \xrightarrow{\sim} \mathrm{pr}_1^* \, \underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$ lifting the identity morphism on $\underline{H}^1_{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$. Since s^* has a simple definition, we can interpret Id^* (whose definition as in [29, Prop. 2.1.6.4] is far from simple) as induced by the Gauss–Manin connection (and s^*). The same is true if we base change (horizontally) from R_1 to any R_1-algebra R. By construction of $\mathcal{E}_{\mathrm{G}_1,R}(\,\cdot\,)$ (cf. (1.17)), for any $W \in \mathrm{Rep}_R(\mathrm{G}_1)$, the two isomorphisms above induce two isomorphisms $\mathrm{Id}^*, s^* : \mathrm{pr}_2^*(\mathcal{E}_{\mathrm{G}_1,R}(W)) \xrightarrow{\sim} \mathrm{pr}_1^*(\mathcal{E}_{\mathrm{G}_1,R}(W))$ lifting the identity morphism on $\mathcal{E}_{\mathrm{G}_1,R}(W)$. Hence the difference $s^* - \mathrm{Id}^*$ induces an integrable connection

$$\nabla : \mathcal{E}_{\mathrm{G}_1,R}(W) \to \mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}. \tag{1.23}$$

Definition 1.24. *The integrable connection ∇ in (1.23) above is called the* **Gauss–Manin connection** *for $\mathcal{E}_{\mathrm{G}_1,R}(W)$. The complex $(\mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$ it induces is called the* **de Rham complex** *for $\mathcal{E}_{\mathrm{G}_1,R}(W)$.*

2. Representation theory

2.1. **Decomposition of reductive groups.** Using the decomposition of $\mathcal{O}_{F,1}$ in (1.5), we obtain a corresponding decomposition

$$\mathcal{O}_1 \cong \prod_{\tau \in \Upsilon} \mathcal{O}_\tau, \tag{2.1}$$

where $\mathcal{O}_F$ acts on the factor $\mathcal{O}_\tau$ via the homomorphism $\mathcal{O}_F \to \mathcal{O}_{F,\tau}$ defined by τ.

By [29, Lem. 1.1.3.4], there is a unique (up to isomorphism) indecomposable projective $\mathcal{O}_\tau$-module for each $\tau \in \Upsilon$, which we shall denote by V_τ. When $\mathcal{O}_\tau \cong$

$\mathrm{M}_{t_\tau}(\mathcal{O}_{F,\tau})$ for some t_τ, we can take V_τ to be $\mathcal{O}_{F,\tau}^{\oplus t_\tau}$. Moreover, every finitely generated projective $\mathcal{O}_1$-module is isomorphic to a direct sum $\underset{\tau\in\Upsilon}{\oplus} V_\tau^{\oplus m_\tau}$ for some integers m_τ. We call the tuple $(m_\tau)_{\tau\in\Upsilon}$ of integers the *multi-rank* of such an $\mathcal{O}\underset{\mathbb{Z}}{\otimes} R_1$-module. (See [29, Def. 1.1.3.5].)

Let $(p_\tau)_{\tau\in\Upsilon}$ (resp. $(q_\tau)_{\tau\in\Upsilon}$) be the multi-rank of $L_{0,1}$ (resp. $L_{0,1}^\vee(1)$). Then $q_\tau = p_{\tau\circ c}$, where $c : \mathcal{O}_F \xrightarrow{\sim} \mathcal{O}_F$ is the restriction of $\star : \mathcal{O} \xrightarrow{\sim} \mathcal{O}$. Then the multi-rank of L_1 is $(p_\tau + q_\tau)_{\tau\in\Upsilon}$, because we have the isomorphism (1.6) over R_1.

Choose and fix an isomorphism $L_{0,1} \cong \underset{\tau\in\Upsilon}{\oplus} V_\tau^{\oplus p_\tau}$, as well as the isomorphisms $V_{\tau\circ c}^\vee(1) := \mathrm{Hom}_{R_1}(V_{\tau\circ c}, R_1(1)) \cong V_\tau$ (for $\tau \in \Upsilon$). These chosen isomorphisms canonically induce an isomorphism

$$L_1 \cong \Big(\underset{\tau\in\Upsilon}{\oplus} V_\tau^{\oplus p_\tau}\Big) \oplus \Big(\underset{\tau\in\Upsilon}{\oplus} (V_{\tau\circ c}^\vee(1))^{\oplus q_\tau}\Big) \cong \underset{\tau\in\Upsilon}{\oplus} V_\tau^{\oplus(p_\tau+q_\tau)} \tag{2.2}$$

by (1.6), matching the pairing $\langle\,\cdot\,,\,\cdot\,\rangle$ with the pairing

$$(((x_{1,\tau}, f_{1,\tau\circ c}))_{\tau\in\Upsilon}, ((x_{2,\tau}, f_{2,\tau\circ c}))_{\tau\in\Upsilon}) \mapsto \sum_{\tau\in\Upsilon} (f_{2,\tau}(x_{1,\tau}) - f_{1,\tau}(x_{2,\tau})). \tag{2.3}$$

Lemma 2.4. *There exists a cocharacter $\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1 \to \mathrm{G}_1$ splitting the similitude character $\nu : \mathrm{G}_1 \to \mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1$ which acts trivially on the submodule $L_{0,1}^\vee(1)$ of L_1 (under the identification* (1.6)*).*

Proof. Let R be any R_1-algebra. Let t_0 be any element in $(\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1)(R) = R^\times$. In the decomposition (2.2), if we let t_0 act as t_0 on $V_\tau^{\oplus p_\tau}$, and act trivially on $(V_{\tau\circ c}^\vee(1))^{\oplus q_\tau}$, for any $\tau \in \Upsilon$, then the pairing (2.3) is multiplied by t_0. This gives an element in $\mathrm{G}_1(R)$ with similitude t_0 and with trivial action on $L_{0,1}^\vee(1)$, as desired. $\square$

For each $\tau \in \Upsilon$, set $L_\tau := V_\tau^{\oplus p_\tau} \oplus (V_{\tau\circ c}^\vee(1))^{\oplus q_\tau}$, and define the canonical pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\tau : L_\tau \times L_{\tau\circ c} \to R_1(1)$ by $((x_{1,\tau}, f_{1,\tau\circ c}), (x_{2,\tau\circ c}, f_{2,\tau})) \mapsto f_{2,\tau}(x_{1,\tau}) - f_{1,\tau\circ c}(x_{2,\tau\circ c})$. Then the pairing (2.3) is simply the sum of $\langle\,\cdot\,,\,\cdot\,\rangle_\tau$ over $\tau \in \Upsilon$. Note that $\mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes} R}(L_\tau \underset{R_1}{\otimes} R) \cong \mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes} R}(L_{\tau\circ c} \underset{R_1}{\otimes} R)$ for any R_1-algebra R. If we define

$$\mathrm{G}_\tau(R) := \Big\{ g \in \mathrm{GL}_{\mathcal{O}\underset{\mathbb{Z}}{\otimes} R}(L_\tau \underset{R_1}{\otimes} R) : \langle gx, gy\rangle_\tau = \langle x, y\rangle_\tau,\ \forall x \in L_\tau, \forall y \in L_{\tau\circ c} \Big\}$$

for each R_1-algebra R, then we obtain a group functor G_τ over $\mathrm{Spec}(R_1)$, which falls into only three possible cases:

(1) $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, where $r_\tau = p_\tau = q_\tau$ and Sp_{2r_τ} is the (split) symplectic group of rank r_τ over $\mathrm{Spec}(\mathbb{Z})$. (This is a factor of type C.)

(2) $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, where $r_\tau = p_\tau = q_\tau$ and O_{2r_τ} is the (split) even orthogonal group of rank r_τ over $\mathrm{Spec}(\mathbb{Z})$. (This is a factor of type D.)

(3) $\mathrm{G}_\tau \cong \mathrm{GL}_{r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, where $r_\tau = p_\tau + q_\tau$ and GL_{r_τ} is the general linear group of rank r_τ over $\mathrm{Spec}(\mathbb{Z})$. (This is a factor of type A.)

Thus we obtain a decomposition

$$\mathrm{G}_1 \cong \Big(\prod_{\tau\in\Upsilon/c} \mathrm{G}_\tau\Big) \rtimes (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1), \tag{2.5}$$

where $\tau \in \Upsilon / c$ means (by abuse of language) we pick exactly one representative τ in its c-orbit in Υ, and where the last factor $(\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1)$ is given by the cocharacter given by Lemma 2.4 splitting the similitude character.

2.2. **Decomposition of parabolic subgroups.** Under the identification (1.6), the submodule $L_{0,1}^{\vee}(1)$ of L_1 can be identified with the submodule

$$0 \oplus \Big(\underset{\tau \in \Upsilon}{\oplus} (V_{\tau \circ c}^{\vee}(1))^{\oplus q_\tau} \Big) \tag{2.6}$$

of the second member in (2.2). For each $\tau \in \Upsilon$, define group functors P_τ and M_τ over $\mathrm{Spec}(R_1)$ by setting for each R_1-algebra R

(2.7)

$$\mathrm{P}_\tau(R) := \left\{ \begin{matrix} g \in \mathrm{G}_\tau(R) : g(0 \oplus (V_{\tau \circ c}^{\vee}(1))^{\oplus q_\tau} \underset{R_1}{\otimes} R)) = (0 \oplus (V_{\tau \circ c}^{\vee}(1))^{\oplus q_\tau} \underset{R_1}{\otimes} R)) \\ \text{in } L_\tau \underset{R_1}{\otimes} R = (V_\tau^{\oplus p_\tau} \underset{R_1}{\otimes} R) \oplus ((V_{\tau \circ c}^{\vee}(1))^{\oplus q_\tau} \underset{R_1}{\otimes} R) \end{matrix} \right\}$$

and

$$\mathrm{M}_\tau(R) := \left\{ \begin{matrix} g \in \mathrm{P}_\tau(R) : g((V_\tau^{\oplus p_\tau} \underset{R_1}{\otimes} R) \oplus 0) = ((V_\tau^{\oplus p_\tau} \underset{R_1}{\otimes} R) \oplus 0) \\ \text{in } L_\tau \underset{R_1}{\otimes} R = (V_\tau^{\oplus p_\tau} \underset{R_1}{\otimes} R) \oplus ((V_{\tau \circ c}^{\vee}(1))^{\oplus q_\tau} \underset{R_1}{\otimes} R) \end{matrix} \right\}. \tag{2.8}$$

Then the subgroup P_1 of G_1 can be identified with the subgroup

$$\Big(\prod_{\tau \in \Upsilon / c} \mathrm{P}_\tau \Big) \rtimes (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1) \subset \Big(\prod_{\tau \in \Upsilon / c} \mathrm{G}_\tau \Big) \rtimes (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1),$$

and the canonical surjection $\mathrm{P}_1 \twoheadrightarrow \mathrm{M}_1$ has a splitting $\mathrm{M}_1 \subset \mathrm{P}_1$ given by

$$\Big(\prod_{\tau \in \Upsilon / c} \mathrm{M}_\tau \Big) \rtimes (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1) \subset \Big(\prod_{\tau \in \Upsilon / c} \mathrm{P}_\tau \Big) \rtimes (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1).$$

For each $\tau \in \Upsilon$, we have $\mathrm{Hom}_{\mathcal{O}_1}(V_\tau, V_\tau) \cong \mathrm{Hom}_{\mathcal{O}_1}(V_{\tau \circ c}^{\vee}(1), V_{\tau \circ c}^{\vee}(1)) \cong \mathcal{O}_{F,\tau} \cong R_1$. Therefore, we have diagonal actions of $\mathbf{G}_{\mathrm{m}}^{p_\tau}(R)$ on $V_\tau^{\oplus p_\tau} \underset{R_1}{\otimes} R$ and of $\mathbf{G}_{\mathrm{m}}^{q_\tau}(R)$ on $(V_{\tau \circ c}^{\vee}(1))^{\oplus q_\tau} \underset{R_1}{\otimes} R$, which are functorial in R and hence define a homomorphism $(\mathbf{G}_{\mathrm{m}}^{p_\tau} \times \mathbf{G}_{\mathrm{m}}^{q_\tau}) \underset{\mathbb{Z}}{\otimes} R_1 \to \mathrm{M}_\tau$.

2.3. **Hodge filtration.** Let R be any R_1-algebra. Fix any choice of a cocharacter as in Lemma 2.4, and consider its reciprocal $H : \mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1 \to \mathrm{G}_1$. Note that by definition H factors through P_1.

Definition 2.9. *Given any object $W \in \mathrm{Rep}_R(\mathrm{P}_1)$, the induced action of $\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1$ decomposes W into weight spaces $W^{(a)}$ for $\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1$, indexed by integers. Then the Hodge filtration $\mathtt{F}$ on W is the decreasing filtration $\mathtt{F}(W) = \{\mathtt{F}^a(W)\}_{a \in \mathbb{Z}}$ defined by $\mathtt{F}^a(W) := \underset{b \geq a}{\oplus} W^{(b)}$.*

Example 2.10. Since the cocharacter H acts with weight 0 on $L_{0,1}^{\vee}(1)$ (as a submodule of L_1) and with weight -1 on $L_{0,1}$ (as a quotient module of L_1), the Hodge filtration $\mathtt{F}$ on L_1 is given by $\mathtt{F}^{-1}(L_1) = L_1$, $\mathtt{F}^0(L_1) = L_{0,1}^{\vee}(1)$, and $\mathtt{F}^1(L_1) = \{0\}$. Thus the only possibly nonzero graded pieces are $\mathrm{Gr}_{\mathtt{F}}^{-1}(L_1) = L_{0,1}$ and $\mathrm{Gr}_{\mathtt{F}}^0(L_1) = L_{0,1}^{\vee}(1)$. Note that the choice of H is not unique, but the resulting filtration is independent of this choice.

Lemma 2.11. *Let $W \in \mathrm{Rep}_R(\mathrm{P}_1)$ and let $\{\mathtt{F}^a(W)\}_{a\in\mathbb{Z}}$ denote the Hodge filtration defined in Definition 2.9. Then the unipotent radical U_1 of P_1 acts trivially on $\mathrm{Gr}^a_{\mathtt{F}}(W)$ for any $a \in \mathbb{Z}$. In other words, each graded piece $\mathrm{Gr}^a_{\mathtt{F}}(W)$ can be identified with an object in $\mathrm{Rep}_R(\mathrm{M}_1)$.*

Proof. Since the adjoint action of H on $\mathrm{Lie}(\mathrm{U}_1)$ has weight -1, the action of $\mathrm{Lie}(\mathrm{U}_1)$ decreases the H-weights by 1, as desired. □

By Corollary 1.21, the Hodge filtration on W defines submodules of $\mathcal{E}_{\mathrm{P}_1,R}(W)$, which we denote by $\mathtt{F}^a(\mathcal{E}_{\mathrm{P}_1,R}(W))$ for $a \in \mathbb{Z}$.

Definition 2.12. *The filtration $\mathtt{F}(\mathcal{E}_{\mathrm{P}_1,R}(W)) = \{\mathtt{F}^a(\mathcal{E}_{\mathrm{P}_1,R}(W))\}_{a\in\mathbb{Z}}$ is called the* **Hodge filtration** *on $\mathcal{E}_{\mathrm{P}_1,R}(W)$.*

By Corollary 1.21, we have $\mathrm{Gr}^a_{\mathtt{F}}(\mathcal{E}_{\mathrm{P}_1,R}(W)) \cong \mathcal{E}_{\mathrm{M}_1,R}(\mathrm{Gr}^a_{\mathtt{F}}(W))$.

Definition 2.13. *Let $W \in \mathrm{Rep}_R(\mathrm{G}_1)$. By considering W as an object of $\mathrm{Rep}_R(\mathrm{P}_1)$ by restriction from G_1 to P_1, we can define the Hodge filtration on $\mathcal{E}_{\mathrm{G}_1,R}(W) \cong \mathcal{E}_{\mathrm{P}_1,R}(W)$ (see Lemma 1.19) as in Definition 2.12. The Hodge filtration on the de Rham complex $\mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ is defined by*

$$\mathtt{F}^a(\mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}) := \mathtt{F}^{a-\bullet}\mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$$

It is a subcomplex of $\mathcal{E}_{\mathrm{G}_1,R}(W) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}$ for the Gauss–Manin connection thanks to the Griffiths transversality. (The only de Rham complexes we will need for our main results are those realized by geometric plethysm as in Lemma 4.7 below, for which the Griffiths transversality is clear. For de Rham complexes attached to an arbitrary $W \in \mathrm{Rep}_R(\mathrm{G}_1)$, see [30].)

Lemma 2.14. *Suppose W_1 and W_2 are two objects in $\mathrm{Rep}_R(\mathrm{G}_1)$ such that the induced actions of P_1 and $\mathrm{Lie}(\mathrm{G}_1)$ on them satisfy $W_1|_{\mathrm{P}_1} \cong W_2|_{\mathrm{P}_1}$ and $W_1|_{\mathrm{Lie}(\mathrm{G}_1)} \cong W_2|_{\mathrm{Lie}(\mathrm{G}_1)}$. Then we have a canonical isomorphism $(\mathcal{E}_{\mathrm{G}_1,R}(W_1) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla) \cong (\mathcal{E}_{\mathrm{G}_1,R}(W_2) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}, \nabla)$ respecting the Hodge filtrations on both sides.*

Proof. By Lemma 1.19, we have isomorphisms $\mathtt{F}^a(\mathcal{E}_{\mathrm{G}_1,R}(W_1) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^b_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R}) \cong \mathtt{F}^a(\mathcal{E}_{\mathrm{G}_1,R}(W_2) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^b_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R})$ between the individual terms because they are defined by P_1-modules. Then the lemma is true because the definition of the connections only involves differentials on $\mathsf{M}_{\mathcal{H},R}$ and $\mathrm{G}_1 \underset{R_1}{\otimes} R$ (relative to R). □

Remark 2.15. Lemma 2.14 will be needed only when G_1 is not connected, i.e. when $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ involves simple factors of type D (as in [29, Def. 1.2.1.15]).

While we claim that the two automorphic bundles in Lemma 2.14 are isomorphic as abstract vector bundles with integrable connections, we do not claim that the Hecke operators on their cohomology are identical. This is harmless for our purpose, but the reader should not make similar identifications for questions about the Galois or Hecke actions.

2.4. **Roots and weights.** We shall choose a maximal torus T_τ of M_τ by choosing a subgroup of $(\mathbf{G}_{\mathrm{m}}^{p_\tau} \times \mathbf{G}_{\mathrm{m}}^{q_\tau}) \underset{\mathbb{Z}}{\otimes} R_1$ that embeds into M_τ under the natural homomorphism $(\mathbf{G}_{\mathrm{m}}^{p_\tau} \times \mathbf{G}_{\mathrm{m}}^{q_\tau}) \underset{\mathbb{Z}}{\otimes} R_1 \to \mathrm{M}_\tau$ defined at the end of Section 2.2. There are two cases:

(1) If $\tau = \tau \circ c$, then $p_\tau = q_\tau$ and we take $\mathrm{T}_\tau = \{t_\tau = (t_{\tau,i_\tau})_{1 \leq i_\tau \leq r_\tau}\}$, embedded in $(\mathbf{G}_{\mathrm{m}}^{p_\tau} \times \mathbf{G}_{\mathrm{m}}^{q_\tau}) \underset{\mathbb{Z}}{\otimes} R_1$ by $t_\tau \mapsto (t_\tau^{-1}, t_\tau)$.

(2) If $\tau \neq \tau \circ c$, then we take $\mathrm{T}_\tau = \{t_\tau = (t_{\tau,i_\tau})_{1 \leq i_\tau \leq r_\tau}\}$ and identify it with $(\mathbf{G}_{\mathrm{m}}^{p_\tau} \times \mathbf{G}_{\mathrm{m}}^{q_\tau}) \underset{\mathbb{Z}}{\otimes} R_1$ by sending $(t_{\tau,i_\tau})_{1 \leq i_\tau \leq r_\tau}$ to $((t_{\tau,q_\tau+i_\tau}^{-1})_{1 \leq i_\tau \leq p_\tau}, (t_{\tau,i_\tau})_{1 \leq i_\tau \leq q_\tau})\}$.

We take $\mathrm{T}_1 \subset \mathrm{M}_1$ to be the subgroup corresponding to

$$\Big(\prod_{\tau \in \Upsilon/c} \mathrm{T}_\tau \Big) \times (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1) \subset \Big(\prod_{\tau \in \Upsilon/c} \mathrm{M}_\tau \Big) \rtimes (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1). \tag{2.16}$$

Then the split torus T_1 is a maximal torus in both M_1 and G_1 (this can be seen by comparing the ranks).

Elements in T_1 can be written as $t = ((t_\tau)_{\tau \in \Upsilon}; t_0) = (((t_{\tau,i_\tau})_{1 \leq i_\tau \leq r_\tau})_{\tau \in \Upsilon}; t_0)$, and hence elements in the character group $\mathrm{X} := \mathrm{Hom}_{R_1}(\mathrm{T}_1, \mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1)$ of T_1 are of the from $\mu = ((\mu_\tau)_{\tau \in \Upsilon/c}; \mu_0) = (((\mu_{\tau,i_\tau})_{1 \leq i_\tau \leq r_\tau})_{\tau \in \Upsilon/c}; \mu_0)$, sending $t \mapsto (\prod_{\tau \in \Upsilon/c} \mu_\tau(t_\tau))\, \mu_0(t_0) = (\prod_{\tau \in \Upsilon/c} \prod_{1 \leq i_\tau \leq r_\tau} t_{\tau,i_\tau}^{\mu_{\tau,i_\tau}})\, t_0^{\mu_0}$.

Let $\mathrm{X}^\vee := \mathrm{Hom}_{R_1}(\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1, \mathrm{T}_1)$ be the cocharacter group of T_1, and let $(\,\cdot\,,\,\cdot\,) : \mathrm{X} \times \mathrm{X}^\vee \to \mathbb{Z}$ be the canonical pairing between X and $\mathrm{X}^\vee$ defined by sending $(\mu, \nu^\vee) \in \mathrm{X} \times \mathrm{X}^\vee$ to $\mu \circ \nu^\vee \in \mathrm{Hom}_{R_1}(\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1, \mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1) \cong \mathbb{Z}$ (matching the identity morphism with 1). Let $\Phi_{\mathrm{G}_1} \subset \mathrm{X}$ (resp. $\Phi_{\mathrm{G}_1}^\vee \subset \mathrm{X}^\vee$) be the roots (resp. coroots) of the split reductive group scheme G_1 over $\mathrm{Spec}(R_1)$. For any root $\alpha \in \Phi_{\mathrm{G}_1}$, we shall denote by $\alpha^\vee \in \Phi_{\mathrm{G}_1}^\vee$ the associated coroot.

The choice of the positive roots $\Phi_{\mathrm{G}_1}^+$ in Φ_{G_1} corresponds to the choice of a Borel subgroup B_1 in G_1. By choosing B_1 to contain the unipotent radical U_1 of P_1 (using the explicit identifications in (2.5), (2.7), (2.8), and (2.16)), we can choose $\Phi_{\mathrm{G}_1}^+$ such that the set $\mathrm{X}_{\mathrm{G}_1}^+$ of dominant weights of G_1 consists of those $\mu \in \mathrm{X}$ as above with $\mu_{\tau,i_\tau} \geq \mu_{\tau,i_\tau+1}$ for any $\tau \in \Upsilon/c$ and for any $1 \leq i_\tau < r_\tau$, satisfying in addition:

(1) If $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, then $\mu_{\tau,r_\tau} \geq 0$.

(2) If $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, then $\mu_{\tau,r_\tau-1} \geq |\mu_{\tau,r_\tau}|$.

(If $\mathrm{G}_\tau \cong \mathrm{GL}_{r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, then there is no other condition on μ_τ.)

Remark 2.17. When $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ for some $\tau \in \Upsilon$, irreducible algebraic representations of G_τ are not exactly parameterized by dominant weights, due to the presence of an element in $\mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ flipping the two weights $\mu_\tau = (\mu_{\tau,1}, \ldots, \mu_{\tau,r_\tau-1}, \mu_{\tau,r_\tau})$ and $(\mu_{\tau,1}, \ldots, \mu_{\tau,r_\tau-1}, -\mu_{\tau,r_\tau})$. (A concise discussion on this matter can be found in [17, §5.5.5].) By Lemma 2.14, two representations of $\mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ will serve the same purpose for us if their restrictions to $\mathrm{SO}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ are isomorphic. Therefore, in what follows, we will denote by $[\mu]$ the set of highest dominant weights that appear in the irreducible representation of G_1 containing the dominant weight μ. This does not, for example, distinguish the determinant representation of $\mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ from the trivial representation, but will be sufficient for our purpose. Then there is always a unique μ' in $[\mu]$ satisfying the additional condition that $\mu'_{\tau,r_\tau} \geq 0$ for any $\tau \in \Upsilon$ such that $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$.

Let Φ_{M_1} be the roots of the split reductive group scheme M over $\mathrm{Spec}(R_1)$. Then intersection of M_1 (realized as a subgroup in P_1 as above) with the B_1 chosen above determines a set of positive roots $\Phi^+_{\mathrm{M}_1}$ in Φ_{M_1}, so that $\Phi^+_{\mathrm{M}_1} = \Phi_{\mathrm{M}_1} \cap \Phi^+_{\mathrm{G}_1}$. The set $\mathrm{X}^+_{\mathrm{M}_1}$ of dominant weights of M_1 consists of those $\mu \in \mathrm{X}$ as above with $\mu_{\tau,i_\tau} \geq \mu_{\tau,i_\tau+1}$ for any $\tau \in \Upsilon/c$ and for any $1 \leq i_\tau < q_\tau$ or $q_\tau < i_\tau < r_\tau$.

It is conventional to say that a root $\alpha \in \Phi_{\mathrm{G}_1}$ is *compact* if it is an element of Φ_{M_1}, and that α is *non-compact* otherwise. We denote the non-compact roots of Φ_{G_1} by Φ^{M_1}, and denote the collection of positive non-compact roots by $\Phi^{\mathrm{M}_1,+}$. For negative roots, we replace $+$ with $-$ in the above notation.

Let $\mathrm{W}_{\mathrm{G}_1}$ (resp. $\mathrm{W}_{\mathrm{M}_1}$) be the Weyl group of G_1 (resp. of M_1). The realization of M_1 as a subgroup of G_1 containing T_1 identifies $\mathrm{W}_{\mathrm{M}_1}$ as a subgroup of $\mathrm{W}_{\mathrm{G}_1}$. We define

$$\mathrm{W}^{\mathrm{M}_1} := \{w \in \mathrm{W}_{\mathrm{G}_1} : w(\mathrm{X}^+_{\mathrm{G}_1}) \subset \mathrm{X}^+_{\mathrm{M}_1}\}.$$

Then any element w in $\mathrm{W}_{\mathrm{G}_1}$ has a unique expression as $w = w_1 w_2$ with $w_1 \in \mathrm{W}_{\mathrm{M}_1}$ and $w_2 \in \mathrm{W}^{\mathrm{M}_1}$. Let $\rho := \frac{1}{2} \sum\limits_{\alpha \in \Phi^+_{\mathrm{G}_1}} \alpha$. The *dot action* of $\mathrm{W}_{\mathrm{G}_1}$ (and hence the subset $\mathrm{W}^{\mathrm{M}_1}$ of it) is defined by $w \cdot \mu := w(\mu + \rho) - \rho$ for any $w \in \mathrm{W}_{\mathrm{G}_1}$.

2.5. **Plethysm for representations.** In this subsection, we denote by GL_r, Sp_{2r}, O_{2r}, etc, the split forms of the groups over $\mathbb{Z}$, and we denote the base change to other rings by subscripts. We shall explain in our context the construction of representations of classical groups using Weyl's invariant theory. (It may be helpful to consult [15], [17], [20], and [47] for more information.)

Let $r \geq 0$ be any integer, and let $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$ be any tuple of integers satisfying $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_r$. We know that ν is the weight of an algebraic irreducible $\mathbb{Q}$-representation of $\mathrm{GL}_{r,\mathbb{Q}}$. Let us define $|\nu| := \sum\limits_{1 \leq i \leq r} \nu_i$. If $\nu_r \geq 0$, we say the tuple ν and the corresponding $\mathbb{Q}$-representation are *polynomial*, and write $\nu \geq 0$.

For any polynomial weight ν, we plot the so-called *Young diagram* by putting ν_1 blocks in the first row, ν_2 in the second rows, and so on. By filling in numbers (in arbitrary order) from 1 to $|\nu|$, we obtain a so-called *Young tableau* for ν. (See, e.g., [15, p. 45].) We shall denote a particular choice of Young tableau of ν by D_ν. Let $\mathfrak{S}_{|\nu|}$ denote the symmetric group of permutations on $\{1, 2, \ldots, |\nu|\}$. Based on the choice of D_ν, we define $\mathfrak{P}_{\mathrm{D}_\nu}$ (resp. $\mathfrak{Q}_{\mathrm{D}_\nu}$) to be the subgroup of $\mathfrak{S}_{|\nu|}$ consisting of elements permuting numbers in each row (resp. column) of D_ν. Let $\mathbb{Z}[\mathfrak{S}_{|\nu|}]$ be the group algebra with generators e_h for each $h \in \mathfrak{S}_{|\nu|}$. Let us define $\mathrm{a}_{\mathrm{D}_\nu} := \sum\limits_{h \in \mathfrak{P}_{\mathrm{D}_\nu}} \mathrm{e}_h$ and $\mathrm{b}_{\mathrm{D}_\nu} := \sum\limits_{h \in \mathfrak{Q}_{\mathrm{D}_\nu}} \mathrm{sgn}(h)\, \mathrm{e}_h$. Then the *Young symmetrizer* is $\mathrm{c}_{\mathrm{D}_\nu} := \mathrm{a}_{\mathrm{D}_\nu} \mathrm{b}_{\mathrm{D}_\nu}$.

Lemma 2.18. *Let $n = |\nu|$. Then we have the following facts in $\mathbb{Z}[\mathfrak{S}_n]$:*

(1) $\mathrm{c}_{\mathrm{D}_\nu} \mathbb{Z}[\mathfrak{S}_n] \mathrm{c}_{\mathrm{D}_\nu} \subset \mathbb{Z}\mathrm{c}_{\mathrm{D}_\nu}$.

(2) $c_{D_\nu} c_{D_\nu} = d_{D_\nu} c_{D_\nu}$ *for some integer* d_{D_ν} *dividing* $n!$ (*i.e. factorial*).
(3) *Let* $D_{\nu'}$ *be the Young tableau for some* $\nu' \geq 0$ *with* $|\nu'| = n$. *Then* $c_{D_\nu} c_{D_{\nu'}} = 0$ *if* $\nu \neq \nu'$.
(4) *Let* $\mathbf{V}_{D_\nu} := \mathbb{Z}[\mathfrak{S}_n] c_{D_\nu}$. *Then* $\mathbf{V}_{D_\nu, \mathbb{Q}}$ *is an irreducible* $\mathbb{Q}$*-representation of* $\mathfrak{S}_n$, *and* $\mathbf{V}_{D_\nu, \mathbb{Q}} \cong \mathbf{V}_{D_{\nu'}, \mathbb{Q}}$ (*for some* $D_{\nu'}$ *with* $|\nu'| = n$) *if and only if* $\nu = \nu'$.

Proof. In [47, Ch. IV, §3] or [15, §4.2, Lem. 4.23, 4.25, and 4.26], variants of these are stated over $\mathbb{C}$, but the proofs are valid for our statements above over $\mathbb{Z}$ or $\mathbb{Q}$. $\square$

Let $\mathbb{V}_{\mathrm{std},r} := \mathbb{Z}^{\oplus r}$ be the standard representation of GL_r. Let $n \geq 0$ be any integer. Then $(g, h) \in \mathrm{GL}_r \times \mathfrak{S}_n$ acts on $\mathbb{V}_{\mathrm{std},r}^{\otimes n}$ by

$$g(v_1 \otimes v_2 \otimes \ldots \otimes v_n) := g(v_1) \otimes g(v_2) \otimes \ldots \otimes g(v_n)$$
$$h(v_1 \otimes v_2 \otimes \ldots \otimes v_n) := v_{h^{-1}(1)} \otimes v_{h^{-1}(2)} \otimes \ldots \otimes v_{h^{-1}(n)}$$

for any $v_1, v_2, \ldots, v_n \in \mathbb{V}_{\mathrm{std},r}$. (These relations are interpreted functorially.)

Proposition 2.19 (cf. [20, 2.4.3]). *There is an isomorphism*

$$\mathbb{V}_{\mathrm{std},r,\mathbb{Q}}^{\otimes n} \cong \bigoplus_{\nu \geq 0, |\nu| = n} (\mathbb{V}_{\nu,\mathbb{Q}} \underset{\mathbb{Q}}{\otimes} \mathbf{V}_{D_\nu}) \tag{2.20}$$

between $\mathbb{Q}$*-representations of* $\mathrm{GL}_{r,\mathbb{Q}} \times \mathfrak{S}_n$, *called* **Schur duality**, *where* $V_{\nu,\mathbb{Q}}$ *is the algebraic* $\mathbb{Q}$*-representation of* $\mathrm{GL}_{r,\mathbb{Q}}$ *of highest weight* ν, *and where* D_ν *is any Young tableau for* ν. *As a result, we obtain* **Weyl's construction**, *an isomorphism*

$$\mathbb{V}_{\nu,\mathbb{Q}} \cong c_{D_\nu} \mathbb{V}_{\mathrm{std},r,\mathbb{Q}}^{\otimes |\nu|} \tag{2.21}$$

between $\mathbb{Q}$*-representations of* $\mathrm{GL}_{r,\mathbb{Q}}$ *for any polynomial weight* ν *of* $\mathrm{GL}_{r,\mathbb{Q}}$.

Proof. The proof of (2.20) in [20, 2.4.3] is carried out over $\mathbb{C}$. Once (2.20) is known over $\mathbb{C}$, we know (2.21) over $\mathbb{C}$ by Lemma 2.18. Then (2.21) is true over $\mathbb{Q}$ because both sides of (2.21) are absolutely irreducible and defined over $\mathbb{Q}$, and hence (2.20) is also true over $\mathbb{Q}$. $\square$

Definition 2.22. *Let* $r \geq 0$ *be any integer. Let* $\mathbb{V}_{\mathrm{std},2r} = \mathbb{Z}^{\oplus 2r} \cong \mathbb{Z}^{\oplus r} \oplus \mathbb{Z}^{\oplus r}$ *be equipped with the standard symplectic pairing* $\langle \cdot, \cdot \rangle_{\mathrm{std}}$ *with matrix* $\left(\begin{smallmatrix} 0 & 1_r \\ -1_r & 0 \end{smallmatrix}\right)$, *and with the standard symmetric pairing* $[\cdot, \cdot]_{\mathrm{std}}$ *with matrix* $\left(\begin{smallmatrix} 0 & 1_r \\ 1_r & 0 \end{smallmatrix}\right)$. *Then we have a canonical action of* Sp_{2r} *on* $\mathbb{V}_{\mathrm{std},2r}$ *preserving* $\langle \cdot, \cdot \rangle_{\mathrm{std}}$, *and a canonical action of* O_{2r} *on* $\mathbb{V}_{\mathrm{std},2r}$ *preserving* $[\cdot, \cdot]_{\mathrm{std}}$. *For any integer* $n \geq 0$, *and for any* $1 \leq i < j \leq n$, *we define* $\phi_{i,j}^{\langle \cdot, \cdot \rangle} : \mathbb{V}_{\mathrm{std},2r}^{\otimes n} \to \mathbb{V}_{\mathrm{std},2r}^{\otimes (n-2)}$ *by*

$$\phi_{i,j}^{\langle \cdot, \cdot \rangle}(v_1 \otimes v_2 \otimes \ldots \otimes v_n) := \langle v_i, v_j \rangle (v_1 \otimes \ldots \otimes \hat{v}_i \otimes \ldots \otimes \hat{v}_j \otimes \ldots \otimes v_n),$$

and define similarly $\phi_{i,j}^{[\cdot, \cdot]} : \mathbb{V}_{\mathrm{std},2r}^{\otimes n} \to \mathbb{V}_{\mathrm{std},2r}^{\otimes (n-2)}$ *by replacing* $\langle v_i, v_j \rangle$ *with* $[v_i, v_j]$ *in the above expression.* (*Here* $\hat{v}_i$ *and* $\hat{v}_j$ *denote omissions of entries as usual. When* $n < 2$, *we declare* $\mathbb{V}_{\mathrm{std},2r}^{\otimes (n-2)} = 0$ *and hence* $\phi_{i,j}^{\langle \cdot, \cdot \rangle} = 0 = \phi_{i,j}^{[\cdot, \cdot]}$.) *Then we define* $\mathbb{V}_{\mathrm{std},2r}^{\langle n \rangle} := \bigcap_{1 \leq i < j \leq n} \ker(\phi_{i,j}^{\langle \cdot, \cdot \rangle})$ *and* $\mathbb{V}_{\mathrm{std},2r}^{[n]} := \bigcap_{1 \leq i < j \leq n} \ker(\phi_{i,j}^{[\cdot, \cdot]})$.

Note that $\mathbb{V}_{\mathrm{std},2r}$ is its own dual under either $\langle \cdot, \cdot \rangle$ or $[\cdot, \cdot]$. Therefore, the maps $\phi_{i,j}^{\langle \cdot, \cdot \rangle}$ and $\phi_{i,j}^{[\cdot, \cdot]}$ define, by duality, the maps $\psi_{i,j}^{\langle \cdot, \cdot \rangle} : \mathbb{V}_{\mathrm{std},2r}^{\otimes (n-2)} \to \mathbb{V}_{\mathrm{std},2r}^{\otimes n}$ and $\psi_{i,j}^{[\cdot, \cdot]} : \mathbb{V}_{\mathrm{std},2r}^{\otimes (n-2)} \to \mathbb{V}_{\mathrm{std},2r}^{\otimes n}$, respectively, by inserting the pairings into the i-th and j-th factors. (See [15, §17.3 and §19.5].) By taking a standard symplectic basis as in the proof of [15, (17.12)], we see that $\phi_{i,j}^{\langle \cdot, \cdot \rangle} \psi_{i,j}^{\langle \cdot, \cdot \rangle} = 2r$, and hence $(2r)\ker(\phi_{i,j}^{\langle\,\cdot\,,\,\cdot\,\rangle}) \subset ((2r)\operatorname{Id}-\psi_{i,j}^{\langle\,\cdot\,,\,\cdot\,\rangle}\phi_{i,j}^{\langle\,\cdot\,,\,\cdot\,\rangle})(\mathbb{V}_{\mathrm{std},2r}^{\otimes|\nu|}) \subset \ker(\phi_{i,j}^{\langle\,\cdot\,,\,\cdot\,\rangle})$. Similarly, $\phi_{i,j}^{[\,\cdot\,,\,\cdot\,]}\psi_{i,j}^{[\,\cdot\,,\,\cdot\,]} = 2r$ and hence $(2r)\ker(\phi_{i,j}^{[\,\cdot\,,\,\cdot\,]}) \subset ((2r)\operatorname{Id}-\psi_{i,j}^{[\,\cdot\,,\,\cdot\,]}\phi_{i,j}^{[\,\cdot\,,\,\cdot\,]})(\mathbb{V}_{\mathrm{std},2r}^{\otimes|\nu|}) \subset \ker(\phi_{i,j}^{[\,\cdot\,,\,\cdot\,]})$. These relations will be especially useful when $2r$ is invertible in the rings we consider. (See Section 3.4 below.)

Proposition 2.23. *Let $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$ be the weight of an irreducible algebraic $\mathbb{Q}$-representation $\mathbb{V}_{\nu,\mathbb{Q}}^{\langle\,\cdot\,,\,\cdot\,\rangle}$ of $\mathrm{Sp}_{2r,\mathbb{Q}}$ satisfying $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_r \geq 0$. We view ν as a polynomial weight of GL_{2r} by supplying zeros in the end. Then we have an isomorphism $\mathbb{V}_{\nu,\mathbb{Q}}^{\langle\,\cdot\,,\,\cdot\,\rangle} \cong \mathbb{V}_{\mathrm{std},2r,\mathbb{Q}}^{\langle|\nu|\rangle} \cap (\mathrm{c}_{\mathrm{D}_\nu}\mathbb{V}_{\mathrm{std},2r,\mathbb{Q}}^{\otimes|\nu|})$ between $\mathbb{Q}$-representations of $\mathrm{Sp}_{2r,\mathbb{Q}}$ for any choice of Young tableau D_ν for ν.*

Proof. This is stated (without proof) in [47, Ch. VI, §3] and proved in [15, Thm. 17.11] over $\mathbb{C}$. It is then valid over $\mathbb{Q}$ because both sides of the isomorphism are absolutely irreducible and defined over $\mathbb{Q}$. □

Proposition 2.24. *Let γ_r be the element of $\mathrm{O}_{2r,\mathbb{Q}}$ flipping the two weights $(\mu_1, \mu_2, \ldots, \mu_r)$ and $(\mu_1, \mu_2, \ldots, -\mu_r)$ of $\mathrm{O}_{2r,\mathbb{Q}}$ for any integers $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_r \geq 0$ (cf. Remark 2.17). Let $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$ be the weight of an irreducible algebraic $\mathbb{Q}$-representation $\mathbb{V}_{\nu,\mathbb{Q}}^{[\,\cdot\,,\,\cdot\,]}$ of $\mathrm{O}_{2r,\mathbb{Q}}$ satisfying $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_{r-1} \geq \nu_r \geq 0$. When $\nu_r = 0$, we require moreover that the action of γ_r is trivial on $\mathbb{V}_{\nu,\mathbb{Q}}^{[\,\cdot\,,\,\cdot\,]}$. We view ν as a polynomial weight of GL_{2r} by supplying zeros in the end. Then we have an isomorphism $\mathbb{V}_{\nu,\mathbb{Q}}^{[\,\cdot\,,\,\cdot\,]} \cong \mathbb{V}_{\mathrm{std},2r,\mathbb{Q}}^{[|\nu|]} \cap (\mathrm{c}_{\mathrm{D}_\nu}\mathbb{V}_{\mathrm{std},2r,\mathbb{Q}}^{\otimes|\nu|})$ between $\mathbb{Q}$-representations of $\mathrm{O}_{2r,\mathbb{Q}}$ for any choice of Young tableau D_ν for ν.*

Proof. This is proved in [47, Ch. V, §7] and stated (without proof) in [15, Thm. 19.19] over $\mathbb{C}$. A modern treatment can be found in [17, §10.2.5]. It is then valid over $\mathbb{Q}$ because both sides of the isomorphism are absolutely irreducible and defined over $\mathbb{Q}$. □

Remark 2.25. When $\nu_r = 0$, there is another irreducible representation of $\mathrm{O}_{2r,\mathbb{Q}}$ containing the weight ν, on which γ_r acts nontrivially. According to [17, §10.2.5], it is isomorphic to $\mathbb{V}_{\mathrm{std},2r,\mathbb{Q}}^{[|\nu^\natural|]} \cap (\mathrm{c}_{\mathrm{D}_{\nu^\natural}}\mathbb{V}_{\mathrm{std},2r,\mathbb{Q}}^{\otimes|\nu^\natural|})$, where $\nu^\natural = (\nu_1^\natural, \ldots, \nu_{2r}^\natural)$ is the polynomial weight of GL_{2r} such that, for $1 \leq i \leq r$, $\nu_i^\natural := \nu_i$ and $\nu_{2r+1-i}^\natural := 0$ when $\nu_i > 0$, while $\nu_i^\natural := \nu_{2r+1-i}^\natural := 1$ when $\nu_i = 0$. In other words, it can be constructed by a variant of the isomorphism in Proposition 2.24. However, for simplicity, we shall ignore these representations. (As in Remark 2.17, this is justified by Lemma 2.14.)

As in [42, 1.5], a $\mathbb{Z}$-lattice in a $\mathbb{Q}$-representation of a group scheme over $\mathbb{Z}$ is called *admissible* if it is stable under the action of the group scheme.

Definition 2.26. *Let $\nu = (\nu_1, \nu_2, \ldots, \nu_r)$ be a weight satisfying $\nu_1 \geq \nu_2 \geq \ldots \geq \nu_r$.*

(1) *Let ν_{r+1} be any integer such that $\nu_r \geq \nu_{r+1}$, put $\nu' := (\nu_1 - \nu_{r+1}, \nu_2 - \nu_{r+1}, \ldots, \nu_r - \nu_{r+1})$, and choose any Young tableau $\mathrm{D}_{\nu'}$ for ν'. Then we define $\mathbb{V}_{\nu,\nu_{r+1}}$ to be the admissible $\mathbb{Z}$-lattice*

$$\mathbb{V}_{\nu,\nu_{r+1}} := (\mathrm{c}_{\mathrm{D}_{\nu'}}\mathbb{V}_{\mathrm{std},r}^{\otimes|\nu'|}) \otimes (\wedge^r \mathbb{V}_{\mathrm{std},r})^{\otimes \nu_{r+1}}$$

in $\mathbb{V}_{\nu,\mathbb{Q}} \cong \mathbb{V}_{\nu',\mathbb{Q}} \underset{\mathbb{Q}}{\otimes} \det^{\otimes \nu_{r+1}} \cong (\mathrm{c}_{\mathrm{D}_{\nu'}}\mathbb{V}_{\mathrm{std},r,\mathbb{Q}}^{\otimes|\nu'|}) \otimes (\wedge^r \mathbb{V}_{\mathrm{std},r})^{\otimes \nu_{r+1}}$. (Here $\mathbb{V}_{\nu,\nu_{r+1}}$ depends on the choice of ν_{r+1}, but $\mathbb{V}_{\nu,\nu_{r+1}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \cong \mathbb{V}_{\nu,\mathbb{Q}}$ does not.)

(2) *If $\nu_r \geq 0$, we can view ν as a polynomial weight of GL_{2r} by supplying zeros in the end, and choose a Young tableau D_ν for ν. Then we define $\mathbb{V}_\nu^{\langle\,\cdot\,,\,\cdot\,\rangle}$ to be the admissible $\mathbb{Z}$-lattice*

$$\mathbb{V}_\nu^{\langle\,\cdot\,,\,\cdot\,\rangle} := \mathbb{V}_{\mathrm{std},2r}^{\langle|\nu|\rangle} \cap (\mathrm{c}_{\mathrm{D}_\nu} \mathbb{V}_{\mathrm{std},2r}^{\otimes\,|\nu|}) = \mathrm{c}_{\mathrm{D}_\nu} \mathbb{V}_{\mathrm{std},2r}^{\langle|\nu|\rangle}$$

in $\mathbb{V}_{\nu,\mathbb{Q}}^{\langle\,\cdot\,,\,\cdot\,\rangle} \cong \mathbb{V}_{\mathrm{std},2r,\mathbb{Q}}^{\langle|\nu|\rangle} \cap (\mathrm{c}_{\mathrm{D}_\nu} \mathbb{V}_{\mathrm{std},2r,\mathbb{Q}}^{\otimes\,|\nu|})$, and we define $\mathbb{V}_\nu^{[\,\cdot\,,\,\cdot\,]}$ to be the admissible $\mathbb{Z}$-lattice

$$\mathbb{V}_\nu^{[\,\cdot\,,\,\cdot\,]} := \mathbb{V}_{\mathrm{std},2r}^{[|\nu|]} \cap (\mathrm{c}_{\mathrm{D}_\nu} \mathbb{V}_{\mathrm{std},2r}^{\otimes\,|\nu|}) = \mathrm{c}_{\mathrm{D}_\nu} \mathbb{V}_{\mathrm{std},2r}^{[|\nu|]}$$

in $\mathbb{V}_{\nu,\mathbb{Q}}^{[\,\cdot\,,\,\cdot\,]} \cong \mathbb{V}_{\mathrm{std},2r,\mathbb{Q}}^{[|\nu|]} \cap (\mathrm{c}_{\mathrm{D}_\nu} \mathbb{V}_{\mathrm{std},2r,\mathbb{Q}}^{\otimes\,|\nu|})$.

The admissibility of these $\mathbb{Z}$-lattices is clear because the constructions using Young symmetrizers, using $\mathbb{V}_{\mathrm{std},2r}^{\langle|\nu|\rangle}$, and using $\mathbb{V}_{\mathrm{std},2r}^{[|\nu|]}$ are all compatible with the actions of the group schemes (over $\mathbb{Z}$).

Definition 2.27. *Suppose $\mu = ((\mu_\tau)_{\tau\in\Upsilon/c}; \mu_0) = ((((\mu_{\tau,i_\tau})_{1\leq i_\tau\leq r_\tau})_{\tau\in\Upsilon/c}; \mu_0) \in \mathrm{X}_{\mathrm{G}_1}^+$. By replacing μ with another element in $[\mu]$ (see Remark 2.17) if necessary, we shall assume that $\mu_{\tau,r_\tau} \geq 0$ for any $\tau \in \Upsilon$ such that $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$. There are three cases for factors G_τ of G_1:*

(1) *If $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, then we set $V_{\mu_\tau} := \mathbb{V}_{\mu_\tau}^{\langle\,\cdot\,,\,\cdot\,\rangle} \underset{\mathbb{Z}}{\otimes} R_1$.*
(2) *If $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, then we set $V_{\mu_\tau} := \mathbb{V}_{\mu_\tau}^{[\,\cdot\,,\,\cdot\,]} \underset{\mathbb{Z}}{\otimes} R_1$.*
(3) *If $\mathrm{G}_\tau \cong \mathrm{GL}_{r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, and if $\mu_{\tau,r_\tau+1}$ is the even integer such that $1 \geq \mu_{\tau,r_\tau} - \mu_{\tau,r_\tau+1} \geq 0$, then we set $V_{\mu_\tau} := \mathbb{V}_{\mu_\tau,\mu_{\tau,r_\tau+1}} \underset{\mathbb{Z}}{\otimes} R_1$.*

Here the modules $\mathbb{V}_{\mu_\tau}^{\langle\,\cdot\,,\,\cdot\,\rangle}$, $\mathbb{V}_{\mu_\tau}^{\langle\,\cdot\,,\,\cdot\,\rangle}$, and $\mathbb{V}_{\mu_\tau,\mu_{\tau,r_\tau+1}}$ are defined in Definition 2.26. Then we set

$$V_{[\mu]} := \Big(\underset{\tau\in\Upsilon/c}{\otimes} V_{\mu_\tau} \Big) \underset{R_1}{\otimes} \upsilon^{\otimes\,\mu_0},$$

where υ is the free rank one R_1-module on which G_1 acts via the similitude character.

Definition 2.28. *Suppose $\mu = ((\mu_\tau)_{\tau\in\Upsilon/c}; \mu_0) = ((((\mu_{\tau,i_\tau})_{1\leq i_\tau\leq r_\tau})_{\tau\in\Upsilon/c}; \mu_0) \in \mathrm{X}_{\mathrm{M}_1}^+$. There are two cases for factors M_τ of M_1:*

(1) *If $\tau = \tau \circ c$, then $\mathrm{M}_\tau \cong \mathrm{GL}_{r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, and we take $W_{\mu_\tau} := \mathbb{V}_{\mu_\tau,\mu_{\tau,r_\tau}} \underset{\mathbb{Z}}{\otimes} R_1$.*
(2) *If $\tau \neq \tau \circ c$, then $\mathrm{M}_\tau \cong (\mathrm{GL}_{q_\tau} \times \mathrm{GL}_{p_\tau}) \underset{\mathbb{Z}}{\otimes} R_1$, and we take*

$$W_{\mu_\tau} := (\mathbb{V}_{(\mu_{\tau,1},\mu_{\tau,2},\ldots,\mu_{\tau,q_\tau}),\mu_{\tau,q_\tau}} \underset{\mathbb{Z}}{\otimes} \mathbb{V}_{(\mu_{\tau,q_\tau+1},\mu_{\tau,q_\tau+2},\ldots,\mu_{\tau,r_\tau}),\mu_{\tau,r_\tau}}) \underset{\mathbb{Z}}{\otimes} R_1.$$

Then we set

$$W_\mu := \Big(\underset{\tau\in\Upsilon/c}{\otimes} W_{\mu_\tau} \Big) \underset{R_1}{\otimes} \upsilon^{\otimes\,\mu_0},$$

where υ is the free rank one R_1-module on which M_1 acts via the similitude character.

2.6. **p-small weights and Weyl modules.**

Definition 2.29. *We say $\mu \in \mathrm{X}$ is **p-small for** G_1 if $(\mu+\rho,\alpha^\vee) \leq p$ for every $\alpha \in \Phi_{\mathrm{G}_1}$. We say $\mu \in \mathrm{X}$ is **p-small for** M_1 if $(\mu+\rho,\alpha^\vee) \leq p$ for every $\alpha \in \Phi_{\mathrm{M}_1}$. We denote the subset of X that are p-small for G_1 (resp. M_1) by $\mathrm{X}_{\mathrm{G}_1}^{<p}$ (resp. $\mathrm{X}_{\mathrm{M}_1}^{<p}$), and we set $\mathrm{X}_{\mathrm{G}_1}^{+,<p} := \mathrm{X}_{\mathrm{G}_1}^{+} \cap \mathrm{X}_{\mathrm{G}_1}^{<p}$ (resp. $\mathrm{X}_{\mathrm{M}_1}^{+,<p} := \mathrm{X}_{\mathrm{M}_1}^{+} \cap \mathrm{X}_{\mathrm{M}_1}^{<p}$).*

Remark 2.30 (cf. [42, 1.9]). The dot action of $\mathrm{W}_{\mathrm{G}_1}$ sends a p-small weight of G_1 to a p-small weight of M_1. The second statement in Definition 2.29 makes sense because $\rho_{\mathrm{M}_1} := \frac{1}{2} \sum\limits_{\alpha \in \Phi_{\mathrm{M}_1}^+} \alpha$ satisfies $(\rho - \rho_{\mathrm{M}_1}, \alpha^\vee) = 0$ for any $\alpha \in \Phi_{\mathrm{M}_1}$. Thus, if $\mu \in \mathrm{X}$ is p-small for G_1, then $w \cdot \mu$ is p-small for M_1 for any $w \in \mathrm{W}_{\mathrm{G}_1}$.

Since G_1 (resp. M_1) is split over R_1, there exists a split reductive algebraic group $\mathrm{G}_{\mathrm{split}}$ (resp. $\mathrm{M}_{\mathrm{split}}$) over $\mathbb{Z}_{(p)}$ such that $\mathrm{G}_1 \cong \mathrm{G}_{\mathrm{split}} \underset{\mathbb{Z}_{(p)}}{\otimes} R_1$ (resp. $\mathrm{M}_1 \cong \mathrm{M}_{\mathrm{split}} \underset{\mathbb{Z}_{(p)}}{\otimes} R_1$). Note that $\mathrm{G}_{\mathrm{split}}$ (resp. $\mathrm{M}_{\mathrm{split}}$) has the same roots and weights as G_1 (resp. M_1), and is a (semi-direct) product of $\mathbf{G}_{\mathrm{m}}$ with groups of the three types in Propositions 2.19, 2.23, and 2.24 over $\mathbb{Z}_{(p)}$. For $\mu \in \mathrm{X}_{\mathrm{G}_1}^+$ (resp. $\mu \in \mathrm{X}_{\mathrm{M}_1}^+$), let $V_{[\mu],\mathbb{Q}}$ (resp. $W_{\mu,\mathbb{Q}}$) be *the* irreducible representation of $\mathrm{G}_{\mathrm{split}} \underset{\mathbb{Z}_{(p)}}{\otimes} \mathbb{Q}$ (resp. $\mathrm{M}_{\mathrm{split}} \underset{\mathbb{Z}_{(p)}}{\otimes} \mathbb{Q}$) containing the dominant weight μ (see Remark 2.17 for the meaning of $[\mu]$) with simple factors (modulo the similitude character) of the forms given in Propositions 2.19, 2.23, and 2.24. (See also Remark 2.25.) Let $V_{[\mu],\mathbb{Z}_{(p)}} \subset V_{[\mu],\mathbb{Q}}$ (resp. $W_{\mu,\mathbb{Z}_{(p)}} \subset W_{\mu,\mathbb{Q}}$) be the *Weyl module* over $\mathbb{Z}_{(p)}$ defined as in [42, 1.3], (namely the span of a highest weight vector under the action of the group scheme over $\mathbb{Z}_{(p)}$,) which is minimal among admissible $\mathbb{Z}_{(p)}$-lattices in $V_{[\mu],\mathbb{Q}}$ (resp. $W_{\mu,\mathbb{Q}}$) containing the same highest weight vector. (See [42, 1.5].)

According to [42, Cor. 1.9], if $\mu \in \mathrm{X}_{\mathrm{G}_1}^{+,<p}$ (resp. $\mu \in \mathrm{X}_{\mathrm{M}_1}^{+,<p}$), then all admissible $\mathbb{Z}_{(p)}$-lattices in $V_{[\mu],\mathbb{Q}}$ (resp. $W_{\mu,\mathbb{Q}}$) are isomorphic to each other. Therefore, it necessarily follows (cf. [42, Cor. 5]) that $V_{[\mu]} \cong V_{[\mu],\mathbb{Z}_{(p)}} \underset{\mathbb{Z}_{(p)}}{\otimes} R_1$ (resp. $W_\mu \cong W_{\mu,\mathbb{Z}_{(p)}} \underset{\mathbb{Z}_{(p)}}{\otimes} R_1$), regardless of the artificial choices made in Definitions 2.27 and 2.28. We set $V_{[\mu],R} := V_{[\mu]} \underset{R_1}{\otimes} R$ (resp. $W_{\mu,R} := W_\mu \underset{R_1}{\otimes} R$) for any R_1-algebra R.

3. Geometric realizations of automorphic bundles

The aim of this and the next sections is to explain how automorphic bundles and their cohomology can be realized geometrically using the cohomology of fiber products of $A \to \mathsf{S}_1$ (with trivial coefficients).

3.1. **Standard representations.** Consider the decomposition (2.1) induced by (1.5). By [29, Prop. 1.1.1.17], we have $\mathcal{O}_\tau \cong \mathrm{M}_{t_\tau}(\mathcal{O}_{F,\tau})$ for some integer $t_\tau \geq 1$. There are three possibilities, depending on the classification of the group G_τ, or rather the restriction of $\star$ to $\mathcal{O}_\tau$. (See [29, Lem. 1.2.3.2] and its proof, with several misleading typos corrected in the revision.)

Suppose $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$. This happens exactly when $\tau = \tau \circ c$ and the restriction of $\star$ to $\mathcal{O}_\tau$ is of the form $x \mapsto c\,{}^t x c^{-1}$ for some element $c \in \mathcal{O}_\tau$ satisfying ${}^t c = c$. Let us take $\varepsilon_\tau \in \mathcal{O}_\tau \cong \mathrm{M}_{t_\tau}(\mathcal{O}_{F,\tau})$ to be the elementary idempotent matrix E_{11} with unique nonzero entry 1 at the most upper-left corner.

Then we have ${}^t\varepsilon_\tau = \varepsilon_\tau$, $\mathcal{O}_\tau \varepsilon_\tau \mathcal{O}_\tau = \mathcal{O}_\tau$, and $L_{\mathrm{std},\tau} := \varepsilon_\tau(L_1) \subset L_1$ is a free R_1-module of rank $2r_\tau$ whose $\mathcal{O}_\tau$-span in L_1 is L_τ (under the identification (2.2)). For any R_1-algebra R, to check if $g \in \mathrm{GL}_{\mathcal{O}_\tau}(L_\tau)$ lies in G_τ, we need to verify if $\langle gx, gy\rangle = \langle x, y\rangle$ for $x, y \in L_\tau \underset{R_1}{\otimes} R$. We may assume that $x \in \varepsilon_\tau(L_1 \underset{R_1}{\otimes} R)$. Let us write $x = \varepsilon_\tau x_0$ and $y = cy_0$ for some $x_0, y_0 \in L_\tau$. Then $x = \varepsilon_\tau x$, and $\langle x, y\rangle = \langle \varepsilon_\tau x, y\rangle = \langle x, \varepsilon_\tau^\star y\rangle = \langle x, c\,{}^t\varepsilon_\tau c^{-1} y\rangle = \langle x, c\varepsilon_\tau c^{-1} y\rangle = \langle x, c\varepsilon_\tau y_0\rangle$ shows that it suffices to check if the action induced by g on $L_{\mathrm{std},\tau}$ preserves the pullback to R of the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},\tau} : L_{\mathrm{std},\tau} \times L_{\mathrm{std},\tau} \to R_1(1)$ defined by $\langle x, z\rangle_{\mathrm{std},\tau} := \langle x, cz\rangle$ for any $x, z \in L_{\mathrm{std},\tau}$. (This pairing is alternating because $c^\star = c\,{}^t cc^{-1} = c$.) Then we view $(L_{\mathrm{std},\tau}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},\tau})$ as the standard representation of $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$.

Suppose $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$. This happens exactly when $\tau = \tau\circ c$ and the restriction of $\star$ to $\mathcal{O}_\tau$ is of the form $x \mapsto d\,{}^t x d^{-1}$ for some element $d \in \mathcal{O}_\tau$ satisfying ${}^t d = -d$. Let us take $\varepsilon_\tau \in \mathcal{O}_\tau \cong \mathrm{M}_{t_\tau}(\mathcal{O}_{F,\tau})$ to be the elementary idempotent matrix E_{11} with unique nonzero entry 1 at the most upper-left corner. Then we have ${}^t\varepsilon_\tau = \varepsilon_\tau$, $\mathcal{O}_\tau\varepsilon_\tau\mathcal{O}_\tau = \mathcal{O}_\tau$, and $L_{\mathrm{std},\tau} := \varepsilon_\tau(L_1) \subset L_1$ is a free R_1-module of rank $2r_\tau$ whose $\mathcal{O}_\tau$-span in L_1 is L_τ (under the identification (2.2)). By an analogous procedure as in the symplectic case, we define the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},\tau} : L_{\mathrm{std},\tau} \times L_{\mathrm{std},\tau} \to R_1(1)$ by $\langle x, z\rangle_{\mathrm{std},\tau} := \langle x, dz\rangle$ for any $x, z \in L_{\mathrm{std},\tau}$. (This pairing is symmetric because $d^\star = d\,{}^t dd^{-1} = -d$.) Then we view $(L_{\mathrm{std},\tau}, \langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},\tau})$ as the standard representation of $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$.

Suppose $\mathrm{G}_\tau \cong \mathrm{GL}_{r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$. This happens exactly when $\tau \neq \tau \circ c$. Then $\star$ switches the two factors $\mathcal{O}_\tau$ and $\mathcal{O}_{\tau\circ c}$ in (2.1). Let us take $\varepsilon_\tau \in \mathcal{O}_\tau \cong \mathrm{M}_{t_\tau}(\mathcal{O}_{F,\tau})$ to be the elementary idempotent matrix E_{11} with unique nonzero entry 1 at the most upper-left corner. Then we have $\mathcal{O}_\tau\varepsilon_\tau\mathcal{O}_\tau = \mathcal{O}_\tau$, $\mathcal{O}_{\tau\circ c}\varepsilon_\tau^\star\mathcal{O}_{\tau\circ c} = \mathcal{O}_{\tau\circ c}$, and $L_{\mathrm{std},\tau} := \varepsilon_\tau(L_1) \subset L_1$ and $L^\star_{\mathrm{std},\tau} := \varepsilon^\star_\tau(L_1) \subset L_1$ are free R_1-modules of rank r_τ whose $\mathcal{O}_\tau$-spans in L_1 are respectively L_τ and $L_{\tau\circ c}$ (under the identification (2.2)). Then the restriction of $\langle\,\cdot\,,\,\cdot\,\rangle$ to $L_\tau \times L_\tau$ is determined by its restriction to $L_{\mathrm{std},\tau} \times L^\star_{\mathrm{std},\tau}$, so that the action of G_τ on $L_{\mathrm{std},\tau}$ is determined by its action on $L^\star_{\mathrm{std},\tau}$, and we view $L_{\mathrm{std},\tau}$ as the standard representation of $\mathrm{G}_\tau \cong \mathrm{GL}_{r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$.

Any element $b \otimes r \in \mathcal{O}_1 = \mathcal{O} \underset{\mathbb{Z}}{\otimes} R_1$ acts on $\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$ by

$$(b \otimes r)_* := r\, i(b)_* : \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}) \to \underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}),$$

where $i : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathsf{M}_{\mathcal{H},1}}(A)$ is the $\mathcal{O}$-endomorphism structure inducing $i(b)_*$ by functoriality, and where r acts via the R_1-module structure. (Similar actions work for any reasonable homology or cohomology of A with coefficients in R_1-modules.) Since ε_τ is an idempotent, we obtain an R_1-module summand

$$\underline{L}_{\mathrm{std},\tau} := (\varepsilon_\tau)_*(\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1}))$$

of $\underline{H}_1^{\mathrm{dR}}(A/\mathsf{M}_{\mathcal{H},1})$. By functoriality and exactness of $\mathcal{E}_{\mathrm{G}_1}(\,\cdot\,)$, we have

$$\mathcal{E}_{\mathrm{G}_1}(L_{\mathrm{std},\tau}) \cong \underline{L}_{\mathrm{std},\tau}.$$

3.2. **Lieberman's trick.** Let $m, n \geq 0$ be two integers. Let $\mathbb{Z}$ denote the multiplicative semi-group of integers, and let $\mathbb{Z}^n$ denote its n-fold product. Then $\mathbb{Z}^n$ has a natural componentwise action on $L_1^{\oplus n}$, inducing canonically an action on

$$\wedge^m (L_1^{\oplus n}) \cong \bigoplus_{\substack{i_1,i_2,\ldots,i_n\geq 0\\ i_1+i_2+\ldots+i_n=m}} \Big((\wedge^{i_1}L_1)\underset{R_1}{\otimes}(\wedge^{i_2}L_1)\underset{R_1}{\otimes}\ldots\underset{R_1}{\otimes}(\wedge^{i_n}L_1)\Big), \tag{3.1}$$

with $(l_1,l_2,\ldots,l_n)$ acting as $l_1^{i_1}l_2^{i_2}\ldots l_n^{i_n}$ on $(\wedge^{i_1}L_1)\underset{R_1}{\otimes}(\wedge^{i_2}L_1)\underset{R_1}{\otimes}\ldots\underset{R_1}{\otimes}(\wedge^{i_n}L_1)$. When $m=n$, the summand with $i_1=i_2=\ldots=i_n=1$ is just $L_1^{\otimes n}$.

Suppose $m<p$. For each $0\leq i\leq m$ except $i=1$, choose an integer $1\leq l(i)<p$ such that $l(i)^i-l(i)$ is a unit in $\mathbb{Z}_{(p)}$. This is always possible because $m<p$. Let $\varepsilon^{\mathrm{L}}_{n,i,j}$ denote the element $l(i)^i(1,1,1,\ldots,1)-(1,\ldots,l(i),\ldots,1)$ in $\mathbb{Z}[\mathbb{Z}^n]$ with $l(i)$ appearing in the j-th entry in the second term (with all the other entries 1). Then $\varepsilon^{\mathrm{L}}_{n,i,j}$ acts as zero on all summands in (3.1) labeled by $(i_1,i_2,\ldots,i_n)$ with $i_j=i$, and acts as the unit $l(i)^i-l(i)$ in $\mathbb{Z}_{(p)}$ on all summands with $i_j=1$. If we take the element

$$\varepsilon^{\mathrm{L}}_{n,m} := \prod_{1\leq j\leq n}\ \prod_{0\leq i\leq m, i\neq 1} ((l(i)^i-l(i))^{-1}\varepsilon^{\mathrm{L}}_{n,i,j})$$

in $\mathbb{Z}_{(p)}[\mathbb{Z}^n]$, then $\varepsilon^{\mathrm{L}}_{n,m}$ acts as zero on all summands in (3.1) except for $L_1^{\otimes n}$ when $m=n$, on which it acts as 1 instead. This shows that $\varepsilon^{\mathrm{L}}_{n,m}$ acts as an idempotent on $\wedge^m(L_1^{\oplus n})$, defining a projection to $L_1^{\otimes n}$ when $m=n$. We shall denote $\varepsilon^{\mathrm{L}}_{n,n}$ by $\varepsilon^{\mathrm{L}}_n$ for simplicity.

Now suppose we have a tuple $\underline{n}=(n_\tau)_{\tau\in\Upsilon/c}$ such that $n=|\underline{n}|:=\sum\limits_{\tau\in\Upsilon/c} n_\tau$ satisfies $n<p$. Consider the componentwise action of $\mathcal{O}_1^n$ on $L_1^{\oplus n}$. To be precise, we shall denote elements in $\mathcal{O}_1^n$ by $\underline{b}=((b_{\tau,i_\tau})_{1\leq i_\tau\leq n_\tau})_{\tau\in\Upsilon/c}$. Consider the idempotent $\varepsilon_{\underline{n}}=(\varepsilon_{\tau,n_\tau})_{\tau\in\Upsilon/c}=((\varepsilon_{\tau,n_\tau,i_\tau})_{1\leq i_\tau\leq n_\tau})_{\tau\in\Upsilon/c}$ in $\mathcal{O}_1^n$ with $\varepsilon_{\tau,n_\tau,i_\tau}=\varepsilon_\tau$ for any $\tau\in\Upsilon/c$ and any $1\leq i_\tau\leq n_\tau$. Then we have

$$\underset{\tau\in\Upsilon/c}{\otimes} L^{\otimes n_\tau}_{\mathrm{std},\tau} \cong \varepsilon_{\underline{n}}\,\varepsilon^{\mathrm{L}}_n\,(\wedge^n(L_1^{\oplus n})).$$

Geometrically, we can realize $\wedge^m(L_1^{\oplus n})$ by taking the n-fold fiber product A^n of A over $\mathsf{M}_{\mathcal{H},1}$ and then taking the m-th relative de Rham homology

$$\underline{H}^{\mathrm{dR}}_m(A^n/\mathsf{M}_{\mathcal{H},1}) \cong \wedge^m(\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\oplus n}).$$

Then we obtain natural isomorphisms

$$\mathcal{E}_{\mathrm{G}_1}(\underset{\tau\in\Upsilon/c}{\otimes} L^{\otimes n_\tau}_{\mathrm{std},\tau}) \cong \underset{\tau\in\Upsilon/c}{\otimes} \underline{L}^{\otimes n_\tau}_{\mathrm{std},\tau} \cong (\varepsilon_{\underline{n}})_*\,(\varepsilon^{\mathrm{L}}_n)_*\,\underline{H}^{\mathrm{dR}}_n(A^n/\mathsf{M}_{\mathcal{H},1}).$$

3.3. **Young symmetrizers.** Now suppose we have an element $\mu\in\mathrm{X}^+_{\mathrm{G}_1}$ such that $\mu=((\mu_\tau)_{\tau\in\Upsilon/c};\mu_0)=(((\mu_{\tau,i_\tau})_{1\leq i_\tau\leq r_\tau})_{\tau\in\Upsilon/c};\mu_0)$. As always, up to replacing μ with another element in $[\mu]$ (see Remark 2.17), we shall assume that $\mu_{\tau,r_\tau}\geq 0$ for any $\tau\in\Upsilon$ such that $\mathrm{G}_\tau\cong\mathrm{O}_{2r_\tau}\underset{\mathbb{Z}}{\otimes}R_1$. For each $\tau\in\Upsilon/c$, we have two possibilities:

(1) If $\mathrm{G}_\tau\cong\mathrm{Sp}_{2r_\tau}\underset{\mathbb{Z}}{\otimes}R_1$ or $\mathrm{G}_\tau\cong\mathrm{O}_{2r_\tau}\underset{\mathbb{Z}}{\otimes}R_1$, we view μ_τ as a polynomial weight μ'_τ of GL_{2r_τ} by supplying zeros in the end. We set $t_{\mu_\tau}:=0$ in this case.

(2) If $\mathrm{G}_\tau\cong\mathrm{GL}_{r_\tau}\underset{\mathbb{Z}}{\otimes}R_1$, we take $\mu_{\tau,r_\tau+1}$ to be the unique *even* integer such that $1\geq\mu_{\tau,r_\tau}-\mu_{\tau,r_\tau+1}\geq 0$, and take the polynomial weight $\mu'_\tau=(\mu_{\tau,1}-\mu_{\tau,r_\tau+1},\mu_{\tau,2}-\mu_{\tau,r_\tau+1},\ldots,\mu_{\tau,r_\tau}-\mu_{\tau,r_\tau+1})$ of $\mathrm{GL}_{r_\tau}\underset{\mathbb{Z}}{\otimes}R_1$. We set $t_{\mu_\tau}:=(1/2)r_\tau\mu_{\tau,r_\tau+1}$ in this case.

In either case, we take a Young tableau $\mathrm{D}_{\mu'_\tau}$ for μ'_τ, and define the Young symmetrizer $\mathrm{c}_{\mathrm{D}_{\mu'_\tau}}$ in $\mathbb{Z}[\mathfrak{S}_{|\mu'_\tau|}]$. By Lemma 2.18, $\mathrm{c}_{\mathrm{D}_{\mu'_\tau}}\mathrm{c}_{\mathrm{D}_{\mu'_\tau}} = \mathrm{d}_{\mathrm{D}_{\mu'_\tau}}\mathrm{c}_{\mathrm{D}_{\mu'_\tau}}$ for some integer $\mathrm{d}_{\mathrm{D}_{\mu'_\tau}}$ dividing $|\mu'_\tau|!$ (i.e. factorial).

Definition 3.2. *Set* $|\mu|_{\mathrm{Y}} := \max_{\tau\in\Upsilon/c} |\mu'_\tau|$ *and* $|\mu|_{\mathrm{L}} := \sum_{\tau\in\Upsilon/c} |\mu'_\tau|$. (*Here* μ'_τ *is defined after replacing* μ *with the element in* $[\mu]$ (*see Remark* 2.17) *satisfying* $\mu_{\tau,r_\tau} \geq 0$ *for any* $\tau \in \Upsilon$ *such that* $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$.) *By abuse of notation, we shall also write* $|\mu_\tau|_{\mathrm{L}} = |\mu'_\tau|$. *We say a weight* μ *in* $\mathrm{X}^+_{\mathrm{G}_1}$ *is* p**-small for Young symmetrizers** (*resp.* **for Lieberman's trick**) *if* $|\mu|_{\mathrm{Y}} < p$ (*resp.* $|\mu|_{\mathrm{L}} < p$). *Obviously,* $|\mu|_{\mathrm{L}} < p$ *implies* $|\mu|_{\mathrm{Y}} < p$, *and they coincide when* Υ/c *is a singleton. If* $|\mu|_{\mathrm{L}} < p$ *and* $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$, *we say* μ *is* p**-small for the geometric realization of Weyl's construction**. *We denote by* $\mathrm{X}^{+,<_{\mathrm{Y}}p}_{\mathrm{G}_1}$ (*resp.* $\mathrm{X}^{+,<_{\mathrm{L}}p}_{\mathrm{G}_1}$, *resp.* $\mathrm{X}^{+,<_{\mathrm{W}}p}_{\mathrm{G}_1}$) *the set of weights* p*-small for Young symmetrizers* (*resp. for Lieberman's trick, resp. the geometric realization of Weyl's construction*).

Now suppose $\mu \in \mathrm{X}^{+,<_{\mathrm{L}}p}_{\mathrm{G}_1}$ (and hence $\mu \in \mathrm{X}^{+,<_{\mathrm{Y}}p}_{\mathrm{G}_1}$). Then $\mathrm{d}^{-1}_{\mathrm{D}_{\mu'_\tau}}\mathrm{c}_{\mathrm{D}_{\mu'_\tau}} \in \mathbb{Z}_{(p)}[\mathfrak{S}_{|\mu'_\tau|}]$ for each $\tau \in \Upsilon/c$, and we define

$$\varepsilon^{\mathrm{Y}}_\mu := \underset{\tau\in\Upsilon/c}{\otimes}(\mathrm{d}^{-1}_{\mathrm{D}_{\mu'_\tau}}\mathrm{c}_{\mathrm{D}_{\mu'_\tau}}) \in \underset{\tau\in\Upsilon/c}{\otimes}\mathbb{Z}_{(p)}[\mathfrak{S}_{|\mu'_\tau|}] \overset{\text{can.}}{\hookrightarrow} \mathbb{Z}_{(p)}[\mathfrak{S}_{|\mu|_{\mathrm{L}}}],$$

which acts on $\underset{\tau\in\Upsilon/c}{\otimes} L^{\otimes|\mu'_\tau|}_{\mathrm{std},\tau}$ as an idempotent. Since $\mathfrak{S}_{|\mu|_{\mathrm{L}}}$ acts naturally on $A^{|\mu|_{\mathrm{L}}}$ by permutations, we can realize the geometric action $(\varepsilon^{\mathrm{Y}}_\mu)_*$ on $\underline{H}^{\mathrm{dR}}_m(A^{|\mu|_{\mathrm{L}}}/\mathsf{M}_{\mathcal{H},1})$ by functoriality.

We shall denote by $\varepsilon^{\mathrm{S}}_\mu$ the $\varepsilon_{\underline{n}}$ in Section 3.2 with $\underline{n} = (|\mu'_\tau|)_{\tau\in\Upsilon/c}$.

3.4. **Poincaré bundles.** We retain the setting in the previous section.

Suppose $\tau \in \Upsilon/c$ satisfies $\tau = \tau \circ c$. Suppose $\langle x, y\rangle_{\mathrm{std},\tau} = \langle x, c_\tau y\rangle$ for some $c_\tau \in \mathcal{O}_\tau$ (which was either c or d in Section 3.1, depending on whether we were in the symplectic or orthogonal case) such that $\varepsilon^\star_\tau = c_\tau\varepsilon_\tau c^{-1}_\tau$, for any $x, y \in L_{\mathrm{std},\tau} = \varepsilon_\tau(L_1)$.

For any $1 \leq i < j \leq |\mu'_\tau|$, we define $\phi^{\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},\tau}}_{i,j} : L^{\otimes|\mu'_\tau|}_{\mathrm{std},\tau} \to L^{\otimes(|\mu'_\tau|-2)}_{\mathrm{std},\tau}(1)$ by

$$\phi^{\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},\tau}}_{i,j}(v_1 \otimes v_2 \otimes \ldots \otimes v_{|\mu'_\tau|}) := \langle v_i, v_j\rangle_{\mathrm{std},\tau}(v_1 \otimes \ldots \otimes \hat{v}_i \otimes \ldots \otimes \hat{v}_j \otimes \ldots \otimes v_{|\mu'_\tau|})$$

for $v_1, \ldots, v_{|\mu'_\tau|} \in L_{\mathrm{std},\tau}$, and define $\phi^{\langle\,\cdot\,,c_\tau\varepsilon_\tau\,\cdot\,\rangle}_{i,j} : L^{\otimes|\mu'_\tau|}_1 \to L^{\otimes(|\mu'_\tau|-2)}_1(1)$ by

$$\phi^{\langle\,\cdot\,,c_\tau\varepsilon_\tau\,\cdot\,\rangle}_{i,j}(v_1 \otimes v_2 \otimes \ldots \otimes v_{|\mu'_\tau|}) := \langle v_i, c_\tau\varepsilon_\tau v_j\rangle(v_1 \otimes \ldots \otimes \hat{v}_i \otimes \ldots \otimes \hat{v}_j \otimes \ldots \otimes v_{|\mu'_\tau|})$$

for $v_1, \ldots, v_{|\mu'_\tau|} \in L_1$. (Here $\hat{v}_i$ and $\hat{v}_j$ denote omissions of entries as usual.)

Lemma 3.3. *We have* $\ker(\phi^{\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},\tau}}_{i,j}) = \varepsilon_{\tau,|\mu'_\tau|}\ker(\phi^{\langle\,\cdot\,,c_\tau\varepsilon_\tau\,\cdot\,\rangle}_{i,j})$ *in* $L^{\otimes|\mu'_\tau|}_1$, *where* $\varepsilon_{\tau,|\mu'_\tau|} \in \mathcal{O}^{|\mu'_\tau|}_\tau$ *has all its entries equal to* ε_τ.

Proof. This is because $\langle x, c_\tau\varepsilon_\tau y\rangle = \langle x, c_\tau\varepsilon^2_\tau y\rangle = \langle x, (c_\tau\varepsilon_\tau c^{-1}_\tau)c_\tau\varepsilon_\tau y\rangle = \langle \varepsilon_\tau x, c_\tau\varepsilon_\tau y\rangle$ for any $x, y \in L_1$. (See Section 3.1.) □

Now let us turn to geometric realizations. The first Chern class $c_1((\mathrm{Id}_A \times \lambda)^*\mathcal{P}_A) \in \underline{H}^2_{\mathrm{dR}}(A^2/\mathsf{M}_{\mathcal{H},1})(1)$ induces, by Künneth decomposition, the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda : \underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1}) \times \underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1}) \to \mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)$, which is the geometric realization of $\langle\,\cdot\,,\,\cdot\,\rangle : L_1 \times L_1 \to R_1(1)$. Thus, if $c_\tau \varepsilon_\tau = \sum_{\alpha\in I} b_\alpha \otimes r_\alpha \in \mathcal{O}_\tau = \mathcal{O} \underset{\mathbb{Z}}{\otimes} R_1$, then $\langle\,\cdot\,, c_\tau\varepsilon_\tau\,\cdot\,\rangle$ is realized geometrically by

$$c_\tau^\lambda := \sum_{\alpha\in I} r_\alpha (\mathrm{Id}_A \times i(b_\alpha))^*(c_1((\mathrm{Id}_A \times \lambda)^*\mathcal{P}_A)) \in \underline{H}^2_{\mathrm{dR}}(A^2/\mathsf{M}_{\mathcal{H},1})(1)$$

For any $1 \le i < j \le |\mu'_\tau|$, consider the Künneth morphisms

$$\mathrm{K}_\tau^{i,j} : \underline{H}^{|\mu'_\tau|-2}_{\mathrm{dR}}(A^{|\mu'_\tau|-2}/\mathsf{M}_{\mathcal{H},1}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \underline{H}^2_{\mathrm{dR}}(A^2/\mathsf{M}_{\mathcal{H},1}) \hookrightarrow \underline{H}^{|\mu'_\tau|}_{\mathrm{dR}}(A^{|\mu'_\tau|}/\mathsf{M}_{\mathcal{H},1})$$

corresponding to the i-th and j-th factors in $A^{|\mu'_\tau|}$. (Note that the image of $\mathrm{K}_\tau^{i,j}$ can also be cut out by a variant of Lieberman's trick.) Then the composition

$$\underline{H}^{|\mu'_\tau|-2}_{\mathrm{dR}}(A^{|\mu'_\tau|-2}/\mathsf{M}_{\mathcal{H},1}) \cong \underline{H}^{|\mu'_\tau|-2}_{\mathrm{dR}}(A^{|\mu'_\tau|-2}/\mathsf{M}_{\mathcal{H},1}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \underline{H}^0_{\mathrm{dR}}(A^2/\mathsf{M}_{\mathcal{H},1})$$

$$\overset{\mathrm{Id}\otimes(\cup c_\tau^\lambda)}{\hookrightarrow} \underline{H}^{|\mu'_\tau|-2}_{\mathrm{dR}}(A^{|\mu'_\tau|-2}/\mathsf{M}_{\mathcal{H},1}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \underline{H}^2_{\mathrm{dR}}(A^2/\mathsf{M}_{\mathcal{H},1})(1) \overset{\mathrm{K}_\tau^{i,j}}{\hookrightarrow} \underline{H}^{|\mu'_\tau|}_{\mathrm{dR}}(A^{|\mu'_\tau|}/\mathsf{M}_{\mathcal{H},1})(1)$$

is dual to the morphism $\underline{H}^{\mathrm{dR}}_{|\mu'_\tau|}(A^{|\mu'_\tau|}/\mathsf{M}_{\mathcal{H},1}) \to \underline{H}^{\mathrm{dR}}_{|\mu'_\tau|-2}(A^{|\mu'_\tau|-2}/\mathsf{M}_{\mathcal{H},1})(1)$ inducing the geometric realization

$$\phi^\lambda_{\tau,i,j} : \underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\otimes\,|\mu'_\tau|} \to \underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\otimes(|\mu'_\tau|-2)}(1) \tag{3.4}$$

of $\phi_{i,j}^{\langle\,\cdot\,,c_\tau\varepsilon_\tau\,\cdot\,\rangle}$. That is, we take the cup product of the image of $\underline{H}^{|\mu'_\tau|-2}_{\mathrm{dR}}(A^{|\mu'_\tau|-2}/\mathsf{M}_{\mathcal{H},1})$ under $\mathrm{K}_\tau^{i,j}$ in $\underline{H}^{|\mu'_\tau|-2}_{\mathrm{dR}}(A^{|\mu'_\tau|}/\mathsf{M}_{\mathcal{H},1})$ with the pull-back of c_τ^λ to $A^{|\mu'_\tau|}$.

On the other hand, the pairing $\langle\,\cdot\,,\,\cdot\,\rangle_\lambda$ identifies $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})$ with its own dual, with values in $\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}(1)$. Therefore we obtain a morphism

$$\psi^\lambda_{\tau,i,j} : \underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\otimes(|\mu'_\tau|-2)}(1) \to \underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\otimes\,|\mu'_\tau|} \tag{3.5}$$

geometrically realizing the map $\psi_{i,j}^{\langle\,\cdot\,,c_\tau\varepsilon_\tau\,\cdot\,\rangle} : L_1^{\otimes(|\mu'_\tau|-2)}(1) \to L_1^{\otimes\,|\mu'_\tau|}$ inserting $\langle\,\cdot\,, c_\tau\varepsilon_\tau\,\cdot\,\rangle$ into the i-th and j-th component. Since $\varepsilon_\tau^\star = c_\tau\varepsilon_\tau c_\tau^{-1}$, the geometric action of $\varepsilon_{\tau,|\mu'_\tau|}$ commutes with $\phi^\lambda_{\tau,i,j}$ and $\psi^\lambda_{\tau,i,j}$, and induces $\phi^\lambda_{\tau,i,j} : \underline{L}^{\otimes\,|\mu'_\tau|}_{\mathrm{std},\tau} \to \underline{L}^{\otimes(|\mu'_\tau|-2)}_{\mathrm{std},\tau}(1)$ and $\psi^\lambda_{\tau,i,j} : \underline{L}^{\otimes(|\mu'_\tau|-2)}_{\mathrm{std},\tau}(1) \to \underline{L}^{\otimes\,|\mu'_\tau|}_{\mathrm{std},\tau}$.

Now assume that either $r_\tau = 0$ or $p \nmid 2r_\tau$. This is true, for example, if $\max(2, r_\tau) < p$. As explained in the paragraph following Definition 2.22, we have $\mathcal{E}_{\mathrm{G}_1}(\ker(\phi_{i,j}^{\langle\,\cdot\,,\,\cdot\,\rangle_{\mathrm{std},\tau}})) \cong (\mathrm{Id} - (2r_\tau)^{-1}\psi^\lambda_{\tau,i,j}\phi^\lambda_{\tau,i,j})(\underline{L}^{\otimes\,|\mu'_\tau|}_{\mathrm{std},\tau})$. (We cannot define $(2r_\tau)^{-1}$ when $r_\tau = 0$, but at the same time $\underline{L}^{\otimes\,|\mu'_\tau|}_{\mathrm{std},\tau}$ is trivial. In this case, we shall maintain the abuse of language that $\mathrm{Id} - (2r_\tau)^{-1}\psi^\lambda_{\tau,i,j}\phi^\lambda_{\tau,i,j}$ and similar operators below are defined symbolically and act trivially.) Combining all possible $1 \le i < j \le |\mu'_\tau|$, we define $\varepsilon^\lambda_{\tau,|\mu'_\tau|}$ to be the R_1-linear combination of algebraic correspondences on $A^{|\mu'_\tau|}$ acting as the idempotent

$$(\varepsilon^\lambda_{\tau,|\mu'_\tau|})_* = \prod_{1\le i<j\le|\mu'_\tau|} (\mathrm{Id} - (2r_\tau)^{-1}\psi^\lambda_{\tau,i,j}\phi^\lambda_{\tau,i,j}) \tag{3.6}$$

on $\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\otimes\,|\mu'_\tau|}$. Then $(\varepsilon^\lambda_{\tau,|\mu'_\tau|})_*(\underline{L}^{\otimes\,|\mu'_\tau|}_{\mathrm{std},\tau})$ is isomorphic to $\mathcal{E}_{\mathrm{G}_1}(L^{\langle|\mu'_\tau|\rangle}_{\mathrm{std},\tau})$ (resp. $\mathcal{E}_{\mathrm{G}_1}(L^{[|\mu'_\tau|]}_{\mathrm{std},\tau})$) when $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ (resp. $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$).

Finally, in the case $\tau \neq \tau \circ c$ we set $\varepsilon^{\lambda}_{\tau,|\mu'_\tau|}$ to be trivial, so that $(\varepsilon^{\lambda}_{\tau,|\mu'_\tau|})_* = \mathrm{Id}$. Using the Künneth morphisms, we define $\varepsilon^{\lambda}_{\mu}$ to be the product of pullbacks of $\varepsilon^{\lambda}_{\tau,|\mu'_\tau|}$, so that $(\varepsilon^{\lambda}_{\mu})_*$ acts on $\underset{\tau\in\Upsilon/c}{\otimes} \underline{L}^{\otimes n_\tau}_{\mathrm{std},\tau}$ as the idempotent

$$(\varepsilon^{\lambda}_{\mu})_* = \underset{\tau\in\Upsilon/c}{\otimes} (\varepsilon^{\lambda}_{\tau,|\mu'_\tau|})_*.$$

3.5. **Geometric plethysm.** We can summarize our constructions as follows:

Proposition 3.7. *Suppose $\mu \in \mathrm{X}^{+,<\mathrm{w}p}_{\mathrm{G}_1}$, with $0 \leq n := |\mu|_{\mathrm{L}} < p$, as in Definition 3.2. Then $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$ as well, so that the Weyl module $V_{[\mu]}$ is defined. (See Section 2.6.) Suppose moreover that $\max(2, r_\tau) < p$ whenever $\tau = \tau\circ c$. Consider the n-fold fiber product A^n of A over $\mathsf{M}_{\mathcal{H},1}$. Consider the coherent sheaf $\underline{H}^{\mathrm{dR}}_n(A^n/\mathsf{M}_{\mathcal{H},1}) \cong \wedge^n(\underline{H}^{\mathrm{dR}}_1(A/\mathsf{M}_{\mathcal{H},1})^{\oplus n})$ equipped with the canonical action of $R_1[\mathcal{O}^n_1 \rtimes \mathfrak{S}_n]$ induced functorially by the $\mathcal{O}$-endomorphism structure $i : \mathcal{O} \hookrightarrow \mathrm{End}_{\mathsf{M}_{\mathcal{H},1}}(A)$ and by permuting factors. Let $\varepsilon^{\mathrm{L}}_n$, $\varepsilon^{\mathrm{S}}_\mu$, and $\varepsilon^{\mathrm{Y}}_\mu$ be the elements in $R_1[\mathcal{O}^n_1 \rtimes \mathfrak{S}_n]$ defined in Sections 3.2–3.3, and let $\varepsilon^{\lambda}_\mu$ be the one defined in Section 3.4, all acting as idempotents on $\underline{H}^{\mathrm{dR}}_n(A^n/\mathsf{M}_{\mathcal{H},1})$. Put $\varepsilon_\mu := \varepsilon^{\lambda}_\mu\, \varepsilon^{\mathrm{Y}}_\mu\, \varepsilon^{\mathrm{S}}_\mu\, \varepsilon^{\mathrm{L}}_n$, so that*

$$(\varepsilon_\mu)_* = (\varepsilon^{\lambda}_\mu)_*\, (\varepsilon^{\mathrm{Y}}_\mu)_*\, (\varepsilon^{\mathrm{S}}_\mu)_*\, (\varepsilon^{\mathrm{L}}_n)_*,$$

and let

$$t_\mu := \mu_0 + \sum_{\tau\in\Upsilon/c} t_{\mu_\tau}$$

be the total number of Tate twists. (The order of $\varepsilon^{\lambda}_\mu$, $\varepsilon^{\mathrm{Y}}_\mu$, $\varepsilon^{\mathrm{S}}_\mu$, and $\varepsilon^{\mathrm{L}}_n$ in the definition of ε_μ does not matter, and their product ε_μ acts as an idempotent, because they commute with one another by definition.) Then we have canonical isomorphisms

$$\underline{V}_{[\mu]} := \mathcal{E}_{\mathrm{G}_1}(V_{[\mu]}) \cong (\varepsilon_\mu)_*\, \underline{H}^{\mathrm{dR}}_n(A^n/\mathsf{M}_{\mathcal{H},1})(t_\mu).$$

and (by duality)

$$\underline{V}^\vee_{[\mu]} := \mathcal{E}_{\mathrm{G}_1}(V^\vee_{[\mu]}) \cong (\varepsilon_\mu)^*\, \underline{H}^n_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1})(-t_\mu).$$

Moreover, the $\mathtt{F}$-filtration on $\mathcal{E}_{\mathrm{G}_1}(V_{[\mu]})$ coincides with the Hodge filtration on $\underline{H}^n_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1})(t_\mu)$. The duality between $\mathcal{E}_{\mathrm{G}_1}(V_{[\mu]})$ and $\mathcal{E}_{\mathrm{G}_1}(V^\vee_{[\mu]})$ is obvious.

Proof. Since $\mu \in \mathrm{X}^{+,<\mathrm{w}p}_{\mathrm{G}_1}$, the construction in Section 2.5 shows that $V_{[\mu]}$ can be constructed using the same collection of idempotents. Hence the result follows from the identifications in Example 1.22, and from the matching between powers of the similitude character υ and Tate twists. □

Remark 3.8. Since ε_μ acts as an idempotent, the vector bundle $\underline{V}_{[\mu]}$ (resp. $\underline{V}^\vee_{[\mu]}$) is a direct summand of $\underline{H}^{\mathrm{dR}}_n(A^n/\mathsf{M}_{\mathcal{H},1})(t_\mu)$ (resp. $\underline{H}^n_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1})(-t_\mu)$).

Definition 3.9. *We set $d := \dim_{\mathsf{S}_1}(\mathsf{M}_{\mathcal{H},1})$, $|\mu|_{\mathrm{re}} := d + |\mu|_{\mathrm{L}}$, and $|\mu|_{\mathrm{tot}} := \dim_{\mathsf{S}_1}(A^n) = d + \dim_{\mathsf{M}_{\mathcal{H},1}}(A)\ |\mu|_{\mathrm{L}}$. We call $|\mu|_{\mathrm{re}}$ (resp. $|\mu|_{\mathrm{tot}}$) the* **realization size** *(resp.* **total size***) of μ.*

Remark 3.10. According to Remark 1.10, we have the simple formula $d = \dim_{R_1}(\mathrm{G}_1) - \dim_{R_1}(\mathrm{P}_1)$.

Remark 3.11. Note that $|\mu|_{\mathrm{re}}$ and $|\mu|_{\mathrm{tot}}$ are always non-negative and are insensitive to the entry μ_0 in μ. In particular, they are different from the so-called *motivic weight* of the local system $\underline{V}_{[\mu]}^{\vee}$. (Nowhere in the various bounds in our results on torsion coefficients will appear the motivic weight.)

3.6. **Construction without Poincaré duality.** We retain the assumptions of Proposition 3.7 in this subsection.

The definition of the idempotent ε_μ, which we have employed to realize $\underline{V}_{[\mu]}^{\vee}$ as a direct summand of $\underline{H}_{\mathrm{dR}}^n(A^n/\mathsf{M}_{\mathcal{H},1})(-t_\mu)$ (see Remark 3.8), relies on Poincaré duality when $\tau = \tau \circ c$ (i.e., for types C and D). For technical reasons (that will be clarified in Section 5.2), it is preferable to avoid this dependence, and here is how it can be done.

So suppose $\tau \in \Upsilon$ satisfies $\tau = \tau \circ c$. For simplicity, let us assume that $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$. (The case when $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$ is similar.) Then we know that $(\varepsilon^{\lambda}_{\tau,|\mu'_\tau|})_*(\underline{L}_{\mathrm{std},\tau}^{\otimes|\mu'_\tau|}) \cong \mathcal{E}_{\mathrm{G}_1}(L_{\mathrm{std},\tau}^{\langle|\mu'_\tau|\rangle})$ is the kernel of

$$\underset{1\leq i<j\leq n}{\oplus} \phi^{\lambda}_{\tau,i,j} : \underline{L}_{\mathrm{std},\tau}^{\otimes|\mu'_\tau|} \to \underset{1\leq i<j\leq n}{\oplus} \underline{L}_{\mathrm{std},\tau}^{\otimes|\mu'_\tau|-2}(1). \tag{3.12}$$

(See the paragraph containing (3.4).) Here the notation $\phi^{\lambda}_{\tau,i,j}$ makes sense (as a restriction) because the geometric action of $\varepsilon_{\tau,|\mu'_\tau|}$ commutes with $\phi^{\lambda}_{\tau,i,j}$. Equivalently, $(\varepsilon^{\lambda}_{\tau,|\mu'_\tau|})^*((\underline{L}_{\mathrm{std},\tau}{}^{\vee})^{\otimes|\mu'_\tau|})$ is the cokernel of

$$\sum_{1\leq i<j\leq n} (\phi^{\lambda}_{\tau,i,j})^{\vee} : \underset{1\leq i<j\leq n}{\oplus} (\underline{L}^{\vee}_{\mathrm{std},\tau})^{\otimes|\mu'_\tau|-2}(-1) \to (\underline{L}^{\vee}_{\mathrm{std},\tau})^{\otimes|\mu'_\tau|}. \tag{3.13}$$

As explained in Section 3.4, the definition of each $(\phi^{\lambda}_{\tau,i,j})^{\vee}$ involves only functoriality and cup product with the pullback of c^{λ}_τ.

Lemma 3.14. *The image of the morphism* (3.13) *is globally a direct summand* (*as a coherent module with connection*).

Proof. This is because it is the kernel of the idempotent $(\varepsilon^{\lambda}_{\tau,|\mu'_\tau|})^*$. □

Remark 3.15. The point is that, while we use $(\varepsilon^{\lambda}_{\tau,|\mu'_\tau|})^*$ in the proof, we do not need it in the definition using the morphism (3.13).

Lemma 3.16. *If* $|\mu|_{\mathrm{re}} < p$, *then the kernel of the morphism* (3.13) *is globally a direct summand.*

Proof. Equivalently, we can show that the image of the dual morphism (3.12) is a direct summand. Without using a convenient idempotent like $(\varepsilon^{\lambda}_{\tau,|\mu'_\tau|})^*$, it suffices to notice that (3.12) is the functorial image under $\mathcal{E}_{\mathrm{G}_1,R_1}(\,\cdot\,)$ of a similar morphism in $\mathrm{Rep}_R(\mathrm{G}_1)$. The question is whether the surjection from the source to the image of this morphism (in $\mathrm{Rep}_R(\mathrm{G}_1)$) *splits* (non-canonically). By [42, 1.10, Cor.] (or rather by the same proof there), it suffices to show that all the objects in (3.13) lie in the image under $\mathcal{E}_{\mathrm{G}_1,R_1}(\,\cdot\,)$ of representations with p-small weights, between which there cannot be any nontrivial extension classes. Since $|\mu|_{\mathrm{re}} = d + |\mu|_{\mathrm{L}} < p$, it suffices to check that, for any integer m such that $0 \leq m < p - d$, all the weights of the representation $L_{\mathrm{std},\tau}^{\otimes m}$ of G_1 (or rather of G_τ) are p-small. Any weight ν of $L_{\mathrm{std},\tau}^{\otimes m}$ satisfies $|\nu_\tau| := \sum_{1\leq i_\tau\leq r_\tau} |\nu_{\tau,i_\tau}| \leq m < p - d$. Then, for any $1 \leq i_\tau < j_\tau \leq r_\tau$, we have $|\nu_{\tau,i_\tau} + i_\tau| + |\nu_{\tau,j_\tau} + j_\tau| \leq m + d < p$. This implies that $(\nu + \rho, \alpha^\vee) \leq p$, i.e. ν is p-small, as desired. □

Remark 3.17. Lemma 3.16 is needed only in Section 5.2.

4. Cohomology of automorphic bundles

In this section, we fix a choice of $\mu \in \mathrm{X}^{+,<\mathrm{w}p}_{\mathrm{G}_1}$ and take $n = |\mu|_{\mathrm{L}}$. We shall maintain the running assumption that $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$, so that the element $\varepsilon_\mu = \varepsilon^\lambda_\mu\, \varepsilon^{\mathrm{Y}}_\mu\, \varepsilon^{\mathrm{S}}_\mu\, \varepsilon^{\mathrm{L}}_n$ in Proposition 3.7 is defined. Let $f_n : A^n \to \mathsf{M}_{\mathcal{H},1}$ be the structural morphism.

4.1. **Koszul and Hodge filtrations.** By smoothness of f_n, we have the exact sequence $0 \to f_n^*(\Omega^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}) \to \Omega^1_{A^n/\mathsf{S}_1} \to \Omega^1_{A^n/\mathsf{M}_{\mathcal{H},1}} \to 0$, which induces the *Koszul filtration* [24, 1.2, 1.3] $\mathtt{K}^a(\Omega^\bullet_{A^n/\mathsf{S}_1}) := \mathrm{image}(\Omega^{\bullet-a}_{A^n/\mathsf{S}_1} \underset{\mathscr{O}_{A^n}}{\otimes} f_n^*(\Omega^a_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}) \to \Omega^\bullet_{A^n/\mathsf{S}_1})$ on $\Omega^\bullet_{A^n/\mathsf{S}_1}$, with graded pieces $\mathrm{Gr}^a_{\mathtt{K}}(\Omega^\bullet_{A^n/\mathsf{S}_1}) \cong \Omega^{\bullet-a}_{A^n/\mathsf{M}_{\mathcal{H},1}} \underset{\mathscr{O}_{A^n}}{\otimes} f_n^*(\Omega^a_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1})$.

On the other hand, we have the Hodge filtration $\mathtt{F}^a(\Omega^\bullet_{A^n/\mathsf{M}_{\mathcal{H},1}}) := \Omega^{\bullet\geq a}_{A^n/\mathsf{M}_{\mathcal{H},1}}$ on $\Omega^\bullet_{A^n/\mathsf{M}_{\mathcal{H},1}}$, giving the Hodge filtration $\mathtt{F}^a(\underline{H}^i_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1})) := \mathrm{image}(R^i(f_n)_*(\mathtt{F}^a(\Omega^\bullet_{A^n/\mathsf{M}_{\mathcal{H},1}})) \to R^i(f_n)_*(\Omega^\bullet_{A^n/\mathsf{M}_{\mathcal{H},1}}))$ on $\underline{H}^i_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1})$. By applying $R^\bullet(f_n)_*$ to the short exact sequence

$$0 \to \Omega^{\bullet-1}_{A^n/\mathsf{M}_{\mathcal{H},1}} \underset{\mathscr{O}_{A^n}}{\otimes} f_n^*(\Omega^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}) \to \mathtt{K}^0/\mathtt{K}^2 \to \Omega^\bullet_{A^n/\mathsf{M}_{\mathcal{H},1}} \to 0, \tag{4.1}$$

we obtain in the long exact sequence the connecting homomorphisms $\underline{H}^i_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1}) = R^i(f_n)_*(\Omega^\bullet_{A^n/\mathsf{M}_{\mathcal{H},1}}) \xrightarrow{\nabla} R^{i+1}(f_n)_*(\Omega^{\bullet-1}_{A^n/\mathsf{M}_{\mathcal{H},1}} \underset{\mathscr{O}_{A^n}}{\otimes} f_n^*(\Omega^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1})) \cong \underline{H}^i_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}$, which is the Gauss–Manin connection. If we take the $\mathtt{F}$-filtration on (4.1), we obtain $0 \to (\mathtt{F}^{a-1}(\Omega^\bullet_{A^n/\mathsf{M}_{\mathcal{H},1}}) \underset{\mathscr{O}_{A^n}}{\otimes} f_n^*(\Omega^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}))[-1] \to \mathtt{F}^a(\mathtt{K}^0/\mathtt{K}^2) \to \mathtt{F}^a(\Omega^\bullet_{A^n/\mathsf{M}_{\mathcal{H},1}}) \to 0$ and hence the *Griffiths transversality* (as in [24, Prop. 1.4.1.6]) $\nabla(\mathtt{F}^a(\underline{H}^i_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1}))) \subset \mathtt{F}^{a-1}(\underline{H}^i_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1})) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}$. Since $A^n \to \mathsf{M}_{\mathcal{H},1}$ is an abelian scheme, the Hodge to de Rham spectral sequence $E_1^{a,i-a} := R^{i-a}(f_n)_*(\Omega^a_{A^n/\mathsf{M}_{\mathcal{H},1}}) \Rightarrow \underline{H}^i_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1})$ degenerates at E_1. (See for example [4, Prop. 2.5.2].) Then $\mathrm{Gr}^a_{\mathtt{F}}(\underline{H}^i_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1})) \cong R^{i-a}(f_n)_*(\Omega^a_{A^n/\mathsf{M}_{\mathcal{H},1}})$, and we can conclude (as in [24, Prop. 1.4.1.7]) that the induced morphism $\nabla : \mathrm{Gr}^a_{\mathtt{F}}\, \underline{H}^i_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1}) \to \mathrm{Gr}^{a-1}_{\mathtt{F}}\, \underline{H}^i_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}$ agrees with the morphism $R^{i-a}(f_n)_*(\Omega^a_{A^n/\mathsf{M}_{\mathcal{H},1}}) \to R^{i-a+1}(f_n)_*(\Omega^{a-1}_{A^n/\mathsf{M}_{\mathcal{H},1}}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^1_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}$ defined by cup product with the Kodaira–Spencer class.

The Koszul filtration gives a spectral sequence

$$E_1^{a,b} := R^{a+b}(f_n)_*(\mathrm{Gr}^a_{\mathtt{K}}(\Omega^\bullet_{A^n/\mathsf{S}_1})) \Rightarrow R^{a+b}(f_n)_*(\Omega^\bullet_{A^n/\mathsf{S}_1}), \tag{4.2}$$

where each $E_1^{a,b}$ can be canonically identified with $\underline{H}^b_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^a_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}$. As in [23, (3.2.5)] (with the notation $\mathtt{K}$ here being $\mathtt{F}$ there), the de Rham complex $(\underline{H}^b_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}, \nabla)$ is the complex $(E_1^{\bullet,b}, \mathrm{d}_1^{\bullet,b})$ in the b-th row of the

E_1-terms of the above spectral sequence (4.2). By taking cohomology over $\mathsf{M}_{\mathcal{H},1}$, we obtain the Leray spectral sequence (cf. [23, Rem. 3.3])

$$E_2^{a,b} := H^a_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{H}^b_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1})) \Rightarrow H^{a+b}_{\mathrm{dR}}(A^n/\mathsf{S}_1). \tag{4.3}$$

(The left-hand side of (4.3) stands for $H^a(\mathsf{M}_{\mathcal{H},1}, \underline{H}^b_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1})$, for simplicity.)

For any integer l, we denote by $[l]$ the multiplication by l morphism on the abelian scheme A^n over $\mathsf{M}_{\mathcal{H},1}$. This lets the algebra $\mathbb{Z}_{(p)}[\mathbb{Z}]$ (spanned by the symbols $[l]$) act on the (relative and absolute) cohomology groups of A^n. Essential in the Lieberman's trick is the observation that $[l]$ acts as multiplication by l^i on the relative cohomology $\underline{H}^i$ of any abelian scheme.

Proposition 4.4. *Suppose $2d < p$. Then the Leray spectral sequence (4.3) degenerates at E_2.*

Proof. The algebra $\mathbb{Z}_{(p)}[\mathbb{Z}]$ acts on the spectral sequence (4.3) by functoriality. Let l_0 be an integer reducing modulo p to a generator of $\mathbb{F}_p^\times$. Then for any pair of integers i and j, the integer $l_0^i - l_0^j$ is invertible in $\mathbb{Z}_{(p)}$ unless $i \equiv j \mod p-1$. For an integer b_0 such that $0 \leq b_0 \leq N := 2n \dim(A/\mathsf{M}_{\mathcal{H},1})$, put

$$\varepsilon^{\deg}_{b_0} := \prod_{0 \leq i \leq N,\ i \not\equiv b_0 \bmod p-1} (l_0^{b_0} - l_0^i)^{-1}([l_0] - l_0^i[1]) \in \mathbb{Z}_{(p)}[\mathbb{Z}]. \tag{4.5}$$

It annihilates $E_2^{a,b}$ unless $b \equiv b_0 \mod p-1$, acts as a unit on $E_2^{a,b}$ when $b \equiv b_0 \mod p-1$, and acts as 1 on E_2^{a,b_0}. Already from the terms on the E_2 page of (4.3), we have $E_r^{a,b} = 0$ for all $r \geq 2$, unless $a \in [0, 2d]$ and $b \in [0, N]$. Any differential between terms in two rows of E_r with the vertical distance at least $p-1$ is zero, since $p - 1 \geq 2d$. With varying b_0, we obtain the degeneration of (4.3). □

Remark 4.6. The degeneration itself is not strictly necessary in the main line of proofs of our results. However, we *will* make use of the element (4.5).

4.2. **De Rham cohomology.**

Lemma 4.7. *With the assumptions as in the beginning of Section 4, the application of $(\varepsilon_\mu)^*$ and the Tate twist in Proposition 3.7 gives*

$$(\underline{V}^\vee_{[\mu]} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}, \nabla) \cong (\varepsilon_\mu)^* \, (\underline{H}^n_{\mathrm{dR}}(A^n/\mathsf{M}_{\mathcal{H},1}) \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^a_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1}, \nabla)(-t_\mu)$$

and respects the Hodge filtrations on both sides.

Proof. The operator ε_μ was defined using the product of certain R_1-linear combinations of pullbacks via morphisms between $\mathsf{M}_{\mathcal{H},1}$-schemes, the first Chern class of the Poincaré line bundle, the cup product, and the Künneth decomposition. As such, $(\varepsilon_\mu)^*$ is horizontal with respect to the Gauss–Manin connection. The Hodge filtrations are respected because they are so when $V_{[\mu]} \cong L_1$ as in Example 2.10. □

Proposition 4.8. *With the assumptions as in the beginning of Section 4, suppose moreover that $2d < p$. Let $\varepsilon^{\deg}_n \in \mathbb{Z}_{(p)}[\mathbb{Z}]$ be defined by (4.5) (with some choice of l_0 and with $b_0 = n$). Then we have a canonical isomorphism*

$$H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]}) \cong (\varepsilon_\mu)^* \, (\varepsilon^{\deg}_n)^* \, H^{i+n}_{\mathrm{dR}}(A^n/\mathsf{S}_1)(-t_\mu). \tag{4.9}$$

for every integer i.

Proof. According to the proof of Proposition 4.4, under the application of $(\varepsilon_n^{\deg})^*$, only the term $E_2^{i,n}$ survives among the terms $E_2^{a,b}$ with $a+b=i+n$ in (4.3). Therefore the result follows from Lemma 4.7. □

Remark 4.10. Everything in Sections 4.1–4.2 remains valid if we base change (horizontally) from R_1 to an R_1-algebra R.

4.3. **Étale and Betti cohomology.** Let F_0^{ac} be the algebraic closure of F_0 in $\mathbb{C}$. By abuse of notation, we shall write $\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}} := \mathsf{M}_{\mathcal{H},0} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} F_0^{\mathrm{ac}}$ and denote by $A_{F_0^{\mathrm{ac}}}$ the pullback (to $\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}$) of the universal family from $\mathsf{M}_{\mathcal{H},0}$, rather than from $\mathsf{M}_{\mathcal{H},1}$. Let $f_{n,F_0^{\mathrm{ac}}} : A^n_{F_0^{\mathrm{ac}}} \to \mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}$ denote the structural morphism. We shall use similar notation for pullbacks to $\mathbb{C}$.

Let Λ be an integral domain, finite flat over the p-adic completion of R_1 (and hence finite flat over $\mathbb{Z}_p$). Then $(\varepsilon_\mu)_*$ acts naturally on the relative étale cohomology $R^n(f_{n,F_0^{\mathrm{ac}}})_{*,\text{ét}}(\Lambda) \cong R^n(f_{n,F_0^{\mathrm{ac}}})_{*,\text{ét}}(\mathbb{Z}_p) \underset{\mathbb{Z}_p}{\otimes} \Lambda$ and the relative Betti cohomology $R^n(f_{n,\mathbb{C}})_{*,\mathrm{B}}(\Lambda) \cong R^n(f_{n,\mathbb{C}})_{*,\mathrm{B}}(\mathbb{Z}) \underset{\mathbb{Z}}{\otimes} \Lambda$, and we define

$$ {}_{\text{ét}}\underline{V}_{[\mu]}^{\vee} := (\varepsilon_\mu)^*\, R^n(f_{n,F_0^{\mathrm{ac}}})_{*,\text{ét}}(\Lambda)(-t_\mu) $$

and

$$ {}_{\mathrm{B}}\underline{V}_{[\mu]}^{\vee} := (\varepsilon_\mu)^*\, R^n(f_{n,\mathbb{C}})_{*,\mathrm{B}}(\Lambda)(-t_\mu). $$

Remark 4.11. For the same reason as in Remark 3.8, the sheaf ${}_{\text{ét}}\underline{V}_{[\mu]}^{\vee}$ (resp. ${}_{\mathrm{B}}\underline{V}_{[\mu]}^{\vee}$) is a direct summand of $R^n(f_{n,F_0^{\mathrm{ac}}})_{*,\text{ét}}(\Lambda)(-t_\mu)$ (resp. $R^n(f_{n,\mathbb{C}})_{*,\mathrm{B}}(\Lambda)(-t_\mu)$).

Proposition 4.12. *With the assumptions as in the beginning of Section* 4*, suppose moreover that* $2d < p$*. Let* $\varepsilon_n^{\deg} \in \mathbb{Z}_{(p)}[\mathbb{Z}]$ *be as in Proposition* 4.8*. Then, for any* i*, we have canonical isomorphisms*

$$ H^i_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}_{[\mu]}^{\vee}) \cong (\varepsilon_\mu)^*\, (\varepsilon_n^{\deg})^*\, H^{i+n}_{\text{ét}}(A^n_{F_0^{\mathrm{ac}}}, \Lambda)(-t_\mu) \tag{4.13} $$

and

$$ H^i_{\mathrm{B}}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, {}_{\mathrm{B}}\underline{V}_{[\mu]}^{\vee}) \cong (\varepsilon_\mu)^*\, (\varepsilon_n^{\deg})^*\, H^{i+n}_{\mathrm{B}}(A^n_{\mathbb{C}}, \Lambda)(-t_\mu). $$

Proof. The same argument as in the proof of Proposition 4.8 using a Leray spectral sequence analogous to (4.3) works here. □

Proposition 4.14. *Let* K^{ac} *be any algebraically closed subfield of* $\mathbb{C}$ *containing* F_0^{ac}*. The embeddings* $F_0^{\mathrm{ac}} \overset{\text{can.}}{\hookrightarrow} K^{\mathrm{ac}} \hookrightarrow \mathbb{C}$ *determine canonical isomorphisms* $H^i_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}_{[\mu]}^{\vee}) \overset{\sim}{\to} H^i_{\text{ét}}(\mathsf{M}_{\mathcal{H},K^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}_{[\mu]}^{\vee}) \overset{\sim}{\to} H^i_{\mathrm{B}}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, {}_{\mathrm{B}}\underline{V}_{[\mu]}^{\vee})$ *for all* i.

Proof. By [8, Arcata, V, Cor. 3.3], the embeddings between separably closed fields determine canonical isomorphisms $H^{i+n}_{\text{ét}}(A^n_{F_0^{\mathrm{ac}}}, \Lambda) \overset{\sim}{\to} H^{i+n}_{\text{ét}}(A^n_{K^{\mathrm{ac}}}, \Lambda) \overset{\sim}{\to} H^{i+n}_{\text{ét}}(A^n_{\mathbb{C}}, \Lambda)$. By [2, XI, Thm. 4.4], there is a canonical isomorphism $H^{i+n}_{\text{ét}}(A^n_{\mathbb{C}}, \Lambda) \overset{\sim}{\to} H^{i+n}_{\mathrm{B}}(A^n_{\mathbb{C}}, \Lambda)$. Thus the result follows from Proposition 4.12 by applying $(\varepsilon_\mu)^*$ and Tate twists. □

Thus Proposition 4.14 relates the Betti cohomology in the Question of the Introduction with the étale cohomology, which might be more interesting because it realizes Galois representations. Moreover, for our purpose, the main technical advantage of the (torsion) étale cohomology is that (with the reduction steps to be introduced in later sections) it can be studied using techniques only available in positive characteristics via p-adic comparison theorems.

5. Crystalline comparison isomorphisms

To prove the vanishing and the torsion-freeness of the Betti (or étale) cohomology in the Introduction, we will first prove the corresponding statements for the de Rham (or crystalline) cohomology, and apply the crystalline comparison isomorphism. We will only use the basic case of a projective smooth scheme over an absolutely unramified p-adic base ring.

First, let us fix the notation. The structural homomorphism $\mathcal{O}_{F_0} \to R_1$ determines a p-adic place of F_0, and we will denote the completion of $\mathcal{O}_{F_0}$ at this place by W; recall that p is unramified in $\mathcal{O}_{F_0}$, and we will identify W with the ring of Witt vectors of its residue field. By passing to the completions, W embeds canonically into the p-adic completion of R_1. Let $K := \mathrm{Frac}(W)$, and fix an algebraic closure K^{ac} of K. We also fix an isomorphism $\iota : K^{\mathrm{ac}} \xrightarrow{\sim} \mathbb{C}$ of F_0-algebras, and identify F_0^{ac} (under ι) with the algebraic closure of F_0 in K^{ac}.

We let $\mathsf{M}_{\mathcal{H},W} := \mathsf{M}_{\mathcal{H},0} \underset{\mathcal{O}_{F_0,(p)}}{\otimes} W$ and denote by A_W the pullback (to $\mathsf{M}_{\mathcal{H},W}$) of the universal family from $\mathsf{M}_{\mathcal{H},0}$ (rather than from $\mathsf{M}_{\mathcal{H},1}$). We shall use similar notations for pullbacks to K and K^{ac}.

5.1. **Constant coefficients.**

For an integer $s \geq 1$, we write $W_s = W/p^sW$ and use the abelian category $\underline{MF}_{\mathrm{tor}}^{f,r}$ defined in [6, 3.1.1]. For the sake of brevity, we shall refer to an object $(M, (Fil^a(M))_{0\leq a\leq r}, (\varphi_a)_{0\leq a\leq r})$ of $\underline{MF}_{\mathrm{tor}}^{f,r}$ simply by the underlying W-module M when there is no ambiguity about additional data.

Let Z be a proper smooth scheme over W. For any integer $s \geq 1$, put $Z_s := Z \underset{W}{\otimes} W_s$. Then [14, II, Cor. 2.7] shows that for $0 \leq j \leq r \leq p-1$, the de Rham cohomology $H^j(Z_s, \Omega^\bullet_{Z_s})$ (with its Hodge filtration and its crystalline Frobenius, which we omit from the notation) defines an object of the category $\underline{MF}_{\mathrm{tor}}^{f,r}$.

Recall $\mathrm{A}_{\mathrm{cr}} := \varprojlim_s H^0_{\mathrm{cr}}((\mathcal{O}_{K^{\mathrm{ac}}}/(p\mathcal{O}_{K^{\mathrm{ac}}}))/W_s)$. (See [6, 3.1.2] or [13, p. 242].)

Definition 5.1 (see [14, II, Cor. 2.7])**.** *For an object M of $\underline{MF}_{\mathrm{tor}}^{f,r}$ and an integer $s \geq 1$ such that $p^sM = 0$, we put $\mathrm{T}^*_{\mathrm{cr}}(M) := \mathrm{Hom}_{W,Fil^\bullet,\varphi_\bullet}(M, \mathrm{A}_{\mathrm{cr}}/p^s\mathrm{A}_{\mathrm{cr}})$. (We suppress s from the notation since the result is independent of the choice of s.) It defines a contravariant functor from $\underline{MF}_{\mathrm{tor}}^{f,r}$ to the category of continuous $\mathrm{Gal}(K^{\mathrm{ac}}/K)$-modules. We also define a covariant functor by putting $\mathrm{T}_{\mathrm{cr}}(M) := \mathrm{T}^*_{\mathrm{cr}}(M)^\vee \cong Fil^r(\mathrm{A}_{\mathrm{cr}} \underset{W}{\otimes} M)^{\varphi_r=1}(-r)$.*

By [6, Thm. 3.1.3.1], for $0 \leq r \leq p-2$, the functor $\mathrm{T}^*_{\mathrm{cr}}$ is fully faithful.

Theorem 5.2 (see [6, Thm. 3.2.3], [14, III, 6.3], and [12, Thm. 5.3])**.** *Let Z be a proper smooth scheme over W, and let s be an integer ≥ 1. For $0 \leq j \leq r \leq p-2$, we have a natural isomorphism $\mathrm{T}_{\mathrm{cr}}(H^j_{\mathrm{dR}}(Z_s/W_s)) \cong H^j_{\text{ét}}(Z \underset{W}{\otimes} K^{\mathrm{ac}}, \mathbb{Z}/p^s\mathbb{Z})$, compatible with the action of $\mathrm{Gal}(K^{\mathrm{ac}}/K)$. The isomorphism is functorial in the proper smooth W-scheme Z and is compatible with the cup product structures and with the formation of the Chern classes of line bundles over Z.*

5.2. **Automorphic coefficients.**

Let Λ be an integral domain, finite flat over the p-adic completion of R_1 (and hence finite flat over $\mathbb{Z}_p$). (See the second paragraph of Section 4.3.) Assume moreover that the set $\Omega := \mathrm{Hom}_{\mathbb{Z}_p\text{-alg.}}(W, \Lambda)$ has cardinality

$[F_0 : \mathbb{Q}]$, so that there is a natural decomposition

$$W \underset{\mathbb{Z}_p}{\otimes} \Lambda \cong \prod_{\sigma\in\Omega} W_\sigma, \tag{5.3}$$

where each W_σ is a copy of Λ on which W acts via $\sigma : W \to \Lambda$.

Let $\mu \in \mathrm{X}^{+,<\mathrm{w}p}_{\mathrm{G}_1}$ with $n := |\mu|_{\mathrm{L}}$. According to Theorem 5.2, for any integer $s \geq 1$ and any $0 \leq j \leq p-2$, we have a natural isomorphism

$$\mathrm{T}_{\mathrm{cr}}(H^j_{\mathrm{dR}}(A^n_{W_s}/W_s)) \cong H^j_{\text{ét}}(A^n_{K^{\mathrm{ac}}}, \mathbb{Z}/p^s\mathbb{Z}). \tag{5.4}$$

Let $\Lambda_s := \Lambda/p^s\Lambda$, and apply $\underset{\mathbb{Z}/p^s\mathbb{Z}}{\otimes} \Lambda_s$ to both sides of (5.4). Then we obtain

$$\mathrm{T}_{\mathrm{cr}}((H^j_{\mathrm{dR}}(A^n_{W_s}/\operatorname{Spec}(W_s)) \underset{\mathbb{Z}/p^s\mathbb{Z}}{\otimes} \Lambda_s) \cong H^j_{\text{ét}}(A^n_{K^{\mathrm{ac}}}, \Lambda_s). \tag{5.5}$$

By taking reduction modulo p^s of (5.3), we obtain a similar decomposition $W_s \underset{\mathbb{Z}/p^s\mathbb{Z}}{\otimes} \Lambda_s \cong \prod_{\sigma\in\Omega} W_{\sigma,s}$ for each integer $s \geq 1$. By the base change property of the de Rham cohomology, the isomorphism (5.5) can be rewritten as

$$\mathrm{T}_{\mathrm{cr}}\Big(\underset{\sigma\in\Omega}{\oplus} H^j_{\mathrm{dR}}(A^n_{W_{\sigma,s}}/W_{\sigma,s})\Big) \cong H^j_{\text{ét}}(A^n_{K^{\mathrm{ac}}}, \Lambda_s). \tag{5.6}$$

Suppose $2d < p$, and $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Let $\varepsilon_\mu = \varepsilon^\lambda_\mu\, \varepsilon^{\mathrm{Y}}_\mu\, \varepsilon^{\mathrm{S}}_\mu\, \varepsilon^{\mathrm{L}}_n$ be defined in Proposition 3.7, and let $\varepsilon^{\deg}_n$ be defined as in (4.5) with $b_0 = n$. Then the sheaves ${}_{\text{ét}}\underline{V}^\vee_{[\mu]}$ and ${}_{\mathrm{B}}\underline{V}^\vee_{[\mu]}$ are defined as in Section 4.3, and Propositions 4.8 and 4.12 relate the cohomology of automorphic sheaves to those of the fiber products of A.

Suppose moreover that $|\mu|_{\mathrm{re}} < p$. Then Lemmas 3.14 and 3.16 imply that the action of the idempotent $(\varepsilon^\lambda_\mu)^*$ can be achieved by taking cokernels of morphisms from cohomology groups of lower degrees, defined by functoriality and by cup products with Chern classes of line bundles. (We use Lemma 3.16 to ensure that the cohomology of the cokernel of (3.13) is the cokernel of the induced morphism between cohomology groups.) On the other hand, all the actions of $\varepsilon^{\mathrm{Y}}_\mu$, $\varepsilon^{\mathrm{S}}_\mu$, $\varepsilon^{\mathrm{L}}_n$, and $\varepsilon^{\deg}_n$ involve only functoriality. Therefore, by (4.9) and (4.13), the natural properties satisfied by the comparison isomorphism in Theorem 5.2 imply that

$$\mathrm{T}_{\mathrm{cr}}\Big(\underset{\sigma\in\Omega}{\oplus} H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},W_{\sigma,s}}/\mathsf{S}_{W_{\sigma,s}}, \underline{V}^\vee_{[\mu],W_{\sigma,s}})\Big) \cong H^i_{\text{ét}}(\mathsf{M}_{\mathcal{H},K^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_s}) \tag{5.7}$$

for any $0 \leq i \leq 2d$ such that $j = i + n \leq p-2$.

Proposition 5.8. *With the assumptions on μ and p above, if $H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},1}, \underline{V}^\vee_{[\mu],\kappa_1}) = 0$ for some integer i such that $i + n \leq p-2$, then $H^i_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1}) = 0$ for the same i.*

Proof. This follows from (5.7) and Proposition 4.14. □

Definition 5.9. *We set $|\mu|_{\mathrm{comp}} := 2d + n$, called the* **comparison size** *of μ.*

Remark 5.10. The definition of $|\mu|_{\mathrm{comp}}$ depends on the comparison theorem we use. Using the crystalline comparison that allows non-constant coefficients, $|\mu|_{\mathrm{comp}}$ can be made smaller.

6. Illusie's vanishing theorem

6.1. **Statement.** We use Illusie's notation in this subsection, which is somewhat different from ours. As we will rely on the vanishing theorem only in the form of Corollary 6.2 in the next subsection, this should not create any confusion.

Let k be a perfect field of characteristic $p > 0$, and let (X, D) and (Y, E) be pairs of smooth schemes over k endowed with simple normal crossings divisors. Suppose $f : (X, D) \to (Y, E)$ is a proper semistable morphism (see [22, §1]), and consider the relative logarithmic de Rham cohomology sheaves $H^m(f) = R^m f_*(\omega^\cdot_{X/Y})$ for integers $m \geq 0$, equipped with the Hodge filtration and the Gauss–Manin connection (the two satisfying the Griffiths transversality).

Theorem 6.1 (Illusie; cf. [22, Cor. 4.16])**.** *Assume that f lifts to $\tilde{f}$ over $W_2(k)$ in the obvious sense* (*see* [22, §2])*, that Y is proper over k of pure dimension e, and that L is an ample line bundle over Y. Then, for every integer $m < p - e$, we have*

1. $H^{i+j}(Y, L \otimes \mathrm{gr}^i \, \omega^\cdot_Y(H^m(f))) = 0$ *for $i + j > e$; and*
2. $H^{i+j}(Y, L^{-1} \otimes \mathrm{gr}^i \, \omega^\cdot_Y(H^m(f))) = 0$ *for $i + j < e$.*

Proof. The assumptions imply that the conclusion of [22, Thm. 4.7] is true, namely that there is a decomposition in the derived category

$$\mathop{\oplus}_j \mathrm{gr}^j \, \omega^\cdot_{Y_1}(H_1) \overset{\sim}{\to} F_{Y/k*}\omega^\cdot_Y(H),$$

where we abbreviated $H = H^m(f)$, and where the subscript 1 denotes the base change by the absolute Frobenius on k. The condition (*) in [22, Thm. 4.7] is verified for $i+j < p$ by [22, Cor. 2.4] in view of our assumptions, and this suffices for the calculations and constructions in [22, §§3–4]. Moreover, the condition $m+e < p$ implies that the subcomplex G_{p-1} is the whole complex.

From this decomposition, we get our first vanishing statement just as Illusie got [22, (4.16.1)], using Serre vanishing.

The second statement is different from (4.16.2) in *loc. cit.*, when E is nonempty. Instead of applying duality, we directly apply the inequality (4.16.3) in *loc. cit.* to $M = L^{-1}$ repeatedly, and use Serre vanishing for high tensor powers of anti-ample line bundles. ☐

6.2. **Application to automorphic bundles.** Applying Theorem 6.1 to the Shimura variety and automorphic bundles, we immediately deduce:

Corollary 6.2. *Suppose $\mu \in \mathrm{X}^{+,<_{\mathrm{w}}p}_{\mathrm{G}_1}$ with $n := |\mu|_{\mathrm{L}}$, and $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Recall that $d = \dim_{\mathsf{S}_1}(\mathsf{M}_{\mathcal{H},1})$.* (*See Definition* 3.9.) *Suppose moreover that $|\mu|_{\mathrm{re}} = d+n < p$. Let $\mathcal{L}$ be an ample line bundle over $\mathsf{M}_{\mathcal{H},1}$. Let $\mathcal{L}_{\kappa_1} := \mathcal{L} \underset{R_1}{\otimes} \kappa_1$. Then we have:*

1. $H^i(\mathsf{M}_{\mathcal{H},\kappa_1}, \mathcal{L}_{\kappa_1} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\kappa_1}}}{\otimes} \mathrm{Gr}_{\mathsf{F}}(\underline{V}^\vee_{[\mu],\kappa_1} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\kappa_1}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},\kappa_1}/\mathsf{S}_{\kappa_1}})) = 0$ *for every $i > d$.*
2. $H^i(\mathsf{M}_{\mathcal{H},\kappa_1}, \mathcal{L}^\vee_{\kappa_1} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\kappa_1}}}{\otimes} \mathrm{Gr}_{\mathsf{F}}(\underline{V}^\vee_{[\mu],\kappa_1} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\kappa_1}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},\kappa_1}/\mathsf{S}_{\kappa_1}})) = 0$ *for every $i < d$.*

Definition 6.3. *We say $\mu \in \mathrm{X}^{+,<_{\mathrm{w}}p}_{\mathrm{G}_1}$ is p-***small for Illusie's theorem** *if $|\mu|_{\mathrm{re}} = d + |\mu|_{\mathrm{L}} < p$.* (*See Definition* 3.9.) *We write in this case that $\mu \in \mathrm{X}^{+,<_{\mathrm{re}}p}_{\mathrm{G}_1}$.*

6.3. **Reformulations using dual BGG complexes.** For any $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ (as in Definition 2.29), and for any R_1-algebra R, we define $\underline{W}_{\nu,R} := \mathcal{E}_{\mathrm{M}_1,R}(W_{\nu,R}) \cong \mathcal{E}_{\mathrm{P}_1,R}(W_{\nu,R})$ (see Lemma 1.20). For any $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$ and any $w \in \mathrm{W}^{\mathrm{M}_1}$, we define $W_{w\cdot[\mu],R} := \underset{\nu \in w\cdot[\mu]}{\oplus} W_{\nu,R}$, and define $W^\vee_{w\cdot[\mu],R}$ and $\underline{W}^\vee_{w\cdot[\mu],R}$ in the similar, obvious way.

For any integer $a \geq 0$, we denote by $\mathrm{W}^{\mathrm{M}_1}(a)$ the elements w in $\mathrm{W}^{\mathrm{M}_1}$ with length $l(w) = a$.

Theorem 6.4 (Faltings; cf. [11, §3], [13, Ch. VI, §5], and [37, §5])**.** *Let R be any R_1-algebra. For any $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$, there is an $\mathtt{F}$-filtered complex $\mathrm{BGG}^\bullet(\underline{V}^\vee_{[\mu],R})$, with trivial differentials on $\mathtt{F}$-graded pieces, such that*

$$\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}^a(\underline{V}^\vee_{[\mu],R})) \cong \underset{w \in \mathrm{W}^{\mathrm{M}_1}(a)}{\oplus} \underline{W}^\vee_{w\cdot[\mu],R}$$

as $\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}$-modules, together with a canonical quasi-isomorphic embedding

$$\mathrm{Gr}_{\mathtt{F}}(\mathrm{BGG}^\bullet(\underline{V}^\vee_{[\mu],R})) \hookrightarrow \mathrm{Gr}_{\mathtt{F}}(\underline{V}^\vee_{[\mu],R} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R})$$

(of complexes of $\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}$-modules) between $\mathtt{F}$-graded pieces.

If G_1 has no type D factors, then this is well known. The same method in [13, Ch. VI, §5] and [37, §5], using [42, Thm. D] as the main representation-theoretic input, carries over with little modification. However, after consulting Patrick Polo and after checking the details more carefully, we realize that the method involves only the (compatible) actions of P_1 and $\mathrm{Lie}(\mathrm{G}_1)$ (cf. Lemma 2.14), and that, if one use a simple variant of [42, Thm. A] instead of [42, Thm. D], the method also works when G_1 has type D factors. For more detailed explanations, see [30].

Corollary 6.5. *For any $\mu \in \mathrm{X}^{+,<p}_{\mathrm{G}_1}$ and any R_1-algebra R,*

$$H^i(\mathsf{M}_{\mathcal{H},R}, \mathrm{Gr}_{\mathtt{F}}(\underline{V}^\vee_{[\mu],R} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},R}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R})) \cong \underset{w \in \mathrm{W}^{\mathrm{M}_1}}{\oplus} H^{i-l(w)}(\mathsf{M}_{\mathcal{H},R}, \underline{W}^\vee_{w\cdot[\mu],R}). \tag{6.6}$$

Combining Corollary 6.2 and Theorem 6.4, we obtain:

Corollary 6.7. *Suppose $\mu \in \mathrm{X}^{+,<_{\mathrm{re}}p}_{\mathrm{G}_1}$ (see Definition 6.3), and $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Let $\mathcal{L}$ be an ample line bundle over $\mathsf{M}_{\mathcal{H},1}$. Let $\mathcal{L}_{\kappa_1} := \mathcal{L} \underset{R_1}{\otimes} \kappa_1$. Then, for any $w \in \mathrm{W}^{\mathrm{M}_1}$, we have:*

1. $H^{i-l(w)}(\mathsf{M}_{\mathcal{H},\kappa_1}, \mathcal{L}_{\kappa_1} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\kappa_1}}}{\otimes} \underline{W}^\vee_{w\cdot[\mu],\kappa_1}) = 0$ *for $i > d$.*
2. $H^{i-l(w)}(\mathsf{M}_{\mathcal{H},\kappa_1}, \mathcal{L}^\vee_{\kappa_1} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\kappa_1}}}{\otimes} \underline{W}^\vee_{w\cdot[\mu],\kappa_1}) = 0$ *for $i < d$.*

Clearly, Corollary 6.7 will be more useful if $\mathcal{L}$ is an *automorphic bundle* (in the sense of Definition 1.16). We shall investigate this possibility in Section 7.

7. Ample automorphic line bundles

7.1. Automorphic line bundles.

Definition 7.1. *Any weight $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ such that W_ν is a rank one free R_1-module is called a* **generalized parallel weight**. *We say in this case that $\underline{W}_\nu$ is an* **automorphic line bundle**. *For simplicity, we say ν is* **positive** *if the associated automorphic line bundle $\underline{W}_\nu$ is ample over $\mathsf{M}_{\mathcal{H},1}$.*

According to (2.8), we have $\mathrm{M}_1 \cong \Big(\prod_{\tau\in\Upsilon/c} \mathrm{M}_\tau\Big) \times (\mathbf{G}_{\mathrm{m}} \underset{\mathbb{Z}}{\otimes} R_1)$, with two possibilities for the factors M_τ:

(1) If $\tau = \tau\circ c$, then $\mathrm{M}_\tau \cong \mathrm{GL}_{p_\tau} \underset{\mathbb{Z}}{\otimes} R_1 = \mathrm{GL}_{r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$.

(2) If $\tau \neq \tau\circ c$, then $\mathrm{M}_\tau \cong (\mathrm{GL}_{p_\tau} \times \mathrm{GL}_{q_\tau}) \underset{\mathbb{Z}}{\otimes} R_1$.

This shows that:

Lemma 7.2. *The generalized parallel weights ν in $\mathrm{X}^{+,<p}_{\mathrm{M}_1}$ are exactly those $\nu = ((\nu_\tau)_{\tau\in\Upsilon/c};\nu_0) = (((\nu_{\tau,i_\tau})_{1\leq i_\tau\leq r_\tau})_{\tau\in\Upsilon/c};\nu_0)$ satisfying the following conditions:*

(1) *If $\tau = \tau\circ c$, then $\nu_\tau = k_\tau(1,1,\ldots,1)$, where $k_\tau \in \mathbb{Z}$.*

(2) *If $\tau \neq \tau\circ c$, then $\nu_\tau = k_\tau(1,1,\ldots,1,0,0,\ldots,0) - k_{\tau\circ c}(0,0,\ldots,0,1,1,\ldots,1)$, where $k_\tau, k_{\tau\circ c} \in \mathbb{Z}$, where the first term has 1's in the first q_τ entries, and where the second term has 1's in the last p_τ entries. (We place the minus sign in front of $k_{\tau\circ c}$ so that the value of ν_τ and $\nu_{\tau\circ c}$ are independent of the choice of representatives in Υ/c.)*

(There are no restrictions on the sizes of k_τ or $k_{\tau\circ c}$. Weights ν of the above form are all p-small.)

Definition 7.3. *The integers $(k_\tau)_{\tau\in\Upsilon}$ in Lemma 7.2 are called the* **coefficients** *of the generalized parallel weight ν.*

An important feature of a generalized parallel weight is that $W_\nu \underset{R_1}{\otimes} W^\vee_\mu \cong W^\vee_{\mu-\nu}$ and $W^\vee_\nu \underset{R_1}{\otimes} W^\vee_\mu \cong W^\vee_{\mu+\nu}$ for any $\mu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$. (Adding or subtracting a generalized parallel weight does not affect p-smallness of a weight in $\mathrm{X}^+_{\mathrm{M}_1}$.) Therefore, tensoring with an automorphic line bundle simply shifts the weight of an automorphic vector bundle.

Corollary 6.7 implies in particular that:

Corollary 7.4. *Suppose $\mu \in \mathrm{X}^{+,<_{\mathrm{re}}p}_{\mathrm{G}_1}$, and $\max(2,r_\tau) < p$ whenever $\tau = \tau\circ c$. Suppose $w \in \mathrm{W}^{\mathrm{M}_1}$, and $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ is a positive generalized parallel weight. Then we have:*

(1) *$H^{i-l(w)}(\mathsf{M}_{\mathcal{H},\kappa_1}, \underline{W}^\vee_{w\cdot[\mu]-\nu,\kappa_1}) = 0$ for every $i > d$.*

(2) *$H^{i-l(w)}(\mathsf{M}_{\mathcal{H},\kappa_1}, \underline{W}^\vee_{w\cdot[\mu]+\nu,\kappa_1}) = 0$ for every $i < d$.*

Changing our perspective a little bit:

Corollary 7.5. *Suppose $\mu \in \mathrm{X}^{+,<_{\mathrm{w}}p}_{\mathrm{G}_1}$, $w \in \mathrm{W}^{\mathrm{M}_1}$, and $\max(2,r_\tau) < p$ whenever $\tau = \tau\circ c$. Suppose that, for each $\mu' \in [\mu]$, there exist positive generalized parallel weights $\nu_+,\nu_- \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ such that the condition $\mu' \pm w^{-1}(\nu_\pm) \in \mathrm{X}^{+,<_{\mathrm{re}}p}_{\mathrm{G}_1}$ is satisfied. (The choices of $\nu_\pm$ may depend on μ'.) Then $H^{i-l(w)}(\mathsf{M}_{\mathcal{H},\kappa_1}, \underline{W}^\vee_{w\cdot[\mu],\kappa_1}) = 0$ for every $i \neq d$.*

Combining Corollaries 6.5 and 7.5, we obtain:

Theorem 7.6. *Suppose $\mu \in \mathrm{X}^{+,<_{\mathrm{w}}p}_{\mathrm{G}_1}$, and $\max(2,r_\tau) < p$ whenever $\tau = \tau\circ c$. Suppose that, for each $w \in \mathrm{W}^{\mathrm{M}_1}$ and each $\mu' \in [\mu]$, there exist positive generalized parallel weights $\nu_+,\nu_- \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ such that the condition $\mu' \pm w^{-1}(\nu_\pm) \in \mathrm{X}^{+,<_{\mathrm{re}}p}_{\mathrm{G}_1}$ is satisfied. Then we have $H^i(\mathsf{M}_{\mathcal{H},\kappa_1}, \mathrm{Gr}_{\mathsf{F}}(\underline{V}^\vee_{[\mu],\kappa_1} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\kappa_1}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},\kappa_1}/\mathsf{S}_{\kappa_1}})) = 0$ and $H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},\kappa_1}/\mathsf{S}_{\kappa_1}, \underline{V}^\vee_{[\mu],\kappa_1}) = 0$ for every $i \neq d$.*

Proof. The first statement follows from Corollaries 6.5 and 7.5. The second statement then follows from the Hodge to de Rham spectral sequence

$$E_1^{a,b} := H^{a+b}(\mathsf{M}_{\mathcal{H},1}, \mathrm{Gr}^a_{\mathsf{F}}(\underline{V}^\vee_{[\mu]} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1})) \Rightarrow H^{a+b}_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1, \underline{V}^\vee_{[\mu]}) \tag{7.7}$$

associated with the hypercohomology of filtered complexes. □

7.2. **Ampleness.** The most well-known (and perhaps the only known) way to produce ample automorphic line bundles is to use variants of the *Hodge line bundle*:

Proposition 7.8. *The line bundle* $\omega := \wedge^{\mathrm{top}} \underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_\mathcal{H}}$ *is ample over* $\mathsf{M}_\mathcal{H}$.

Proof. For the case of Siegel moduli schemes with principal levels at least 3, this is recorded in [38, IX, Thm. 3.1; cf. VII, Def. 4.3.3]. The case for $\mathsf{M}_\mathcal{H}$ can be deduced in two ways. The first way is, by replacing $\mathcal{H}$ with a finite index subgroup (which results in passing to a finite cover of $\mathsf{M}_\mathcal{H}$, which does not affect ampleness of line bundles), we may assume that there exists some finite forgetful morphism (defined by the universal polarized abelian scheme) from $\mathsf{M}_\mathcal{H}$ to a Siegel moduli scheme with principal level at least 3. The second way is to refer to [29, Thm. 7.2.4.1] (following and generalizing [13, Thm. 2.5]). □

Lemma 7.9. *The line bundle* ω *is isomorphic to* $\underline{W}_\nu$ *with coefficients* $(k_\tau)_{\tau\in\Upsilon}$ *of* ν *satisfying* $k_\tau = \mathrm{rk}_{R_1}(V_\tau)$. *(See Section* 2.1 *for the definition of* V_τ.*)*

Proof. This is because $\underline{\mathrm{Lie}}^\vee_{A/\mathsf{M}_{\mathcal{H},1}} \cong \underline{\mathrm{Lie}}^\vee_{A^\vee/\mathsf{M}_{\mathcal{H},1}} \cong \mathcal{E}_{\mathrm{M}_1,R}(L^\vee_{0,1})$ as vector bundles over $\mathsf{M}_{\mathcal{H},1}$ (ignoring Tate twists). (See Definition 1.13 and Example 1.22.) □

Proposition 7.10 (Correction of the originally published version)**.** *An automorphic line bundle* $\underline{W}_\nu$ *defines a torsion element in the Picard group of* $\mathsf{M}_{\mathcal{H},1}$ *if its coefficients* $(k_\tau)_{\tau\in\Upsilon}$ *of* ν *satisfy the condition that* $k_\tau + k_{\tau\circ c} = 0$ *for all* $\tau \in \Upsilon$.

Proof. Suppose that the condition in the proposition holds. Then the representation W_ν is trivial after pullback to the complexification of the maximal compact subgroup of $\mathrm{G}(\mathbb{R})$, and hence the pullback $\underline{W}_{\nu,\mathbb{C}}$ of $\underline{W}_\nu$ under any ring homomorphism $R_1 \to \mathbb{C}$ is trivial, by the comparison in [27, §5.2]. Suppose R is any discrete valuation ring finite flat over R_1 such that $K := \mathrm{Frac}(R)$ is Galois over $K_1 = \mathrm{Frac}(R_1)$, and such that the connected components of $\mathsf{M}_{\mathcal{H},K} = \mathsf{M}_{\mathcal{H},1} \underset{R_1}{\otimes} K$ are geometrically connected. Let k and ϖ denote the residue field and uniformizer of R, respectively. Let M to be any connected component of $\mathsf{M}_{\mathcal{H},1} \underset{R_1}{\otimes} R$, and let $\underline{W}$ denote the pullback of $\underline{W}_\nu$ to M. By taking norms with respect to the action of $\mathrm{Gal}(K/K_1)$, it suffices to show that $\underline{W}$ is trivial. Since the structural morphism $\mathsf{M}_\mathcal{H} \to \mathsf{S}_0 = \mathrm{Spec}(\mathcal{O}_{F_0,(p)})$ is proper and smooth, all fibers of $\mathsf{M} \to \mathrm{Spec}(R)$ are geometrically integral, so that $H^0(\mathsf{M}, \mathscr{O}_\mathsf{M}) \cong R$. Since $\underline{W}_{\nu,\mathbb{C}}$ is trivial, both $H^0(\mathsf{M}, \underline{W})$ and $H^0(\mathsf{M}, \underline{W}^\vee)$ are nonzero. Suppose s and t are nonzero elements of these two groups, respectively, whose product st defines an element of $H^0(\mathsf{M}, \mathscr{O}_\mathsf{M}) \cong R$. Let $V(s)$ (resp. $V(t)$) denote the closed subsets of M where the morphism $\mathscr{O}_\mathsf{M} \to \underline{W}$ (resp. $\underline{W} \to \mathscr{O}_\mathsf{M}$) defined by s (resp. t) fails to be an isomorphism. Suppose $st = \varpi r$ for some $r \in R$, so that $\mathsf{M} \underset{R}{\otimes} k \subset V(s) \cup V(t)$. Since $\mathsf{M} \underset{R}{\otimes} k$ is integral, either $\mathsf{M} \underset{R}{\otimes} k \subset V(s)$ and $s = \varpi s'$ for some $s' \in H^0(\mathsf{M}, \underline{W})$, or $\mathsf{M} \underset{R}{\otimes} k \subset V(t)$ and $t = \varpi t'$ for some $t' \in H^0(\mathsf{M}, \underline{W}^\vee)$. Up to replacing s with s' or t with t', and by repeating this process, we may assume that $st \in R^\times$, in which case $V(s) = \emptyset = V(t)$, and so $\underline{W}$ is trivial, as desired. □

Definition 7.11. *We say our linear algebraic data* $(\mathcal{O}, \star, L, \langle\cdot\,,\cdot\rangle, h_0)$ *is* $\mathbb{Q}$*-simple if* F *is a field, or equivalently if* $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ *is a simple algebra.*

Definition 7.12. *We say that two elements* $\tau, \tau' : \mathcal{O}_F \to R_1$ *in* $\Upsilon = \mathrm{Hom}_{\mathbb{Z}\text{-alg.}}(\mathcal{O}_F, R_1)$ *are equivalent over* $\mathbb{Q}$*, and write* $\tau \sim_{\mathbb{Q}} \tau'$*, if they factor through the same simple factor of* $\mathcal{O}_F$*. The equivalence class containing* τ *is denoted by* $[\tau]_{\mathbb{Q}}$*.*

Then our linear algebraic data is simple if and only if Υ has a single equivalence class under $\sim_{\mathbb{Q}}$.

Lemma 7.13. *If our linear algebraic data is simple, then* $\mathrm{rk}_{R_1}(V_\tau)$ *is a constant independent of* $\tau \in \Upsilon$*.*

Proof. Since we assumed that $\mathcal{O}_F$ is split over R_1, if our linear algebraic data is simple, then $\mathcal{O}_\tau$ is abstractly the same algebra over R_1 for all $\tau \in \Upsilon$. Hence $\mathrm{rk}_{R_1}(V_\tau)$ is a constant independent of τ, as desired. □

Definition 7.14. *We say that a generalized parallel weight* ν *with coefficients* $(k_\tau)_{\tau\in\Upsilon}$ *is* **parallel** *if* $[k]_\tau := k_\tau + k_{\tau\circ c}$ *satisfies* $[k]_\tau = [k]_{\tau'}$ *whenever* $\tau \sim_{\mathbb{Q}} \tau'$*.*

Proposition 7.15. *Let* ν *be a generalized parallel weight with coefficients* $(k_\tau)_{\tau\in\Upsilon}$*. Then the automorphic line bundle* $\underline{W}_\nu$ *over* $\mathsf{M}_{\mathcal{H},1}$ *is* **ample** *if it is parallel (as in Definition 7.14), and if all the numbers* $[k]_\tau$ *are positive.*

Proof. By decomposing F into simple factors over $\mathbb{Q}$, by decomposing our linear algebraic data accordingly, and by replacing $\mathcal{H}$ with a finite index subgroup (which is harmless as in the proof of Proposition 7.8), we may assume that there exists a finite morphism from $\mathsf{M}_{\mathcal{H},0}$ to a product of (base changes from possibly smaller rings of) analogous moduli problems defined by simple linear algebraic data. Since the conditions we listed respect this decomposition, we may assume that our moduli problem is defined by a simple linear algebraic data. By Proposition 7.8, Lemma 7.9, and Lemma 7.13, we know that an automorphic line bundle with coefficients $(k_\tau)_{\tau\in\Upsilon}$ is ample when k_τ is positive and independent of $\tau \in \Upsilon$. Then the result follows from Proposition 7.10. □

7.3. **Positive parallel weights of minimal size.** For each $\tau \in \Upsilon$, let $d_\tau := \dim_{R_1}(\mathrm{G}_\tau) - \dim_{R_1}(\mathrm{P}_\tau)$, and let $d_{[\tau]_{\mathbb{Q}}} := \max\limits_{\tau' \in [\tau]_{\mathbb{Q}}}(d_{\tau'})$. Note that $d_{[\tau]_{\mathbb{Q}}} = d_\tau$ whenever $\tau = \tau \circ c$.

Definition 7.16. *We say that a parallel weight* $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ *(as in Definition 7.14) is* **positive of minimal size** *if its coefficients* $(k_\tau)_{\tau\in\Upsilon}$ *satisfy the following conditions:*

(1) *If* $d_{[\tau]_{\mathbb{Q}}} = 0$*, then* $k_\tau = 0$*.*
(2) *If* $d_{[\tau]_{\mathbb{Q}}} > 0$ *and* $\tau = \tau \circ c$*, then* $k_\tau = 1$*.*
(3) *If* $d_{[\tau]_{\mathbb{Q}}} > 0$ *and* $\tau \neq \tau \circ c$*, then* $(k_\tau, k_{\tau\circ c})$ *is either* $(1, 0)$ *or* $(0, 1)$*.*

Using (2.5), we can say if a root $\alpha \in \Phi^+_{\mathrm{G}_1}$ comes from G_τ for some $\tau \in \Upsilon/c$.

Proposition 7.17. *Suppose* $\mu \in \mathrm{X}^+_{\mathrm{G}_1}$*, and suppose* $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ *is parallel and positive of minimal size as in Definition 7.16. Then the condition* $\mu' \pm w^{-1}(\nu) \in \mathrm{X}^+_{\mathrm{G}_1}$ *is satisfied for every* $\mu' \in [\mu]$ *and* $w \in \mathrm{W}^{\mathrm{M}_1}$ *if the following conditions are satisfied for all* $\alpha \in \Phi^+_{\mathrm{G}_1}$*:*

(1) *If α comes from G_τ such that $\tau = \tau \circ c$, then $(\mu', \alpha^\vee) \geq \min(|\alpha^\vee|, d_{[\tau]_\mathbb{Q}})$. (Here the norm $|\alpha^\vee|$ defined by the Killing form is at most 2.)*
(2) *If α comes from G_τ such that $\tau \neq \tau \circ c$, then $(\mu', \alpha^\vee) \geq \min(1, d_{[\tau]_\mathbb{Q}})$.*

Proof. If α comes from G_τ, then $(\mu', \alpha^\vee) = (\mu'_\tau, \alpha^\vee)$. If $\tau = \tau \circ c$, then $w^{-1}(\nu_\tau) = (\nu_{\tau,i_\tau})_{1 \leq i_\tau \leq r_\tau}$ has entries either ± 1 or 0. Hence $|(w^{-1}(\nu_\tau), \alpha^\vee)| \leq \min(|\alpha^\vee|, d_{[\tau]_\mathbb{Q}})$ for $\alpha \in \Phi^+_{\mathrm{G}_\tau}$. If $\tau \neq \tau \circ c$, then $w^{-1}(\nu_\tau)$ has entries either 0 or 1 (resp. either 0 or -1, resp. all 0) when the coefficients $(k_\tau)_{\tau \in \Upsilon}$ of ν (see Definition 7.3) satisfies $(k_\tau, k_{\tau \circ c}) = (1, 0)$ (resp. $(k_\tau, k_{\tau \circ c}) = (0, 1)$, resp. $(k_\tau, k_{\tau \circ c}) = (0, 0)$). Hence $|(w^{-1}(\nu_\tau), \alpha^\vee)| \leq \min(1, d_{[\tau]_\mathbb{Q}})$ for $\alpha \in \Phi^+_{\mathrm{G}_\tau}$. In both cases, we have $(\mu' \pm w^{-1}(\nu_\tau), \alpha^\vee) \geq (\mu', \alpha^\vee) - |(w^{-1}(\nu_\tau), \alpha^\vee)| \geq 0$, as desired. □

Definition 7.18. *We say that $\mu \in \mathrm{X}^+_{\mathrm{G}_1}$ is* **sufficiently regular** *if it satisfies the conditions (1) and (2) in Proposition 7.17. We shall denote the set of sufficiently regular elements in $\mathrm{X}^+_{\mathrm{G}_1}$ (resp. $\mathrm{X}^{+,<p}_{\mathrm{G}_1}$) by $\mathrm{X}^{++}_{\mathrm{G}_1}$ (resp. $\mathrm{X}^{++,<p}_{\mathrm{G}_1}$).*

Remark 7.19. If $\tau \neq \tau \circ c$ for all $\tau \in \Upsilon$, which implies that G_1 has only type A factors, then being regular implies being sufficiently regular.

Lemma 7.20. *Suppose that a weight $\mu \in \mathrm{X}^{++,<p}_{\mathrm{G}_1}$ satisfies $|\mu|_{\mathrm{re}} \leq p - \min(2, d)$, and that $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ is a positive parallel weight of minimal size. Then $(\mu' \pm w^{-1}(\nu) + \rho, \alpha^\vee) \leq p$ for any $w \in \mathrm{W}^{\mathrm{M}_1}$, any $\mu' \in [\mu]$, and any $\alpha \in \Phi^+_{\mathrm{G}_1}$.*

Proof. By Definition 3.9, $|\mu|_{\mathrm{re}} = d + |\mu|_{\mathrm{L}}$. By Definition 3.2, $|\mu|_{\mathrm{L}} = \sum_{\tau \in \Upsilon / c} |\mu''_\tau|$, where μ''_τ means μ'_τ in Section 3.3 (see in particular the explanation in Definition 3.2); we modified the notation here simply to avoid a conflict with the $\mu' \in [\mu]$ in the statement of this lemma. Since $d = \sum_{\tau \in \Upsilon / c} d_\tau$ with $d_\tau = \dim_{R_1}(\mathrm{G}_\tau) - \dim_{R_1}(\mathrm{P}_\tau)$, it suffices to prove the inequalities for each individual τ-factor.

If $\tau = \tau \circ c$, then $\mu''_{\tau,i_\tau} = \mu'_{\tau,i_\tau} \geq \mu''_{\tau,i_\tau+1} = |\mu'_{\tau,i_\tau+1}| \geq 0$ for every $1 \leq i_\tau < r_\tau$. The condition $|\mu|_{\mathrm{re}} \leq p - \min(2, d)$ implies that $d_\tau + |\mu''_\tau|_{\mathrm{L}} = d_\tau + \sum_{1 \leq i_\tau \leq r_\tau} \mu''_{\tau,i_\tau} = d_\tau + \sum_{1 \leq i_\tau \leq r_\tau} |\mu'_{\tau,i_\tau}| \leq p - \min(2, d)$. Therefore, $0 \leq (\mu'_\tau, \alpha^\vee) \leq \sum_{1 \leq i_\tau \leq r_\tau} |\mu'_{\tau,i_\tau}| \leq p - \min(2, d) - d_\tau$, and hence $(\mu'_\tau + \rho, \alpha^\vee) \leq p - \min(2, d)$ for any $\alpha \in \Phi^+_{\mathrm{G}_\tau}$, because $(\rho, \alpha^\vee) \leq d_\tau$. Then the result is true because $|(w^{-1}(\nu_\tau), \alpha^\vee)| \leq \min(|\alpha^\vee|, d_{[\tau]_\mathbb{Q}}) \leq \min(2, d)$. (We use sufficient regularity of μ when $\tau = \tau \circ c$ only to make sure that the condition $\mu' \pm w^{-1}(\nu) \in \mathrm{X}^+_{\mathrm{G}_1}$ is satisfied for every $w \in \mathrm{W}^{\mathrm{M}_1}$.)

If $\tau \neq \tau \circ c$, then the sufficient regularity of μ' implies that $\mu''_{\tau,i_\tau} = \mu'_{\tau,i_\tau} - \mu'_{\tau,r_\tau+1} \geq 0$ is a strictly decreasing sequence of integers for $1 \leq i_\tau \leq r_\tau$ except when $d_{[\tau]_\mathbb{Q}} = 0$. We may assume that $d_{[\tau]_\mathbb{Q}} \neq 0$, because otherwise $\nu_\tau = 0$ by Definition 7.16, in which case there is nothing to prove. Since $|\mu|_{\mathrm{re}} = d + |\mu|_{\mathrm{L}} < p$ and μ''_{τ,i_τ} is strictly decreasing (because $d_{[\tau]_\mathbb{Q}} \neq 0$), we have, for any $1 \leq a < b \leq r_\tau$, $(\mu''_{\tau,a} - \mu''_{\tau,b}) + \frac{1}{2}(b-a)(b-a-1) \leq \sum_{a \leq i_\tau \leq b} \mu''_{\tau,i_\tau} \leq |\mu'_\tau|_{\mathrm{L}} \leq p - 1 - d$, and hence $(\mu''_{\tau,a} - \mu''_{\tau,b}) + (b-a) \leq p - d$. This implies that $(\mu'_\tau + \rho, \alpha^\vee) = (\mu''_\tau + \rho, \alpha^\vee) \leq p - d$ for any $\alpha \in \Phi^+_{\mathrm{G}_\tau}$. Then the result follows from $|(w^{-1}(\nu_\tau), \alpha^\vee)| \leq \min(1, d_{[\tau]_\mathbb{Q}}) \leq \min(1, d)$. □

Proposition 7.21. *Suppose that a weight* $\mu \in \mathrm{X}^{++,<p}_{\mathrm{G}_1}$ *satisfies the condition*

$$|\mu|_{\mathrm{re},+} := |\mu|_{\mathrm{re}} + \sum_{\tau\in\Upsilon/c} \min(1, d_\tau)\max(p_\tau, q_\tau) < p. \tag{7.22}$$

Then μ *belongs to* $\mathrm{X}^{+,<_{\mathrm{w}}p}_{\mathrm{G}_1}$ *and satisfies the condition in Theorem* 7.6*; that is, for each* $w \in \mathrm{W}^{\mathrm{M}_1}$ *and each* $\mu' \in [\mu]$*, there exist positive parallel weights* $\nu_+, \nu_- \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ *such that the condition* $\mu' \pm w^{-1}(\nu_\pm) \in \mathrm{X}^{+,<_{\mathrm{re}}p}_{\mathrm{G}_1}$ *is satisfied.*

Proof. Under the condition (7.22), we claim that, for each $w \in \mathrm{W}^{\mathrm{M}_1}$ and each $\mu' \in [\mu]$, there exist positive parallel weights ν_+ and ν_- of minimal size such that:

(1) $(\mu' \pm w^{-1}(\nu_\pm) + \rho, \alpha^\vee) \leq p$ for all $\alpha \in \mathrm{X}^+_{\mathrm{G}_1}$.
(2) $|\mu'|_{\mathrm{re}} < p$ and $|\mu' \pm w^{-1}(\nu_\pm)|_{\mathrm{re}} < p$.
(3) $|\mu'|_{\mathrm{L}} < p$ and $|\mu' \pm w^{-1}(\nu_\pm)|_{\mathrm{L}} < p$.

Since p_τ and q_τ cannot be both zero when $d_\tau \geq 1$, the condition $|\mu'|_{\mathrm{re},+} < p$ implies $|\mu'|_{\mathrm{re}} \leq p - \min(2, d)$. Hence (1) follows from Lemma 7.20. Moreover, (3) follows from (2) because $|\mu'|_{\mathrm{re}} = d + |\mu'|_{\mathrm{L}}$.

Let us verify (2) by bounding $|\mu' \pm w^{-1}(\nu_\pm)|_{\mathrm{L}} - |\mu'|_{\mathrm{L}}$. As always, it suffices to prove the inequalities for each individual τ-factor. We may assume that $d_{[\tau]_\mathbb{Q}} > 0$, because otherwise $\nu_{\pm,\tau} = 0$. If $\tau = \tau \circ c$, then $p_\tau = q_\tau = r_\tau$ and $|\mu'_\tau \pm w^{-1}(\nu_{\pm,\tau})|_{\mathrm{L}} \leq |\mu'_\tau|_{\mathrm{L}} + r_\tau$, because r_τ entries in μ'_τ are added or subtracted by 1. If $\tau \neq \tau \circ c$, then the definition of $|\cdot|_{\mathrm{L}}$ (in Section 3.3) depends on the parity of the last entry. Since the two choices of positive parallel weights of minimal size have disjoint nonzero entries, we can choose $\nu_{\pm,\tau}$ such that $\mu'_\tau \pm w^{-1}(\nu_{\pm,\tau})$ have the same last entry as μ'_τ. Therefore, in the calculation of $|\mu'_\tau \pm w^{-1}(\nu_{\pm,\tau})|_{\mathrm{L}}$ and $|\mu'_\tau|_{\mathrm{L}}$, at most $\max(p_\tau, q_\tau)$ entries in μ'_τ are added or subtracted by 1. Hence $|\mu'_\tau \pm w^{-1}(\nu_{\pm,\tau})|_{\mathrm{L}} \leq |\mu'_\tau|_{\mathrm{L}} + \max(p_\tau, q_\tau)$, as desired. □

Remark 7.23. Although the number $\sum_{\tau\in\Upsilon/c} \min(1, d_\tau)\max(p_\tau, q_\tau)$ in (7.22) can be large, it depends only on the real group $\mathrm{G} \underset{\mathbb{Z}}{\otimes} \mathbb{R}$.

Lemma 7.24. *Suppose that* $\mu \in \mathrm{X}^+_{\mathrm{G}_1}$ *satisfies the condition* (7.22)*. Then* $\max(2, r_\tau) < p$ *whenever* $\tau = \tau \circ c$*.*

Proof. If $\mathrm{G}_\tau \cong \mathrm{Sp}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, then $\max(2, r_\tau) \leq \frac{1}{2} r_\tau(r_\tau + 1) + r_\tau = d_\tau + r_\tau \leq |\mu|_{\mathrm{re},+}$. If $\mathrm{G}_\tau \cong \mathrm{O}_{2r_\tau} \underset{\mathbb{Z}}{\otimes} R_1$, then $r_\tau \leq \frac{1}{2} r_\tau(r_\tau + 1) = d_\tau + r_\tau \leq |\mu|_{\mathrm{re},+}$ unless $d_\tau = 0$, in which case $r_\tau < 2$. On the other hand, $2 < p$ because we assume (see Section 1.1) that $p \neq 2$ if $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}$ involves simple factors of type D (as in [29, Def. 1.2.1.15]). Hence $\max(2, r_\tau) < p$ in all cases. □

8. Main results and consequences

8.1. De Rham and Hodge cohomology.

Theorem 8.1. *Suppose* $\mu \in \mathrm{X}^{++,<p}_{\mathrm{G}_1}$ *satisfies* $|\mu|_{\mathrm{re},+} < p$*. (See Definition* 7.18 *and* (7.22)*.) Then, for every* $i \neq d$*,* $H^i(\mathsf{M}_{\mathcal{H},\kappa_1}, \mathrm{Gr}_{\mathrm{F}}(\underline{V}^\vee_{[\mu],\kappa_1} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\kappa_1}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},\kappa_1}/\mathsf{S}_{\kappa_1}})) \cong \bigoplus_{w\in\mathrm{W}^{\mathrm{M}_1}} H^{i-l(w)}(\mathsf{M}_{\mathcal{H},\kappa_1}, \underline{W}^\vee_{w\cdot[\mu],\kappa_1}) = 0$ *and* $H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},\kappa_1}/\mathsf{S}_{\kappa_1}, \underline{V}^\vee_{[\mu],\kappa_1}) = 0$*.*

Proof. This follows from Theorem 7.6 (and its proof using Corollaries 6.5 and 7.5), Proposition 7.21, and Lemma 7.24. ☐

Theorem 8.2. *With the assumptions of Theorem* 8.1*, the Hodge to de Rham spectral sequence* (7.7) *degenerates at* E_1 *and defines the Hodge decomposition*

$$\mathrm{Gr}_{\mathsf{F}}(H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})) \cong \bigoplus_{w\in \mathrm{W}^{\mathrm{M}_1}} H^{i-l(w)}(\mathsf{M}_{\mathcal{H},R}, \underline{W}^\vee_{w\cdot[\mu],R})$$

for any R_1*-algebra* R*. The two sides are zero unless* $i = d$*, and each summand* $H^{i-l(w)}(\mathsf{M}_{\mathcal{H},R}, \underline{W}^\vee_{w\cdot[\mu],R})$ *on the right-hand side is a free* R*-module of finite rank that surjects onto* $H^{i-l(w)}(\mathsf{M}_{\mathcal{H},R}, \underline{W}^\vee_{w\cdot[\mu],\kappa_R})$*, where* $\kappa_R := R \underset{R_1}{\otimes} \kappa_1$*, under the canonical homomorphism* $R_1 \to \kappa_1$ *given by reduction modulo* p*.*

Proof. Let us begin with the case $R = R_1$. Then (6.6) gives a decomposition

$$H^i(\mathsf{M}_{\mathcal{H},1}, \mathrm{Gr}_{\mathsf{F}}(\underline{V}^\vee_{[\mu]} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},1}}}{\otimes} \Omega^\bullet_{\mathsf{M}_{\mathcal{H},1}/\mathsf{S}_1})) \cong \bigoplus_{w\in \mathrm{W}^{\mathrm{M}_1}} H^{i-l(w)}(\mathsf{M}_{\mathcal{H},1}, \underline{W}^\vee_{w\cdot[\mu]}).$$

Because $\mathsf{M}_{\mathcal{H},1} \to \mathsf{S}_1$ is proper and flat, and because the sheaves $\underline{W}^\vee_{w\cdot[\mu]}$ are locally free, the upper semi-continuity of dimensions of cohomology (cf. [39, §5, Cor. (a)]) and Theorem 8.1 show that the summands $H^{i-l(w)}(\mathsf{M}_{\mathcal{H},1}, \underline{W}^\vee_{w\cdot[\mu]})$ on the right-hand side are zero unless $i = d$. A similar semi-continuity argument (cf. [39, §5, Cor. 2]) proves that these summands are free and that they surject onto the similar cohomology groups over κ_1 when $i = d$. All the cohomology groups being free over R_1, these statements remain true after base change from R_1 to any R_1-algebra R. Finally, the degeneration of (7.7) is trivial because $E_1^{a,b} = 0$ whenever $a+b \neq d$. ☐

Corollary 8.3. *With the assumptions of Theorem* 8.1*, the following are true for any* R_1*-algebra* R*:*

1. $H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) = 0$ *for every* $i \neq d$*.*
2. $H^d_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R})$ *is a free* R*-module of finite rank.*
3. *The tensor product of the de Rham complex of* $\underline{V}^\vee_{[\mu]}$ *with the canonical short exact sequence* $0 \to pR \to R \to \kappa_R = R \underset{R_1}{\otimes} \kappa_1 \to 0$ *induces an exact sequence*

$$0 \to H^d_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, p(\underline{V}^\vee_{[\mu],R})) \to H^d_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^\vee_{[\mu],R}) \\ \to H^d_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},\kappa_R}/\mathsf{S}_{\kappa_1}, \underline{V}^\vee_{[\mu],\kappa_R}) \to 0.$$

Proof. By [23, Thm. 8.0], it suffices to treat the case $R = R_1$. We have already seen (1) in Theorem 8.2, but here is another argument to prove it and the other two statements. Since all terms in the long exact sequence associated with the short exact sequence in (3) are finitely generated R_1-modules, and since $H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},\kappa_1}/\mathsf{S}_{\kappa_1}, \underline{V}^\vee_{[\mu],\kappa_1}) = 0$ for all $i \neq d$ by Theorem 8.1, we obtain (1) by Nakayama's lemma. Then (2) and (3) follow tautologically. ☐

8.2. **Cohomological automorphic forms.** Let w_0 be the unique Weyl element in $\mathrm{W}_{\mathrm{M}_1}$ such that $w_0\Phi^+_{\mathrm{M}_1} = \Phi^-_{\mathrm{M}_1}$ and $W_\nu \cong W^\vee_{-w_0(\nu)}$ for any $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$.

Definition 8.4. *We say that a weight* $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ *is* **cohomological** *if there exist* $\mu \in \mathrm{X}^+_{\mathrm{G}_1}$ *and* $\mu' \in [\mu]$ *such that* $-w_0(\nu) = w \cdot \mu'$ *for some* $w \in \mathrm{W}^{\mathrm{M}_1}$*. (Here* w*,* μ'*, and hence* $[\mu]$ *are unique if they exist.) We write in this case that* $\mu' = \mu(\nu)$*,* $[\mu] = [\mu(\nu)]$*, and* $w = w(\nu)$*.*

Definition 8.5. *Let $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$. Let R be any R_1-algebra. An R-valued* **algebraic automorphic form of weight** *ν is an element of the graded R-module*

$$A^{\bullet}_{\nu}(\mathcal{H}; R) := H^{\bullet}(\mathsf{M}_{\mathcal{H},R}, \underline{W}_{\nu,R}).$$

It is convenient to also introduce, for any R_1-module E, the E-valued forms $A^{\bullet}_{\nu}(\mathcal{H}; E) := H^{\bullet}(\mathsf{M}_{\mathcal{H},1}, \underline{W}_{\nu} \underset{R_1}{\otimes} E)$. (This is compatible with Definition 8.5 when $E = R$.)

Proposition 8.6. *Let R be any R_1-algebra. If $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$ is cohomological and satisfies $\mu(\nu) \in \mathrm{X}^{++,<p}_{\mathrm{G}_1}$ and $|\mu(\nu)|_{\mathrm{re},+} < p$, then $A^{\bullet}_{\nu}(\mathcal{H}; R)$ is concentrated in degree $d - l(w(\nu))$, and $A^{d-l(w(\nu))}_{\nu}$ is a direct summand of $\mathrm{Gr}_{\mathrm{F}}(H^d_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},R}/\mathsf{S}_R, \underline{V}^{\vee}_{[\mu(\nu)],R}))$.*

Proof. This is a special case of Theorem 8.2. □

Theorem 8.7. *Let R be any R_1-module, and let $\kappa_R := R \underset{R_1}{\otimes} \kappa_1$. Let $\nu \in \mathrm{X}^{+,<p}_{\mathrm{M}_1}$. For simplicity, let us assume that $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Then $A^{\bullet}_{\nu}(\mathcal{H}; R)$ has the following properties:*

1. *If there exists a positive parallel weight ν_+ (resp. ν_-) such that $\nu - \nu_+$ (resp. $\nu + \nu_-$) is cohomological and $\mu(\nu - \nu_+) \in \mathrm{X}^{+,<_{\mathrm{re}}p}_{\mathrm{G}_1}$ (resp. $\mu(\nu + \nu_-) \in \mathrm{X}^{+,<_{\mathrm{re}}p}_{\mathrm{G}_1}$), then $A^i_{\nu}(\mathcal{H}; R) = 0$ for every $i > d - l(w(\nu - \nu_+))$ (resp. $i < d - l(w(\nu + \nu_-))$).*
2. *If R is flat over R_1, and if $A^{i-1}_{\nu}(\mathcal{H}; \kappa_1) = 0$ for some degree i, then $A^{i-1}_{\nu}(\mathcal{H}; R) = 0$ and $A^i_{\nu}(\mathcal{H}; R)$ is a free R-module of finite rank.*
3. *If $A^{i+1}_{\nu}(\mathcal{H}; R_1) = 0 = \mathrm{Tor}^{R_1}_1(A^{i+2}_{\nu}(\mathcal{H}; R_1), pR)$ for some degree i, then $A^{i+1}_{\nu}(\mathcal{H}; pR) = 0$ and the natural morphism $A^i_{\nu}(\mathcal{H}; R) \to A^i_{\nu}(\mathcal{H}; \kappa_R)$ induced by $R_1 \twoheadrightarrow \kappa_1$ is* **surjective***; in other words, any section of $A^i_{\nu}(\mathcal{H}; \kappa_R)$ is* **liftable***, in the sense that it is the reduction modulo p of some section in $A^i_{\nu}(\mathcal{H}; R)$. (The condition $\mathrm{Tor}^{R_1}_1(A^{i+2}_{\nu}(\mathcal{H}; R_1), pR) = 0$ holds, for example, when either $A^{i+2}_{\nu}(\mathcal{H}; R_1)$ or pR is flat over R_1. In particular, by (2), the full condition $A^{i+1}_{\nu}(\mathcal{H}; R_1) = 0 = \mathrm{Tor}^{R_1}_1(A^{i+2}_{\nu}(\mathcal{H}; R_1), pR)$ holds when $A^{i+1}_{\nu}(\mathcal{H}; \kappa_1) = 0$.)*
4. *If $A^{i-1}_{\nu}(\mathcal{H}; \kappa_1) = 0$ and $A^{i+1}_{\nu}(\mathcal{H}; R_1) = 0 = \mathrm{Tor}^{R_1}_1(A^{i+2}_{\nu}(\mathcal{H}; R_1), pR)$ for some degree i, then $A^i_{\nu}(\mathcal{H}; R)$ is a free R-module of finite rank, and we have a canonical exact sequence*

$$0 \to A^i_{\nu}(\mathcal{H}; pR) \to A^i_{\nu}(\mathcal{H}; R) \to A^i_{\nu}(\mathcal{H}; \kappa_R) \to 0.$$

Proof. Let us first treat the case $R = R_1$ (and hence $\kappa_R = \kappa_1$). The statements for $A^i_{\nu}(\mathcal{H}; \kappa_1)$ in (1) follows from a reformulation of Corollary 7.4, and the corresponding statements for $A^i_{\nu}(\mathcal{H}; R_1)$ follows from upper semi-continuity of dimensions of cohomology, as in the proof of Theorem 8.2. Then (2), (3), and (4) all follow from taking the long exact sequence induced by the canonical short exact sequence $0 \to \underline{W}_{\nu} \overset{[p]}{\to} \underline{W}_{\nu} \to \underline{W}_{\nu,\kappa_1} \to 0$, as in the proof of Corollary 8.3.

For a general R_1-algebra R, essentially by [39, §5, Thm.], there exists a bounded complex $\mathscr{L}$ whose components are free R_1-modules of finite type (a strictly perfect complex) such that $\mathscr{H}^i(\mathscr{L} \underset{R_1}{\otimes} E) = A^i_{\nu}(\mathcal{H}; E)$ for any R_1-module E, where $\mathscr{H}^i$ denotes the i-th cohomology of the complex. Consequently, since R_1 is a discrete valuation ring, we obtain an exact sequence (the "universal coefficients theorem")

$$0 \to A^i_\nu(\mathcal{H}; R_1) \underset{R_1}{\otimes} E \to A^i_\nu(\mathcal{H}; E) \to \mathrm{Tor}_1^{R_1}(A^{i+1}_\nu(\mathcal{H}; R_1), E) \to 0. \tag{8.8}$$

To show (1) and (2), we use the vanishing and the freeness statements we have already proved over R_1. For example, if $A^i_\nu(\mathcal{H}; R_1) = 0$ for every $i < d - l(w(\nu))$, then $A^{d-l(w(\nu))}_\nu(\mathcal{H}; R_1)$ is free over R_1, and consequently (8.8) with $E = R$ implies that $A^i_\nu(\mathcal{H}; R) = 0$ for every $i < d - l(w(\nu))$. To show (3), we take the cohomology long exact sequence attached to the canonical short exact sequence $0 \to p(\underline{W}_{\nu,R}) \to \underline{W}_{\nu,R} \to \underline{W}_{\nu,\kappa_R} \to 0$, and deduce the vanishing $A^{i+1}_\nu(\mathcal{H}; pR) = 0$ from (8.8) with $E = pR$. Finally, to show (4), we deduce the isomorphism $A^i_\nu(\mathcal{H}; R) \cong A^i_\nu(\mathcal{H}; R_1) \underset{R_1}{\otimes} R$ from (8.8) with $E = R$ (and the assumption that $A^{i+1}_\nu(\mathcal{H}; R_1) = 0$), and combine this with (2) and (3). □

Remark 8.9. One can show using Serre vanishing that, for any positive parallel weight ν_+, there exists an integer $N_0 \geq 0$ such that for all $N \geq N_0$ sections of $A^0_{\nu+N\nu_+}(\mathcal{H}; \kappa_R)$ (zero or not) are liftable to $A^0_{\nu+N\nu_+}(\mathcal{H}; R)$. However, Serre vanishing does not give an effective bound for N_0, and N_0 might have to increase with the level $\mathcal{H}$.

Remark 8.10 (cf. [32, Rem. 4.5]). One cannot expect the statements of Theorem 8.7 to be true for all weights, even for compact Picard modular surfaces. See [45, Thm. 3.4] for counterexamples to liftability of sections of $A^0_\nu(\mathcal{H}; \kappa_1)$ to $A^0_\nu(\mathcal{H}; R_1)$ with $\mu(\nu) = 0$ and $l(w(\nu)) = d$ (so for this ν there cannot be a positive parallel weight ν_+ such that $\nu - \nu_+$ is cohomological). Over such surfaces, there are global sections of the canonical bundle (the bottom Hodge piece of the de Rham cohomology with trivial coefficients) that cannot be lifted to characteristic zero. (The fact that the Hodge to de Rham spectral sequence degenerates by [9] does not help.) Similarly, there are nontrivial p-torsion Betti and étale cohomology classes.

8.3. Étale and Betti cohomology.

Let Λ be an integral domain, finite flat over the p-adic completion of R_1 (and hence finite flat over $\mathbb{Z}_p$). (See the second paragraph of Section 4.3.) Let $\Lambda_1 = \Lambda/p\Lambda$ (as in Section 5.2).

Lemma 8.11. *Suppose there is a* $\mu \in \mathrm{X}^{++}_{\mathrm{G}_1}$ *such that* $|\mu|_{\mathrm{re}} < p$. *Then* $2d < p$ *holds automatically.* (*See Proposition* 4.8.)

Proof. Since $|\mu|_{\mathrm{re}} = d + |\mu|_{\mathrm{L}}$, it suffices to show that $d_\tau \leq |\mu_\tau|_{\mathrm{L}}$ for any $\tau \in \Upsilon/c$. If $d_\tau = 0$, then this is obvious. Otherwise, since $\mu \in \mathrm{X}^{++}_{\mathrm{G}_1}$, we may assume that entries of μ'_τ are strictly decreasing integers for any $\mu' \in [\mu]$. If $\tau = \tau \circ c$, then $d_\tau \leq |\mu_\tau|_{\mathrm{L}}$. If $\tau \neq \tau \circ c$, then $d_\tau = p_\tau q_\tau \leq \frac{1}{2}(p_\tau + q_\tau)(p_\tau + q_\tau - 1) \leq |\mu_\tau|_{\mathrm{L}}$. □

Theorem 8.12. *Suppose that* $\mu \in \mathrm{X}^{++,<p}_{\mathrm{G}_1}$ *satisfies* $|\mu|_{\mathrm{re},+} < p$ *and* $|\mu|_{\mathrm{comp}} \leq p-2$ (*see Definition* 5.9). *Then the following are true:*

(1) $H^i_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1}) = 0$ *for every* $i \neq d$.
(2) $H^i_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]}) = 0$ *for every* $i \neq d$.
(3) $H^d_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]})$ *is a free* Λ*-module of finite rank.*
(4) *The canonical exact sequence* $0 \to p({}_{\text{ét}}\underline{V}^\vee_{[\mu]}) \to {}_{\text{ét}}\underline{V}^\vee_{[\mu]} \to {}_{\text{ét}}\underline{V}^\vee_{[\mu]} \underset{\Lambda}{\otimes} \Lambda_1 \to 0$ *induces an exact sequence*

$$0 \to H^d_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, p({}_{\text{ét}}\underline{V}^\vee_{[\mu]})) \to H^d_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu]}) \to H^d_{\text{ét}}(\mathsf{M}_{\mathcal{H},F_0^{\mathrm{ac}}}, {}_{\text{ét}}\underline{V}^\vee_{[\mu],\Lambda_1}) \to 0.$$

The same are true if we base change the coefficient Λ *to any* Λ*-algebra.*

Proof. As in the proof of Corollary 8.3, by taking the long exact sequence induced by the short exact sequence $0 \to p({}_{\text{ét}}\underline{V}^\vee_{[\mu]}) \to {}_{\text{ét}}\underline{V}^\vee_{[\mu]} \to {}_{\text{ét}}\underline{V}^\vee_{[\mu]} \underset{\Lambda}{\otimes} \Lambda_1 \to 0$, the statements (2), (3), and (4) all follow from (1). (The base change statement follows from the "universal coefficient theorem" for étale cohomology; cf. the proof of Theorem 8.7.) To prove (1), we may replace Λ with a domain finite flat over Λ and assume that the set $\Omega := \mathrm{Hom}_{\mathbb{Z}_p\text{-alg.}}(W, \Lambda)$ has cardinality $[F_0 : \mathbb{Q}]$, so that the results in Section 5.2 apply. By Lemmas 8.11 and 7.24, $|\mu|_{\mathrm{re},+} < p$ implies that $2d < p$ and that $\max(2, r_\tau) < p$ whenever $\tau = \tau \circ c$. Since $|\mu|_{\mathrm{comp}} \leq p - 2$, Proposition 5.8 applies for any i (from 0 to $2d$), and (1) follows from Theorem 8.1, as desired. □

Corollary 8.13. *Theorem* 8.12 *remains true if we replace the étale cohomology with the Betti cohomology* (*over the complex numbers instead of* F_0^{ac}).

Proof. This follows from Proposition 4.14. □

8.4. **Comparison with transcendental results.** Just as Deligne and Illusie deduced vanishing theorems of Kodaira type in characteristic zero from the vanishing statements in positive characteristic (see [9] and [22]), we now obtain *purely algebraic* proofs of (the crudest form of) certain vanishing theorems that have so far been proven only by transcendental methods.

Lemma 8.14. *Suppose* $\mu \in \mathrm{X}^{+,<\mathrm{w}p}_{\mathrm{G}_1}$, *and* $\max(2, r_\tau) < p$ *whenever* $\tau = \tau \circ c$. *Then* ${}_{\mathrm{B}}\underline{V}^\vee_{[\mu],\mathbb{C}}$ (*resp. the analytification of* $\underline{V}^\vee_{[\mu],\mathbb{C}}$) *over the analytification of* $\mathsf{M}_{\mathcal{H},\mathbb{C}}$ *can be canonically identified with the sheaf of locally constant* (*resp. holomorphic*) *sections of*

$$\mathrm{G}(\mathbb{Q})\backslash(\mathsf{X} \times V^\vee_{[\mu],\mathbb{C}}) \times \mathrm{G}(\mathbb{A}^\infty)/H \to \mathrm{G}(\mathbb{Q})\backslash\mathsf{X} \times \mathrm{G}(\mathbb{A}^\infty)/H,$$

so that ${}_{\mathrm{B}}\underline{V}^\vee_{[\mu],\mathbb{C}}$ *is canonically isomorphic to the sheaf of horizontal sections in the analytification of* $(\underline{V}^\vee_{[\mu],\mathbb{C}}, \nabla)$. *A similar statement holds for* $\nu \in \mathrm{X}^{+,<\mathrm{w}p}_{\mathrm{M}_1}$ *and* $\underline{W}^\vee_{\nu,\mathbb{C}}$, *and the identifications respect the Hodge filtrations.*

Proof. It suffices to verify this for $V^\vee_{[\mu],\mathbb{C}} = L \underset{\mathbb{Z}}{\otimes} \mathbb{C}$, together with its filtration defined by V_0^c in (1.2), which can be canonically identified with the relative H_1 of the universal abelian scheme, together with its Hodge filtration. Then the result follows because this is exactly how we identify PEL-type Shimura varieties (and their universal objects) with their complex versions, as explained in, e.g., [27, §2]. □

Corollary 8.15. *The objects* ${}_{\mathrm{B}}\underline{V}^\vee_{[\mu],\mathbb{C}}$, $\underline{V}^\vee_{[\mu],\mathbb{C}}$, *and* $\underline{W}^\vee_{\nu,\mathbb{C}}$ *in Lemma* 8.14 *can be defined independently of* p, *and we have a canonical isomorphism* $H^i_{\mathrm{B}}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{[\mu],\mathbb{C}}) \cong H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, \underline{V}^\vee_{[\mu],\mathbb{C}})$ *for each* i. *By abuse of language, we shall extend the definition of these objects to all dominant weights.*

Note that $\mathrm{X}^{++,<p}_{\mathrm{G}_\mathbb{C}} = \mathrm{X}^{++,<p}_{\mathrm{G}_1}$ has an unambiguous meaning for any valid choices of p and R_1. We shall write $\mathrm{G}_\mathbb{C}$ in place of G_1 in what follows in this subsection.

Theorem 8.16. *Suppose* $\mu \in \mathrm{X}^{++}_{\mathrm{G}_\mathbb{C}}$. *Then the following are true:*

(1) $H^i_{\mathrm{B}}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, {}_{\mathrm{B}}\underline{V}^\vee_{[\mu],\mathbb{C}}) = 0$ *for every* $i \neq d$.

(2) *The Hodge to de Rham spectral sequence for the de Rham cohomology of* $\underline{V}^{\vee}_{[\mu],\mathbb{C}}$ *degenerates at* E_1 *and defines by taking* $\mathrm{Gr}_{\mathtt{F}}$ *a Hodge decomposition*

$$\mathrm{Gr}_{\mathtt{F}}(H^i_{\mathrm{dR}}(\mathsf{M}_{\mathcal{H},\mathbb{C}}/\mathsf{S}_{\mathbb{C}}, \underline{V}^{\vee}_{[\mu],\mathbb{C}})) \cong H^i(\mathsf{M}_{\mathcal{H},\mathbb{C}}, \mathrm{Gr}_{\mathtt{F}}(\underline{V}^{\vee}_{[\mu],\mathbb{C}} \underset{\mathscr{O}_{\mathsf{M}_{\mathcal{H},\mathbb{C}}}}{\otimes} \Omega^{\bullet}_{\mathsf{M}_{\mathcal{H},\mathbb{C}}/\mathsf{S}_{\mathbb{C}}})) \cong \underset{w\in \mathrm{W}^{\mathrm{M}_{\mathbb{C}}}}{\oplus} H^{i-l(w)}(\mathsf{M}_{\mathcal{H},\mathbb{C}}, \underline{W}^{\vee}_{w\cdot[\mu],\mathbb{C}}).$$

Combining (1) *with* (2)*, we see that every summand on the right-hand side is zero when* $i \neq d$*.*

Proof. By Corollary 8.15, we can choose a good prime p (see Section 1.1) so large that $\mu \in \mathrm{X}^{++,<p}_{\mathrm{G}_{\mathbb{C}}}$ and $|\mu|_{\mathrm{re},+} < p$. Then the results follow from Theorem 8.2. □

Remark 8.17*.* To the best of our knowledge, the simplest (analytic) proof of Theorem 8.16 is given by Faltings in [11], using his construction of dual BGG complexes (based on older ideas in [3]). It is perhaps not a coincidence that our method uses this BGG idea as well. However, the proof in [11] uses C^∞-resolutions of vector bundles and harmonic forms, and as such looks inadequate for dealing with torsion coefficients. In this sense, the (purely algebraic, characteristic $p > 0$) theory developed by Deligne and Illusie is as indispensable in our proof as Hodge theory is in that of Faltings.

Remark 8.18*.* A more general theory of vanishing theorems from the perspective of automorphic representations and group cohomology of arithmetic groups (for general reductive groups) has a good modern summary in [33, §2], with major inputs from [46], and with some updates in [34] (concerning Eisenstein cohomology classes absent in our compact case).

Remark 8.19*.* In works mentioned in Remarks 8.17 and 8.18, it suffices to assume that μ is *regular*, a weaker (and hence better) condition than ours when $\mathrm{G}_{\mathbb{C}}$ has factors of types C or D. (See Remark 7.19.) This is a fundamental restriction of our technique, relying on the positive parallel weights of minimal size.

Similarly (to the case of G_1), $\mathrm{X}^+_{\mathrm{M}_{\mathbb{C}}} = \mathrm{X}^+_{\mathrm{M}_1}$ has an unambiguous meaning, and we shall write $\mathrm{M}_{\mathbb{C}}$ in place of M_1 in the remainder of this subsection.

We can extend the definition of $A^{\bullet}_{\nu}(\mathcal{H},\mathbb{C})$ to all $\nu \in \mathrm{X}^+_{\mathrm{M}_{\mathbb{C}}}$, and deduce from Theorem 8.7 that:

Theorem 8.20. *If there exists a positive parallel weight* ν_+ *(resp.* ν_-*) such that* $\nu-\nu_+$ *(resp.* $\nu+\nu_-$*) is cohomological and* $\mu(\nu-\nu_+) \in \mathrm{X}^+_{\mathrm{G}_{\mathbb{C}}}$ *(resp.* $\mu(\nu+\nu_-) \in \mathrm{X}^+_{\mathrm{G}_{\mathbb{C}}}$*), then* $A^i_{\nu}(\mathcal{H};\mathbb{C}) = 0$ *for every* $i > d - l(w(\nu-\nu_+))$ *(resp.* $i < d - l(w(\nu+\nu_-))$*).*

Remark 8.21*.* When ν is cohomological and $\mu(\nu)$ is regular, the simplest analytic result is the same work of Faltings mentioned in Remark 8.17.

In the general non-compact case, there is a much longer story for analytic results on vanishing. We defer such discussions to [31], where we will present their algebraic (and torsion) analogues.

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